

# The Golomb space is topologically rigid

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*Abstract.* The Golomb space  $\mathbb{N}_\tau$  is the set  $\mathbb{N}$  of positive integers endowed with the topology  $\tau$  generated by the base consisting of arithmetic progressions  $\{a + bn: n \geq 0\}$  with coprime  $a, b$ . We prove that the Golomb space  $\mathbb{N}_\tau$  is topologically rigid in the sense that its homeomorphism group is trivial. This resolves a problem posed by T. Banakh at Mathoverflow in 2017.

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## 1. Introduction

In the AMS Meeting announcement [3] M. Brown introduced an amusing topology  $\tau$  on the set  $\mathbb{N}$  of positive integers turning it into a connected Hausdorff space. The topology  $\tau$  is generated by the base consisting of arithmetic progressions  $a + b\mathbb{N}_0 := \{a + bn: n \in \mathbb{N}_0\}$  with coprime parameters  $a, b \in \mathbb{N}$ . Here by  $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$  we denote the set of nonnegative integer numbers.

In [15] the topology  $\tau$  is called the *relatively prime integer topology*. This topology was popularized by S. Golomb in [7], [8], who observed that the classical Dirichlet theorem on primes in arithmetic progressions is equivalent to the density of the set  $\Pi$  of prime numbers in the topological space  $(\mathbb{N}, \tau)$ . As a by-product of such popularization efforts, the topological space  $\mathbb{N}_\tau := (\mathbb{N}, \tau)$  is known in general topology as the *Golomb space*, see [16], [17].

The topological structure of the Golomb space  $\mathbb{N}_\tau$  was studied by T. Banakh, J. Mioduszewski and S. Turek in [2], who proved that the space  $\mathbb{N}_\tau$  is not topologically homogeneous (by showing that 1 is a fixed point of any homeomorphism of  $\mathbb{N}$ ). Motivated by this results, the authors of [2] posed a problem of the topological rigidity of the Golomb space. This problem was also repeated by the first author at [Mathoverflow](#), see [1]. A topological space  $X$  is defined to be *topologically rigid* if its homeomorphism group is trivial.

The main result of this note is the following theorem answering the above problem.

**Theorem 1.** *The Golomb space  $\mathbb{N}_\tau$  is topologically rigid.*

The proof of this theorem will be presented in Section 5 after some preparatory work made in Sections 3 and 4. The idea of the proof belongs to the second author who studied in [13] the rigidity properties of the Golomb topology on a Dedekind ring with removed zero, and established in [13, Theorem 6.7] that the homeomorphism group of the Golomb topology on  $\mathbb{Z} \setminus \{0\}$  consists of two homeomorphisms. The proof of Theorem 1 is a modified (and simplified) version of the proof of Theorem 6.7 given in [13]. It should be mentioned that the Golomb topology on Dedekind rings with removed zero was studied by J. Knopfmacher, Š. Porubský in [11], P.L. Clark, N. Lebowitz-Lockard, P. Pollack in [4], and D. Spirito in [13], [14].

**2. Preliminaries and notations**

In this section we fix some notation and recall some known results on the Golomb topology. For a subset  $A$  of a topological space  $X$ , by  $\bar{A}$  we denote the closure of  $A$  in  $X$ .

A *poset* is a set  $X$  endowed with a partial order “ $\leq$ ”. A subset  $L$  of a partially ordered set  $(X, \leq)$  is called

- *linearly ordered* (or else a *chain*) if any points  $x, y \in L$  are *comparable* in the sense that  $x \leq y$  or  $y \leq x$ ;
- an *antichain* if any two distinct elements  $x, y \in A$  are not comparable.

By  $\Pi$  we denote the set of prime numbers. For a number  $x \in \mathbb{N}$  we denote by  $\Pi_x$  the set of all prime divisors of  $x$ . Two numbers  $x, y \in \mathbb{N}$  are *coprime* if and only if  $\Pi_x \cap \Pi_y = \emptyset$ . For a number  $x \in \mathbb{N}$  let  $x^{\mathbb{N}} := \{x^n : n \in \mathbb{N}\}$  be the set of all powers of  $x$ .

For a number  $x \in \mathbb{N}$  and a prime number  $p$  let  $l_p(x)$  be the largest integer number such that  $p^{l_p(x)}$  divides  $x$ . The function  $l_p(x)$  plays the role of logarithm with base  $p$ .

The following formula for the closures of basic open sets in the Golomb topology was established in [2, 2.2].

**Lemma 2** (T. Banakh, J. Mioduszewski, S. Turek). *For any  $a, b \in \mathbb{N}$*

$$\overline{a + b\mathbb{N}_0} = \mathbb{N} \cap \bigcap_{p \in \Pi_b} (p\mathbb{N} \cup (a + p^{l_p(b)}\mathbb{Z})).$$

Also we shall heavily exploit the following lemma, proved in [2, 5.1].

**Lemma 3** (T. Banakh, J. Mioduszewski, S. Turek). *Each homeomorphism  $h: \mathbb{N}_\tau \longrightarrow \mathbb{N}_\tau$  of the Golomb space has the following properties:*

- (1)  $h(1) = 1$ ;
- (2)  $h(\mathbb{N}) = \mathbb{N}$ ;
- (3)  $\Pi_{h(x)} = h(\Pi_x)$  for every  $x \in \mathbb{N}$ ;
- (4)  $h(x^{\mathbb{N}}) = h(x)^{\mathbb{N}}$  for every  $x \in \mathbb{N}$ .

Let  $p$  be a prime number and  $k \in \mathbb{N}$ . Let  $\mathbb{Z}$  be the ring of integer numbers,  $\mathbb{Z}_{p^k}$  be the residue ring  $\mathbb{Z}/p^k\mathbb{Z}$ , and  $\mathbb{Z}_{p^k}^\times$  be the multiplicative group of invertible elements of the ring  $\mathbb{Z}_{p^k}$ . It is well-known that  $|\mathbb{Z}_{p^k}^\times| = \varphi(p^k) = p^{k-1}(p-1)$ . The structure of the group  $\mathbb{Z}_{p^k}^\times$  was described by Gauss in [6, art. 52–56] (see also Theorems 2 and 2' in Chapter 4 of [9]).

