The Number of Star Operations on Numerical Semigroups and on Related Integral Domains

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Abstract We study the cardinality of the set $\mathrm{Star}(S)$ of star operations on a numerical semigroup S; in particular, we study ways to estimate $\mathrm{Star}(S)$ and to bound the number of nonsymmetric numerical semigroups such that $|\mathrm{Star}(S)| \leq n$. We also study this problem in the setting of analytically irreducible, residually rational rings whose integral closure is a fixed discrete valuation ring.

Keywords Numerical semigroups · Star operations · Residually rational rings

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1 Introduction

A *star operation* on an integral domain D is a particular closure operation on the set of fractional ideals of D; this notion was defined to generalize the *divisorial closure* [4, 13] and has been further generalized to the notion of *semistar operation* [17]. Star operations have also been defined on cancellative semigroups in order to obtain semigroup-theoretic analogues of some ring-theoretic (multiplicative) definitions [11]. A classical result characterizes the Noetherian domains D in which every ideal is divisorial or, equivalently, which Noetherian domains admit only one star operation: this happens if and only if D is Gorenstein of dimension one [2]. Recently, this result has been a starting point of a deeper investigation on the cardinality of the set $\mathrm{Star}(D)$ of the star operations on D, obtaining a precise counting for h-local Prüfer domains [7] (and, more generally, an algorithm to calculate their number for semilocal Prüfer domains [23]), some pseudo-valuation domains [18, 26] and some Noetherian one-dimensional domains [6, 8, 25]. In particular, for Noetherian

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domains, a rich source of examples are *numerical semigroup rings*, that is, rings in the form $K[[S]] := K[[X^s \mid s \in S]]$, where K is a field and S is a numerical semigroup.

Inspired by this example, the study of star operations on numerical semigroups (and, in particular, of their cardinality) was initiated in [21]. In particular, the main problem that was tackled was the following: given a (fixed) integer n, how many numerical semigroups have exactly n star operations? By estimating the cardinality of Star(S), it was shown that this number is always finite, and that the same holds for residually rational rings (see Sect. 10 for a precise statement). Subsequently, in [27], better estimates on $|\text{Star}(S)|$ allowed to give a much better bound the number of semigroups with at most n star operations, while in [22] the set Star(S) was described in a very precise way when the semigroup S has multiplicity 3.

In this paper, we give a unified treatment of the study of Star(S), surveying the main results of [21, 22, 27] and [24] and deepening them. In particular, we give a rather precise asymptotic expression for the number of semigroups of multiplicity 3 with less than n star operations (Theorem 6.4), an $O(n^\epsilon)$ bound for the semigroups of prime multiplicity (Theorem 7.4), we list all nonsymmetric numerical semigroups with 150 or less star operations (Table 4), and prove an explicit bound for residually rational rings (Theorem 10.5).

The structure of the paper is as follows: Sects. 2 and 3 present basic material; Sects. 4 and 5 present estimates already present in [21] and [27]; Sect. 6 deepens the analysis of [22] on semigroups of multiplicity 3; Sect. 7 studies the case where the multiplicity is prime (and bigger than 3); Sect. 8 introduces the concept of *linear families* (one example of which was analyzed in [24]); Sect. 9 is devoted to algorithms to calculate $|\text{Star}(S)|$ and to determine all the nonsymmetric semigroups with at most n star operations; Sect. 10 studies the domain case, and contains analogues of the results of Sect. 4 for residually rational domains.

2 Notation

For all unreferenced results on numerical semigroups we refer the reader to [19].

A *numerical semigroup* is a set $S \subseteq \mathbb{N}$ that contains 0, is closed by addition and such that $\mathbb{N} \setminus S$ is finite. If $a_1, \dots, a_n$ are coprime positive integers, the numerical semigroup *generated* by $a_1, \dots, a_n$ is $\langle a_1, \dots, a_n \rangle := \left\{ \sum_{i=1}^n t_i a_i \mid t_i \in \mathbb{N} \right\}$. The notation $S = \{0, b_1, \dots, b_n, \rightarrow\}$ indicates that S is the set containing $0, b_1, \dots, b_n$ and all integers bigger than b_n.

To any numerical semigroup S are associated some natural numbers:

- the *genus* of S is $g(S) := |\mathbb{N} \setminus S|$;
- the *Frobenius number* of S is $F(S) := \sup(\mathbb{N} \setminus S)$;
- the *multiplicity* of S is $m(S) := \inf(S \setminus \{0\})$.

The *Apéry set* of S with respect to $n \in S$ is the set $\mathrm{Ap}(S, n) := \{x \in S \mid x - n \notin S\}$. Without specifications, the *Apéry set* of S is the Apéry set with respect to the multiplicity; we write $\mathrm{Ap}(S) := \mathrm{Ap}(S, m(S))$.

A *hole* of S is an integer $x \in \mathbb{N} \setminus S$ such that $F(S) - x \notin S$. A semigroup S is *symmetric* if it has no holes, while it is *pseudosymmetric* if $g(S)$ is even and $g(S)/2$ is its only hole. Setting $T(S) := \{x \in \mathbb{N} \setminus S \mid x + (S \setminus \{0\}) \subseteq S\}$, we also have that S is symmetric if and only if $T(S) = \{F(S)\}$ [19, Corollary 4.11].

An *integral ideal* of S is a nonempty subset $I \subseteq S$ such that $I + S \subseteq I$, i.e., such that $i + s \in I$ for all $i \in I$, $s \in S$. A *fractional ideal* of S is a subset $I \subseteq \mathbb{Z}$ such that $d + I$ is an integral ideal for some $d \in \mathbb{Z}$, or equivalently an $I \subsetneq \mathbb{Z}$ such that $I + S \subseteq I$. We shall use the term "ideal" as a shorthand for "fractional ideal".

If $\{I_\alpha\}_{\alpha \in A}$ is a family of ideals, then its intersection (if nonempty) is an ideal, while its union is an ideal if and only if there is a $d \in \mathbb{Z}$ such that $d < i$ for all i in the union. If I, J are ideals, the set $(I - J) := \{x \in \mathbb{Z} \mid x + J \subseteq I\}$ is still an ideal of S.

We denote by $\mathcal{F}(S)$ the set of fractional ideals of S, and by $\mathcal{F}_0(S)$ the set of fractional ideals contained between S and $\mathbb{N}$; equivalently, $\mathcal{F}_0(S) = \{I \in \mathcal{F}(S) \mid 0 = \inf(I)\}$. For every ideal I, there is a unique d such that $-d + I \in \mathcal{F}_0(S)$ (namely, $d = \inf(I)$).

If a, b are integers, we use (a, b) to indicate their greatest common divisor. If f, g are functions of n, we use $f = O(g)$ to mean that there is a constant C such that $f(n) \leq C \cdot g(n)$ for all $n \geq 0$.

3 Star Operations

Definition 3.1 ([21, Definition 3.1]) Let S be a numerical semigroup. A *star operation* on S is a map $* : \mathcal{F}(S) \longrightarrow \mathcal{F}(S)$, $I \mapsto I^*$, that satisfies the following properties:

- $*$ is *extensive*: $I \subseteq I^*$ for every $I \in \mathcal{F}(S)$;
- $*$ is *order-preserving*: if $I, J \in \mathcal{S}$ and $I \subseteq J$, then $I^* \subseteq J^*$;
- $*$ is *idempotent*: $(I^*)^* = I^*$ for every $I \in \mathcal{F}(S)$;
- $*$ fixes S, that is, $S^* = S$;
- $*$ is *translation-invariant*: $d + I^* = (d + I)^*$ for every $I \in \mathcal{F}(S)$ and every $d \in \mathbb{Z}$.

We denote by $\mathrm{Star}(S)$ the set of star operations on S.

If $I = I^*$, we say that I is $*$-*closed*; we denote the set of $*$-closed ideals by $\mathcal{F}^*(S)$.

The set $\mathrm{Star}(S)$ can be endowed with a natural partial order: we say that $*_1 \leq *_2$ if $I^{*_1} \subseteq I^{*_2}$ for every ideal I, or equivalently if $\mathcal{F}^{*_2}(S) \subseteq \mathcal{F}^{*_1}(S)$. Under this order, $\mathrm{Star}(S)$ is a complete lattice: its minimum is the identity, while its maximum is the v-*operation* (or *divisorial closure*) $v : I \mapsto (S - (S - I))$.

Since $\mathbb{N}$ is v-closed, any star operation restricts to a map $*_0 : \mathcal{F}_0(S) \longrightarrow \mathcal{F}_0(S)$; furthermore, $*_0$ uniquely determines $*$ (since every ideal can be translated into $\mathcal{F}_0(S)$). We define $\mathcal{G}_0(S) := \mathcal{F}_0(S) \setminus \mathcal{F}^v(S)$, that is, $\mathcal{G}_0(S)$ is the set of ideals I of S such that $0 = \inf I$ and $I \neq I^v$.

Since $\mathcal{F}_0(S)$ is finite, Star(S) is a finite set for all numerical semigroups S [21, Proposition 3.2]. Furthermore, $|\mathrm{Star}(S)| = 1$ if and only if v is the identity, which happens if and only if S is symmetric [1, Proposition I.1.15].

4 Estimates Through the Genus

Our main interest in this paper will be the function $\Xi(n)$ that associates to every integer $n > 1$ the number of numerical semigroups S such that $2 \leq |\mathrm{Star}(S)| \leq n$. More generally, if $\mathcal{S}$ is a set of numerical semigroups, we define $\Xi_{\mathcal{S}}(n)$ as the number of semigroups $S \in \mathcal{S}$ such that $2 \leq |\mathrm{Star}(S)| \leq n$. We will mainly be interested in the asymptotic growth and in asymptotic bounds of Ξ and $\Xi_{\mathcal{S}}$, for some distinguished sets $\mathcal{S}$ of semigroups.

