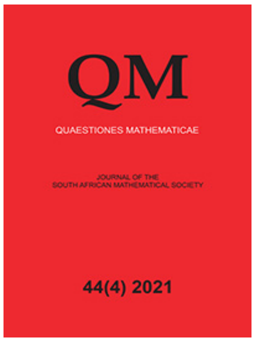

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THE GOLOMB TOPOLOGY OF POLYNOMIAL RINGS

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Abstract. We study properties of the Golomb topology on polynomial rings over fields, in particular trying to determine conditions under which two such spaces are not homeomorphic. We show that if K is an algebraic extension of a finite field and K' is a field of the same characteristic, then the Golomb spaces of $K[X]$ and $K'[X]$ are homeomorphic if and only if K and K' are isomorphic.

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Key words: Golomb topology, Dedekind domains, polynomial rings.

1. Introduction. Let R be an integral domain. The *Golomb space* of R is the topological space $G(R)$ having $R^\bullet := R \setminus \{0\}$ as its underlying set, and whose topology is generated by the coprime cosets. This topology, introduced by Brown [3] on $\mathbb{Z}^+$ and later studied by Golomb [9, 10], is one of many coset topologies [13], and it can be used to generalize Furstenberg's "topological" proof of the infinitude of primes [8, 4].

Recently two papers, the first one by Banakh, Mioduszewski and Turek [1] and the second one by Clark, Lebowitz-Lockard and Pollack [5], have started studying more deeply the topology on $G(R)$ and the continuous maps between these spaces, with the former concentrating on the "classical" case of $\mathbb{Z}^+$ and the latter generalizing several results to integral domains and, in particular, to Dedekind domains. A central problem of both is the *isomorphism problem*: if $G(R)$ and $G(S)$ are homeomorphic topological spaces, must R and S be isomorphic rings? More generally, how much do the continuous maps (and, in particular, homeomorphisms and self-homeomorphisms) of Golomb spaces respect the algebraic structure of the underlying rings? In [16], it was shown that the unique self-homeomorphisms of $h : G(\mathbb{Z}) \longrightarrow G(\mathbb{Z})$ are the identity or the multiplication by -1; the proof of this result relies crucially on the fact that the groups of units of the quotients $\mathbb{Z}/p^n\mathbb{Z}$ (where p is a prime number) are very close to being cyclic.

In this paper, we study the isomorphism problem in the context of polynomial rings over fields; in particular, we are interested in the more restricted problem of determining if the existence of a homeomorphism between $G(K[X])$ and $G(K'[X])$ implies that K and K' are isomorphic as fields. To do so, we study the closure of several sets under the Golomb topology and under the P-adic topologies (which can be reconstructed from the Golomb topology), obtaining several results that allow to determine algebraic properties of K from the topological properties of $G(K[X])$.

While we aren't able to solve the isomorphism problem in full generality, we show that if K is an algebraic extension of a finite field, K' has the same characteristic of K and $G(K[X]) \simeq G(K'[X])$ then K' must be isomorphic to K (Theorem 7.5); in particular, this implies that the number of distinct Golomb topologies associated to countable domains is the cardinality of the continuum, answering a question posed in [5, Section 3.1].

The structure of the paper is as follows. In Section 2, we fix the notation and recall some results that will be used throughout the paper. In Section 3 we give a few results about some distinguished subsets of $G(R)$, for an arbitrary Dedekind domain R. The rest of the paper can be divided into three parts that are essentially autonomous one from each other.

In Section 4 we show that, for polynomial rings, the Golomb topology allows to distinguish between zero and positive characteristic (Proposition 4.1), and study $G(K[X])$ when K has characteristic 0.

In Section 5 we study the case of separably closed fields in positive characteristic: we show that we can distinguish them from the other fields (Proposition 5.1) and that we can recover the characteristic of K from $G(K[X])$ (Theorem 5.11).

Sections 6 and 7 provide a proof of the main theorem. In Section 6 we generalize a result of [1] on the image of prime elements under a homeomorphism, while in Section 7 we use this result to link a (topologically distinguished) subgroup of self-homeomorphisms of $G(K[X])$ with the multiplicative group of K (Proposition 7.4), which allows to prove the aforementioned main theorem (Theorem 7.5).

2. Preliminaries and notation.

Let R be an integral domain; we shall always suppose that R is not a field. Given a set $I \subseteq R$, we set $I^\bullet := I \setminus \{0\}$. We denote by $U(R)$ the set of units of R (both as a set and as a group).

The *Golomb space* of R is the topological space having $R^\bullet$ as underlying set and whose topology is generated by the coprime cosets of R, that is, by the sets $x + I$ where $x \in R^\bullet$, I is a nonzero ideal of R and $\langle x, I \rangle = R$. We denote by $G(R)$ the Golomb space of R, and call the topology the *Golomb topology* on R. When R is an integral domain with zero Jacobson radical,[1] $G(R)$ is a Hausdorff space that is not regular; furthermore, $G(R)$ is not compact, and is a connected space that is totally disconnected at each of its points [5, Theorems 5, 8 and 9 and Proposition 10].

Suppose from now on that R is a Dedekind domain.

Given a subset $A \subseteq R^\bullet$, we denote by $\overline{A}$ the closure of A in the Golomb topology. Let $x + I$ be a coprime coset. If $I = P_1^{e_1} \cdots P_n^{e_n}$ is the factorization of I into prime ideals, then [5, Lemma 15]

$$\overline{x + I} = \bigcap_{i=1}^{n} (P_i^\bullet \cup (x + P_i^{e_i})).$$

If $h : G(R) \longrightarrow G(S)$ is a homeomorphism, then h sends units to units (i.e., $h(U(R)) = U(S)$) [5, Theorem 13]. If the class group of R is torsion then h sends

[1] The *Jacobson radical* of R is the intersection of the maximal ideals of R.

prime ideals to prime ideals, that is, if P is a prime ideal of R then $h(P^\bullet) \cup \{0\}$ is a prime ideal of S; more generally, h takes radical ideals to radical ideals [16, Theorem 2.8].

For every $x \in R$, let $V(x) := \{P \in \mathrm{Spec}(R) \mid x \in P\}$. Given a subset Δ of $\mathrm{Max}(R)$, we denote by $G_\Delta(R)$ the set of all $x \in R^\bullet$ such that $V(x) = \Delta$; note that $G_\Delta(R)$ is empty if Δ is infinite. If the class group of R is torsion, this set is again preserved by homeomorphisms: if h is a homeomorphism and $x \in G_\Delta(R)$, then $h(x) \in G_\Lambda(R)$, where Λ contains the images under h of the elements of Δ [16, Proposition 2.7]. Given $a \in R$, we set $\mathrm{pow}(a) := \{ua^n \mid u \in U(R), n \geq 1\}$; if a generates P, then $\mathrm{pow}(a)$ is exactly $G_{\{P\}}(R)$.

Let now R be a Dedekind domain with torsion class group and P be a prime ideal of R. The *P-topology* to $R \setminus P$ is the topology generated by the sets $a + P^n$, for all $a \in R \setminus P$ and all $n \geq 1$; this is exactly the restriction of the P-adic topology to $R \setminus P$. The P-topology can be recovered from the Golomb topology by considering only the clopen subset of $R \setminus P$, and thus every homeomorphism $h : G(R) \longrightarrow G(S)$ in the Golomb topology restricts to a homeomorphism between $R \setminus P$ and $S \setminus Q$ (with $Q := h(P^\bullet) \cup \{0\}$), where the former is endowed with the P-topology and the latter with the Q-topology [16, Section 3].

We denote by $\mathrm{char}K$ the characteristic of the field K. If q is a prime power, we denote by $\mathbb{F}_q$ the finite field with q elements. If p is a prime number, we denote by $\overline{\mathbb{F}_p}$ the algebraic closure of $\mathbb{F}_p$.

3. The spaces $G_n(R)$.

Let R be an integral domain. We denote by $G_0(R)$ the set of units of R endowed with the Golomb topology; this space is rather more well-behaved than the whole Golomb space.

PROPOSITION 3.1. *Let R be an integral domain.*

(a) $G_0(R)$ is homogeneous.

(b) Suppose the Jacobson radical of R is zero. Then, $G_0(R)$ is discrete if and only if there is an ideal I such that the restriction $G_0(R) \longrightarrow R/I$ of the canonical quotient is injective.

(c) $G_0(R[X])$ is discrete.

Proof. Since multiplication by units is a homeomorphism, we can always send x to y by multiplying by yx^{-1}; hence $G_0(R)$ is homogeneous.

For the second claim, suppose first that $G_0(R) \longrightarrow R/I$ is injective: then, for every unit u the coset $u + I$ meets $G_0(R)$ only in u, and thus $G_0(R)$ is discrete. Conversely, suppose $G_0(R)$ is discrete: then, there is an ideal I such that $(1 + I) \cap G_0(R) = \{1\}$. For every other unit u of R, $u + I = u(1 + I)$; hence, u is the only unit in $(u + I) \cap G_0(R)$. Thus, $G_0(R) \longrightarrow R/I$ is injective.

The last claim follows taking $I = XR[X]$. □

When R is a Dedekind domain we can say more.