**Lemma 4** (C. F. Gauss). *Let  $p$  be a prime number and  $k \in \mathbb{N}$ .*

- (1) *If  $p$  is odd, then the group  $\mathbb{Z}_{p^k}^\times$  is cyclic.*
- (2) *If  $p = 2$  and  $k \geq 2$ , then the element  $-1 + 2^k\mathbb{Z}$  generates a two-element cyclic group  $C_2$  in  $\mathbb{Z}_{2^k}^\times$  and the element  $5 + 2^k\mathbb{Z}$  generates a cyclic subgroup  $C_{2^{k-2}}$  of order  $2^{k-2}$  in  $\mathbb{Z}_{2^k}^\times$  such that  $\mathbb{Z}_{2^k}^\times = C_2 \oplus C_{2^{k-2}}$ .*

**Lemma 5.** *If  $H$  is a non-cyclic subgroup of the multiplicative group  $\mathbb{Z}_{2^k}^\times$  for some  $k \geq 3$ , then  $H$  contains the Boolean subgroup*

$$V = \{1 + 2^k\mathbb{Z}, -1 + 2^k\mathbb{Z}, 1 + 2^{k-1} + 2^k\mathbb{Z}, -1 + 2^{k-1} + 2^k\mathbb{Z}\}.$$

PROOF: Observe that the multiplicative group  $\mathbb{Z}_{2^k}^\times$  has order  $2^{k-1}$ , which implies that the order of every element of  $\mathbb{Z}_{2^k}^\times$  is a power of 2. The Gauss Lemma 4 implies that the multiplicative group  $\mathbb{Z}_{2^k}^\times$  has exactly 4 elements of order less than or equal to 2 and those elements form the 4-element Boolean subgroup  $V$ .

Applying the Frobenius–Stickelberger theorem 4.2.6, see [12], we conclude that the finite subgroup  $H \subseteq \mathbb{Z}_{2^k}^\times$  is the direct sum of finite cyclic groups whose orders are powers of 2. Since  $H$  is not cyclic, at least two cyclic groups in this direct sum are not trivial, which implies that  $H$  contains at least four element of order less than or equal to 2. Taking into account that the elements of the subgroup  $V$  are the only elements of order less than or equal to 2 in the group  $\mathbb{Z}_{2^k}^\times$ , we conclude that  $V \subseteq H$ . □

### 3. Golomb topology versus the $p$ -adic topologies on $\mathbb{N}$

Let  $p$  be any prime number. Let us recall that the  $p$ -adic topology on  $\mathbb{Z}$  is generated by the base consisting of the sets  $x + p^n\mathbb{Z}$ , where  $x \in \mathbb{Z}$  and  $n \in \mathbb{N}$ . This topology induces the  $p$ -adic topology on the subset  $\mathbb{N}$  of  $\mathbb{Z}$ . It is generated by the base consisting of the sets  $x + p^n\mathbb{N}_0$  where  $x, n \in \mathbb{N}$ . It is easy to see that  $\mathbb{N}$  endowed with the  $p$ -adic topology is a regular second-countable space

without isolated points. So, by Sierpiński theorem, see [5, 6.2.A (d)], this space is homeomorphic to the space of rationals and hence is topologically homogeneous. Consequently, any nonempty open subspace of  $\mathbb{N}$  with the  $p$ -adic topology (in particular,  $\mathbb{N} \setminus p\mathbb{N}$ ) also is homeomorphic to  $\mathbb{Q}$  and hence is topologically homogeneous.

The following lemma is a special case of Proposition 3.1 in [13].

**Lemma 6.** *For any clopen subset  $\Omega$  of  $\mathbb{N}_\tau \setminus p\mathbb{N}$ , and any  $x \in \Omega$ , there exists  $n \in \mathbb{N}$  such that  $x + p^n\mathbb{N}_0 \subseteq \Omega$ .*

PROOF: Since the set  $p\mathbb{N}$  is closed in  $\mathbb{N}_\tau$ , the set  $\Omega$  is open in  $\mathbb{N}_\tau$  and hence  $x + p^n b\mathbb{N}_0 \subseteq \Omega$  for some  $n \in \mathbb{N}$  and  $b \in \mathbb{N}$  which is coprime with  $px$ . We claim that  $x + p^n\mathbb{N}_0 \subseteq \Omega$ . To derive a contradiction, assume that  $x + p^n\mathbb{N}_0 \setminus \Omega$  contains some number  $y$ . Since  $\Omega$  is closed in  $\mathbb{N}_\tau \setminus p\mathbb{N}$ , there exist  $m \geq n$  and  $d \in \mathbb{N}$  such that  $d$  is coprime with  $p$  and  $y$ , and  $(y + p^m d\mathbb{N}_0) \cap \Omega = \emptyset$ . It follows that  $y + p^m\mathbb{N}_0 \subseteq (x + p^n\mathbb{N}_0) + p^m\mathbb{N}_0 \subseteq x + p^n\mathbb{N}_0$ . Since  $p \notin \Pi_b \cup \Pi_d$ , we can apply the Chinese remainder theorem [10, 3.12] and conclude that  $\emptyset \neq (y + p^m\mathbb{N}) \cap \bigcap_{q \in \Pi_b \cup \Pi_d} q\mathbb{N}$ . Applying Lemma 2 and taking into account that the set  $\Omega$  is clopen in  $\mathbb{N}_\tau \setminus p\mathbb{N}$ , we conclude that

$$\begin{aligned} \emptyset &\neq (y + p^m\mathbb{N}_0) \cap \left( \bigcap_{q \in \Pi_b \cup \Pi_d} q\mathbb{N} \right) \\ &= (x + p^n\mathbb{N}_0) \cap \left( \bigcap_{q \in \Pi_b} q\mathbb{N} \right) \cap (y + p^m\mathbb{N}_0) \cap \left( \bigcap_{q \in \Pi_d} q\mathbb{N} \right) \\ &\subseteq \overline{x + p^n b\mathbb{N}_0} \cap \overline{y + p^m d\mathbb{N}_0} \subseteq \overline{\Omega} \cap \overline{(\mathbb{N} \setminus p\mathbb{N}) \setminus \Omega} \subseteq p\mathbb{N}, \end{aligned}$$

which is not possible as the sets  $x + p^n\mathbb{N}_0$  and  $p\mathbb{N}$  are disjoint. This contradiction shows that  $x + p^n\mathbb{N}_0 \subseteq \Omega$ . □

A subset of a topological space is *clopen* if it is closed and open. By the *zero-dimensional reflection* of a topological space  $X$  we understand the space  $X$  endowed with the topology generated by the base consisting of clopen subsets of the space  $X$ .