It is very difficult to determine precisely the number of star operations on a numerical semigroup S, while it is easier to find estimates for $|\mathrm{Star}(S)|$: for this reason, we work with Ξ instead of the function that counts the number of semigroups with exactly n star operations. Most of the bounds proven in the paper will be obtained in a two-step process:

1. find a function ϕ (depending on some of the invariants of S) such that $|\mathrm{Star}(S)| \geq \phi(S)$ for all $S \in \mathcal{S}$;
2. estimate the number of $S \in \mathcal{S}$ satisfying $\phi(S) \leq n$.

In this way, we obtain an estimate on the number of semigroups $S \in \mathcal{S}$ satisfying $|\mathrm{Star}(S)| \leq n$: indeed, if $|\mathrm{Star}(S)| \leq n$ then we must also have $\phi(S) \leq n$.

The first important result is to prove that Ξ is actually well-defined, that is, that there are only a finite number of numerical semigroups satisfying $2 \leq |\mathrm{Star}(S)| \leq n$. To do so, the first estimate involves the genus of S.

Theorem 4.1 ([27, Proposition 8.1]) *Let S be a nonsymmetric numerical semigroup. Then,* $|\mathrm{Star}(S)| \geq g(S) + 1$.

Sketch of Proof For every ideal $I \in \mathcal{G}_0(S)$, we define $*_I$ as the largest star operation $*$ such that $I = I^*$; equivalently, $*_I$ is the map such that

$$J^{*_I} = J^v \cap (I - (I - J))$$

for every ideal J [21, Proposition 3.6]. Then, $*_I = *_J$ if and only if $I = J$ [21, Theorem 3.8]. Since S is nonsymmetric, there is a $\tau \in T(S) \setminus \{F(S)\}$ [19, Corollary 4.11]; let $\lambda := \min\{\tau, F(S) - \tau\}$. If $x \in \mathbb{N} \setminus S$, let $M_x := \{z \in \mathbb{N} \mid x - z \notin S\}$; then, M_x is an ideal (which is not always divisorial). We associate to each $x \in \mathbb{N} \setminus S$ an ideal I_x:

- if $x < \lambda$ and $\lambda - x \notin S$, then $I_x := S \cup \{z \in \mathbb{N} \mid z > x, z \in M_\lambda\}$;
- if $x < \lambda$ and $\lambda - x \in S$, then $I_x := S \cup \{z \in \mathbb{N} \mid z > g - (\lambda - x)\}$;
- if $x \geq \lambda$, then $I_x := M_x$.

Then, $I_x \neq I_y$ if $x \neq y$. Each I_x is not divisorial: in the first case because $\sup(\mathbb{N} \setminus S) = \lambda \notin S$ and by [21, Lemma 4.7], in the second case because $\tau \in I_x^v \setminus I^v$ (and by [21, Proposition 3.11]), in the third case by [21, Lemma 4.8]. Hence, they generate $g(S)$ different star operations, all different from the divisorial closure. Thus, $|\mathrm{Star}(S)| \geq g(S) + 1$. □

We now translate this estimate to a bound on Ξ.

Theorem 4.2 ([27, Section 8]) *Preserve the notation above.*

(a) $\Xi(n) < \infty$ *for every* $n > 1$.
(b) If $\varphi := \frac{\sqrt{5}+1}{2}$ *is the golden ratio, then*

$$\Xi(n) = O(\varphi^n) = O(\exp(n \log \varphi)).$$

Proof By [32], the number of numerical semigroups of genus at most n is $O(\varphi^n)$. The claim follows from Theorem 4.1. □

5 Estimates Through the Multiplicity

The proof of Theorem 4.1 involves star operations generated by a single ideal (called *principal* star operations). In general, not all star operations have this form; to work more generally we define, given $\Delta \subseteq \mathcal{G}_0(S)$, the star operation *induced by* Δ as

$$*_\Delta := \inf\{*_I \mid I \in \Delta\}.$$

Every star operation can be represented in this form [27, Section 3], but in general we may have $*_\Delta = *_\Lambda$ even if $\Delta \neq \Lambda$. To obtain better estimates, we want to identify special subsets of $\mathcal{G}_0(S)$ that induce pairwise different star operations. We introduce the following definitions.

Definition 5.1 ([27, Definition 3.1]) The $*$*-order* on $\mathcal{G}_0(S)$ is the partial order $\leq_*$ defined by $I \leq_* J$ if and only if $*_I \geq *_J$; equivalently, $I \leq_* J$ if I is $*_J$-closed.

The fact that, for $I, J \in \mathcal{G}_0(S)$, $*_I = *_J$ if and only if $I = J$ guarantees that the $*$-order is really a partial order (see [21, Corollary 3.9] or the proof of Theorem 4.1); on the other hand, the same relation defined on the whole $\mathcal{F}(S)$ is only a preorder (see the discussion after [27, Definition 3.1]).

Definition 5.2 Let $(\mathcal{P}, \leq)$ be a partially ordered set. An *antichain* of $\mathcal{P}$ is a subset of pairwise noncomparable elements.

Definition 5.3 Let $a \in \mathbb{N} \setminus S$. Then, $\mathcal{Q}_a$ is the set of ideals $I \in \mathcal{G}_0(S)$ such that $a = \sup(\mathbb{N} \setminus I)$ and such that $a \in I^v$.

The set $\mathcal{Q}_a$ is nonempty if and only if M_a is nondivisorial (in which case $M_a \in \mathcal{Q}_a$) [27, Proposition 5.2].

Proposition 5.4 ([27, Proposition 5.11]) *Let $a, b \in \mathbb{N} \setminus S$, and let $\Delta \subseteq \mathcal{Q}_a$, $\Lambda \subseteq \mathcal{Q}_b$ be two nonempty sets of ideals that are antichains with respect to set inclusion. If $\Delta \neq \Lambda$, then $*_\Delta \neq *_\Lambda$.*

Given $\mathcal{P} \subseteq \mathcal{G}_0(S)$, we denote by $\omega_i(\mathcal{P})$ the number of antichains of $\mathcal{P}$ with respect to set inclusion.

Corollary 5.5 ([27, Corollary 5.12]) *For every numerical semigroup S, we have*

$$|\mathrm{Star}(S)| \geq 1 + \sum_{a \in \mathbb{N} \setminus S} (\omega_i(\mathcal{Q}_a) - 1).$$

Remark 5.6 In [27], the notation $\omega(\mathcal{P})$ was used for the number of antichain of $\mathcal{P}$ with respect to the $*$-order, and $\omega_i(\mathcal{P})$, with "i" standing for "inclusion", was used to distinguish the two quantities. In this paper we do not use directly the antichains in the $*$-order, but we preserve the notation $\omega_i(\mathcal{P})$ for the sake of consistency.

Corollary 5.5 allows a relatively quick estimate of $\mathrm{Star}(S)$ when S is a fixed semigroup, since finding $\mathcal{Q}_a$ and counting the antichains with respect to inclusion is much quicker than determining and comparing star operations. From a theoretical point of view, it can be used through the following construction.

Suppose a is a hole of S. Let $J := S \cup \{x \in \mathbb{N} \mid x > a\}$, and let $Z(a) := \{a - m + 1, \ldots, a - 1\} \setminus S$. For every $A \subseteq Z(a)$, the set $I_A := J \cup A$ is an ideal of S, and it belongs to $\mathcal{Q}_a$ since $g - a \notin S$ [21, Lemma 4.7]. Furthermore, $I_A \subseteq I_B$ if and only if $A \subseteq B$; hence, the set of the I_A (under the containment order) is isomorphic to the power set of $Z(a)$. The number of antichains of the power set of a set with n elements is called the n-th *Dedekind number*, and we denote it by $\omega(n)$. The sequence $\{\omega(n)\}$ grows extremely quickly (as an exponential of an exponential), and for this reason it is known only up to $n = 8$ [12, 31].

A similar construction can be done if $a < m(S)$ is not a hole, but there is a hole $b < a$; in this case, we consider $Z(a) = \{1, \ldots, a - 2\}$, and the best estimate is obtained with $a = m(S) - 1$. Using these constructions (and some variants), we can prove the following.

Proposition 5.7 ([27, Propositions 5.19 and 5.21]) *Let S be a nonsymmetric numerical semigroup, and let $\nu(S) := \left\lceil \frac{m(S)-1}{2} \right\rceil$. Let $a \in \mathbb{N} \setminus S$.*

(a) If $m(S) < a \leq g/2$ and $g - a \notin S$ then $\omega_i(\mathcal{Q}_a) \geq \omega(\nu(S))$.
(b) If $2m(S) < a \leq g/2$ and $g - a \notin S$ then $\omega_i(\mathcal{Q}_a) \geq 2\omega(\nu(S)) - 2$.
(c) If $a < m(S)$ and $g - a \notin S$ then $\omega_i(\mathcal{Q}_a) \geq \omega(a - 1)$.
(d) If $a < m(S)$ and there is a hole $b < a$ of S, then $\omega_i(\mathcal{Q}_a) \geq \omega(a - 2)$.

In particular, $|\mathrm{Star}(S)| \geq \omega(\nu(S))$.

As in Sect. 4, we can use the last estimate to obtain a bound on Ξ.