PROPOSITION 3.2. *Let R be a Dedekind domain with zero Jacobson radical.*

(a) *$G_0(R)$ has a basis of clopen sets.*

(b) *$G_0(R)$ is regular.*

(c) *If R is countable, then $G_0(R)$ is either discrete or homeomorphic to $\mathbb{Q}$ (endowed with the Euclidean topology).*

(d) *If R is countable, $U(R)$ is infinite and every residue field of R is finite, then $G_0(R) \simeq \mathbb{Q}$.*

Proof. (a) We need to show that $(x+I) \cap G_0(R)$ is clopen in $G_0(R)$ for every $x \in G_0(R)$ and every ideal I. Indeed, let $I = \prod_i P_i^{e_i}$ be the factorization of I; then, by [5, Lemma 15],

$$\overline{x+I} \cap G_0(R) = \bigcap_i (P_i^\bullet \cup (x + P_i^{e_i})) \cap G_0(R).$$

Since $P_i^\bullet \cap G_0(R) = \emptyset$, we have $\overline{x+I} \cap G_0(R) = \bigcap_i ((x+P_i^{e_i}) \cap G_0(R)) = (x+I) \cap G_0(R)$, i.e., $(x+I) \cap G_0(R)$ is clopen in $G_0(R)$.

(b) Let $x \in G_0(R)$ and $V \subseteq G_0(R)$ be a closed set not containing x; then, $G_0(R) \setminus V$ is open, and thus it contains a basic clopen set $(x+I) \cap G_0(R)$. Therefore, x and V are separated by $(x+I) \cap G_0(R)$ and $G_0(R) \setminus (x+I)$, and so $G_0(R)$ is regular.

(c) If R is countable, then it has only countably many ideals, and thus R and $G_0(R)$ are second countable. Hence, it is metrizable [11, e-2]. If $G_0(R)$ is not discrete, then $G_0(R) \simeq \mathbb{Q}$ since $G_0(R)$ is homogeneous [15, 6]. Finally, (d) follows from this and Proposition 3.1. □

We now introduce a sequence $\{G_n(R)\}_{n \in \mathbb{N}}$ of subspaces of $G(R)$ generalizing $G_0(R)$.

DEFINITION 3.3. Let R be a Dedekind domain. For every $n \geq 0$, define

$$G_n(R) := \bigcup \{G_\Delta(R) \mid \Delta \subseteq \operatorname{Max}(R), |\Delta| = n\}.$$

By [16, Proposition 2.7], if R has torsion class group then the topology of the $G_n(R)$ is uniquely determined by the Golomb topology, in the sense that if $h : G(R) \longrightarrow G(S)$ is a homeomorphism then $h(G_n(R)) = G_n(S)$ and thus $G_n(R)$ and $G_n(S)$ are homeomorphic.

The results proved above for $n = 0$ do not generalize to arbitrary n. When $n = 1$, we can prove a partial analogue of Proposition 3.2(b) by extending the proof of [1, Theorem 3.1].

PROPOSITION 3.4. *Let R be a Dedekind domain that is not a field, and suppose that R has finitely many units. Then, $G_1(R)$ is a regular space.*

Proof. Let Ω be an open set of $G(R)$ and let $x \in G_1(R) \cap \Omega$; we need to show that there is an open neighborhood O of x such that $\overline{O} \cap G_1(R) \subseteq \Omega \cap G_1(R)$. Without loss of generality, we can suppose that $\Omega = x + bR$ for some b coprime with x.

Let $P_1,\dots,P_n$ be the prime ideals containing b; then, the set Λ of the prime elements contained in some P_i is finite (as R has finitely many units). Thus, the set $x-\Lambda := \{x-p \mid p \in \Lambda\}$ is finite too, and so there are only finitely many prime ideals that contain some element of $x-\Lambda$.

Since R has finitely many units, it has infinitely many maximal ideals; thus, there is a prime ideal Q that is distinct from each P_i and that do not contain x nor any element of $x-\Lambda$. Consider $O := x+bQ$: then, O is a coprime coset, and thus it is open. By [5, Lemma 15],

$$\overline{O} = \bigcap_i (P_i^\bullet \cup (x+P_i^{e_i})) \cap (Q^\bullet \cup (x+Q)),$$

where e_i is the exponent of P_i in the factorization of bR.

Let $p \in \overline{O} \cap G_1(R)$. By definition, p can be contained in at most one of $P_1,\dots,P_n,Q$. We distinguish three cases.

- If p is not contained in any of them, then $p \in \bigcap_i (x+P_i^{e_i}) \cap (x+Q) = (x+bR)\cap(x+Q) = x+bQ = O \subseteq \Omega$.
- If p is contained in P_i for some i, then it should be contained in $x+Q$, that is, $p-x \in Q$. However, this contradicts the choice of Q.
- If $p \in Q$, then we must have $p \in \bigcap_i (x+P_i^{e_i}) = x+bR = \Omega$.

Hence, $\overline{O} \cap G_1(R) \subseteq \Omega \cap G_1(R)$, as claimed. Thus, $G_1(R)$ is regular. □

Like for $G_0(R)$, this implies that if R is countable then $G_1(R)$ is second countable and thus metrizable.

A homeomorphism of Golomb spaces preserves whether $G_1(R)$ is dense in $G(R)$ or not, and both possibilities can happen (see Propositions 4.3, 5.2 and 6.3); in particular, for polynomial rings $K[X]$, this property can be used in some cases to distinguish between an algebraically closed and a non-algebraically closed K (see Corollary 6.4 or the proof of Theorem 7.5). When $G_1(R)$ is dense, we can prove some properties of $G_n(R)$; we need a topological lemma.

LEMMA 3.5. *Let X be a topological space, $Y \subseteq X$ a dense subset and Ω an open subset of X. Then, $\overline{\Omega} \cap Y = \overline{\Omega \cap Y} \cap Y$.*

Proof. Clearly, $\overline{\Omega \cap Y} \cap Y \subseteq \overline{\Omega} \cap Y$. On the other hand, let $x \in \overline{\Omega} \cap Y$. If $x \notin \overline{\Omega \cap Y}$, then there is an open set O of X containing x but disjoint from $\Omega \cap Y$, that is, $O \cap \Omega \cap Y = \emptyset$. However, since Y is dense and $O \cap \Omega$ is open it follows that $O \cap \Omega = \emptyset$, and thus $x \notin \overline{\Omega}$, a contradiction. It follows that $\overline{\Omega} \cap Y \subseteq \overline{\Omega \cap Y} \cap Y$. The claim is proved. □

PROPOSITION 3.6. *Let R be a Dedekind domain with torsion class group such that $G_1(R)$ is dense in $G(R)$. Then, for every $n \geq 2$,*

(a) $G_n(R)$ is dense in $G(R)$;

(b) $G_n(R)$ is not regular.

Proof. (a) If $x+bR$ is a coprime coset, we can find $p_1 \in (x+bR)\cap(1+xR)\cap G_1(R)$; then, as p_1 is coprime with x, the set $x+p_1bR$ is open, and thus we can find $p_2 \in (1+p_1bR)\cap G_1(R)$, then $p_3 \in (1+p_1p_2bR)\cap G_1(R)$, and so on. Then, $c := p_1\cdots p_n$ will be an element of $G_n(R)$ (as each p_i is in $G_1(R)$ and p_i and p_j are coprime for $i \neq j$) such that $c \equiv x\cdot 1\cdots 1 = x \bmod bR$, i.e., $c \in (x+bR)\cap G_n(R)$. Hence, $G_n(R)$ is dense.

(b) Fix $n \geq 2$, and let $p \in G_1(R)$. Let $\Omega := 1+pR$, take $x \in \Omega\cap G_n(R)$, and let O be an open set such that $x \in O$ and $O\cap G_n(R) \subseteq \Omega\cap G_n(R)$. We claim that $\overline{O}\cap G_n(R) \nsubseteq \Omega$. Without loss of generality we can take $O = x+bR$, with b coprime to x; furthermore, passing if needed to a power b^k we can suppose that b is a product of primary elements.

If $x+b \in pR$, then we can write $x+b = py$ for some $y \in R$, and $py+pbR \subseteq O$ since $x+b+pbR \subseteq x+bR$. Let $O' := y+bR$; then, O' is open (if y and b have a common factor s, then s would divide also x, a contradiction). Since $G_{n-1}(R)$ is dense, we can find $q \in O'\cap G_{n-1}(R)$; then, $pq \in O\cap G_n(R)$, while $pq \notin \Omega$ as $pq \in pR$. This contradicts $O\cap G_n(R) \subseteq \Omega\cap G_n(R)$.