**Lemma 7.** *The  $p$ -adic topology on  $\mathbb{N} \setminus p\mathbb{N}$  coincides with the zero-dimensional reflection of the subspace  $\mathbb{N}_\tau \setminus p\mathbb{N}$  of the Golomb space  $\mathbb{N}_\tau$ .*

PROOF: Lemma 6 implies that the  $p$ -adic topology  $\tau_p$  on  $\mathbb{N} \setminus p\mathbb{N}$  is stronger than the topology  $\zeta$  of zero-dimensional reflection on  $\mathbb{N}_\tau \setminus p\mathbb{N}$ . To see that the  $\tau_p$  coincides with  $\zeta$ , it suffices to show that for every  $x \in \mathbb{N} \setminus p\mathbb{N}$  and  $n \in \mathbb{N}$  the basic open set  $\mathbb{N} \cap (x + p^n\mathbb{Z})$  in the  $p$ -adic topology is clopen in the subspace topology of  $\mathbb{N}_\tau \setminus p\mathbb{N} \subset \mathbb{N}_\tau$ . By the definition, the set  $\mathbb{N} \cap (x + p^n\mathbb{Z})$  is open in the Golomb

topology. To see that it is closed in  $\mathbb{N}_\tau \setminus p\mathbb{N}$ , take any point  $y \in (\mathbb{N} \setminus p\mathbb{N}) \setminus (x + p^n\mathbb{Z})$  and observe that the Golomb-open neighborhood  $y + p^n\mathbb{N}_0$  of  $y$  is disjoint with the set  $\mathbb{N} \cap (x + p^n\mathbb{Z})$ .  $\square$

For every prime number  $p$ , consider the countable family

$$\mathcal{X}_p = \{\overline{a^\mathbb{N}} : a \in \mathbb{N} \setminus p\mathbb{N}, a \neq 1\},$$

where the closure  $\overline{a^\mathbb{N}}$  is taken in the  $p$ -adic topology on  $\mathbb{N} \setminus p\mathbb{N}$ , which coincides with the topology of zero-dimensional reflection of the Golomb topology on  $\mathbb{N} \setminus p\mathbb{N}$  according to Lemma 7.

The family  $\mathcal{X}_p$  is endowed with the partial order “ $\leq$ ” defined by  $X \leq Y$  if and only if  $Y \subseteq X$ . So,  $\mathcal{X}_p$  is a poset carrying the partial order of reverse inclusion.

**Lemma 8.** *For any prime number  $p$ , any homeomorphism  $h$  of the Golomb space  $\mathbb{N}_\tau$  induces an order isomorphism*

$$h : \mathcal{X}_p \longrightarrow \mathcal{X}_{h(p)}, \quad h : \overline{a^\mathbb{N}} \mapsto h(\overline{a^\mathbb{N}}) = \overline{h(a)^\mathbb{N}}$$

of the posets  $\mathcal{X}_p$  and  $\mathcal{X}_{h(p)}$ .

PROOF: By Lemma 3,  $h(1) = 1$  and  $h(p)$  is a prime number. First we show that  $h(p\mathbb{N}) = h(p)\mathbb{N}$ . Indeed, for any  $x \in p\mathbb{N}$  we have  $p \in \Pi_x$  and by Lemma 3,  $h(p) \in h(\Pi_x) = \Pi_{h(x)}$  and hence  $h(x) \in h(p)\mathbb{N}$  and  $h(p\mathbb{N}) \subseteq h(p)\mathbb{N}$ . Applying the same argument to the homeomorphism  $h^{-1}$ , we obtain  $h^{-1}(h(p)\mathbb{N}) \subseteq p\mathbb{N}$ , which implies the desired equality  $h(p\mathbb{N}) = h(p)\mathbb{N}$ . The bijectivity of  $h$  ensures that  $h$  maps homeomorphically the space  $\mathbb{N}_\tau \setminus p\mathbb{N}$  onto the space  $\mathbb{N}_\tau \setminus h(p)\mathbb{N}$ .

Then  $h$  also is a homeomorphism of the spaces  $\mathbb{N} \setminus p\mathbb{N}$  and  $\mathbb{N} \setminus h(p)\mathbb{N}$  endowed with the zero-dimensional reflections of their subspace topologies inherited from the Golomb topology of  $\mathbb{N}_\tau$ . By Lemma 7, these reflection topologies on  $\mathbb{N} \setminus p\mathbb{N}$  and  $\mathbb{N} \setminus h(p)\mathbb{N}$  coincide with the  $p$ -adic and  $h(p)$ -adic topologies on  $\mathbb{N} \setminus p\mathbb{N}$  and  $\mathbb{N} \setminus h(p)\mathbb{N}$ , respectively.

By Lemma 3, for any  $a \in \mathbb{N} \setminus (\{1\} \cup p\mathbb{N})$  we have

$$h(a)^\mathbb{N} = h(a^\mathbb{N}) \subseteq h(\mathbb{N} \setminus p\mathbb{N}) = \mathbb{N} \setminus h(p)\mathbb{N}$$

and by the fact that  $h : \mathbb{N} \setminus p\mathbb{N} \longrightarrow \mathbb{N} \setminus h(p)\mathbb{N}$  is a homeomorphism in the topologies of zero-dimensional reflections, we get  $h(\overline{a^\mathbb{N}}) = \overline{h(a^\mathbb{N})} = \overline{h(a)^\mathbb{N}}$ . The same argument applies to the homeomorphism  $h^{-1}$ . This implies that

$$h : \mathcal{X}_p \longrightarrow \mathcal{X}_{h(p)}, \quad h : \overline{a^\mathbb{N}} \mapsto h(\overline{a^\mathbb{N}}) = \overline{h(a)^\mathbb{N}},$$

is a well-defined bijection. It is clear that this bijection preserves the inclusion order and hence it is an order isomorphism between the posets  $\mathcal{X}_p$  and  $\mathcal{X}_{h(p)}$ .  $\square$

#### 4. The order structure of the posets $\mathcal{X}_p$

In this section, given a prime number  $p$ , we investigate the order-theoretic structure of the poset  $\mathcal{X}_p$ .

