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The polynomial closure is not topological

Giulio Peruginelli, Dario Spirito *,[1]

Dipartimento di Matematica, Università di Padova, Padova, Italy

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We characterize the polynomial closure of a pseudo-convergent sequence in a valuation domain V of arbitrary rank, and then we use this result to show that the polynomial closure is never topological when V has rank at least 2.

1. Introduction

Let D be an integral domain with quotient field K and let $S \subseteq K$ be a subset. The ring of *integer-valued polynomial* over S is

$$\mathrm{Int}(S, D) := \{f \in K[X] \mid f(S) \subseteq D\}.$$

The *polynomial closure* of S, denoted by $\overline{S}$, is the largest subset of K for which the equality $\mathrm{Int}(S, D) = \mathrm{Int}(\overline{S}, D)$ holds, and a subset S is *polynomially closed* if $S = \overline{S}$.

Chabert studied in [3] conditions under which the polynomial closure is topological, i.e., when there is a topology on K whose closure operator is the polynomial closure; he showed that for this to happen D must be a local domain, and $D = V$ a valuation domain of rank 1 is a sufficient condition. The purpose of this paper is to complement the latter result by showing that, when V is a valuation domain of rank bigger than 1, the polynomial closure is *never* topological.

* Corresponding author.
E-mail addresses: gperugin@math.unipd.it (G. Peruginelli), spirito@math.unipd.it, dario.spirito@uniud.it (D. Spirito).
[1] *Current address:* Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Università di Udine, Udine, Italy.

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We prove this result by means of the subsets that Chabert used for his own. Indeed, Chabert described the polynomial closure of a generic subset S of V by using *pseudo-convergent sequences*, originally introduced by Ostrowski to study extensions of valued fields [6] and later used by Kaplansky in the study of maximal fields [5] (see below for the definitions), as well as new related classes of *pseudo-divergent* and *pseudo-stationary* sequences which he introduced; more precisely, he showed that the polynomial closure of S can be described by adding all the *pseudo-limits* of the sequences of these kinds contained in S [3, Theorem 5.2]; these three types of sequences can also be used to generalize the work of Ostrowski [8]. In this paper, we completely describe the polynomial closure of a pseudo-convergent sequence for valuation domains of arbitrary rank; this will allow to show that, for some explicitly constructed pseudo-convergent sequence $E := \{s_n\}_{n\in\mathbb{N}}$, we have $\overline{E} \neq \overline{\{s_1\}} \cup \overline{E \setminus \{s_1\}}$, and thus that the polynomial closure is not topological.

Throughout the article, we assume that V is a valuation domain with quotient field K. We denote by v the valuation associated to V and by Γ_v the value group of V. We denote by M the maximal ideal of V. The *rank* of V is the rank of its value group, which is equal to the Krull dimension of V.

Let $E := \{s_n\}_{n\in\mathbb{N}}$ be a sequence of elements of K. We say that E is a *pseudo-convergent sequence* if the sequence $\boldsymbol{\delta}(E) := \{\delta_n := v(s_{n+1} - s_n)\}_{n\in\mathbb{N}} \subseteq \Gamma_v$ (called the *gauge* of E) is strictly increasing. The *breadth ideal* of E is

$$\mathrm{Br}(E) := \{x \in K \mid v(x) > \delta_n \text{ for all } n \in \mathbb{N}\};$$

the breadth ideal is always a fractional ideal of V. An element $\alpha \in K$ is a *pseudo-limit* of E if $v(\alpha - s_n) = \delta_n$ for all $n \in \mathbb{N}$; we denote the set of pseudo-limits of E by $\mathcal{L}_E$. If $\mathcal{L}_E$ is nonempty, then $\mathcal{L}_E = \alpha + \mathrm{Br}(E)$ for any pseudo-limit α ([5, Lemma 3]). We note that, in general, pseudo-convergent sequences can be indexed by any well-ordered set Λ but that for our purposes it suffices to consider only those indexed by $\mathbb{N}$ (see Remark 2.5).

2. The polynomial closure of a pseudo-convergent sequence

The following lemma shows that, given a pseudo-convergent sequence $E = \{s_n\}_{n\in\mathbb{N}} \subset K$, an element $t \in K$ can be close to at most one of the elements of E (with respect to the gauge).