Therefore, $x+b \notin pR$. Let $b := b_1\cdots b_n$, where each b_i belongs to $G_1(R)$ and b_i and b_j are coprime if $i \neq j$. If $b_i \in pR$ for some i, let $b' := b/b_i$; otherwise, set $b' := b$. Then, p is coprime with b', and thus there is a $z \in R$, coprime with p, such that $pz \equiv x \bmod b'R$. By density, there is a $q \in (z+bR)\cap G_{n-1}(R)$; we claim that $pq \in (\overline{O}\cap G_n(R))\setminus\Omega$. Indeed, it is clear that $pq \in G_n(R)$ (since $p \in G_1(R)$, $q \in G_{n-1}(R)$ and p and q are coprime), and $pq \notin \Omega$ since $pq \in pR$. By [5, Lemma 15],

$$\overline{O} = \bigcap_i (P_i^\bullet \cup (x+b_iR)),$$

where P_i is the prime ideal containing b_i. If b_i is not coprime with p, then $pq \in P_i^\bullet \subseteq \overline{O}$. If b_i is coprime with p, then b_i divides b' and

$$pq \in p(z+bR) = pz+pbR \subseteq pz+b'R = x+b'R \subseteq x+b_iR \subseteq \overline{O}.$$

Hence, $pq \in (\overline{O}\cap G_n(R))\setminus\Omega$.

Let $V := G_n(R)\setminus\Omega$: then, V is a closed set of $G_n(R)$. If $G_n(R)$ were regular, then there would be disjoint open sets O_1, O_2 such that $x \in O_1\cap G_n(R)$ and $V \subseteq O_2\cap G_n(R)$. In particular, $O_1\cap G_n(R) \subseteq G_n(R)\setminus(O_2\cap G_n(R))$, and the latter is a closed set; therefore, the closure V' of $O_1\cap G_n(R)$ inside $G_n(R)$ would be disjoint from V. However, by Lemma 3.5,

$$V' = \overline{O_1\cap G_n(R)}\cap G_n(R) = \overline{O_1}\cap G_n(R);$$

by the previous part of the proof, $\overline{O_1}\cap G_n(R)$ is not contained in Ω, i.e., it meets V. This is a contradiction, and thus $G_n(R)$ is not regular. $\square$

4. Characteristic 0. We now start studying the Golomb spaces $G(K[X])$ of polynomial rings over fields. In this section, we analyze what happens when the characteristic of the field is 0. The first result is that we can actually distinguish this case from the positive characteristic case.

Proposition 4.1. *Let K be a field. Then, K has characteristic 0 if and only if there is an irreducible polynomial $g \in K[X]$ such that $\mathrm{pow}(g)$ is closed in the P-topology for every prime ideal P not containing g.*

Proof. Suppose K has characteristic 0, and choose $g(X) := X$. Let $P = (f)$ be a prime ideal not containing g, and let $\lambda \notin (P \cup \mathrm{pow}(g))$: suppose that λ is in the closure of $\mathrm{pow}(g)$ in the P-topology. Then, for every $n \in \mathbb{N}^+$ the open set $\lambda + P^n$ contains an element of $\mathrm{pow}(g)$. Take $n > \deg\lambda + 1$: then, there are $m \in \mathbb{N}^+$ and $u \in K^\bullet$ such that $ug^m \in \lambda + P^n$, i.e., f^n divides $h := \lambda - ug^m$. Since $\lambda \notin \mathrm{pow}(g)$, $h \neq 0$, and thus $m \geq n$. Let H the $(\deg\lambda+1)$-th derivative of h: then, λ goes to 0, and thus H is the $(\deg\lambda + 1)$-th derivative of $-ug^m = -uX^m$, that is, $H(X) = cX^{m-\deg\lambda-1}$ for some $c \in K$. Since $\mathrm{char}K = 0$ and $m > \deg\lambda + 1$, we have $H \neq 0$, and thus its unique zero is 0. This contradicts the facts that $f|H$ and that f is coprime with X. Hence, $\mathrm{pow}(g)$ is closed in the (f)-topology.

Conversely, suppose there is a polynomial g with this property, and suppose that $\mathrm{char}K = p > 0$. Let $a \in K$ be such that $g(a) \neq 0$ (which exists since g is irreducible). Then, $f(X) := X - a$ divides $1 - \frac{g(X)}{g(a)}$, and thus f^{p^n} divides $\left(1 - \frac{g(X)}{g(a)}\right)^{p^n} = 1 - \frac{g(X)^{p^n}}{g(a)^{p^n}}$, that is, $1 + (f)^{p^n}$ meets $\mathrm{pow}(g)$. Therefore, $1 + (f)^k$ meets $\mathrm{pow}(g)$ for every $k \in \mathbb{N}^+$, i.e., 1 is in the closure of $\mathrm{pow}(g)$ in the (f)-topology. This contradicts the choice of g, and thus the characteristic of K must be 0, as claimed. □

Corollary 4.2. *Let K_1, K_2 be fields. If $\mathrm{char}K_1 = 0$ and $\mathrm{char}K_2 > 0$, then the Golomb spaces $G(K_1[X])$ and $G(K_2[X])$ are not homeomorphic.*

Proof. If g is an irreducible polynomial of $K[X]$, then $\mathrm{pow}(g) = G_{\{(g)\}}(K[X])$. By the previous proposition, $\mathrm{char}K = 0$ if and only if there is a prime ideal P such that $G_{\{P\}}(K[X])$ is closed in the Q-topology for every prime ideal $Q \neq P$. Since any homeomorphism of Golomb spaces sends prime ideals into prime ideals, this property is preserved by homeomorphisms. In particular, if $G(K_1[X]) \simeq G(K_2[X])$ then $\mathrm{char}K_1 = 0$ if and only if $\mathrm{char}K_2 = 0$. □

Note that the proof of Proposition 4.1 is qualitative, and thus cannot be readily applied to distinguish different positive characteristics. We shall do this in the algebraically closed case in Theorem 5.11.

We now study the algebraically closed and the real closed case.

Proposition 4.3. *Let K be an algebraically closed field of characteristic 0. For every $n \geq 0$, $G_n(K[X])$ is discrete and closed in $G(K[X])$.*

Proof. Let $p(X) \in K[X]$, and let $b \in K$ be such that $p(b) \neq 0$ (which exists since K is infinite). We claim that, for large N, the only possible element of $(p(X) + (X-b)^N K[X]) \cap G_n(K[X])$ is $p(X)$.

Indeed, suppose that $q(X) \in (p(X) + (X-b)^N K[X]) \cap G_n(K[X])$ is different from $p(X)$: then, we have

$$\begin{cases} q(X) = p(X) + (X-b)^N a(X) \\ q(X) = u(X-a_1)^{e_1} \cdots (X-a_n)^{e_n}, \end{cases}$$

where $a(X) \neq 0$, $a_1, \ldots, a_n$ are distinct, $e_1, \ldots, e_n \geq 1$ and $u \in K$. Let $d := \deg p$, and apply $d+1$ times the derivative process. In the first equation, $p^{(d+1)}$ becomes 0, and thus (since $a(X) \neq 0$) $q^{(d+1)}$ has a zero of multiplicity $N-d-1$ in b. In the second equation, at each step the multiplicity of each a_i is lowered by 1, and thus each a_i is a zero of multiplicity at least $e_i - d - 1$ (this holds even if $e_i < d+1$). Since $p(X)$ and $X-b$ are coprime, it follows that $b \neq a_i$ for each i; hence, the total multiplicities of the zeros of $q^{(d+1)}$ is at least

$$N-d-1+\sum_i (e_i-d-1) = N+\sum_i e_i-(n+1)(d+1) = N+\deg q-(n+1)(d+1).$$

Both n and d are fixed; hence, choosing $N > n(d+1)$, we have (using the fact that K has characteristic 0)

$$\deg q^{(d+1)} > n(d+1) + \deg q - (n+1)(d+1) = \deg q - (d+1) = \deg q^{(d+1)},$$

a contradiction. Hence, $(p(X)+(X-b)^N K[X]) \cap G_n(K[X])$ contains at most $p(X)$.

Therefore, if $p(X) \notin G_n(K[X])$ then $p(X) + (X-b)^N K[X]$ is disjoint from $G_n(K[X])$, and thus $p(X)$ is not in the closure of $G_n(K[X])$; on the other hand, if $p(X) \in G_n(K[X])$ then $(p(X) + (X-b)^N K[X]) \cap G_n(K[X]) = \{p(X)\}$ is an open set of $G_n(K[X])$. Hence, $G_n(K[X])$ is discrete and closed in $G(K[X])$, as claimed. □

COROLLARY 4.4. *Let K be a real closed field. For every $n \geq 0$, $G_n(K[X])$ is discrete and closed in $G(K[X])$.*

Proof. Let K' be the algebraic closure of K, and let $G' := G_n(K'[X]) \cup \cdots \cup G_{2n}(K'[X])$; then, $G_n(K[X]) \subseteq G'$. Take $p(X) \in G(K[X])$. By Proposition 4.3, there is a polynomial $r(X) \in K'[X]$, coprime with $p(X)$, such that $(p(X) + r(X)K'[X]) \cap G' \subseteq \{p(X)\}$.

Take the conjugate polynomial $\overline{r}(X)$ of $r(X)$ over $K[X]$. Then, $s(X) := r(X)\overline{r}(X)$ belongs to $K[X]$ and is coprime with $p(X)$ (its roots are the roots of $r(X)$ and their conjugates). Therefore, $p(X)+s(X)K[X]$ is an open subset of $G(K[X])$, and its intersection with $G_n(K[X])$ is contained in $(p(X) + r(X)K'[X]) \cap G' \subseteq \{p(X)\}$. Hence, $G_n(K[X])$ is discrete and closed in $G(K[X])$. □

These results can be used, for example, to distinguish $G(\mathbb{Q}[X])$ from $G(\overline{\mathbb{Q}}[X])$, see Section 6.