Theorem 5.8 ([27, Theorem 8.4]) *For every* $\epsilon > 0$,

$$\Xi(n) = O\left[\exp\left(\left(\frac{2}{\log 2} + \epsilon\right)\log(n)\log\log(n)\right)\right].$$

Sketch of Proof Let $A_\epsilon := \frac{2}{\log 2} + \epsilon$. Using Proposition 5.7 and the estimates in [12], we have that if $|\mathrm{Star}(S)| \leq n$ then (for any $\epsilon' > 0$ and $n \geq n_0(\epsilon')$)

$$n \geq \omega(\nu(S)) \geq 2^{\binom{\nu(S)}{\lceil \nu(S)/2 \rceil}} \geq 2^{2^{(1-\epsilon')\nu(S)}}$$

when $\nu(S)$ is large. Writing it as a function of $m(S)$, we get $m(S) \leq A_\epsilon \log\log n$.

Let $\Xi_\mu(n)$ be the number of nonsymmetric numerical semigroups of multiplicity μ with at most n star operations: then, using Theorem 4.1, $\Xi_\mu(n)$ is at most equal to the number of numerical semigroups of multiplicity μ of genus $\leq n$, which is at most $(n-1)^{\mu-1}$. It follows that

$$\Xi(n) \leq \sum_{\mu=3}^{A_\epsilon \log\log n} (n-1)^{\mu-1} \leq n^{A_\epsilon \log\log(n)} \leq \exp(A_\epsilon \log(n)\log\log(n)),$$

as claimed. □

6 Multiplicity 3

In the last passage of the proof of Theorem 5.8, we needed to estimate the function $\Xi_\mu(n)$ counting the nonsymmetric numerical semigroups of multiplicity μ with at most n star operations. While a very crude bound was enough to obtain the theorem, it is reasonable to ask for more precise estimates: in this section we analyze the case of multiplicity 3, while in the next one we study the case where $m(S) > 3$ is prime.

The case of numerical semigroups of multiplicity 3 can be analyzed very thoroughly, obtaining a complete solution to the problem of finding the set of star operations on S.

Theorem 6.1 *Let* $S := \langle 3, 3\alpha+1, 3\beta+2\rangle$ *be a numerical semigroup of multiplicity 3, where* $\mathrm{Ap}(S) = \{3, 3\alpha+1, 3\beta+2\}$.

(a) *[22, Theorem 7.4]* $(\mathcal{G}_0(S), \leq_*)$ *is order-isomorphic to the direct product* $\{1, \dots, 2\alpha-\beta\} \times \{1, \dots, 2\beta-\alpha+1\}$.
(b) *[22, Corollary 6.5]* $\mathrm{Star}(S)$ *is order-isomorphic to the set of antichains of* $(\mathcal{G}_0(S), \leq_*)$.
(c) *[22, Theorem 7.6]* $|\mathrm{Star}(S)| = \binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{\alpha+\beta+1}{2\beta-\alpha+1} = \binom{g(S)+1}{F(S)-g(S)+2}$.

Using Proposition 5.4, we can also improve [22, Proposition 7.8].

Proposition 6.2 *Let S be a nonsymmetric numerical semigroup. Then, the following are equivalent:*

(i) *S is a pseudosymmetric semigroup of multiplicity 3;*
(ii) *$(\mathcal{G}_0(S), \leq_*)$ is linearly ordered;*
(iii) *$\mathrm{Star}(S)$ is linearly ordered.*

Proof If $m(S) = 3$, the result is exactly [22, Proposition 7.8]. Suppose thus $m(S) > 3$; we need to show that $(\mathcal{G}_0(S), \leq_*)$ is not linearly ordered, and to do so it is enough (by Proposition 5.4) to find two ideals J_1, J_2 in some $\mathcal{Q}_a$ that are not comparable. Let τ be a hole of S such that $\tau \leq g/2$ (it exists because S is not symmetric). We distinguish several cases.

If $\tau \geq 3$, then by [21, Lemma 4.13] we can find $a_1, a_2 \in (\{\tau - m + 1, \dots, \tau - 1\} \cap \mathbb{N}) \setminus S$; then, we set $J_i := S \cup \{x \in \mathbb{N} \mid x > \tau\} \cup \{a_i\}$.

If $\tau < 3$ and $m(S) > 4$, consider $b := 4$: then, the set $\{1, 2, 3\} \setminus \{3 - \tau\}$ contains two different elements, say x_1 and x_2, and we take $J_i := S \cup \{x \in \mathbb{N} \mid x > 3\} \cup \{3 - \tau, x_i\}$ (they belong to $\mathcal{Q}_3$ by the proof of [27, Proposition 5.20]).

Suppose $m(S) = 4$ and $\tau \leq 2$. If $\tau = 1$ then one between $g := g(S)$ and $g - 1$ is even; call it e. Then, $e/2$ is a hole of S which is not bigger then $g/2$; in particular, if $\frac{e}{2} \geq 3$ we are in the case above. If $\frac{e}{2} \leq 2$, then $g \leq 5$, and so either $g = 3$ or $g = 5$. In the latter case we would have $g - 1 = 4 \notin S$, a contradiction; in the former case, $S = \langle 4, 5, 6, 7\rangle$, and by direct inspection $\mathcal{G}_0(S)$ is not linearly ordered (see [27, Example 5.21]).

If $\tau = 2$, consider $J_1 := S \cup \{g - 2\}$ and $J_2 := S \cup (2 + S)$. Then, both are elements of $\mathcal{Q}_g$, and $g - 2 \notin J_2$ (otherwise $g - 2 - 2 = g - 4 = g - m \in S$, which is absurd); furthermore, $J_1 \neq J_2$ since otherwise $2 = g - 2$, i.e., $g = 4$, a contradiction, and so they are noncomparable.

Therefore, if $m(S) > 3$ the $*$-order on $\mathcal{G}_0(S)$ is not total, as claimed. □

We now want to use Theorem 6.1 to calculate $\Xi_3(n)$. The idea is to divide the set of semigroups of multiplicity 3 in sets defined by the relation $2\alpha - \beta = k$ (if $\alpha \leq \beta$) or $2\beta - \alpha + 1 = k$ (if $\alpha > \beta$), and then estimate $\Xi_{\mathcal{S}}(n)$ for each of these families.

Lemma 6.3 *Let k, n be integers, and define*

$$p_{k,n}(X) := \frac{X(X-1)\cdots(X-k+1)}{k!} - n.$$

Then:

(a) *$p_{k,n}$ has a unique zero $x_{k,n}$ that satisfies $x_{k,n} > k - 1$;*
(b) *for all k, there is a $n_0(k)$ such that, for all $n \geq n_0(k)$,*

$$(k!n)^{1/k} - 1 < x_{k,n} < (k!n)^{1/k} + k - 1.$$

Proof

(a) Let $\widetilde{p}_{k,n}(X) := p_{k,n}(X+k-1) = \frac{X(X+1)\cdots(X+k-1)}{k!} - n$: then, $\widetilde{p}_{k,n}$ is a polynomial whose coefficients are all positive, and thus $\widetilde{p}_{k,n}$ is increasing for $X > 0$, i.e., $p_{k,n}$ is increasing for $X > k-1$. Furthermore, $p_{k,n}(k-1) = \widetilde{p}_n(0) = -n$, and thus $p_{k,n}$ has a unique zero $x_{k,n} > k-1$.

(b) We have

$$p_{k,n}((k!n)^{1/k}+k-1) = \widetilde{p}_{k,n}((k!n)^{1/k}) > \frac{((k!n)^{1/k})^k}{k!} - n = n - n = 0,$$

and thus $x_{k,n} < (k!n)^{1/k}+k-1$. On the other hand, write $k!\widetilde{p}_{k,n}(X) = \sum_{t=0}^{k} \lambda_t X^t$: then, $\lambda_k = 1$ and $\lambda_0 = -k!n$. We have

$$\lambda_t \cdot ((k!n)^{1/k} - k)^t = \lambda_t \sum_{i=0}^{t} \binom{t}{i} (-1)^{t-i} (k!n)^{i/k} k^{(t-i)/k}.$$

Adding all these terms, we see that $k!\widetilde{p}_{k,n}(X)$ is a sum of monomials (with fractional exponent) in n. The maximal exponent is 1, which appears twice: for $t = k = i$ and for $t = 0$. The former is equal to $k!n$ and the latter to $-k!n$, and so their sum is zero. The next term is the one with exponent $(k-1)/k$, and again we have two monomials: for $t = k$ and $i = 1$ and for $t = k-1 = i$. Hence, the leading term of $k!\widetilde{p}_{k,n}((k!n)^{1/k} - k)$, as a function of n, is

$$-\binom{k}{1}(k!n)^{(k-1)/k} \cdot k + \lambda_{k-1}(k!n)^{(k-1)/k} = k!^{(k-1)/k}(-k^2 + \lambda_{k-1})n^{(k-1)/k}.$$

We have $\lambda_{k-1} = 1+2+\cdots+k-1 = \frac{k(k-1)}{2}$; hence, the sign of $k!\widetilde{p}_n((k!n)^{1/k} - k)$ is equal to the sign of

$$-k^2 + \lambda_{k-1} = -k^2 + \frac{k(k-1)}{2} = -\frac{k^2+k}{2} < 0.$$

Therefore, for large n we have $x_{k,n} > (k!n)^{1/k} - k + (k-1) = (k!n)^{1/k} - 1$, as claimed. □

Theorem 6.4 *For every integer $t > 1$, we have*

$$\Xi_3(n) = \frac{2}{3}\left(\sum_{k=1}^{t-1} (k!)^{1/k} \cdot n^{1/k}\right) + O(n^{1/t}\log^2 n).$$

Proof Given a numerical semigroup $S = \langle 3, 3\alpha+1, 3\beta+2\rangle$ of multiplicity 3, let $p(S) := \alpha+\beta+1$ and $q(S) := 2\alpha-\beta$. Then, $p(S)+q(S) = 3\alpha+1$; we have

$p(S) > q(S)$ for all nonsymmetric semigroups, and furthermore $p(S) \neq 2q(S)$ for all S, which means that $p(S) < 2q(S)$ or $p(S) > 2q(S)$.