For every  $n \in \mathbb{N}$  denote by  $\pi_n: \mathbb{N} \longrightarrow \mathbb{Z}_{p^n}$  the homomorphism assigning to each number  $x \in \mathbb{N}$  the residue class  $x + p^n\mathbb{Z}$ . Also for  $n \leq m$  let

$$\pi_{m,n}: \mathbb{Z}_{p^m} \longrightarrow \mathbb{Z}_{p^n}$$

be the ring homomorphism assigning to each residue class  $x + p^m\mathbb{Z}$  the residue class  $x + p^n\mathbb{Z}$ . It is easy to see that  $\pi_n = \pi_{m,n} \circ \pi_m$ . Observe that the multiplicative group  $\mathbb{Z}_{p^n}^\times$  of invertible elements of the ring  $\mathbb{Z}_{p^n}$  coincides with the set  $\mathbb{Z}_{p^n} \setminus p\mathbb{Z}_{p^n}$  and hence has cardinality  $p^n - p^{n-1} = p^{n-1}(p - 1)$ . Observe that for every  $a \in \mathbb{N} \setminus p\mathbb{Z}$  the set  $\pi_n(a^\mathbb{N}) = \pi_n(a)^\mathbb{N}$  is a multiplicative subgroup of the finite group  $\mathbb{Z}_{p^n}^\times$ .

First we establish the structure of the elements  $\overline{a^\mathbb{N}}$  of the family  $\mathcal{X}_p$ .

**Lemma 9.** *If for some  $a \in \mathbb{N} \setminus p\mathbb{Z}$  and  $n \in \mathbb{N}$  the element  $\pi_n(a)$  has order greater than or equal to  $\max\{p, 3\}$  in the multiplicative group  $\mathbb{Z}_{p^n}^\times$ , then  $\overline{a^\mathbb{N}} = \pi_n^{-1}(\pi_n(a)^\mathbb{N})$ .*

PROOF: Let  $B = b^\mathbb{N}$  be the cyclic group generated by the element  $b = \pi_n(a)$  in the multiplicative group  $\mathbb{Z}_{p^n}^\times$ . Since  $|\mathbb{Z}_{p^n}^\times| = p^{n-1}(p - 1)$ , and  $b$  has order greater than or equal to  $\max\{p, 3\}$ , the cardinality of the group  $B$  is equal to  $p^k d$  for some  $k \in \{1, \dots, n - 1\}$  and some divisor  $d$  of the number  $p - 1$ . Moreover, if  $p = 2$ , then  $2^k \geq 3$  and hence  $k \geq 2$  and  $n \geq 3$ .

For any number  $m \geq n$ , consider the quotient homomorphism

$$\pi_{m,n}: \mathbb{Z}_{p^m} \longrightarrow \mathbb{Z}_{p^n}, \quad \pi_{m,n}: x + p^m\mathbb{Z} \mapsto x + p^n\mathbb{Z}.$$

We claim that the subgroup  $H = \pi_{m,n}^{-1}(B)$  of the multiplicative group  $\mathbb{Z}_{p^m}^\times$  is cyclic. For odd  $p$  this follows from the cyclicity of the group  $\mathbb{Z}_{p^n}^\times$ , see Lemma 4.

For  $p = 2$ , by Lemma 4, the multiplicative group  $\mathbb{Z}_{2^m}^\times$  is isomorphic to the additive group  $\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$ . Assuming that  $H$  is not cyclic and applying Lemma 5, we conclude that  $H$  contains the 4-element Boolean subgroup

$$V = \{1 + 2^m\mathbb{Z}, -1 + 2^m\mathbb{Z}, 1 + 2^{m-1} + 2^m\mathbb{Z}, -1 + 2^{m-1} + 2^m\mathbb{Z}\}$$

of  $\mathbb{Z}_{2^m}^\times$ . Then  $B = \pi_{m,n}(H) \supseteq \pi_{m,n}(V) \ni -1 + 2^n\mathbb{Z}$ . Taking into account that  $-1 + 2^n\mathbb{Z}$  has order 2 in the cyclic group  $B$ , we conclude that  $-1 + 2^n\mathbb{Z} = a^{2^{k-1}} + 2^n\mathbb{Z}$ . Since  $k \geq 2$ , the odd number  $c = a^{2^{k-2}}$  is well-defined and  $c^2 + 4\mathbb{Z} = a^{2^{k-1}} + 4\mathbb{Z} = -1 + 4\mathbb{Z}$ , which is not possible (as squares of odd numbers are equal to 1 modulo 4). This contradiction shows that the group  $H$  is cyclic.

By [12, 1.5.5], the number of generators of the cyclic group  $H$  can be calculated using the Euler totient function as

$$\begin{aligned} \varphi(|H|) &= \varphi(p^{m-n}|B|) = \varphi(p^{m-n}p^k d) = \varphi(p^{m-n+k})\varphi(d) \\ &= p^{m-n+k-1}(p-1)\varphi(d) = p^{m-n}\varphi(p^k)\varphi(d) = p^{m-n}\varphi(p^k d) \\ &= p^{m-n}\varphi(|B|), \end{aligned}$$

which implies that for every generator  $g$  of the group  $B$ , every element of the set  $\pi_{m,n}^{-1}(g)$  is a generator of the group  $H$ . In particular, the element  $\pi_m(a) \in \pi_{m,n}^{-1}(\pi_n(a))$  is a generator of the group  $H$ . By the definition of  $p$ -adic topology,

$$\begin{aligned} \overline{a^{\mathbb{N}}} &= \bigcap_{m \geq n} \pi_m^{-1}(\pi_m(a)^{\mathbb{N}}) = \bigcap_{m \geq n} \pi_m^{-1}(\pi_{m,n}^{-1}(B)) \\ &= \bigcap_{m \geq n} \pi_n^{-1}(B) = \pi_n^{-1}(B) = \pi_n^{-1}(\pi_n(a)^{\mathbb{N}}). \end{aligned}$$

□

- Lemma 10.** (1) For every  $X \in \mathcal{X}_p$  there exists  $n \in \mathbb{N}$  and a cyclic subgroup  $H$  of the multiplicative group  $\mathbb{Z}_p^\times$  such that  $X = \pi_n^{-1}(H)$  and  $|H| \geq \max\{p, 3\}$ .
- (2) For every  $n \in \mathbb{N}$  and cyclic subgroup  $H$  of  $\mathbb{Z}_p^\times$  of order  $|H| \geq \max\{p, 3\}$ , there exists a number  $a \in \mathbb{N} \setminus p\mathbb{N}$  such that  $\pi_n^{-1}(H) = \overline{a^{\mathbb{N}}} \in \mathcal{X}_p$ .