5. Separably closed fields in characteristic p. In this section, we analyze what happens to fields of positive characteristic that are separably or algebraically closed. The first step is distinguishing them from the other fields; the following proof is similar to the proof of Proposition 4.1.

PROPOSITION 5.1. *Let K be a field of characteristic $p > 0$, and suppose that K is transcendental over $\mathbb{F}_p$. Then, K is separably closed if and only if, for every coprime irreducible polynomials f, g of $K[X]$, $G_0(R)$ is contained in the closure of* $\mathrm{pow}(g)$ *in the (f)-topology.*

Proof. Suppose first that K is separably closed; since $\mathrm{pow}(g)$ is invariant under multiplication by units, it is enough to show that 1 is in the closure of $\mathrm{pow}(g)$. Write $f(X) = X^{p^n} - a$, and let α be a root of f in the algebraic closure $\overline{K}$ of K. Then, $h := 1 - \frac{1}{g(\alpha)} g$ is a polynomial over $\overline{K}$ having α as a zero, and thus $X - \alpha$ divides h; hence, $f(X) = (X - \alpha)^{p^n}$ divides

$$h^{p^n} = \left(1 - \frac{1}{g(\alpha)} g\right)^{p^n} = 1 - \frac{1}{g(\alpha)^{p^n}} g^{p^n}$$

in $\overline{K}[X]$. However, $g(\alpha)^{p^n} \in K[X]$, and thus f divides h^{p^n} also in $K[X]$. Therefore, for every power q of p, f^q divides $(h^{p^n})^q = 1 - \frac{1}{g(\alpha)^{qp^n}} g^{qp^n}$, and in particular $1 + f^q K[X]$ contains an element of $\mathrm{pow}(q)$. Thus, 1 is in the closure of $\mathrm{pow}(q)$ under the (f)-topology, as claimed.

Conversely, suppose that K is not separably closed, let f be a separable irreducible polynomial, and let α, β be two distinct roots of f in the algebraic closure of K; since K is transcendental over $\mathbb{F}_p$, we can suppose that α, β are transcendental too. We claim that there is a $t \in K \cap \overline{\mathbb{F}_p}$ such that 1 is not in the closure of $\mathrm{pow}(X - t)$ in the (f)-topology. Indeed, suppose 1 is in the closure for some t. Then, $\mathrm{pow}(X - t)$ meets $1 + fK[X]$, and in particular there are a unit u and an integer m such that f divides $1 - u(X - t)^m$. Hence, we must have $1 - u(\alpha - t)^m = 0 = 1 - u(\beta - t)^m$, and thus $(\alpha - t)/(\beta - t)$ must be a root of unity (of degree at most m), and in particular it must be algebraic over $\mathbb{F}_p$.

Let $r(t) := (\alpha - t)/(\beta - t)$ and $r := r(0) = \alpha/\beta$. Then,

$$r(t) = \frac{\alpha - t}{\beta - t} = \frac{r\beta - t}{\beta - t} \Longrightarrow \beta = \frac{t(r(t) - 1)}{r(t) - r}$$

whenever $t \neq 0$ (which implies $r(t) \neq r$). However, both t and $r(t)$ are algebraic over $\mathbb{F}_p$, and thus β should be algebraic too; this is a contradiction, and thus 1 is not in the closure of any $\mathrm{pow}(X - t)$ with $t \neq 0$. □

The following proposition shows the difference between the behavior of $G_n(K[X])$ in positive characteristic with respect to the characteristic 0 case (Proposition 4.3). Part (a) does not hold without the assumption that K is separably closed; indeed, its failure is critical in the proof of Proposition 7.4.

PROPOSITION 5.2. *Let K be a field of characteristic $p > 0$. Then, the following hold.*

(a) *If K is separably closed, then $G_1(K[X])$ is not dense in $G(K[X])$.*

(b) *If K is algebraic over $\mathbb{F}_p$, then $G_n(K[X])$ has no isolated points for all $n \geq 1$.*

(c) *If K is algebraic over $\mathbb{F}_p$, then $G_m(K[X])$ is contained in the closure of $G_n(K[X])$ for all $n \geq m \geq 0$.*

Note that all three of these hypothesis are fulfilled when K is the algebraic closure of $\mathbb{F}_p$.

Proof. (a) Suppose first $p \geq 3$, and consider the open set $1 + X^2 + X^3K[X]$: if it intersects $G_1(K[X])$ then there are an irreducible polynomial $g(X)$, $u \in K$, $k \in \mathbb{N}$ and $b(X) \in K[X]$ such that $ug(X)^k = 1 + X^2 + X^3b(X)$. Since K is separably closed, we can write $g(X) = X^{p^r} - a$ for some $r \geq 0$ and some $a \in K$. If $r > 0$, then $ug(X)^k$ has no monomial of degree 2, a contradiction; hence, it must be $g(X) = X - a$. Considering the coefficients of degree 1 and 2, we have

$$\begin{cases} 0 = u\binom{k}{1}a^{k-1} = uk(-1)^{k-1}a^{k-1} \\ 1 = u\binom{k}{2}a^{k-2} = u\frac{k(k-1)}{2}(-1)^{k-2}a^{k-2}. \end{cases}$$

The second equality implies that k, $k-1$ and a are all different from 0 in K; however, this implies $uka^{k-1} \neq 0$, a contradiction. Hence, $1 + X^2$ does not belong to the closure of $G_1(K[X])$.

Suppose now $p = 2$, and consider the open set $1+X+X^3+X^4K[X]$. Considering the monomial of degree 1, we see that the irreducible polynomial $g(X)$ must be in the form $X-a$. Suppose thus $1+X+X^3+X^4b(X) = u(X-a)^k$: then, confronting the coefficients of degree 1 we have that k is odd, while confronting the coefficients of degree 2 we get that $(k-1)/2$ is even. The coefficient of degree 3 of $u(X-a)^k$ is thus $uk(k-2)\frac{k-1}{2}(-1)^{k-3}a^{k-3} = 0$, contradicting the presence of X^3. Hence, $1 + X + X^3$ does not belong to the closure of $G_1(K[X])$.

(b) Let $a(X) \in G_n(K[X])$, and let $b(X)$ be a polynomial coprime with $a(X)$. Let q be a prime power such that $\mathbb{F}_q$ contains all the coefficients of $a(X)$ and of $b(X)$. Then, $a(X)$ and $b(X)$ are coprime in $\mathbb{F}_q[X]$; since $\mathbb{F}_q[X]/b(X)\mathbb{F}_q[X]$ is finite, we can find $k > 0$ such that $a(X)^k \equiv 1 \bmod b(X)\mathbb{F}_q[X]$, and thus $a(X)^{k+1} \in G_n(K[X]) \cap (a(X)+b(X)K[X])$ is different from $a(X)$. Hence, $a(X)$ is not isolated in $G_n(K[X])$.

(c) If K is not algebraically closed then $G_1(K[X])$ is dense in $G(K[X])$ (see Proposition 6.3 below) and thus by Proposition 3.6(a) all the $G_n(R)$ are actually dense.

Suppose that K is algebraically closed. Let $a(X) \in G_m(K[X])$, with $m < n$, and let $b(X)$ be coprime with $a(X)$; let $r := n - m$. Choose r distinct elements, $t_1, \ldots, t_r$, such that $b(t_i) \neq 0$ and $a(t_i) \neq 0$ for all i. Let F be the subfield of K generated by the t_i and by the coefficients of b; since K is algebraic over $\mathbb{F}_p$, F is finite. Then, $F[X]/b(X)F[X]$ is finite, and thus we can find $k_1, \ldots, k_r$ such that $(X - t_i)^{k_i} \in 1 + b(X)F[X]$ for all i; in particular, $(X - t_i)^{k_i} \in 1 + b(X)K[X]$. Let $A(X) := a(X)(X - t_1)^{k_1} \cdots (X - t_r)^{k_r}$: by construction, $A(X) \in G_n(K[X])$ and $A(X) \equiv a(X) \bmod b(X)K[X]$, that is, $A(X) \in a(X) + b(X)K[X]$. Hence, all neighborhoods of $a(X)$ intersect $G_n(K[X])$, and thus $a(X)$ is in the closure of

$G_n(K[X])$, as claimed. $\square$

We now deal with the problem of distinguishing separably closed fields of different characteristics, that is, we want to prove that if $G(K[X]) \simeq G(K'[X])$ then K and K' have the same characteristic, extending Proposition 4.1. Until the end of the section the section, K will be a field of characteristic $p > 0$ and $\overline{K}$ a (fixed) algebraic closure of K. We denote by v_p the p-adic valuation on the positive integers.

DEFINITION 5.3. Let $r(X) \in K[X]$ be an irreducible polynomial. An *$r(X)$-sequence* is a sequence $E \subset \mathrm{pow}(r(X))$. If $r(X) \notin (X-1)$, we say that E is *basic* if E converges to 1 in the $(X-1)$-topology.