Lemma 2.1. *Let $E := \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence with gauge $\{\delta_n\}_{n\in\mathbb{N}}$, and let $t \in K$. Then, $v(s_n - t) \leq \delta_n$ for all but at most one $n \in \mathbb{N}$.*

Proof. Suppose $v(s_n - t) > \delta_n$, and let $s_m \in E$. If $m < n$, then

$$v(s_m - t) = v(s_m - s_n + s_n - t) = \delta_m$$

since $v(s_m - s_n) = \delta_m < \delta_n < v(s_n - t)$; on the other hand, if $m > n$ then

$$v(s_m - t) = v(s_m - s_n + s_n - t) = \delta_n < \delta_m$$

since $v(s_m - s_n) = \delta_n < v(s_n - t)$. The claim is proved. □

Lemma 2.2. *Let $I \subset M \subset V$ be an ideal. Then, the largest prime ideal contained in I is equal to*

$$\bigcap_{t\notin I, n\geq 1} t^n V$$

Proof. Let $P(I) := \bigcap_{t \notin I, n \geq 1} t^n V$. Then, $P(I)$ is a prime ideal by [4, Theorem 17.1(3)]. If $\alpha \in P(I) \setminus I$, then $\alpha \in \alpha^n V$ for every n, which is not possible (unless α is a unit, which we can exclude since $I \subset M$). This shows that $P(I) \subseteq I$.

Let $Q \subseteq I$ be a prime ideal. If for some $t \notin I$ there exists $n \in \mathbb{N}$ such that $t^n \in Q$ then $t \in Q \subseteq I$, a contradiction. Thus $Q \subseteq P(I)$, and $P(I)$ is the largest prime ideal contained in I. □

The previous lemma can also be rephrased by saying that $x \in P(I)$ if and only if $v(x) > nv(t)$ for all $t \in V \setminus I$ and all $n \in \mathbb{N}$.

Proposition 2.3. *Let $E := \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence with gauge $\{\delta_n\}_{n\in\mathbb{N}}$; let $c_n := s_{n+1} - s_n$. Let $\alpha \in K$ and take any $k \in \mathbb{N}$; let P_k be the largest prime ideal contained in $c_k^{-1}\mathrm{Br}(E)$. Then the following are equivalent:*

(i) $v(\alpha - s_k) > \lambda(\delta_r - \delta_k) + \delta_k$ for every $r \geq k$ and every $\lambda \in \mathbb{N}$;
(ii) $\alpha \in s_k + c_k P_k$.

Proof. Let $\beta := \frac{\alpha - s_k}{c_k}$; then, $v(\beta) = v(\alpha - s_k) - \delta_k$, and thus we have to show that $\beta \in P_k$ if and only if $v(\beta) > \lambda(\delta_r - \delta_k)$ for every $\lambda \in \mathbb{N}$ and $r \geq k$.

The sequence $F := c_k^{-1}E = \{c_k^{-1}s_n\}_{n\in\mathbb{N}}$ is pseudo-convergent with gauge $\{\delta_n - \delta_k\}_{n\in\mathbb{N}}$, and thus $\mathrm{Br}(F) = c_k^{-1}\mathrm{Br}(E) \subsetneq V$. Hence, by Lemma 2.2, $\beta \in P_k$ if and only if $\beta \in t^\lambda V$ for every $t \in V \setminus \mathrm{Br}(F)$ and every $\lambda \in \mathbb{N}$. By definition, this is equivalent to $v(\beta) > \lambda(\delta_r - \delta_k)$ for every $r, \lambda \in \mathbb{N}$. Hence, the two conditions are equivalent. □

The following lemma is essentially [3, Proposition 4.8]; we prove it explicitly to show that it holds without any hypothesis on the rank of V.

Lemma 2.4. *Let $E := \{s_n\}_{n\in\mathbb{N}}$ be a pseudo-convergent sequence. Then, $\mathcal{L}_E \subseteq \overline{E}$.*

Proof. Let $\alpha \in \mathcal{L}_E$, and let $f \in \mathrm{Int}(E, V)$; we can write it as $f(X) = \sum_j a_j(X - \alpha)^j$. By the proof of [7, Proposition 3.7], there is a k such that, for all large n, $v(f(s_n)) = v(a_k(s_n - \alpha)^k) < v(a_j(s_n - \alpha)^j)$ for all $j \neq k$. Since $v(f(s_n)) \geq 0$ for all n, it follows that $v(f(\alpha)) = v(a_0) \geq 0$. Hence $\alpha \in \overline{E}$. □