Given an integer $k \geq 1$, define the following sets: $\mathcal{S}_k$ is the set of numerical semigroups with $p(S) < 2q(S)$ and $q(S) = k$, while $\mathcal{S}_{-k}$ is the set of semigroups with $p(S) > 2q(S)$ and $p(S) - q(S) = k$. Then, each nonsymmetric semigroup belongs to exactly one $\mathcal{S}_k$ or $\mathcal{S}_{-k}$, and thus

$$\Xi_3(n) = \sum_{k \geq 1} \Xi_{\mathcal{S}_k}(n) + \Xi_{\mathcal{S}_{-k}}(n).$$

We claim that $\Xi_{\mathcal{S}_k}(n) = (k!)^{1/k} \cdot n^{1/k} + O(1)$ for each k.

Indeed, $\Xi_{\mathcal{S}_k}(n)$ is equal to the number of integer solutions of the system

$$\begin{cases} \binom{X}{k} \leq n \\ X + k \equiv 1 \bmod 3 \\ X \geq 2k \end{cases}$$

In the notation of Lemma 6.3, the first equation is exactly $p_{k,n}(X) \leq 0$; hence, the number of solutions is $\frac{1}{3}(x_{k,n} - 2k) + \epsilon$ for some $|\epsilon| \leq 1$ (depending on k and n). For large n, using Lemma 6.3(b) this is equal to

$$\frac{1}{3}k!^{1/k}n^{1/k} - \frac{2}{3}k + O(1) = \frac{1}{3}k!^{1/k}n^{1/k} + O(1)$$

for k fixed, as claimed. A completely analogous reasoning holds for $\mathcal{S}_{-k}$, since also $\binom{X}{X-k} = p_{k,n}(X)$.

Take any integer t and let $\mathcal{S} := \bigcup_{k<t} \mathcal{S}_k \cup \mathcal{S}_{-k}$. Then,

$$\begin{aligned} \Xi_{\mathcal{S}}(n) &= \sum_{i=1}^{t-1} \Xi_{\mathcal{S}_k}(n) + \Xi_{\mathcal{S}_{-k}}(n) = \sum_{i=1}^{t-1} \left(\frac{2}{3}k!^{1/k}n^{1/k} + O(1) \right) \\ &= \frac{2}{3} \left(\sum_{i=1}^{t-1} k!^{1/k}n^{1/k} \right) + O(t). \end{aligned}$$

Let $\mathcal{S}'$ be the complement of $\mathcal{S}$ in the set of all numerical semigroups of multiplicity 3, and consider $\Xi_{\mathcal{S}'}(n)$. Let $G_r(n)$ be the number of binomial coefficients $\binom{a}{b}$ such that $\binom{a}{b} \leq n$, $b \geq t$ and $a \geq 2b$; then, since a binomial coefficient arises from at most one semigroup, we have

$$\Xi_{\mathcal{S}'}(n) \leq 2 \sum_{r=t}^{\infty} G_r(n). \tag{1}$$

If $k > \log_2(n)$, then

$$\binom{2k}{k} \geq \frac{4^{\log_2(n)}}{\sqrt{4\log_2(n)}} \geq \frac{n^2}{\sqrt{4\log_2(n)}} > n$$

for large n. Thus, it is enough to consider the sum in (1) only for k going from t to $\log_2(n)$.

By Lemma 6.3, if $\binom{a}{t} \geq n$ then $a \leq (k!n)^{1/k}$; hence, $G_k(n) \leq (k!n)^{1/k}$ and

$$\Xi_{\mathcal{S}'}(n) \leq 2\sum_{k=t}^{\log_2(n)} (k!n)^{1/k} \leq 2n^{1/t} \sum_{k=t}^{\log_2 n} (k!)^{1/k} = O(n^{1/t}\log^2 n).$$

since $(k!)^{1/k} \leq k$. The claim is proved. □

Note that we *cannot* write Ξ_3 as the series

$$\Xi_3(n) = \frac{2}{3}\sum_{k=1}^{\infty}(k!)^{1/k}\cdot n^{1/k},$$

because at fixed n the terms have limit 1, and so the series does not converge. When n is fixed, a good approximation for $\Xi_3(n)$ is obtained stopping the series at $k = \log_2(n)$; an even better approximation can be obtained stopping it at $k = \frac{1}{2}(\log_2 n + \log_2\log_2 n)$, since also for this value we have $\binom{2k}{k} > n$.

7 Prime Multiplicity

The formula for $|\mathrm{Star}(S)|$ in the previous section was based on an explicit (and very regular) description of $\mathcal{G}_0(S)$. For semigroups of bigger multiplicity, both listing all non-divisorial ideals and understanding the $*$-order becomes much more complicated (see the examples in [24]), and so we need to rely on estimates. In this section, we shall obtain good estimates for some particular classes of semigroups.

The main idea is to generalize the reasoning used to obtain the estimate $|\mathrm{Star}(S)| \geq \omega(\nu(S))$ by considering not only the elements $b \in \{a - m(S) + 1, \dots, a-1\} \setminus S$, but also the integers in the form $b - km$.

Theorem 7.1 *Let S be a nonsymmetric numerical semigroup of multiplicity m, and let $a \in \mathbb{N} \setminus S$ be a hole of S. Suppose that there are $b_1, b_2 \in (a-m, a) \cap \mathbb{N}$ and $\sigma \in \mathbb{N}$ such that:*

- *$b_1, b_2 \notin S$;*
- *for $c \in \{a - b_1, a - b_2, |b_1 - b_2|\}$, the element $a_c \in \mathrm{Ap}(S, m)$ congruent to c modulo m satisfies $a_c \geq \sigma m$.*

Then, $|\mathrm{Star}(S)| \geq \binom{2\sigma}{\sigma}$.

Proof For $0 \leq j, k < \sigma$, let $I(j,k)$ be the ideal

$$I(j,k) := S \cup \{x \in \mathbb{N} \mid x > a\} \cup (b_1 - jm + S) \cup (b_2 - km + S).$$

We first prove that $\max(\mathbb{N} \setminus I(j,k)) = a$. Clearly, every element larger than a is in $I(j,k)$. On the other hand, $a \notin S$, while $a \in b_1 - jm + S$ is equivalent to $a-(b_1-jm) \in S$, and the latter is impossible since $a-(b_1-jm) = (a-b_1)+jm < \sigma m$; hence, $a \notin b_1 - jm + S$, and in the same way $a \notin b_2 - km + S$.

Furthermore, $b_1-jm-m \notin I(j,k)$: the only possibility would be $b_1-jm-m \in b_2 - km + S$, but his would imply

$$b_1 - jm - m - (b_2 - km) = b_1 - b_2 + (k - j - 1)m \in S,$$

which is impossible since $b_1 - b_2 + (k - j - 1)m < \sigma m$. Hence, the Apéry set of $I(j,k)$ contains a, $b_1 - jm$ and $b_2 - km$; in particular, these ideals all distinct.

Since a is a hole of S, all the $I(j,k)$ belong to $\mathcal{Q}_a$, and by Proposition 5.4 every nonempty antichain with respect to containment induces a different star operation on S. Under the containment order, the set of the $I(j,k)$ is isomorphic to the direct product $\{1,\ldots,\sigma\} \times \{1,\ldots,\sigma\}$; by [22, Lemma 7.5], the latter set has $\binom{2\sigma}{\sigma}$ antichains. The claim now follows from Corollary 5.5. □

When instead of b_1 and b_2 we have z elements, say $b_1,\ldots,b_z$, in $(a-m,a)\cap\mathbb{N}$ but out of S, the same reasoning (with the natural modifications to the hypothesis) can be applied, considering the set containing the ideals in the form

$$I(j_1,\ldots,j_z) := S \cup \{x \in \mathbb{N} \mid x > a\} \cup \bigcup_{i=1}^{z} (b_i - j_i m + S),$$

which will be isomorphic to $\{1,\ldots,\sigma\}^z$. Numerically, this version gives a much better bound on $|\mathrm{Star}(S)|$, although there isn't a simple formula to express it; however, the version of the theorem with only b_1 and b_2 will suffice for our purpose.

Lemma 7.2 *If a is a hole of a numerical semigroup S and $a + m(S) \notin S$, then $a + m(S)$ is a hole of S.*

Proof Immediate from the fact that $F(S)-(a+m(S)) = (F(S)-a)-m(S)$ can't belong to S if $F(S) - a \notin S$. □

Lemma 7.3 *Let S be a numerical semigroup with multiplicity m, and let $a \in \mathrm{Ap}(S,m)$. If $(a,m)|(F(S),m)$, then*

$$a \geq \frac{F(S)+m}{m-1}$$

Proof Suppose first that $(a, m) = 1$: then, $S' := \langle m, a\rangle$ is a numerical semigroup, and $F(S) \leq F(S')$. However, $F(S') = am - a - m = a(m-1) - m$; solving for a we have our claim.

If $(a, m) =: d > 1$, we consider the semigroup $S' := S/d := \{x/d \mid x \in S \cap d\mathbb{N}\}$: then, since d divides m and $F(S)$, we have $m(S') = m(S)/d$, $F(S') = F(S)/d$ and $a/d \in S'$. By the previous part of the proof,

$$\frac{a}{d} \geq \frac{F(S') + m(S')}{m(S') - 1} = \frac{F(S) + m(S)}{d} \frac{d}{m(S) + d} = \frac{F(S) + m}{m - d} \geq \frac{F(S) + m}{m - 1},$$

and the claim is proved. □

Theorem 7.4 *Let $m > 3$ be a prime number. Then, for every $\epsilon > 0$,*

$$\Xi_m(n) = O(\log^{m-1} n) = O(n^{\epsilon}).$$

Proof There are only finitely many numerical semigroups of multiplicity m satisfying $F(S) < km$, for every $k \in \mathbb{N}$; hence, we can ignore them and only consider (nonsymmetric) semigroups satisfying $F(S) > m^3$.