Since $E \subseteq \mathrm{pow}(q(X))$, we can always write the elements of an $r(X)$-sequence $E := \{s_n(X)\}_{n\in\mathbb{N}}$ as $s_n(X) := u_n r(X)^{e_n}$, for some $u_n \in K^\bullet$ and some positive integers e_n.

LEMMA 5.4. *Let p be a prime number and e, z be natural numbers such that $p^z < e$. If p divides the binomial coefficient $\binom{e}{p^t}$ for all $1 \leq t \leq z$, then $v_p(e) \geq z+1$.*

Proof. Fixed p and e, we proceed by induction on z. If $z = 0$, then we know that p divides $\binom{e}{p^0} = \binom{e}{1} = e$, and the claim holds.

Suppose we have proved the claim up to $z-1$. Then, $p^z | e$ and p divides

$$\binom{e}{p^z} = \frac{e(e-1)\cdots(e-p^z+1)}{p^z(p^z-1)\cdots 2\cdot 1}.$$

For all $0 < k < p^z$, we have $v_p(k) < v_p(e)$ and thus $v_p(e-k) = \min\{v_p(e), v_p(k)\} = v_p(k)$; hence, the p-valuation of the product $(e-1)\cdots(e-p^z+1)$ is equal to the p-valuation of $(p^z-1)!$. Thus,

$$0 < v_p\left(\binom{e}{p^z}\right) = v_p\left(\frac{e}{p^z}\right) = v_p(e) - v_p(p^z) = v_p(e) - z.$$

It follows that $v_p(e) > z$, i.e., $v_p(e) \geq z+1$. By induction, the claim is proved. $\square$

PROPOSITION 5.5. *Let $r(X), q(X)$ be coprime irreducible polynomials, and let $E = \{s_n(X) := u_n r(X)^{e_n}\}_{n\in\mathbb{N}}$ be an $r(X)$-sequence. Let $s \in K^\bullet$. Then, E converges to s in the $(q(X))$-topology if and only if $v_p(e_n) \to \infty$ and, for every root λ of $q(X)$ in $\overline{K}$, we have $s_n(\lambda) = s$ for all sufficiently large n.*

Proof. Suppose first that $K = \overline{K}$ is algebraically closed. Then, we can write $r(X) := X - t$, $q(X) := X - \lambda$ for some $t, \lambda \in K$. Let $Q := (X - \lambda)$.

Suppose the two conditions hold, and let k be any integer. Then, there is an N such that $v_p(e_n) \geq k$ and $s_n(\lambda) = s$ for every $n \geq N$. Thus,

$$\begin{aligned} s_n(\lambda) = u_n(X-t)^{e_n} &= u_n(X-\lambda+\lambda-t)^{p^k e'_n} = \\ &= u_n((X-\lambda)^{p^k} + (\lambda-t)^{p^k})^{e'_n}. \end{aligned}$$

Untying the binomial, we obtain $u_n((\lambda - t)^{p^k})^{e'_n} = u_n(\lambda - t)^{e_n} = s_n(\lambda) = s$, while the other monomials are all divisible by $(X-\lambda)^{p^k}$. Therefore, $s_n(\lambda) \in s + Q^{p^k}$ for all $n \geq N$. Since $\{s + Q^{p^k}\}$ is a local basis of neighborhoods of s in the Q-topology, E tends to s.

Conversely, if E converges to s in the Q-topology then $s_n(X) \in s + Q$ for all sufficiently large n, i.e., $s_n(X) - s \in Q$, or equivalently $q(X)$ divides $s_n(X) - s$. Hence, $s_n(\lambda) - s = 0$ and $s_n(\lambda) = s$ for all sufficiently large n. We now have

$$s_n(X) = u_n(X - \lambda + \lambda - t)^{e_n} = u_n \sum_i \binom{e_n}{i} (\lambda - t)^{n-i} (X - \lambda)^i.$$

Since E converges to s, the polynomial $s_n(X) - s$ must belong (for large n) to Q^k for every $k > 0$, that is, the coefficients of degree $< k$ in $X - \lambda$ must be equal to 0. Choose $k = p^z + 1$. Then, for large n, we have that $\binom{e_n}{r} = 0$ for all $1 \leq r \leq p^z$; by Lemma 5.4, we have $v_p(e_n) \geq z + 1$. Since z was arbitrary, $v_p(e_n)$ tends to infinity.

If now K is not algebraically closed, it is enough to note that the convergence of E in the $(q(X))$-topology is equivalent to the convergence in $\overline{K}[X]$ of E in the $(X - \lambda)$-topology for every root λ of $q(X)$, and then apply the previous reasoning. □

Let now E be a basic $r(X)$-sequence. We denote by $\mathcal{L}_K(E)$ the set of maximal ideals Q of $K[X]$, different from $(r(X))$, such that E converges to 1 in the Q-topology; furthermore, we denote by $\mathcal{L}_K$ the set of natural numbers n such that there is an irreducible polynomial $r(X)$ and an $(r(X))$-sequence E with $|\mathcal{L}_K(E)| = n$. These sets are determined by the Golomb topology, in the following sense.

Proposition 5.6. *Preserve the notation above.*

(a) Let $h : G(K[X]) \longrightarrow G(K'[X])$ be a homeomorphism such that $h(1) = 1$, and let $s(X)$ be an irreducible polynomial such that $s(X)$ generates $h((r(X))$. Then, $h(E)$ is a $s(X)$-sequence and $h(\mathcal{L}_K(E)) = \mathcal{L}_{K'}(h(E))$.

(b) If $G(K[X])$ and $G(K'[X])$ are homeomorphic, then $\mathcal{L}_K = \mathcal{L}_{K'}$.

Proof. (a) follows from the fact that a homeomorphism of Golomb spaces is also a homeomorphism between the Q-topology and the Q'-topology (where $Q' := h(Q^\bullet) \cup \{0\}$). (b) follows directly from (a). □

To study $\mathcal{L}_K$, we introduce another set associated to an $r(X)$-sequence E: we denote by $\ell(E)$ the subset of $\overline{K}$ formed by the roots of the irreducible polynomials that generate a prime ideal of $\mathcal{L}(E)$, that is, $\ell(E)$ is the set of all λ such that E converges to 1 in the $(X - \lambda)$-topology of $\overline{K}[X]$. Note that $\ell(E)$ does not depend on the field K, i.e., it remains the same also when considering E in $K'[X]$, where K' is an extension of K.

Proposition 5.7. *Let E be a basic X-sequence. If $1 \in \ell(E)$, then $\ell(E)$ is a torsion multiplicative subgroup of $\overline{K}^\bullet$.*

Proof. Let $E = \{s_n(X) := u_n X^{e_n}\}$. If $1 \in \ell(E)$, then $1 = s_n(1)$ for all sufficiently large n, that is, $1 = u_n 1^{e_n} = u_n$ for all large n; without loss of generality we can suppose that $u_n = 1$ for all n. By Proposition 5.5 (and noting that the condition $v_p(e_n) \to \infty$ does not depend on λ) it follows that $\ell(E)$ is the set of all λ such that $\lambda^{e_n} = 1$ for all sufficiently large n, and it is easy to see that this is a subgroup of $K^\bullet$ whose elements are all torsion. $\square$

The previous proposition also has a converse.

PROPOSITION 5.8. *Let H be a torsion multiplicative subgroup of $\overline{K}^\bullet$. Then, there is a basic X-sequence E with $\ell(E) = H$.*

Proof. If H is finite, let $f_n := |H|$ for all n. If H is infinite, let $h_1, h_2, \ldots$ be an enumeration of H (note that, since H is torsion, it is contained in the algebraic closure of $\mathbb{F}_p$ and thus it is countable), and let f_n be the order of the subgroup generated by $h_1, \ldots, h_n$. We claim that the sequence $E = \{s_n(X) := X^{f_n p^n}\}_{n \in \mathbb{N}}$ satisfies the condition: indeed, $v_p(e_n) = n$ for all n, and thus the p-adic valuation of the exponents goes to infinity. Furthermore, if $h \in H$ then $s_n(h) = h^{f_n p^n} = (h^{f_n})^{p^n} = 1^{p^n} = 1$ for all sufficiently large n. Thus $h \in \ell(E)$ and $H \subseteq \ell(E)$.

On the other hand, suppose $h \notin H$. If its order is infinite, then $h^{f_n p^n} \neq 1$ for every n and $h \notin \ell(E)$ by Proposition 5.5. If the order of h is finite, we claim that it does not divide any f_n. Indeed, every finite subgroup of $K^\bullet$ is cyclic, and thus if the order of h divides f_n then h would belong to $\langle h_1, \ldots, h_n \rangle$ and thus to H, a contradiction. Since no element of $K^\bullet$ has order p (or a multiple of p), it follows that the order of H does not divide $f_n p^n$ for every n; thus, again $h^{f_n p^n} \neq 1$ and so $h \notin \ell(E)$. The claim is proved. $\square$

In general, we only know that $|\ell(E)| \leq |\mathcal{L}_K(E)|$; however, when K is algebraically closed the natural map $\lambda \mapsto (X - \lambda)$ from K to $\mathrm{Max}(K[X])$ is a bijection, and thus in particular $|\ell(E)| = |\mathcal{L}_K(E)|$. We now can use the previous propositions to determine $\mathcal{L}_K$.