**PROOF:** (1) Given any  $X \in \mathcal{X}_p$ , find a number  $a \in \mathbb{N} \setminus (\{1\} \cup p\mathbb{N})$  such that  $X = \overline{a^{\mathbb{N}}}$ . Choose any  $n \in \mathbb{N}$  with  $p^n > a^p$  and observe that the cyclic subgroup  $H \subseteq \mathbb{Z}_p^\times$ , generated by the element  $\pi_n(a) = a + p^n\mathbb{Z}$ , has order  $|H| \geq p + 1 \geq \max\{p, 3\}$ .

(2) Fix  $n \in \mathbb{N}$  and a cyclic subgroup  $H$  of  $\mathbb{Z}_p^\times$  of order  $|H| \geq \max\{p, 3\}$ . Find a number  $a \in \mathbb{N}$  such that the residue class  $\pi_n(a) = a + p^n\mathbb{Z}$  is a generator of the cyclic group  $H$ . Then  $\pi_n(a)$  has order  $|H| \geq \max\{p, 3\}$ , Lemma 9 ensures that  $\pi_n^{-1}(H) = \pi_n^{-1}(\pi_n(a)^{\mathbb{N}}) = \overline{a^{\mathbb{N}}} \in \mathcal{X}_p$ . □

For any  $X \in \mathcal{X}_p$ , let

$$n(X) = \min\{n \in \mathbb{N} : X = \pi_n^{-1}(\pi_n(X)), |\pi_n(X)| \geq \max\{p, 3\}\}.$$

Lemmas 9 and 10 imply that the number  $n(X)$  is well-defined and  $\pi_{n(X)}(X)$  is a cyclic subgroup of order greater than or equal to  $\max\{p, 3\}$  in the multiplicative group  $\mathbb{Z}_{p^{n(X)}}^\times$ . Let  $i(X)$  be the index of the cyclic subgroup  $\pi_{n(X)}(X)$  in  $\mathbb{Z}_{p^{n(X)}}^\times$ .

**Lemma 11.** Let  $p = 2$ ,  $a > 1$  be an odd integer, and  $X = \overline{a^{\mathbb{N}}}$  be the closure of the set  $a^{\mathbb{N}}$  in the 2-adic topology of  $\mathbb{N} \setminus 2\mathbb{N}$ . The cyclic subgroup  $\pi_{n(X)}(X)$  of  $\mathbb{Z}_{2^{n(X)}}^\times$  has order 4 and index  $i(X) = 2^{n(X)-3} \geq 2$ .

PROOF: By definition of  $n(X)$  and Lemma 9,  $n(X)$  is the smallest number such that the cyclic subgroup  $\pi_{n(X)}(X) = \pi_{n(X)}(a^{\mathbb{N}})$  of  $\mathbb{Z}_{2^{n(X)}}^\times$  has order greater than or equal to 3. Then  $|\pi_{n(X)}(a^{\mathbb{N}})| = 2^k$  for some  $k \geq 2$ . If  $k \neq 2$ , then we can consider the projection  $\pi_{n(X)-1}(X) = \pi_{n(X),n(X)-1}(\pi_{n(X)}(X))$  and conclude that  $|\pi_{n(X)-1}(X)| \geq |\pi_{n(X)}(X)|/2 \geq 2^{k-1} \geq 4 \geq 3$  (since the homomorphism  $\pi_{n(X),n(X)-1}: \mathbb{Z}_{2^{n(X)}} \longrightarrow \mathbb{Z}_{2^{n(X)-1}}$  has kernel of cardinality 2), but this contradicts the minimality of  $n(X)$ . This contradiction shows that  $|\pi_{n(X)}(X)| = 4$ .

The group  $\mathbb{Z}_{2^{n(X)}}^\times$  has cardinality  $|\mathbb{Z}_{2^{n(X)}}^\times| \geq |\pi_{n(X)}(X)| = 4$  and therefore  $n(X) \geq 3$ . By Lemma 4 (2), the multiplicative group  $\mathbb{Z}_{2^{n(X)}}^\times$  is not cyclic, which implies  $\pi_{n(X)}(X) \neq \mathbb{Z}_{2^{n(X)}}^\times$  and hence  $i(X) \geq |\mathbb{Z}_{2^{n(X)}}^\times / \pi_{n(X)}(X)| = 2^{n(X)-3} \geq 2$ . □

**Lemma 12.** *For any odd prime number  $p$  and two sets  $X, Y \in \mathcal{X}_p$ , the inclusion  $X \subseteq Y$  holds if and only if  $i(Y)$  divides  $i(X)$ .*

PROOF: Let  $m = \max\{n(X), n(Y)\}$ . Then  $X = \pi_m^{-1}(\pi_m(X))$ ,  $Y = \pi_m^{-1}(\pi_m(Y))$  and  $\pi_m(X), \pi_m(Y)$  are subgroups of the multiplicative group  $\mathbb{Z}_{p^m}^\times$ , which is cyclic by the Gauss Lemma 4 (1). It follows that the subgroups  $\pi_m(X)$  and  $\pi_m(Y)$  have indexes  $i(X)$  and  $i(Y)$  in  $\mathbb{Z}_{p^m}^\times$ , respectively. Let  $g$  be a generator of the cyclic group  $\mathbb{Z}_{p^m}^\times$ . It follows that the subgroups  $\pi_m(X)$  and  $\pi_m(Y)$  are generated by the elements  $g^{i(X)}$  and  $g^{i(Y)}$ , respectively. Now we see that  $X \subseteq Y$  if and only if  $\pi_m(X) \subseteq \pi_m(Y)$  if and only if  $g^{i(X)} \in (g^{i(Y)})^{\mathbb{N}}$  if and only if  $i(Y)$  divides  $i(X)$ . □