Remark 2.5. The previous lemma also shows why, in this context, it is enough to consider pseudo-convergent sequences indexed by $\mathbb{N}$. Indeed, let $E := \{s_\nu\}_{\nu\in\Lambda}$ be a pseudo-convergent sequences indexed by a well-ordered set Λ, and let E_{in} be the subsequence $\{s_n\}_{n\in\mathbb{N}}$: then, E_{in} is again pseudo-convergent. Let $\nu \in \Lambda \setminus \mathbb{N}$. Then, $s_\nu \in \mathcal{L}_{E_{\mathrm{in}}} \subseteq \overline{E_{\mathrm{in}}}$, and thus $\overline{E} = \overline{E_{\mathrm{in}}}$; hence, we do not lose anything by considering only $\overline{E_{\mathrm{in}}}$.

For each $n \in \mathbb{N}$, consider the polynomial

$$H_n(X) := \prod_{i=0}^{n-1} \frac{X - s_i}{s_n - s_i}.$$

Note that for each n, $H_n(s_j)$ is zero for $j < n$ and is a unit of V for $j \geq n$, as $v(s_j - s_i) = \delta_i = v(s_n - s_i)$ when $j \geq n > i$. In particular, these polynomials are integer-valued on E, and thus by [2, Proposition 20] they form a *regular basis* for $\mathrm{Int}(E, V)$, that is, a basis for the V-module $\mathrm{Int}(E, V)$ such that $\deg(H_n) = n$ for each $n \in \mathbb{N}$. In particular, an element $\alpha \in K$ is in $\overline{E}$ if and only if $H_n(\alpha) \in V$ for all $n \in \mathbb{N}$.

Theorem 2.6. *Let $E := \{s_n\}_{n\in\mathbb{N}}$ be a pseudo-convergent sequence with gauge $\{\delta_n\}_{n\in\mathbb{N}}$; let $c_n := s_{n+1} - s_n$. Then,*

$$\overline{E} = \mathcal{L}_E \cup \bigcup_{n \geq 1} (s_n + c_n P_n), \tag{1}$$

where P_n is the largest prime ideal contained in $c_n^{-1}\mathrm{Br}(E)$. Furthermore, the union is disjoint.

Proof. Suppose $\alpha \in \overline{E}$.

If $v(\alpha - s_n) = \delta_n$ for every n then $\alpha \in \mathcal{L}_E$, and in particular it is contained in the right hand side of (1). Suppose that is not the case: we distinguish two possibilities.

Suppose that $v(\alpha - s_n) \leq \delta_n$ for every $n \in \mathbb{N}$ and that k is the smallest index for which $v(\alpha - s_k) < \delta_k$; in particular, $v(\alpha - s_i) = \delta_i$ for all $i < k$. We have

$$v(H_{k+1}(\alpha)) = \sum_{i=0}^{k} v(\alpha - s_i) - \sum_{i=0}^{k} \delta_i = v(\alpha - s_k) - \delta_k < 0$$

a contradiction with the fact that $\alpha \in \overline{E}$.

Suppose now that $v(\alpha - s_k) > \delta_k$ for some k; by Lemma 2.1 this k is unique, and for all the other indexes we have

$$v(\alpha - s_i) = v(\alpha - s_k + s_k - s_i) = \begin{cases} \delta_i, & \text{if } i < k \\ \delta_k, & \text{if } i > k \end{cases} \tag{2}$$

In particular, $v(H_{k+1}(\alpha)) = v(\alpha - s_k) - \delta_k > 0$ and if $n > k+1$ by (2) we have

$$\begin{aligned} v(H_n(\alpha)) &= \sum_{i=0}^{k-1}(\delta_i - \delta_i) + v(\alpha - s_k) - \delta_k + \sum_{i=k+1}^{n-1} (\delta_k - \delta_i) = \\ &= v(\alpha - s_k) - \delta_k + \sum_{i=k+1}^{n-1} (\delta_k - \delta_i). \end{aligned} \tag{3}$$

Let now $\lambda, m \in \mathbb{N}$, $m \geq k$, be fixed. Choose n so that $n - m > \lambda$. In particular, $\sum_{i=k+1}^{n-1} \delta_i > \lambda\delta_m$. Hence, by (3) and the fact that $H_n(\alpha) \in V$ we have

$$v(\alpha - s_k) - \delta_k \geq \sum_{i=k+1}^{n-1} (\delta_i - \delta_k) > \lambda(\delta_m - \delta_k)$$

Since λ, m are arbitrary, by Proposition 2.3 it follows that $\alpha \in s_k + c_k P_k$, as we wanted to show.