LEMMA 5.9. *Let K be an algebraically closed field of characteristic $p > 0$, and let n be a positive integer. Then, there is a subgroup of $K^\bullet$ of cardinality n if and only if n is coprime with p.*

Proof. If n is coprime with p, then there is a k such that n divides $p^k - 1$; therefore, the multiplicative group of $\mathbb{F}_{p^k}$ contains a subgroup of cardinality n. Since K is algebraically closed, it contains $\mathbb{F}_{p^k}$, and thus $K^\bullet$ contains a subgroup of cardinality n.

If n is not coprime with p, then p divides n. Thus, if $K^\bullet$ contains a subgroup of cardinality n, it contains also a subgroup of cardinality p. However, no element of $K^\bullet$ has order p. $\square$

PROPOSITION 5.10. *Let K be a separably closed field of characteristic $p > 0$. Then, $\mathcal{L}_K = \mathbb{N} \setminus p\mathbb{N}^+$.*

Proof. Suppose first that K is algebraically closed. If $n > 0$ is coprime with p, then by Lemma 5.9 there is a subgroup of $K^\bullet$ of cardinality n, and thus by Proposition 5.8 there is an X-sequence E with $|\mathcal{L}_K(E)| = n$. Furthermore, the sequence $\{X^k\}_{k\in\mathbb{N}}$ does not converge in any P-topology (as $v_p(k)$ does not tend to infinity) and thus also $0 \in \mathcal{L}_K$. Hence, $\mathbb{N} \setminus p\mathbb{N}^+ \subseteq \mathcal{L}_K$.

Conversely, let E be a $(X-\lambda)$-sequence with $(X-\mu) \in \mathcal{L}_K(E)$. Let ψ be the map

$$\begin{aligned} \psi\colon G(K[X]) &\longrightarrow G(K'[X]),\\ f(X) &\longmapsto f((\mu-\lambda)X+\lambda). \end{aligned}$$

Then, ψ is a ring automorphism of $K[X]$, and thus it is a self-homeomorphism of $G(K[X])$. Furthermore,

$$\psi(X-\lambda) = (\mu-\lambda)X + \lambda - \lambda = (\mu-\lambda)X$$

and thus $\psi((X-\lambda)) = (X)$; on the other hand,

$$\psi(X-\mu) = (\mu-\lambda)X + \lambda - \mu = (\mu-\lambda)(X-1)$$

and thus $\psi((X-\mu)) = (X-1)$. Hence, $\psi(E)$ is a basic X-sequence, and $|\mathcal{L}_K(E)| = |\mathcal{L}_K(\psi(E))|$. By Lemma 5.9, $|\mathcal{L}_K(E)|$ is coprime with p, and thus $\mathcal{L}_K \subseteq \mathbb{N} \setminus p\mathbb{N}^+$. Thus the two sets are equal.

Suppose now that K is separably closed. Then, every irreducible polynomial is either linear or in the form $X^{p^n} - a$ for some $a \in K$ and some $n \geq 1$; hence, every maximal ideal of $K[X]$ is contained in a a single prime ideal of $\overline{K}[X]$. Therefore, a $r(X)$-sequence E in $K[X]$ is a $s(X)$-sequence in $\overline{K}[X]$, where $s(X)$ generates the prime ideal containing $r(X)$. In particular, $|\mathcal{L}_K(E)| = |\mathcal{L}_{\overline{K}}(E)| = \ell(E)$; therefore, $\mathcal{L}_K = \mathcal{L}_{\overline{K}}$ and thus $\mathcal{L}_K = \mathbb{N} \setminus p\mathbb{N}^+$, as claimed. □

THEOREM 5.11. *Let K, K' be two separably closed fields of characteristic p, p' (respectively). If $G(K[X])$ and $G(K'[X])$ are homeomorphic, then $p = p'$.*

Proof. By Corollary 4.2 we can suppose $p, p' > 0$. By Proposition 5.6(b), $\mathcal{L}_K = \mathcal{L}_{K'}$. By Proposition 5.10 $\mathcal{L}_K = \mathbb{N} \setminus p\mathbb{N}^+$ and $\mathcal{L}_{K'} = \mathbb{N} \setminus p'\mathbb{N}^+$. Hence, $p = p'$. □

COROLLARY 5.12. *Let K, K' be algebraically closed fields, and suppose that $G(K[X]) \simeq G(K'[X])$. If one of them is uncountable, then $K \simeq K'$.*

Proof. Since the cardinality of $K[X]$ is the same of K, if $G(K[X]) \simeq G(K'[X])$ then K and K' have the same cardinality. If one of them has characteristic 0, then by Proposition 4.1 so does the other; otherwise, they have the same positive characteristic by Theorem 5.11. Since they have the same uncountable cardinality, and they are algebraically closed and of the same characteristic, by [12, Chapter VI, Theorem 1.12] K and K' are isomorphic, as claimed. □

In the countable case, we need to distinguish fields that have different degree of transcendence over $\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}_p}$. If the characteristic is positive, the following Proposition 7.1 will show that we can distinguish $\overline{\mathbb{F}_p}$ from the other fields, but it is an open question if, for example, the algebraic closure of $\mathbb{F}_p(T)$ and the algebraic closure of $\mathbb{F}_p(T_1, T_2)$ give rise to non-homeomorphic Golomb spaces.

6. Almost prime elements.

Let R be a Dedekind domain. We say that an element $b \in R$ is *almost prime* if it is irreducible and it is contained in a unique prime ideal; this happens if and only if $bR = P^n$ for some prime ideal P, with n being exactly the order of the class of P in the class group.

DEFINITION 6.1. We say that a Dedekind domain R with torsion class group *has the almost Dirichlet property* (or, simply, that R is *almost Dirichlet*) if any coprime coset contains (at least) one almost prime element, that is, if the set of almost prime elements is dense in $G(R)$.

REMARK 6.2.

(1) If R has torsion class group, $h : G(R) \longrightarrow G(S)$ is a homeomorphism of Golomb spaces and $b \in R$ is contained in a unique prime ideal, the same happens for $h(b)$ [16, Proposition 2.7]. However, it is an open question whether h sends irreducible elements into irreducible elements; in particular, we do not know if the almost Dirichlet property is a topological invariant (with respect to the Golomb topology).

(2) If R is almost Dirichlet, then $G_1(R)$ is dense in $G(R)$, as every almost prime element belongs to $G_1(R)$.

(3) By Dirichlet's theorem on primes in arithmetic progressions, the ring $\mathbb{Z}$ of integers is almost Dirichlet. The same happens when $R = F[X]$, where F is a finite field [14, Theorem 4.8] and when $R = \mathbb{Q}[X]$ or, more generally, for $R = K[X]$ when K is a Hilbertian field.

(4) A field F is said to be *pseudo-algebraically closed* (PAC) if every nonempty absolutely irreducible variety defined over F has an F-rational point [7, Chapter 11]. If F is PAC and contains separable irreducible polynomials of arbitrarily large degree, then every coprime coset contains irreducible polynomials, and $F[X]$ has the almost Dirichlet property [2, Theorem A].

PROPOSITION 6.3. *Let F be an algebraic extension of a finite field that is not algebraically closed. Then, $F[X]$ has the almost Dirichlet property.*

Proof. If F is finite, the claim follows from [14, Theorem 4.8]. If not, then F is pseudo-algebraically closed [7, Corollary 11.2.4] and has (simple) separable extensions of arbitrarily large degree, and thus $F[X]$ is almost Dirichlet by [2, Theorem A]. $\square$

A simple consequence of the Remark 6.2(3) and of Proposition 4.3 is the following.

COROLLARY 6.4. $G(\mathbb{Q}[X]) \not\simeq G(\overline{\mathbb{Q}}[X])$.

We now want to prove that, at least in some cases, a homeomorphism of Golomb spaces preserves almost prime elements and, to do so, we shall abstract the proof of [1, Lemmas 5.10 and 5.11].

DEFINITION 6.5. Let R be a Dedekind domain with torsion class group. We say that R is *power separated* if, for every maximal ideal P and every $b \in G_{\{P\}}(R)$, we have $\overline{\mathrm{pow}(b)} \cap G_{\{P\}}(R) = \mathrm{pow}(b)$.

A more explicit sufficient condition is the following.

PROPOSITION 6.6. *Let R be a Dedekind domain with torsion class group, and suppose there is a function $d : R^\bullet \longrightarrow [1, +\infty)$ such that, for all $a, b \in R^\bullet$:*

- *$d(ab) = d(a)d(b)$;*
- *$d(a + b) \leq d(a) + d(b)$ if $a \neq -b$;*
- *$d(a) = 1$ if and only if a is a unit.*

Then, R is power separated.

Proof. Let P be a prime ideal, $b \in G_{\{P\}}(R)$ and $c \in G_{\{P\}}(R) \setminus \mathrm{pow}(b)$. By hypothesis, $d(b) > 1$, and thus we can find an integer t such that $d(b)^t > d(b)^{t-1} + d(c) + 1$. Let $I := (b^t - 1)R$: then, $c + I$ is open (since $b^t - 1 \notin P$), and we claim that $(c + I) \cap \mathrm{pow}(b) = \emptyset$.