**Lemma 13.** *For any odd prime number  $p$ , any  $n \in \mathbb{N}$ , and the number  $a = 1 + p^n$  we have  $\overline{a^{\mathbb{N}}} = 1 + p^n\mathbb{N}_0$  and  $i(\overline{a^{\mathbb{N}}}) = p^{n-1}(p - 1)$ .*

PROOF: Observe that for any  $k < p$  we have  $a^k = (1 + p^n)^k \in 1 + kp^n + p^{n+1}\mathbb{Z} \neq 1 + p^{n+1}\mathbb{Z}$  and  $a^p = (1 + p^n)^p \in 1 + p^{n+1}\mathbb{Z}$ , which means that the element  $\pi_{n+1}(a)$  has order  $p$  in the group  $\mathbb{Z}_{p^{n+1}}^\times$ . By Lemma 9,

$$\overline{a^{\mathbb{N}}} = \pi_{n+1}^{-1}(\{a^k + p^{n+1}\mathbb{Z}: 0 \leq k < p\}) = \bigcup_{k=0}^{p-1} (a^k + p^{n+1}\mathbb{N}_0) = 1 + p^n\mathbb{N}_0.$$

Also  $i(\overline{a^{\mathbb{N}}}) = |\mathbb{Z}_{p^{n+1}}^\times|/p = p^{n-1}(p - 1)$ . □

Lemmas 10 and 12 imply that for an odd prime number  $p$ , the poset  $\mathcal{X}_p$  is order isomorphic to the set

$$\mathcal{D}_p = \{d \in \mathbb{N}: d \text{ divides } p^n(p - 1) \text{ for some } n \in \mathbb{N}_0\},$$

endowed with the divisibility relation.

An element  $t$  of a partially ordered set  $(X, \leq)$  is called  $\uparrow$ -chain if its upper set  $\uparrow t = \{x \in X: x \geq t\}$  is a chain. It is easy to see that the set of  $\uparrow$ -chain





of cardinality  $2^{n(X)-2}$  in  $\mathbb{Z}_{2^{n(X)}}^\times$ . If  $a \in 1 + 4\mathbb{N}$ , the subgroup generated by  $\pi_{n(X)}(a)$  is contained in  $M_X$ . Then  $C_X = \{1 + k \cdot 2^{n(X)-2} + 2^{n(X)}\mathbb{Z} : 0 \leq k < 4\}$  and  $X = \pi_{n(X)}^{-1}(C_X) = 1 + 2^{n(X)-2}\mathbb{N}_0$ .

If  $a \in 3 + 4\mathbb{N}_0$ , then  $C_X$  is not contained in  $M_X$ . By the Gauss Lemma 4 (2), there are two cyclic subgroups of  $\mathbb{Z}_{2^{n(X)}}^\times$  of order 4: one generated by  $g = (5 + 2^{n(X)}\mathbb{Z})^{n(X)-2}$  (which is contained in  $M_X$ ) and the other is generated by  $-g$ , which is not contained in  $M_X$  but contains  $-1 + 2^{n(X)}$ . Therefore,  $C_X$  must be equal to  $C_X = \{(-1)^k + k \cdot 2^{n(X)-2} + 2^{n(X)}\mathbb{Z} : 0 \leq k < 4\}$  and

$$\begin{aligned} X &= \pi_{n(X)}^{-1}(C_X) = \bigcup_{k=0}^3 \pi_{n(X)}^{-1}((-1)^k + k \cdot 2^{n(X)-2} + 2^{n(X)}\mathbb{Z}) \\ &= (1 + 2^{n(X)-1}\mathbb{N}_0) \cup (-1 + 2^{n(X)-2} + 2^{n(X)-1}\mathbb{N}_0). \end{aligned}$$

□

**Lemma 16.** *For every  $n \geq 2$ ,*

- (1) *the set  $X = \overline{(1 + 2^n)\mathbb{N}} \in \mathcal{X}_2$  coincides with  $1 + 2^n\mathbb{N}_0$  and has  $i(X) = 2^{n-1}$ ;*
- (2) *the set  $Y = \overline{(-1 + 2^n)\mathbb{N}} \in \mathcal{X}_2$  coincides with  $(1 + 2^{n+1}\mathbb{N}_0) \cup (2^n - 1 + 2^{n+1}\mathbb{N}_0)$  and has  $i(Y) = 2^{n-1}$ .*

PROOF: (1) Observe that for every positive  $k < 4$  we have  $(1 + 2^n)^k \in 1 + k2^n + 2^{n+2}\mathbb{Z} \neq 1 + 2^{n+2}\mathbb{Z}$  and  $(1 + 2^n)^4 \in 1 + 2^{n+2}\mathbb{Z}$ , which means that the element  $(1 + 2^n) + 2^{n+2}\mathbb{Z}$  has order 4 in the group  $\mathbb{Z}_{2^{n+2}}^\times$ . Then the element  $X = \overline{(1 + 2^n)\mathbb{N}} \in \mathcal{X}_2$  has  $n(X) = n + 2$  and hence  $X = 1 + 2^n\mathbb{N}_0$  and  $i(X) = 2^{n(X)-3} = 2^{n-1}$  by Lemma 15.

(2) Also for every positive  $k < 4$  we have  $(-1 + 2^n)^k \in (-1)^k(1 - k2^n) + 2^{n+2}\mathbb{Z} \neq 1 + 2^{n+2}\mathbb{Z}$  and  $(-1 + 2^n)^4 \in 1 + 2^{n+2}\mathbb{Z}$ , which means that the element  $(-1 + 2^n) + 2^{n+2}\mathbb{Z}$  has order 4 in the group  $\mathbb{Z}_{2^{n+2}}^\times$ . Then the element  $Y = \overline{(-1 + 2^n)\mathbb{N}} \in \mathcal{X}_2$  has  $n(Y) = n + 2$  and hence  $Y = (1 + 2^{n+1}\mathbb{N}_0) \cup (2^n - 1 + 2^{n+1}\mathbb{N}_0)$  and  $i(Y) = 2^{n(Y)-3} = 2^{n-1}$  by Lemma 15. □

**Lemma 17.** *For distinct sets  $X, Y \in \mathcal{X}_2$ , the strict embedding  $X \subset Y$  holds if and only if  $X \subseteq 1 + 4\mathbb{N}_0$  and  $i(Y) < i(X)$ .*