Fix such a semigroup S, and let a be a hole of S satisfying $a \leq F(S)/2$. Applying Lemma 7.2, we see that, for any $k \in \mathbb{N}$, the element $a + km$ is either a hole of S or belongs to S; let h be the largest of such holes that is also smaller or equal than $F(S)/2$. By Lemma 7.3, and since $m > 3$, we must have $h \geq \frac{F(S)+m}{m-1} - m \geq \frac{F(S)-m^2}{m-1}$. Note that, since $F(S) > m^3$, we have $h > m$.

By [21, Lemma 4.13], since $m < h \leq F(S)/2$, there are two elements $b_1, b_2 \in (a - m, m) \setminus S$; taking $\sigma := \left\lfloor \frac{1}{m} \frac{F(S)+m}{m-1} \right\rfloor$, we can apply Theorem 7.1, obtaining $|\text{Star}(S)| \geq \binom{2\sigma}{\sigma}$. Now

$$\left\lfloor \frac{1}{m} \frac{F(S) + m}{m - 1} \right\rfloor \geq \frac{1}{m} \frac{F(S) + m}{m - 1} - 1 = \frac{F(S)}{m(m - 1)} + \frac{1}{m - 1} - 1 \geq \frac{F(S)}{m^2}$$

using $F(S) > m^3$. Setting $\sigma' := \left\lceil \frac{F(S)}{m^2} \right\rceil$, for these semigroups we have

$$|\text{Star}(S)| \geq \binom{2\sigma'}{\sigma'} \geq \frac{2^{2\sigma'-1}}{\sqrt{\sigma'}} \geq 2^{\sigma'}.$$

If $|\text{Star}(S)| \leq n$, this means that $\sigma' \leq \log_2 n$, i.e.,

$$\frac{F(S)}{m^2} < \log_2 n \Longrightarrow F(S) < m^2 \log_2 n.$$

Therefore,

$$\Xi_m(n) \leq C+(m^2 \log_2 n)^{m-1} = C+m^{2(m-1)}(\log_2 n)^{m-1} = O(\log^{m-1} n) = O(n^\epsilon)$$

for every $\epsilon > 0$. □

Corollary 7.5 *Let $\mathcal{S}$ be the set of all numerical semigroups whose multiplicity is a prime number > 3. Then, for every $\epsilon > 0$, we have*

$$\Xi_{\mathcal{S}}(n) = O(n^\epsilon).$$

Proof By [27, Proposition 8.2], we need to consider only semigroups with multiplicity up to $A_\epsilon \log\log n$, where $A_\epsilon := \frac{2}{\log 2} + \epsilon$.

There are at most $(m^2)^{m-1} = m^{2(m-1)}$ semigroups of multiplicity m with $F(S) < m^3$; hence, by the proof of the previous theorem we have

$$\Xi_m(n) \leq m^{2(m-1)} + \frac{2}{\log 2} m^{2(m-1)} \log^{m-1} n \leq \frac{4}{\log 2} \log^{m+2} n$$

for large n, since $m^{2(m-1)} \leq (A_\epsilon \log\log n)^{2A_\epsilon \log\log n} \leq \log^3 n$. Therefore,

$$\Xi_{\mathcal{S}}(n) = \sum_{m>3 \text{ prime}} \Xi_m(n) = \sum_{\substack{m=5 \\ m \text{ prime}}}^{A_\epsilon \log\log n} \Xi_m(n) \leq (A_\epsilon \log\log n) \cdot \frac{4}{\log 2} (\log n)^{A_\epsilon \log\log n},$$

which is $O(n^\epsilon)$. The claim is proved. □

The proof above is based on the fact that if $m(S)$ is prime then no generator of S can be too small. The same happens if we consider only the elements of the Apéry set that are coprime with $m(S)$; however, in this case, we also need to find a large hole. If $F(S)$ is even, one easy solution is using $F(S)/2$.

Theorem 7.6 *Let $\mathcal{S}$ be the set of numerical semigroups of multiplicity $m \geq 4$ such that $3 \nmid m$ and $F(S) \equiv 0 \bmod 2$. Then, for every $\epsilon > 0$,*

$$\Xi_{\mathcal{S}}(n) = O(n^\epsilon).$$

Proof Let $\mathcal{S}_m$ be the set of numerical semigroup with (fixed) multiplicity m satisfying $F(S) \equiv 0 \bmod 2$; for large n, by the proof of Theorem 5.8 we have $\Xi_{\mathcal{S}_m}(n) = 0$ if $m > 2\log\log n$.

As in the previous proof, there are at most m^{2m} semigroups S of multiplicity m with $F(S) \leq 2m^2$.

Fix a semigroup S such that $F(S) > 2m^2$, and let $\tau := F(S)/2$: then, τ is a hole of S and, since $F(S) > 2m^2$, we have $\tau > m^2$. Consider the elements $\tau - 2$ and $\tau - 1$.

If $\tau_1, \tau_2 \notin S$, then we can apply Theorem 7.1 with $b_1 = \tau - 2$, $b_2 = \tau - 1$ and $\sigma = \left\lfloor \frac{F(S)}{m^2} \right\rfloor$, applying Lemma 7.3 (since both $(1, m)$ and $(2, m)$ divide $(m, F(S))$).

If $\tau_1, \tau_2 \in S$, then $\tau + 1$ and $\tau + 2$ cannot belong to S (otherwise $\tau - 1 + \tau + 1 = 2\tau = F(S) \in S$, a contradiction, and analogously for $\tau - 2$). Hence, we can apply Theorem 7.1 with $b_1 = \tau - m + 2$, $b_2 = \tau - m + 1$ and $\sigma = \left\lfloor \frac{F(S)}{m^2} \right\rfloor$.

Suppose that $\tau - 2 \in S$ while $\tau - 1 \notin S$. As before, $\tau + 2 \notin S$, and we take $b_1 := \tau - m + 2$ and $b_2 := \tau - 1$. Then, $b_2 - b_1 = m - 3$, and so $(m, m - 3) = 1$ (since $3 \nmid m$). Using Lemma 7.3 we can apply Theorem 7.1 with $\sigma = \left\lfloor \frac{F(S)}{m^2} \right\rfloor$. Analogously, if $\tau - 2 \notin S$ and $\tau - 1 \in S$ we use $b_1 := \tau - m + 1$ and $b_2 := \tau - 2$.

In all cases, we have $|\mathrm{Star}(S)| \geq \binom{2\sigma}{\sigma} \geq 2^\sigma$. Hence, for large n, is $S \in \mathcal{S}_m$ satisfies $|\mathrm{Star}(S)| \geq n$ we must have $F(S) < m^2 \log_2 n$; as in the proof of Theorem 7.4 it follows that

$$\Xi_{\mathcal{S}_m}(n) \leq m^{2m} + \frac{2}{\log 2} \log^{m+2} n$$

for large n, and summing on m we have

$$\Xi_{\mathcal{S}}(n) \leq (2 \log\log n)^{4 \log\log n + 1} + \frac{2}{\log 2} (\log n)^{A_\epsilon \log\log n} = O(n^\epsilon)$$

for every $\epsilon > 0$. □

Proposition 7.7 *Let $\mathcal{S}$ be the set of numerical semigroups of multiplicity $m \geq 4$ such that $F \equiv 0 \bmod 6$. Then, for every $\epsilon > 0$,*

$$\Xi_{\mathcal{S}}(n) = O(n^\epsilon).$$

Proof The proof is entirely analogous to the proof of Theorem 7.6. □

An interesting point to note is that, if we are interested in an asymptotic bound or expression for $\Xi(n)$, the families considered in Theorems 7.4 and 7.6 or in Proposition 7.7 give a contribution of a lower order than Ξ_3 (for which Theorem 6.4 gives a linear term); hence, these families are irrelevant when considering (the dominant term of) the asymptotic growth for Ξ.

8 Linear Families

In the previous section, Theorem 7.1 has been applied on families where, while the Frobenius number increases, also the generators (or at least some of them) increase; this is then used to prove an exponential bound on $|\mathrm{Star}(S)|$, which in turn gives a bound of type $O(n^\epsilon)$ on $\Xi_{\mathcal{S}}$. In general, however, it is possible to have a family of semigroups where the Frobenius number increases, while some generators remain fixed.

Let S be a numerical semigroup and $d > 1$ be an integer dividing $m(S)$. Let $\{b_1, \dots, b_s\}$ be integers such that $b_i \geq d \cdot (F(S) + m(S))$ and such that each b_i is coprime with $m(S)$. Then, $T := \langle dS, b_1, \dots, b_s \rangle$ is a numerical semigroup. We can divide the Apéry set of T into two parts, $d\mathrm{Ap}(S)$ and a set $A := \{a_1, \dots, a_t\}$ where each a_i is bigger than every element of $d\mathrm{Ap}(S)$.

For every $k \geq 0$, let now $T_k := \langle dS, A + kd \rangle$; then, T_k is still a numerical semigroup, and $T_k = dS \cup (A + kd + m(T)\mathbb{N})$. Considering the family $\{T_k\}_{k \geq 0}$, this means that one part of the semigroup remains fixed for every member of the family, while another part gets smaller and smaller.

We call a family $\mathcal{T} := \{T_k\}_{k \geq 1}$ constructed in this way the *linear family* constructed from S, d and $\{b_1, \dots, b_s\}$.