Indeed, suppose not, and let z be in the intersection. Then, $z = ub^r$ for some $u \in U(R)$, $r \in \mathbb{N}$. Since $b^t \equiv 1 \bmod I$, we see that $z \equiv ub^s \bmod I$ for some $s \in \{0, \dots, t-1\}$ (setting $b^0 := 1$), and thus $c \equiv ub^s \bmod I$, i.e., $c - ub^s \in I$. However, as $c \neq ub^s$ we can calculate

$$d(c - ub^s) \leq d(c) + d(ub^s) = d(c) + d(b)^s \leq d(c) + d(b)^{t-1} < d(b)^t - 1 \leq d(b^t - 1).$$

For all $x \in I$, we have $d(x) \geq d(b^t - 1)$; this is a contradiction, and thus $c + (b^t - 1)R$ does not meet $\mathrm{pow}(b)$. Therefore, $\mathrm{pow}(b)$ is closed in $G_{\{P\}}(R)$, and thus R is power separated. □

COROLLARY 6.7. *The following hold.*

(a) *If R is the integral closure of $\mathbb{Z}$ in an imaginary quadratic extension of $\mathbb{Q}$, then R is power separated.*

(b) *If $R = K[X]$ for some field K, then R is power separated.*

Proof. In the first case, all units of R are roots of unity, and conversely every element of R on the unit circle is a root of unity; hence, we can take the complex modulus as d. For the second case, set $d(p) := 2^{\deg(p)}$. □

THEOREM 6.8. *Let R be a Dedekind domain with torsion class group, and suppose that R is power separated and has the almost Dirichlet property. If S is a Dedekind domain and $h : G(S) \longrightarrow G(R)$ is a homeomorphism, then h sends almost prime elements into almost prime elements.*

Proof. Let $a \in S$ be an element contained in a unique prime ideal, and let $b := h(a)$. We first claim that $h(\mathrm{pow}(a)) \subseteq \mathrm{pow}(b)$.

Fix a unit $u_0 \in S$ and an integer $n \geq 1$. Let $f : G(S) \longrightarrow G(S)$ be the map sending every x to u_0x^n, and let $\phi : G(R) \longrightarrow G(R)$ be the composition $h \circ f \circ h^{-1}$. Then, f is continuous in the Golomb topology, and thus so is ϕ; furthermore, if P is a prime ideal of R, then $h \circ f \circ h^{-1}(P) \subseteq P$ since $h^{-1}(P)$ is a prime ideal of S. Let

$$c := \phi(b) = \phi(h(a)) = (h \circ f \circ h^{-1} \circ h)(a) = h(u_0a^n).$$

Suppose that $c \notin \mathrm{pow}(b)$: then, since R is power separated, we can find an open set $\Omega := c+I$ such that $\Omega \cap \mathrm{pow}(b)$ does not meet $G_{\{Q\}}(R)$ (where Q is the radical of bR). Since ϕ is continuous, $\phi^{-1}(\Omega)$ is an open set containing b; hence, there is a $d \in R$, coprime with b, such that $\phi(b + dR) \subseteq \Omega$.

Since R is almost Dirichlet we can find an almost prime element $p \in b + dI$. Then, $\mathrm{pow}(p) = G_{\{P\}}(R)$, where P is the only prime ideal containing p; hence, $\phi(p) \in \mathrm{pow}(p)$, i.e., there are $u \in U(R)$ and $l \in \mathbb{N}^+$ such that $\phi(p) = up^l$. On the other hand,

$$\phi(p) \in \phi(b + dR) \subseteq \Omega = c + I$$

and, at the same time,

$$up^l \in u(b + dI)^l \subseteq u(b^l + I) = ub^l + I;$$

it follows that $c \equiv ub^l \bmod I$, i.e., $ub^l \in c + I = \Omega$. This contradicts the choice of I; hence, c must be in $\mathrm{pow}(b)$, that is, $h(u_0a^n) = c = ub^l$ for some l. Since this happens for every u_0 and every n, we have $h(\mathrm{pow}(a)) \subseteq \mathrm{pow}(b)$.

Suppose now that a is almost prime, and let P and Q be, respectively, the only prime ideal containing a and the only prime ideal containing b. Then,

$$G_{\{Q\}}(R) = h(G_{\{P\}}(S)) = h(\mathrm{pow}(a)) \subseteq \mathrm{pow}(b) \subseteq G_{\{Q\}}(R).$$

Thus $\mathrm{pow}(b) = G_{\{Q\}}(R)$, i.e., b is almost prime. □

7. Algebraic extensions of $\mathbb{F}_p$. As observed in [5, Corollary 14], a consequence of the fact that a homeomorphism of Golomb spaces sends units to units is that if K, K' are distinct finite fields then the Golomb spaces $G(K[X])$ and $G(K'[X])$ are not homeomorphic. The purpose of this section is to generalize this result, allowing K and K' to be arbitrary algebraic extensions of the same $\mathbb{F}_p$.

The first step is to distinguish algebraic extensions from transcendental extensions.

PROPOSITION 7.1. *Let K be a field of characteristic $p > 0$ and let $g \in K[X]$ be an irreducible polynomial. Then, the following are equivalent.*

(i) $\mathrm{pow}(g)$ *is not discrete in* $G(K[X])$.

(ii) *For every* $s_1, s_2 \in K$, *either* $g(s_1) = 0$, $g(s_2) = 0$ *or* $g(s_1)/g(s_2)$ *is a root of unity.*

(iii) K *is algebraic over* $\mathbb{F}_p$.

Proof. (i) $\Longrightarrow$ (ii) Fix $\lambda := ug^n \in \mathrm{pow}(g)$. Let $s_1, s_2 \in K$ be such that $g(s_1) \neq 0 \neq g(s_2)$, and let I be the ideal of $K[X]$ generated by $(X - s_1)(X - s_2)$: then, I is coprime with g, and thus $\lambda + I$ is an open subset of $G(K[X])$. Since $\mathrm{pow}(g)$ is not discrete, there are infinitely many $\lambda' := u'g^m \in \lambda + I$, with $\lambda' \neq \lambda$.

Therefore, I contains $u'g^m - ug^n = u'g^n(g^r - v)$, where $r := m - n$ and $v := uu'^{-1}$ (with $g^0 := 1$); setting $h := g^r - v$, it follows that $h(s_1) = h(s_2) = 0$, and thus that $r > 0$ (since if $r = 0$ then $v \neq 1$ and h is a nonzero constant) and $g(s_1)^r = v = g(s_2)^r$. Hence, $(g(s_1)/g(s_2))^r = v/v = 1$; that is, $g(s_1)/g(s_2)$ is a root of unity, as claimed.

(ii) $\Longrightarrow$ (iii) Suppose not: then, K is infinite. Let s_1 be any element of K such that $g(s_1) \neq 0$. Let F be field generated by s_1, the coefficients of g and an element of K that is transcendental over the prime field: then, F is infinite and contains only finitely many roots of unity. Hence, there are only finitely many $t \in F$ such that $g(t) = 0$ or $g(t) = ug(s_1)$ for some root of unity u in F. In particular, there is an s_2 which does not satisfy either equality; however, this contradicts the hypothesis, and thus K is algebraic over $\mathbb{F}_p$.

(iii) $\Longrightarrow$ (i) Let $\lambda \in \mathrm{pow}(g)$, and let I be an ideal of $K[X]$ that is coprime with g (and thus with λ); let f be a generator of I. We need to show that the open set $\lambda + I$ contains other elements of $\mathrm{pow}(g)$.

Let F be the subfield of K generated by u, the coefficients of g and by the roots of f: then, F is a finite field, say of cardinality q. For every $\alpha \in F$, $\lambda(\alpha)^{q-1} = 1$; hence, the polynomial $h := 1 - \lambda^{q-1}$ has zeros in every element of F, and in particular all the zeros of f are zeros of λ'. Let q' be a power of q greater than every multiplicity of the roots of f: then, f divides $h^{q'} = (1-\lambda^{q-1})^{q'} = 1-\lambda^{q'(q-1)}$. Therefore,

$$\lambda - \lambda^{q'(q-1)+1} = \lambda(1 - \lambda^{q'(q-1)}) \in I,$$

and thus $\lambda^{q'(q-1)+1} \in \lambda + I$, as claimed. $\square$

COROLLARY 7.2. *Let* K_1, K_2 *be two fields of positive characteristic. If* K_1 *is algebraic over its base field while* K_2 *is not then* $G(K_1[X]) \not\simeq G(K_2[X])$.

Let $\mathrm{Homeo}(G(R))$ be the group of self-homeomorphisms of $G(R)$, and let

$$\Lambda(R) := \{h \in \mathrm{Homeo}(G(R)) \mid h(P^\bullet) = P^\bullet \text{ for every } P \in \mathrm{Spec}(R)\}$$

and

$$\Lambda_1(R) := \{h \in \Lambda(R) \mid h(1) = 1\}.$$

Note that $\Lambda(R)$ does not necessarily contain all self-homeomorphisms of $G(R)$: for example, a ring automorphism ψ of R induces a self-homeomorphism of $\Lambda(R)$, but in general does not fix all prime ideals. (For an example, take $R = \mathbb{Z}[i]$ and let ψ be the complex conjugation.)