Let now α be in the right hand side of (1). If $\alpha \in \mathcal{L}_E$ then $\alpha \in \overline{E}$ by Lemma 2.4. Suppose that $\alpha \notin \mathcal{L}_E$ and $\alpha \in s_k + c_k P_k$ for some $k \geq 1$: then by Proposition 2.3 $v(\alpha - s_k) > \lambda(\delta_r - \delta_k) + \delta_k$ for every $r \geq k$ and every $\lambda \in \mathbb{N}$.

In order to show that $\alpha \in \overline{E}$, it is enough to prove that $H_n(\alpha) \in V$ for all $n \in \mathbb{N}$.

If $n \leq k$, then by (2) we have

$$v(H_n(\alpha)) = \sum_{i=0}^{n-1}(\delta_i - \delta_i) = 0.$$

For $n = k+1$ we have $v(H_{k+1}(\alpha)) > 0$ as we remarked above.

Suppose now that $n > k + 1$. Then by (3) we have

$$v(H_n(\alpha)) > v(\alpha - s_k) - \delta_k + \sum_{i=k+1}^{n-1} (\delta_k - \delta_{n-1}) =$$
$$= v(\alpha - s_k) - \delta_k + (n - k - 1)(\delta_k - \delta_{n-1})$$

and the last quantity is greater than zero by assumption. Hence, $\alpha \in \overline{E}$.

We conclude the proof of the theorem by showing the last claim. For every $t \in s_k + c_k P_k$, we have $t - s_k \in c_k P_k$ and in particular $v(t - s_k) > v(c_k) = \delta_k$. In particular, no such t can be a pseudo-limit of E (since otherwise $v(t - s_k) = \delta_k$ for every k). Moreover, if $t \in (s_k + c_k P_k) \cap (s_{k'} + c_{k'} P_{k'})$ for some $k' > k$, then we should have at the same time $v(t - s_k) > \delta_k$ and $v(t - s_{k'}) > \delta_{k'}$, in contradiction with Lemma 2.1. Hence the union is disjoint, as claimed. □

As a consequence of Theorem 2.6, we have the main result of the paper.

Theorem 2.7. *Let V be a valuation domain of rank > 1. Then, the polynomial closure is not a topological closure.*

Proof. Since V has rank bigger than 1, there is a nonmaximal prime ideal P'; if $t \in V \setminus P'$ is a nonunit, then the largest prime ideal P (strictly) contained in tV is different from the zero ideal. Let $E := \{t^n\}_{n\in\mathbb{N}}$ and let $E' := \{t^n\}_{n\geq 2}$. Then, E and E' are pseudo-convergent sequences with breadth ideal $\mathrm{Br}(E) = \mathrm{Br}(E') = P$ and with $\mathcal{L}_E = \mathcal{L}_{E'}$. Moreover, for every n, we have $(t^{n+1} - t^n)^{-1} P = P$.

By Theorem 2.6, it follows that

$$\overline{E} = \mathcal{L}_E \cup \bigcup_{k\geq 1} (t^k + P) = \overline{E'} \cup (t + P);$$

moreover, the first union is disjoint, and so $(t + P) \cap \overline{E'} = \emptyset$. If the polynomial closure were topological, we would have $\overline{E} = \overline{E' \cup \{t\}} = \overline{E'} \cup \overline{\{t\}} = \overline{E} \cup \{t\}$ (since finite sets are polynomially closed [1, Chapter IV, Example IV.1.3]); however, for every $p \in P \setminus \{0\}$, the element $t + p$ is in $\overline{E}$ but not in $\overline{E'} \cup \overline{\{t\}}$. Thus, the polynomial closure is not topological, as claimed. □

To conclude the paper, we show when two pseudo-convergent sequences have the same polynomial closure.

Proposition 2.8. *Let $E := \{s_n\}_{n\in\mathbb{N}}$ and $F := \{t_n\}_{n\in\mathbb{N}}$ be two pseudo-convergent sequences with gauges $\boldsymbol{\delta}(E) := \{\delta_n\}_{n\in\mathbb{N}}$ and $\boldsymbol{\delta}(F) := \{\eta_n\}_{n\in\mathbb{N}}$, respectively. Then, $\overline{E} = \overline{F}$ if and only if $\delta_t = \eta_t$ for every t and $v(t_k - s_k) > \lambda(\delta_r - \delta_k) + \delta_k$ for every $k \geq r$ and every $\lambda \in \mathbb{Z}$.*

Proof. By Theorem 2.6, we can write

$$\overline{E} = \mathcal{L}_E \cup \bigcup_{k\in\mathbb{N}} (s_k + c_k P_k)$$

and

$$\overline{F} = \mathcal{L}_F \cup \bigcup_{k\in\mathbb{N}} (t_k + d_k Q_k)$$

for some $c_k, d_k \in K$ and prime ideals P_k, Q_k defined as in the theorem. Let $S_k := s_k + c_k P_k$ and $T_k := t_k + d_k Q_k$.