PROOF: If  $X \subseteq 1 + 4\mathbb{N}_0$ , then by Lemma 15,  $X = 1 + 2^{n(X)-2}\mathbb{N}_0$ . If  $i(Y) < i(X)$ , then  $n(Y) < n(X)$  (see Lemma 15). If  $Y \subseteq 1 + 4\mathbb{N}_0$ , then Lemma 15 implies

$$X = 1 + 2^{n(X)-2}\mathbb{N}_0 \subset 1 + 2^{n(Y)-2}\mathbb{N}_0 = Y.$$

If  $Y \not\subseteq 1 + 4\mathbb{N}_0$ , then Lemma 15 ensures that

$$X = 1 + 2^{n(X)-2}\mathbb{N}_0 \subset 1 + (2^{n(Y)-1}\mathbb{N}_0) \cup (-1 + 2^{n(Y)-2} + 2^{n(Y)-1}\mathbb{N}_0) = Y.$$

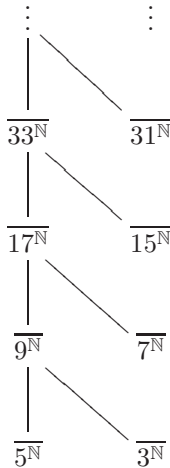
In both cases we have the strict embedding  $X \subset Y$ .

Conversely, assume that  $X \subset Y$ . We should prove that  $X \subseteq 1+4\mathbb{N}_0$  and  $i(Y) < i(X)$ . To derive a contradiction, assume that  $X \not\subseteq 1+4\mathbb{N}_0$ . Applying Lemma 15 and taking into account that  $X \subset Y$ , we conclude that  $X = (1 + 2^{n(X)-1}\mathbb{N}_0) \cup (-1+2^{n(X)-2}+2^{n(X)-1}\mathbb{N}_0)$ ,  $Y = (1+2^{n(Y)-1}\mathbb{N}_0) \cup (-1+2^{n(Y)-2}+2^{n(Y)-1}\mathbb{N}_0)$  and  $n(X) > n(Y)$ . Then  $-1 + 2^{n(X)-2} \in X \subseteq Y$  implies that  $-1 + 2^{n(X)-2}$  belongs either to  $1+2^{n(Y)-1}\mathbb{N}_0$  or to  $-1+2^{n(Y)-2}+2^{n(Y)-1}\mathbb{N}_0$ . In the first case we conclude that  $2 \in 2^{n(Y)-1}\mathbb{Z}$  and hence  $n(Y) \leq 2$ , which contradicts Lemma 11. In the second case, we obtain that  $2^{n(X)-2} \in 2^{n(Y)-2} + 2^{n(Y)-1}\mathbb{N}_0$ . Since  $n(X) > n(Y)$ , this implies  $2^{n(Y)-2} \in 2^{n(Y)-1}\mathbb{Z}$ , which is the final contradiction showing that  $X \subseteq 1 + 4\mathbb{N}_0$ . Then  $X = 1 + 2^{n(X)-2}\mathbb{N}_0$  according to Lemma 15.

Next, we prove that  $i(Y) < i(X)$ . By Lemma 15, two cases are possible:  $Y = 1 + 2^{n(Y)-2}\mathbb{N}_0$  or  $Y = (1 + 2^{n(Y)-1}\mathbb{N}_0) \cup (-1 + 2^{n(Y)-2} + 2^{n(Y)-1}\mathbb{N}_0)$ . In both cases the strict inclusion  $1 + 2^{n(X)-2}\mathbb{N}_0 = X \subset Y$  implies that  $n(X) > n(Y)$  and hence  $i(X) = 2^{n(X)-3} > 2^{n(Y)-3} = i(Y)$ .  $\square$

Lemmas 16 and 17 imply:

**Lemma 18.** *The family  $\min \mathcal{X}_2 = \{X \in \mathcal{X}_2 : X \not\subseteq 1 + 8\mathbb{N}_0\}$  coincides with the set of minimal elements of the poset  $\mathcal{X}_2$  and the set  $\mathcal{X}_2 \setminus \min \mathcal{X}_2 = \{X \in \mathcal{X}_2 : X \subseteq 1 + 8\mathbb{N}_0\}$  is well-ordered and coincides with the set  $\{1 + 2^n\mathbb{N}_0 : n \geq 3\}$ .*



The Hasse diagram of the poset  $\mathcal{X}_2$ .

**Lemma 19.** *For any homeomorphism  $h$  of the Golomb space  $\mathbb{N}_\tau$  and any  $n \in \{1, 2, 3\}$  we have  $h(n) = n$ .*

PROOF: 1. The equality  $h(1) = 1$  follows from Lemma 3 (1).

2. By Lemma 8,  $h$  induces an order isomorphism of the posets  $\mathcal{X}_2$  and  $\mathcal{X}_{h(2)}$ . By Lemmas 16 and 18, the set  $\{\overline{(-1 + 2^n)^{\mathbb{N}}}: n \geq 2\}$  is an infinite antichain in the poset  $\mathcal{X}_2$ . Consequently, the poset  $\mathcal{X}_{h(2)}$  also contains an infinite antichain. On the other hand, for any odd prime number  $p$  the poset  $\mathcal{X}_p$  is order-isomorphic to the poset  $\mathcal{D}_p$ , which contain no infinite antichains. Consequently,  $\mathcal{X}_{h(2)}$  cannot be order isomorphic to  $\mathcal{X}_p$ , and hence  $h(2) = 2$ .