These groups are effectively invariants of the Golomb topology.

PROPOSITION 7.3. *Let R, S be two Dedekind domains, and suppose $G(R)$ and $G(S)$ are homeomorphic. Then, $\Lambda(R) \simeq \Lambda(S)$ and $\Lambda_1(R) \simeq \Lambda_1(S)$.*

Proof. Let $h : G(R) \longrightarrow G(S)$ be a homeomorphism. For every $\psi \in \Lambda(R)$, the map $\overline{\psi} := h \circ \psi \circ h^{-1}$ is a self-homeomorphism of $G(S)$, and if P is a prime ideal of R then $\overline{\psi}(P^\bullet) = h(\psi(h^{-1}(P^\bullet))) = h(h^{-1}(P^\bullet)) = P^\bullet$; thus, $\overline{\psi} \in \Lambda(S)$. Hence, h induces a map $\Lambda(R) \longrightarrow \Lambda(S)$, sending ψ to $\overline{\psi}$, which is easily seen to be a group homomorphism. Likewise, h^{-1} induces a map $\Lambda(S) \longrightarrow \Lambda(R)$ which is the inverse of the previous one. Hence, $\Lambda(R) \simeq \Lambda(S)$.

The reasoning for Λ_1 is the same, using the homeomorphism $h' : G(R) \longrightarrow G(S)$ sending x to $h(1)^{-1}h(x)$ (so that $h'(1) = 1$). □

For any unit u of R, let ψ_u be the multiplication by u, and let $H := \{\psi_u \mid u \in U(R)\}$. Then, H is a subgroup of $\Lambda(R)$ (and thus of $\mathrm{Homeo}(R)$) that is isomorphic to the group of units of R. For every $h \in \mathrm{Homeo}(G(R))$, the map $h_1 := \psi_{h(1)^{-1}} \circ h$ is a self-homeomorphism of $G(R)$ fixing 1; furthermore, if h lies in $\Lambda(R)$ then so does h_1, and thus $h_1 \in \Lambda_1(R)$. It follows that $\Lambda(R)$ is generated by H and $\Lambda_1(R)$, and in particular if $\Lambda_1(R)$ is trivial then $\Lambda(R) = H \simeq U(R)$.

For example, if $R = \mathbb{Z}$ then by [16, Theorem 6.7] $\Lambda_1(\mathbb{Z})$ is trivial and thus $\Lambda(\mathbb{Z})$ is isomorphic to $U(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$. This phenomenon is linked to the hypothesis we worked with in Section 6.

PROPOSITION 7.4. *Let R be a Dedekind domain with torsion class group that has the almost Dirichlet property and is power separated. Suppose that there are infinitely many prime ideals P such that $U(R) \longrightarrow R/P^{n_P}$ is injective for some integer n_P. Then, $\Lambda_1(R)$ is trivial and $\Lambda(R) \simeq U(R)$.*

Proof. Let Δ be the set of all prime ideals for which there is such a n_P, and let $X := \bigcup\{P^\bullet \mid P \in \Delta\}$.

By Theorem 6.8, any self-homeomorphism h of $G(R)$ sends almost prime elements into almost prime elements. Let $h \in \Lambda_1(R)$, and let f be almost prime: then, $h(f)$ is an almost prime element contained in the same prime ideal of f, and thus there is a $u_f \in U(R)$ such that $h(f) = u_f f$.

Let $P \in \Delta$. Then, h is a homeomorphism in the P-topology, and thus in particular it is continuous, i.e., for every n there is an $m = m(n) \geq n$ such that $h(1+P^m) \subseteq 1+P^n$ (using $h(1) = 1$). Choose $n \geq n_P$: then, for every $f \in 1+P^m$ that is almost prime both f and $u_f f$ are in $1+P^n$, and thus $f - u_f f = f(1-u_f) \in P^n$. Since $f \notin P$, it follows that $1-u_f \in P^n$. By the injectivity of $U(R) \longrightarrow R/P^n$ we have $u_f = 1$, i.e., f is a fixed point of h. The closure of $1+P^m$ is the Golomb topology is $(1+P^m) \cup P^\bullet$; hence, also all the elements of $P^\bullet$ are fixed points of h. It follows that $h|_X$ is the identity.

Let now $z \in G(R)$ and let $z + I$ be an open neighborhood of z. Since Δ is infinite, there is a $Q \in \Delta$ that is coprime with I and z; thus, $z + I$ meets Q. Since

I was arbitrary, it follows that z is in the closure of X; thus, X is dense in $G(R)$. Since $h|_X$ is the identity, the whole h is the identity. Hence, $\Lambda_1(R)$ is trivial and $\Lambda(R) \simeq U(R)$. □

Theorem 7.5. *Let K, K' be fields of characteristic $p > 0$. If K is algebraic over $\mathbb{F}_p$ and $G(K[X]) \simeq G(K'[X])$ then $K \simeq K'$.*

Proof. By Corollary 7.2, K' must be algebraic over $\mathbb{F}_p$. If K is algebraically closed, then $G_1(K[X])$ is not dense in $G(K[X])$ (Proposition 5.2(a)); if K' is not algebraically closed, then $K'[X]$ is almost Dirichlet (Proposition 6.3) and thus $G_1(K[X])$ is dense in $G(K[X])$. Therefore, if K is algebraically closed then so is K', and thus $K \simeq K'$.

Suppose now that K is not algebraically closed. By the previous reasoning, neither K' is algebraically closed. By Proposition 6.3, $K[X]$ and $K'[X]$ are almost Dirichlet, and thus by Proposition 7.4 $\Lambda_1(K[X]) \simeq U(K[X]) = K^\bullet$ and $\Lambda_1(K'[X]) \simeq U(K'[X]) = K'^\bullet$. Furthermore, all maps $K^\bullet \longrightarrow K[X]/P$ are injective; by Proposition 7.4, it follows that $K^\bullet \simeq K'^\bullet$.

We can consider K and K' contained in the algebraic closure $\overline{\mathbb{F}_p}$. If K' is not isomorphic to K, then $K \neq K'$, and thus without loss of generality there is a finite extension $\mathbb{F}_{p^n}$ that is contained in K but not in K'. Hence, $K^\bullet$ contains elements of order $p^n - 1$ (the generator of the multiplicative group of $\mathbb{F}_{p^n}$) while K' does not, because $p^m - 1$ is a multiple of $p^n - 1$ only if m is a multiple of n. Therefore, $K^\bullet \simeq K'^\bullet$ implies $K = K'$, as claimed. □

As a corollary, we are able to answer affirmatively to a question posed in [5, Section 3.1]. We denote by $\mathfrak{c}$ the cardinality of the continuum.

Corollary 7.6. *The number of distinct Golomb topologies associated to countably infinite domains is $\mathfrak{c}$.*

Proof. There are $\mathfrak{c}$ possible pairs of binary operations on a countably infinite set; hence, there are at most $\mathfrak{c}$ ring structures and at most $\mathfrak{c}$ distinct Golomb topologies.

To show that there are exactly $\mathfrak{c}$, let p be a prime number and let $\mathcal{C}_p$ be the set of all (isomorphism classes of) algebraic extensions of $\mathbb{F}_p$. By Theorem 7.5, the Golomb topologies relative to the members of $\mathcal{C}_p$ are pairwise non-homeomorphic, and thus we need to show that $\mathcal{C}_p$ has cardinality at least $\mathfrak{c}$.

Let $\{q_1, q_2, \ldots\}$ be the set of prime numbers. To each $A \subseteq \mathbb{N}$, we can associate the field $F(A)$ defined as the composition of the extensions of $\mathbb{F}_p$ of degree q_i, for $i \in A$: then, $F(A) \neq F(A')$ if $A \neq A'$, and thus the cardinality of $\mathcal{C}_p$ is at least the cardinality of the power set of $\mathbb{N}$, i.e., $\mathfrak{c}$. The claim is proved. □

The method used in the proof of Theorem 7.5 does not quite extend to the case in which the characteristic of K and K' are not supposed beforehand to be equal; that is, it is not clear how to prove the analogue of Theorem 5.11 for algebraic extensions of finite fields. We can however say something about the relation between the two characteristics.

PROPOSITION 7.7. *Let K, K' be algebraic extensions of $\mathbb{F}_p$ and $\mathbb{F}_{p'}$, respectively. If p divides $p'-1$, then $G(K[X])$ and $G(K'[X])$ are not homeomorphic.*

Proof. Using Theorem 5.11 we can suppose that K and K' are not algebraically closed. As in the proof of Theorem 7.5, by Propositions 6.3 and 7.4 if $G(K[X]) \simeq G(K'[X])$ then the groups of units $K^\bullet$ and $K'^\bullet$ are isomorphic. However, $p|p'-1$ implies that there is an $u \in K'^\bullet$ of order p, something which cannot happen in $K^\bullet$. Hence, $G(K[X])$ and $G(K'[X])$ are not homeomorphic. $\square$

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