In particular, we have $F(T_k) = F(T) + kd$; furthermore, if $x \in \mathbb{N} \setminus S$ and $x + m(S) \in dS$, then $F(T) - x \in T$ if and only if $F(T_k) - x = F(T) + kd - x \in T_k$. Suppose now that T has only two holes, x and $F(T) - x$, and suppose that $x + m(S) \in dS$. Then, the only holes of T_k will be x and $F(T) + kd - x$; in particular, the method applied in the previous section using Theorem 7.1 can fail badly, in the sense that the integer σ will be the same for all members of the family. In particular, the bound on $|\mathrm{Star}(S)|$ does not increase with k.

Example 8.1 Start from $S = \langle 2, 3 \rangle$ and take $d = 2$. Then, $d(F(S) + m(S)) = 6$, so we can take $\{b_1, b_2\} = \{9, 11\}$. Hence, $T := \langle 4, 6, 9, 11 \rangle$, while $T_k := \langle 4, 6, 9 + 2k, 11 + 2k \rangle$. The only holes of T are 2 and 7, so the holes of T_k are 2 and $7 + 2k$. For the hole $a = 2$, the only possible σ is 0, while for the hole $a = 7 + 2k$ the set $\{a - m + 1, \dots, a - 1\}$ contains a unique element out of S, namely $a - m + 2 = 5 + 2k$, and thus Theorem 7.1 cannot even be applied to $7 + 2k$.

The only estimate we have is thus Theorem 4.1, which gives $|\mathrm{Star}(T_k)| \geq g(T_k) + 1 = k + 5$ and corresponds to a bound $\Xi_{\mathcal{T}}(n) \leq n - 4$, where $\mathcal{T} := \{T_k\}_{k \geq 1}$.

For this particular family, [24, Proposition 5.8] gives the *upper* bound $|\mathrm{Star}(T_k)| \leq 65 + 30k$, which in particular implies $\Xi_{\mathcal{T}}(n) \geq \frac{1}{30}n - \frac{65}{30}$.

A calculation of $|\mathrm{Star}(T_k)|$ for low k suggests that the behavior of $|\mathrm{Star}(T_k)|$ is linear in k; more precisely, that $|\mathrm{Star}(T_k)| = 51 + 20k$, and thus that $\Xi_{\mathcal{T}}(n) = \frac{1}{20}n - \frac{31}{20} = \frac{1}{20}(n - 31)$.

In general, there will be linear families for which $|\mathrm{Star}(T_k)|$ does not exhibit a linear behavior: for example, if $m(S)$ is odd and coprime with 3 (and so d must be odd too) then $F(T_k)$ will be alternatively even and odd, and so for at least one half of the semigroups of the family we can apply Theorem 7.6; the same happens if T has holes that are bigger than the elements of $d\mathrm{Ap}(S)$.

On the other hand, if the behavior of $|\mathrm{Star}(T_k)|$ *is* linear (as it seems to happen in the example), then the contribution of $\Xi_{\mathcal{T}}$ to Ξ has the same asymptotic growth as Ξ_3, contrary to what happens for the families of Sect. 7. In particular, the overall contribution of these families will depend also on the precise value of the linear bounds on $\Xi_{\mathcal{T}}$, which seem difficult to calculate theoretically for all families.

In Table 1, we list the precise value of $|\mathrm{Star}(T_k)|$ for a few families obtained with the above construction and for which the sequence $\{|\mathrm{Star}(T_k)|\}$ exhibits (experimentally) a linear behavior.

Table 1 Linear behavior of $|\mathrm{Star}(S)|$

S	d	$\{b_1, \dots, b_s\}$	T_k	$\lvert\mathrm{Star}(T_k)\rvert$	Range checked
$\langle 2, 3\rangle$	2	$\{9, 11\}$	$\langle 4, 6, 9+2k, 11+2k\rangle$	$51 + 20k$	$0 \leq k \leq 20$
$\langle 2, 5\rangle$	2	$\{15, 21\}$	$\langle 4, 10, 15+2k, 21+2k\rangle$	$1368 + 400k$	$0 \leq k \leq 15$
$\langle 2, 7\rangle$	2	$\{21, 23\}$	$\langle 4, 14, 21+2k, 23+2k\rangle$	$29{,}800 + 6800k$	$0 \leq k \leq 4$

9 Algorithms and Explicit Data

A star operation $*$ is uniquely determined by its restriction $* : \mathcal{F}_0(S) \longrightarrow \mathcal{F}_0(S)$. Since $\mathcal{F}_0(S)$ is a finite set that can be computed explicitly, the set of star operations (and, in particular, its cardinality) can be determined just by listing all maps from $\mathcal{F}_0(S)$ to itself and checking which ones satisfy the properties of a star operation.

An easier way to work algorithmically is to consider the set of closed ideals. Indeed, a star operation $*$ is also uniquely determined by the set $\mathcal{F}_0^*(S) := \{I \in \mathcal{F}_0(S) \mid I = I^*\}$; furthermore, a set $\Delta \subseteq \mathcal{F}_0(S)$ is equal to $\mathcal{F}_0^*(S)$ for some $*$ if and only if it satisfies the following conditions [21, Lemma 3.3]:

- $S \in \Delta$;
- if $I, J \in \Delta$, then $I \cap J \in \Delta$;
- if $I \in \Delta$ and $k \in I$, then $(-k + I) \cap \mathbb{N} \in \Delta$.

In particular, since every star operation is smaller than the divisorial closure, Δ must also contain the set $\mathcal{F}_0^v(S) = \{I \in \mathcal{F}_0(S) \mid I = I^v\}$.

Hence, we can write $\mathcal{F}_0^*(S) = \mathcal{F}_0^v(S) \cup \mathcal{G}_0^*(S)$, where $\mathcal{G}_0^*(S) := \mathcal{G}_0(S) \cap \mathcal{F}_0^*(S)$. By definition, $\mathcal{G}_0^*(S)$ must be downward closed in the $*$-order: thus, we need only to check the subsets of $\mathcal{G}_0(S)$ that are downward closed, and these can be constructed recursively (either directly or by constructing the antichains Θ of $\mathcal{G}_0(S)$ and then considering the sets $\Theta^{\downarrow} := \{J \mid J \leq_* I \text{ for some } I \in \Theta\}$). Furthermore, for any ideal I, the ideals $I \cap J$ (for J divisorial) and $(-k + I) \cap \mathbb{N}$ (for $k \in I$) are always smaller than I in the $*$-order, and thus they do not need to be checked.

Therefore, we can write the following algorithm to calculate the cardinality of $\mathrm{Star}(S)$.

1. Find all ideals in $\mathcal{F}_0(S)$:
   (a) find $\mathrm{Ap}(S) = \{0 = a_0, a_1, \dots, a_{m-1}\}$, where $m = m(S)$ and $a_i \equiv i \bmod m$;
   (b) for each $1 \leq i \leq m - 1$, let $b_i := \lfloor a_i/m \rfloor$;
   (c) for each vector $\mathbf{v} := [c_1, \dots, c_{m-1}]$ such that $0 \leq c_i \leq b_i$ for all i, consider the set $I(\mathbf{v}) := S \cup \bigcup_i (c_i + m\mathbb{N})$;
   (d) if $I(\mathbf{v})$ is an ideal, store it into $\mathcal{F}_0(S)$.
2. Divide $\mathcal{F}_0(S)$ into $\mathcal{F}_0^v(S)$ and $\mathcal{G}_0(S)$ by checking whether $I = I^v$ or $I \neq I^v$ for all $I \in \mathcal{F}_0(S)$.
3. Construct the $*$-order by checking if $I \leq_* J$ or $J \leq_* I$ for every pair (I, J).

4. For all downward closed subsets Λ of $\mathcal{G}_0(S)$:
   (a) consider $\Delta := \Lambda \cup \mathcal{F}_0^v(S)$;
   (b) check if $I \cap J \in \Delta$ for all $I, J \in \Lambda$;
   (c) if this condition holds, $\Delta = \mathcal{F}_0^*(S)$ for some star operation $*$.

This algorithm has been implemented in GAP, using the functions of the package `numericalsgps` [3, 29].

To calculate explicitly $\Xi(n)$ (for some $n \geq 2$), we can use Theorem 4.1 and Proposition 5.7 to limit the calculation to a finite number of semigroups, and the estimates in Sects. 5–7 to greatly shrink the number of semigroups.

1. Find the maximal m such that $\omega\left(\left\lceil \frac{m-1}{2} \right\rceil\right) \leq n$ (call it M);
2. For $m = 3$, calculate how many binomial coefficients $\binom{a}{b}$ satisfy $a+b \equiv 1 \bmod 3$ and $\binom{a}{b} \leq n$.
3. For $4 \leq m \leq M$, find all numerical semigroups S of multiplicity m with $g(S) \leq n-1$.
4. For every such semigroup S:
   (a) for every $a \in \mathbb{N} \setminus S$, bound $\omega_i(Q_a)$ by using Proposition 5.7, Theorem 7.1 or an explicit calculation;
   (b) if their sum is strictly larger than n, by Corollary 5.5 we have $|\mathrm{Star}(S)| > n$;
   (c) if the sum is at most n, calculate explicitly $|\mathrm{Star}(S)|$.

Remark 9.1

(a) The number of numerical semigroups of multiplicity m and genus up to $n-1$ grows polynomially, and M grows very slowly with n (as a double logarithm of n, by [27, Proposition 8.2]/Theorem 5.8 – for example, if $n = 7000$ we have only $M = 7$).
(b) Those semigroups can be found efficiently by solving linear inequalities, using the so-called *Kunz polytope* of S (see [10, 20]).
(c) Step 4 of the algorithm is very flexible, because it allows to use any kind of estimate on $|\mathrm{Star}(S)|$ before calculating it explicitly. For example, it is possible to use first Proposition 5.7 to obtain a quick estimate, and then, for those semigroups whose estimate is below n, calculate explicitly all of the sets Q_a (which is slower, but gives a better bound). It can also be used with other estimates, not necessarily depending on Q_a.