Suppose that the two conditions of the statement hold. Then, by Proposition 2.3, for every k we have $v(c_k) = v(d_k)$, $P_k = Q_k$ and $s_k - t_k \in c_k P_k$, so that $S_k = T_k$. Furthermore, $s_k - t_k \in \mathrm{Br}(E)$ for every k, and thus E and F are equivalent in the sense of [7, Section 5], so $\mathcal{L}_E = \mathcal{L}_F$ by [7, Lemma 5.3] and $\overline{E} = \overline{F}$.

Conversely, suppose $\overline{E} = \overline{F}$. Let $x, y \in \overline{E}$: then

- if $x, y \in S_k$ then $x - y \in c_k P_k \subseteq \mathrm{Br}(E)$;
- if $x \in S_k$ and $y \in S_j$ for $k < j$ then $v(x - y) = \delta_k$;
- if $x \in S_k$ and $y \in \mathcal{L}_E$ then $v(x - y) = \delta_k$;
- if $x, y \in \mathcal{L}_E$ then $x - y \in \mathrm{Br}(E)$.

Let $D(E) := \{v(x - y) \mid x, y \in \overline{E}\}$: then, $D(E) = \boldsymbol{\delta}(E) \cup X_E$, where X_E is an up-closed subset of $\Gamma_v \cup \{\infty\}$ (more precisely, $X = v(\mathrm{Br}(E))$ if $\mathcal{L}_E$ has at least two elements, while $X = \bigcup_i v(c_i P_i)$ otherwise). Analogously, $D(F) = \boldsymbol{\delta}(F) \cup X_F$.

If $\overline{E} = \overline{F}$, then $D(E) = D(F)$. Since X_E is the largest up-closed subset of $D(E)$ (and analogously for $D(F)$), we must have $\boldsymbol{\delta}(E) = \boldsymbol{\delta}(F)$; since the gauges are linearly ordered, it must be $\delta_n = \eta_n$ for every $n \in \mathbb{N}$. In particular, $\mathrm{Br}(E) = \mathrm{Br}(F)$ and $v(c_k) = v(d_k)$ for every k; thus, $P_k = Q_k$ for every k.

Therefore, to prove the statement we only need to show that $s_k \in T_k$ for every k. For $y \in \overline{E}$, let $D(E, y) := \{v(x - y) \mid x \in \overline{E}\}$: then, with the same reasoning as above, we see that

$$D(E, y) = \begin{cases} \{\delta_1, \dots, \delta_n\} \cup X_{E,y} & \text{if } y \in S_k \\ D(E) & \text{if } y \in \mathcal{L}_E, \end{cases}$$

where $X_{E,y}$ is an up-closed subset of $\Gamma_v \cup \{\infty\}$. Clearly, $D(E, y) = D(F, y)$; in particular, $D(F, s_k) = \{\delta_1, \dots, \delta_k\} \cup X_{E,s_k}$, and thus it must be $s_k \in T_k$. The claim is proved. □

When V has rank 1, the two previous propositions have a very simplified form, which can also be obtained from Chabert's paper [3].

Corollary 2.9. *Let V be a valuation ring of rank 1, and let $E := \{s_n\}_{n \in \mathbb{N}}$ and $F := \{t_n\}_{n \in \mathbb{N}}$ be two pseudo-convergent sequences. Then:*

(a) $\overline{E} = E \cup \mathcal{L}_E$;
(b) $\mathrm{Int}(E, V) = \mathrm{Int}(F, V)$ *if and only if* $s_n = t_n$ *for every* $n \in \mathbb{N}$.

Proof. Each $c_k^{-1}\mathrm{Br}(E)$ is a proper, non-maximal ideal of V; therefore, if V has rank 1 then we must have $P_k = (0)$. Hence, (a) follows from Theorem 2.6, while (b) from Proposition 2.8, since we must have $s_k - t_k \in P_k = (0)$. □

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