3. By Lemma 3 (2),  $h(3)$  is a prime number, not equal to  $h(2) = 2$ . By Lemma 8,  $h$  induces an order isomorphism of the posets  $\mathcal{X}_3$  and  $\mathcal{X}_{h(3)}$ . Then the posets  $\mathcal{D}_3$  and  $\mathcal{D}_{h(3)}$  also are order isomorphic. The smallest  $\uparrow$ -chain element of the poset  $\mathcal{D}_3$  is 2 and the set  $\downarrow 2 = \{d \in \mathcal{D}_3: d \text{ divides } 2\}$  has cardinality 2. On the other hand, the smallest  $\uparrow$ -chain element of the poset  $\mathcal{D}_{h(3)}$  is  $h(3) - 1$ . Since the sets  $\mathcal{D}_3$  and  $\mathcal{D}_{h(3)}$  are order-isomorphic, the set  $\downarrow(h(3) - 1) = \{d \in \mathcal{D}_{h(3)}: d \text{ divides } h(3) - 1\}$  has cardinality 2, which means that the number  $h(3) - 1$  is prime. Observing that 3 is a unique odd prime number  $p$  such that  $p - 1$  is prime, we conclude that  $h(3) = 3$ . □

**Lemma 20.** *For any homeomorphism  $h$  of the Golomb space  $\mathbb{N}_\tau$ , and any prime number  $p$  we have  $h(1 + p^n\mathbb{N}_0) = 1 + h(p)^n\mathbb{N}_0$  for all  $n \in \mathbb{N}$ .*

PROOF: By Lemma 8, the homeomorphism  $h$  induces an order isomorphism of the posets  $\mathcal{X}_p$  and  $\mathcal{X}_{h(p)}$ .

If  $p = 2$ , then  $h(p) = 2$  by Lemma 19. Lemma 3 implies  $h(2\mathbb{N}) = h(2) \cdot \mathbb{N} = 2\mathbb{N}$  and hence  $h(1 + 2\mathbb{N}_0) = h(\mathbb{N} \setminus 2\mathbb{N}) = \mathbb{N} \setminus h(2\mathbb{N}) = 1 + 2\mathbb{N}_0$ . By Lemma 8,  $h$  induces an order automorphism of the poset  $\mathcal{X}_2$  and hence  $h$  is identity on the well-ordered set  $\{1 + 2^n\mathbb{N}_0: n \geq 3\}$  of non-minimal elements of  $\mathcal{X}_2$ , see Lemma 18. Consequently,  $h(1 + 2^n\mathbb{N}_0) = 1 + 2^n\mathbb{N}_0$  for all  $n \geq 3$ .

Next, we show that  $h(1 + 4\mathbb{N}_0) = 1 + 4\mathbb{N}_0$ . Observe that for the smallest non-minimal element  $\overline{9^{\mathbb{N}}} = 1 + 8\mathbb{N}_0$  of  $\mathcal{X}_2$  there are only two elements,  $\overline{5^{\mathbb{N}}} = 1 + 4\mathbb{N}_0$  and  $\overline{3^{\mathbb{N}}} = (1 + 8\mathbb{N}_0) \cup (3 + 8\mathbb{N}_0)$ , which are strictly smaller than  $\overline{9^{\mathbb{N}}}$  in the poset  $\mathcal{X}_2$ . Then  $h(\overline{5^{\mathbb{N}}}) \in \{\overline{3^{\mathbb{N}}}, \overline{5^{\mathbb{N}}}\}$ . By Lemma 19,  $h(3) = 3$  and hence  $h(\overline{3^{\mathbb{N}}}) = \overline{3^{\mathbb{N}}}$ , which implies that  $h(1 + 4\mathbb{N}_0) = h(\overline{5^{\mathbb{N}}}) = \overline{5^{\mathbb{N}}} = 1 + 4\mathbb{N}_0$ .

Now assume that  $p$  is an odd prime number. Since  $h(2) = 2$ , the prime number  $h(p) \neq h(2) = 2$  is odd. By Lemma 14, the well-ordered sets  $\{1 + p^n\mathbb{N}_0: n \in \mathbb{N}\}$  and  $\{1 + h(p)^n\mathbb{N}_0: n \in \mathbb{N}\}$  coincide with the sets of  $\uparrow$ -chain elements of the posets  $\mathcal{X}_p$  and  $\mathcal{X}_{h(p)}$ , respectively. Taking into account that  $h$  is an order isomorphism, we conclude that  $h(1 + p^n\mathbb{N}_0) = 1 + h(p)^n\mathbb{N}_0$  for every  $n \in \mathbb{N}$ . □

### 5. Proof of Theorem 1

In this section we present the proof of Theorem 1. Given any homeomorphism  $h$  of the Golomb space  $\mathbb{N}_\tau$ , we need to prove that  $h(n) = n$  for all  $n \in \mathbb{N}$ . This equality will be proved by induction.

For  $n \leq 3$  the equality  $h(n) = n$  is proved in Lemma 19. Assume that for some number  $n \geq 4$  we have proved that  $h(k) = k$  for all  $k < n$ . For every prime number  $p$  let  $\alpha_p$  be the largest integer number such that  $p^{\alpha_p}$  divides  $n - 1$  (so,  $\alpha_p = l_p(n - 1)$ ). For every  $p \in \Pi_{n-1}$  we have  $p \leq n - 1$  and hence  $h(p) = p$  (by the inductive hypothesis). Then  $h(\Pi_{n-1}) = \Pi_{n-1}$  and  $h(\Pi \setminus \Pi_{n-1}) = \Pi \setminus \Pi_{n-1}$ .

Observe that  $n$  is the unique element of the set

$$\bigcap_{p \in \Pi} (1 + p^{\alpha_p} \mathbb{N}_0) \setminus (1 + p^{\alpha_p + 1} \mathbb{N}_0).$$

By Lemma 20,  $h(n)$  coincides with the unique element of the set

$$\begin{aligned} & \bigcap_{p \in \Pi} (1 + h(p)^{\alpha_p} \mathbb{N}_0) \setminus (1 + h(p)^{\alpha_p + 1} \mathbb{N}_0) \\ &= \left( \bigcap_{p \in \Pi_{n-1}} (1 + h(p)^{\alpha_p} \mathbb{N}_0) \setminus (1 + h(p)^{\alpha_p + 1} \mathbb{N}_0) \right) \cap \left( \bigcap_{p \in \Pi \setminus \Pi_{n-1}} \mathbb{N} \setminus (1 + h(p) \mathbb{N}_0) \right) \\ &= \left( \bigcap_{p \in \Pi_{n-1}} (1 + p^{\alpha_p} \mathbb{N}_0) \setminus (1 + p^{\alpha_p + 1} \mathbb{N}_0) \right) \cap \left( \bigcap_{p \in \Pi \setminus \Pi_{n-1}} \mathbb{N} \setminus (1 + p \mathbb{N}_0) \right) = \{n\} \end{aligned}$$

and hence  $h(n) = n$ . □

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