Using this algorithm, I calculated $\Xi(n)$ and $\Xi_m(n)$ for all $n \leq 150$, and $\Xi_m(n)$ for $m \in \{3, 5, 7\}$ and for all $n \leq 2000$ (for $m = 4$ and $m = 6$, the fact that m is not prime introduces linear families, which slow down considerably the calculation). Tables 2 and 3 show these values, and Table 4 lists those semigroups for $m(S) > 3$.

Table 2 $\Xi(n)$ for $n \leq 150$

n	$\Xi(n)$	$\Xi_3(n)$	$\Xi_4(n)$	$\Xi_5(n)$	$\Xi_6(n)$	$\Xi_7(n)$
10	8	7	1	0	0	0
20	18	14	4	0	0	0
30	27	22	4	1	0	0
40	40	31	6	3	0	0
50	46	37	6	3	0	0
60	57	46	8	3	0	0
70	69	54	9	6	0	0
80	76	60	10	6	0	0
90	83	67	10	6	0	0
100	93	75	11	7	0	0
110	101	82	12	7	0	0
120	111	90	13	8	0	0
130	122	98	15	9	0	0
140	131	105	17	9	0	0
150	141	112	17	12	0	0

Table 3 $\Xi_m(n)$ for $n \leq 2000$ and $m \in \{3, 5, 7\}$

n	$\Xi_3(n)$	$\Xi_5(n)$	$\Xi_7(n)$
100	75	7	0
200	148	13	0
300	220	16	0
400	290	21	0
500	361	21	0
600	431	22	0
700	500	22	0
800	570	22	0
900	639	24	0
1000	709	24	0
1100	776	25	0
1200	845	25	1
1300	914	25	1
1400	982	28	1
1500	1050	28	1
1600	1120	28	1
1700	1186	29	1
1800	1257	30	1
1900	1326	30	1
2000	1393	30	1

Table 4 Numerical semigroups with few star operations (with $\lvert\mathrm{Star}(S)\rvert$ in parentheses)

$m(S) = 4$, $\lvert\mathrm{Star}(S)\rvert \leq 150$		
• $\langle 4, 5, 7\rangle$ (7) • $\langle 4, 5, 6, 7\rangle$ (14) • $\langle 4, 5, 11\rangle$ (14) • $\langle 4, 7, 9\rangle$ (15) • $\langle 4, 9, 11\rangle$ (31) • $\langle 4, 6, 7, 9\rangle$ (32)	• $\langle 4, 6, 9, 11\rangle$ (51) • $\langle 4, 7, 17\rangle$ (57) • $\langle 4, 11, 13\rangle$ (63) • $\langle 4, 6, 11, 13\rangle$ (71) • $\langle 4, 6, 13, 15\rangle$ (91) • $\langle 4, 7, 10, 13\rangle$ (105)	• $\langle 4, 6, 15, 17\rangle$ (111) • $\langle 4, 13, 15\rangle$ (127) • $\langle 4, 7, 13\rangle$ (129) • $\langle 4, 6, 17, 19\rangle$ (131) • $\langle 4, 7, 9, 10\rangle$ (131)
$m(S) = 5$, $\lvert\mathrm{Star}(S)\rvert \leq 2000$		
• $\langle 5, 6, 7, 9\rangle$ (21) • $\langle 5, 6, 13\rangle$ (31) • $\langle 5, 6, 7\rangle$ (32) • $\langle 5, 7, 16\rangle$ (63) • $\langle 5, 7, 13\rangle$ (65) • $\langle 5, 6, 8\rangle$ (68) • $\langle 5, 8, 9, 11\rangle$ (96) • $\langle 5, 7, 8\rangle$ (117) • $\langle 5, 8, 19\rangle$ (127) • $\langle 5, 8, 11, 12\rangle$ (141)	• $\langle 5, 7, 9\rangle$ (147) • $\langle 5, 6, 8, 9\rangle$ (148) • $\langle 5, 6, 7, 8, 9\rangle$ (163) • $\langle 5, 6, 14\rangle$ (206) • $\langle 5, 9, 22\rangle$ (255) • $\langle 5, 6, 19\rangle$ (275) • $\langle 5, 7, 9, 13\rangle$ (340) • $\langle 5, 9, 16\rangle$ (351) • $\langle 5, 7, 8, 11\rangle$ (369) • $\langle 5, 6, 9, 13\rangle$ (387)	• $\langle 5, 9, 12, 13\rangle$ (400) • $\langle 5, 7, 11\rangle$ (539) • $\langle 5, 7, 8, 9, 11\rangle$ (824) • $\langle 5, 8, 11\rangle$ (867) • $\langle 5, 11, 28\rangle$ (1023) • $\langle 5, 6, 13, 14\rangle$ (1331) • $\langle 5, 8, 9\rangle$ (1356) • $\langle 5, 11, 12, 14\rangle$ (1363) • $\langle 5, 7, 23\rangle$ (1685) • $\langle 5, 8, 9, 12\rangle$ (1726)
$m(S) = 7$, $\lvert\mathrm{Star}(S)\rvert \leq 2000$		
• $\langle 7, 8, 9, 19\rangle$ (1116)		

10 The Ring Version

Suppose D is an integral domain with quotient field K. A *star operation* on D is a map $* : \mathcal{F}(D) \longrightarrow \mathcal{F}(D)$ that is extensive, order-preserving, idempotent, satisfies $D = D^*$ and such that $x \cdot I^* = (xI)^*$ for all $x \in K$ and all $I \in \mathcal{F}(D)$ (where $\mathcal{F}(D)$ is the set of fractional ideals of D, i.e., of the D-submodules I of the quotient field K of D such that $xI \subseteq D$ for some $x \neq 0$).

The concepts of principal star operations and of the $*$-order can be introduced also for rings; however, in general, there is no set corresponding to $\mathcal{F}_0(S)$ (and so to $\mathcal{G}_0(S)$). Furthermore, we may have $*_I = *_J$ even if I, J are nondivisorial and $I \neq xJ$ for all x.

In this section, we want to study star operations on a class of domains which is close to numerical semigroups. In particular, we shall study domains R satisfying the following conditions:

- R is Noetherian, one-dimensional and local;
- its integral closure V is a discrete valuation ring (DVR);

- the conductor ideal $(R : V)$ is nonzero;
- the extension of residue fields $R/\mathfrak{m}_R \subseteq V/\mathfrak{m}_V$ induced by the extension $R \subseteq V$ is an isomorphism.

Note that, in the previous conditions, we could have dropped "one-dimensional" and "local", since they follow from the fact that the integral closure is a DVR. An equivalent characterization is that the domains we study are the one-dimensional local Noetherian domains that are analytically irreducible and residually rational.

From now on, fix a discrete valuation ring V, and denote by $\mathscr{R}(V)$ the domains of this form whose integral closure is V; R will be a domain in $\mathscr{R}(V)$ and $\mathfrak{m}$ its maximal ideal. We shall use $\mathbf{v}$ to denote the normalized valuation relative to V: then, the set $\mathbf{v}(R) := \{\mathbf{v}(r) \mid r \in R\}$ is a numerical semigroup.

The questions we want to answer in this case are the same as in the numerical semigroup case: is the number of rings in $\mathscr{R}(V)$ with exactly n star operations finite? how many have less than n star operations? How to bound $|\mathrm{Star}(R)|$, for $R \in \mathscr{R}(V)$? For $n = 1$, the answer is well-known: $|\mathrm{Star}(R)| = 1$ if and only if R is Gorenstein, which happens if and only if $\mathbf{v}(R)$ is symmetric, i.e., if and only if $|\mathrm{Star}(\mathbf{v}(R))| = 1$ [2, 14].

Define $\mathcal{F}_0(R) := \{I \in \mathcal{F}(R) \mid R \subseteq I \subseteq V\}$: then, every fractional ideal I is isomorphic to an element of $\mathcal{F}_0(R)$ (just take $x^{-1}I$, where $x \in I$ satisfies $\mathbf{v}(x) = \min \mathbf{v}(I)$). However, unlike the semigroup case, this ideal is not unique: that is, if $y \in I$ is another element of minimal valuation, it may be that $x^{-1}I \neq y^{-1}I$. In particular, we can have $*_{x^{-1}I} = *_{y^{-1}I}$ even if $x^{-1}I \neq y^{-1}I$. However, if I and J are in $\mathcal{F}_0(S)$ and not divisorial, then $*_I = *_J$ implies that $\mathbf{v}(I) = \mathbf{v}(J)$ [21, Proposition 6.4]. We can thus prove an analogue to Theorem 4.1.

If S is a numerical semigroup, a *canonical ideal* of S is a fractional ideal Ω such that $(\Omega - (\Omega - I)) = I$ for every fractional ideal I of S, or equivalently such that $*_{\Omega(S)}$ is the identity. Every canonical ideal is in the form $a + K(S)$, where $K(S) := \{t \in \mathbb{N} \mid F(S) - t \in S\}$ is sometimes called the *standard canonical ideal* of S [9, Section 5]. Likewise, if D is an integral domain, a canonical ideal of D is a fractional ideal Ω such that $(\Omega : (\Omega : I)) = I$ for every fractional ideal I. If $R \in \mathscr{R}(V)$, then R admits canonical ideals [15, Theorem 15.7], and if Ω is one of them then $\mathbf{v}(\Omega)$ is a canonical ideal of $\mathbf{v}(R)$ [9, Satz 5].

Proposition 10.1 *Let $R \in \mathscr{R}(V)$, and suppose that R is not Gorenstein. Then,* $|\mathrm{Star}(R)| \geq g(\mathbf{v}(R)) + 1$.

Proof Let $S := \mathbf{v}(R)$. Since R is not Gorenstein, S is not symmetric, and thus there is a $\tau \in T(S) \setminus \{F(S)\}$; let $\lambda := \min\{\tau, F(S) - \tau\}$. For any positive $a \in \mathbb{N}$, let $T_a := R \cup \{\phi \in V \mid \mathbf{v}(\phi) > a\}$; then, T_a is a ring in $\mathscr{R}(V)$ and $\mathbf{v}(T_a) = \mathbf{v}(R) \cup \{x \in \mathbb{N} \mid x > a\}$, so that $F(\mathbf{v}(T_a)) = a$. For every a, let Ω_a be a canonical ideal of T_a such that $\mathbf{v}(\Omega_a) = \{t \in \mathbb{N} \mid a - t \in \mathbf{v}(T_a)\}$ is the standard canonical ideal of $\mathbf{v}(T_a)$.

Let $x \in \mathbb{N} \setminus S$. We distinguish three cases.

If $x < \lambda$ and $\lambda - x \notin S$, let $I_x := R + \{\phi \in \Omega_\lambda \mid \mathbf{v}(\phi) > x\}$. Then, I_x is an R-module, and $\mathbf{v}(I_x) = \mathbf{v}(R) \cup \{t \in \mathbf{v}(\Omega_\lambda) \mid t > x\}$; in particular, $\lambda \notin \mathbf{v}(I_x)$, and thus $\mathbf{v}(I_x)$ is not divisorial over S, which implies that I_x is not divisorial over R [1, Lemma II.1.22].

If $x < \lambda$ and $\lambda - x \in S$, let $y := g(S) - \lambda + x = g(S) - (\lambda - x)$, and define $I_x := R \cup \{\phi \in V \mid \mathbf{v}(\phi) > y\}$. Then, $\mathbf{v}(I_x)$ is not divisorial since it contains $g(S)$ but not $g(S) - \lambda$, and so I_x is not divisorial.

If $x \geq \lambda$ and $x \neq g(S)$, let $I_x := \Omega_x$: then, I_x is not divisorial since otherwise $T_x = (\Omega_x : \Omega_x)$ would be divisorial, against the fact that $\mathbf{v}(T_x)$ contains $g(S)$ but not λ (if $x = g$, then Ω_x is not divisorial since otherwise S would be symmetric).

It is straightforward to see that $\mathbf{v}(I_x) \neq \mathbf{v}(I_y)$ for $x \neq y$; hence, each one generates a different star operation, and $|\mathrm{Star}(R)| \geq g(\mathbf{v}(R)) + 1$. □

We also note that Proposition 5.7 carries over to the domain case, and in particular $|\mathrm{Star}(R)| \geq \omega(\nu(\mathbf{v}(R)))$. We now prove an analogue of Theorem 4.2, but we have to add an important additional hypothesis.

Theorem 10.2 *Let V be a DVR with finite residue field.*

(a) Every $R \in \mathscr{R}(V)$ has only finitely many star operations.
(b) For every $n > 1$, the set $\{R \in \mathscr{R}(V) \mid 2 \leq |\mathrm{Star}(R)| \leq n\}$ is finite.

Proof The first claim is a special case of [8, Theorem 2.5]. (It follows, for example, from the fact that $\mathcal{F}_0(R)$ is finite.)

For the second claim, we see that if $2 \leq |\mathrm{Star}(R)| \leq n$, then $\mathbf{v}(R)$ is not symmetric and $g(\mathbf{v}(R)) \leq n - 1$; hence, there are only finitely many possible $\mathbf{v}(R)$. Furthermore, since the residue field of V is finite, for any S there are only finitely many R such that $\mathbf{v}(R) = S$ [21, Lemma 5.13(a)]; hence, there are only finitely many $R \in \mathscr{R}(V)$ with $|\mathrm{Star}(R)| \leq n$. The claim is proved. □

In the previous theorem, the restriction to a finite residue field is not really restricting, since otherwise $\mathrm{Star}(R)$ is very often infinite.

Proposition 10.3 *Let $R \in \mathscr{R}(V)$, and suppose that the residue field F of R is infinite; suppose also that R is not Gorenstein. If $m(\mathbf{v}(R)) > 3$, then* $\mathrm{Star}(R)$ *is infinite.*

Proof Let $A := (\mathfrak{m} : \mathfrak{m})$; then, A is a ring, and it is local since its integral closure is V. Since R is not Gorenstein, $\dim_F(A/\mathfrak{m}) > 2$ [2, Theorem 6.3]. If $\dim_F(A/\mathfrak{m}) \geq 4$, then $|\mathrm{Star}(R)| = \infty$ by [8, Corollary 2.8]. If $\dim_F(A/\mathfrak{m}) = 3$, then following [6] let N be the maximal ideal of A and let $B := (N : N)$; by [6, Theorem 2.15], if $\mathrm{Star}(R)$ is finite then $B = V$ and $\dim_F(B/\mathfrak{m}B) = 3$. By [16],

$$\dim_F(B/\mathfrak{m}B) = |\mathbf{v}(B) \setminus \mathbf{v}(\mathfrak{m}B)| = m(\mathbf{v}(R))$$

since $\mathfrak{m}B$ contains all elements of valuation $m(\mathbf{v}(R))$ or more. Hence, if $m(\mathbf{v}(R)) > 3$ then $\mathrm{Star}(R)$ is infinite, as claimed. □

We can also obtain an explicit version of Theorem 10.2.

Lemma 10.4 *Let F be a finite field of cardinality q, and let W be a vector space over F of dimension n. Then, W has at most $2^n q^{n(n-1)/2}$ vector subspaces.*

Proof The number of vector subspaces of W of dimension k is the *q-binomial coefficient* (or *Gaussian binomial coefficient*)

$$\binom{n}{k}_q := \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-t+1}-1)}{(q^t-1)(q^{t-1}-1)\cdots(q-1)}$$

(see e.g. [28, Proposition 1.3.18] or [5, Chapter 13, Proposition 2.1]). Using the q-binomial theorem [28, Chapter 3, Exercise 45] with $y = z = 1$ we have

$$\sum_{k=0}^{n}\binom{n}{k}_q \le \sum_{k=0}^{n} q^{k(k-1)/2}\binom{n}{k}_q = \prod_{k=0}^{n-1}(1+q^k) \le 2^n q^{n(n-1)/2},$$

as claimed. □

Theorem 10.5 *There is a constant C such that, for all discrete valuation rings V with residue field F of finite cardinality q and for all n,*

$$\Xi_V(n) := |\{R \in \mathscr{R}(V) \mid 2 \le |\mathrm{Star}(R)| \le n\}| \le C(4\varphi)^n q^{n(2n-1)}$$

where $\varphi := \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Proof If $|\mathrm{Star}(R)| \le n$, then by Theorem 10.2 we have $g(\mathbf{v}(R)) \le n-1$, and by [32] there are at most $C'\varphi^{n-1}$ semigroups with this property, for some constant C'. If S is a numerical semigroup, then as in the proof of [21, Lemma 5.13(a)] the $R \in \mathscr{R}(V)$ such that $\mathbf{v}(R) = S$ correspond to certain F-vector subspaces of $V/\mathfrak{m}_V^{F(S)+1}$; since $F(S) \le 2g(S)$, using Lemma 10.4 we see that each S gives at most $2^{2n}q^{n(2n-1)}$ rings. Hence,

$$\Xi_V(n) \le C'\varphi^{n-1} \cdot 2^{2n} q^{n(2n-1)} = C(4\varphi)^n q^{n(2n-1)}$$

with $C := C'/\varphi$. □

In this bound, the term φ^n can be substituted by a better bound, using (the analogue of) Proposition 5.7; however, the main term is $q^{n(2n-1)}$, whose lowering hinges on a more precise grasp of how many rings correspond to a given semigroup.

In general, the cardinality of $\mathrm{Star}(R)$ does not depend only on $S = \mathbf{v}(R)$ and on the residue field of V, but also on the precise nature of R itself; as a consequence, while it is possible to calculate explicitly $|\mathrm{Star}(R)|$ for a fixed R, in general there will not be a general formula (valid for each R). Sometimes, however, knowing S and the residue field is everything we need.

Proposition 10.6 *Let V be a DVR with residue field F, and let $q := |F|$. Let $R \in \mathscr{R}(V)$. Then:*

(a) [8, Theorem 3.8] if $\mathbf{v}(R) = \langle 3, 4, 5\rangle$, then $|\mathrm{Star}(R)| = 3$;
(b) [8, Example 3.10] if $\mathbf{v}(R) = \langle 3, 5, 7\rangle$, then $|\mathrm{Star}(R)| = 4$;
(c) [25, Proposition 3.4] if $\mathbf{v}(R) = \langle 4, 5, 7\rangle$, then $|\mathrm{Star}(R)| = 2^{2q+3}$;
(d) [30, Corollary 4.1.2] if $\mathbf{v}(R) = \langle 4, 5, 6, 7\rangle$, then $|\mathrm{Star}(R)| = 2^{2q+1} + 2^{q+1} + 2$.

Remark 10.7

(a) If $q = \infty$, then the last two cases should be interpreted as saying that $\mathrm{Star}(R)$ is infinite.
(b) The proofs given in [8, Example 3.10] and [30, Corollary 4.1.2] for $\mathbf{v}(R) = \langle 3, 5, 7\rangle$ and $\mathbf{v}(R) = \langle 4, 5, 6, 7\rangle$ (respectively) were given only in the case $R = K[[S]]$. However, their proofs can be applied also to the general case.

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