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RESEARCH ARTICLE

Star operations on numerical semigroups: the multiplicity 3 case

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Abstract We prove an explicit formula for the number of star operations on numerical semigroups of multiplicity 3 in terms of the generators of the semigroup. We also estimate the number of semigroups of multiplicity 3 with exactly n star operations.

Keywords Numerical semigroups · Star operations · Pseudosymmetric semigroups

1 Introduction

The notion of star operation was born in the context of the multiplicative theory of ideals, as a generalization of the divisorial closure (or v-operation) [6,11]. The problem of counting the number of star operations on a given domain has been recently solved in some special cases, such as h-local Prüfer domains [7], pseudo-valuation domains [13] and some classes of one-dimensional Noetherian domains [8,9]. In the latter case, there is often much interplay between local rings and their value semigroups (see e.g. [2–4,12]); in particular, semigroup rings in the form $K[[X^S]] := K[[\{X^s : s \in S\}]]$ (where K is a field and S is a numerical semigroup) are a rich source of examples, either for studying star operations [8,9] or the related case of semiprime operations [19].

Star operations were subsequently defined on semigroups as a way to generalize certain ring-theoretic definitions [10]. The study of the case of numerical semigroups was undergone in [18], where it was shown that, if $n > 1$, there are only a finite number of numerical semigroups with exactly n star operations; however, this result

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was obtained not through a precise counting, but through estimates. Like in other cases [5,14,15], the problem of obtaining an exact counting becomes simpler if we fix a low multiplicity: since the cases of multiplicity 1 and 2 are trivial (the former containing only $\mathbb{N}$ and the latter consisting only of symmetric semigroups, which have only one star operation), the goal of this paper is to tackle semigroups of multiplicity 3. We prove (Theorem 7.6) a direct formula for the number of star operations in terms of the generators of the semigroup, which in particular allows, for any integer n, to obtain fairly quickly an explicit list of the semigroups of multiplicity 3 with exactly n star operations.

The structure of the paper is as follows: Sect. 3 introduces an order on the set of non-divisorial ideals of a numerical semigroup S; in Sect. 4 is introduced a graphical representation of the ideals between S and $\mathbb{N}$, which is used in Sect. 6 to find explicitly the set of ideals closed by a principal star operations. Section 7 contains the main theorem of the paper, while Sect. 8 presents some estimates on the number of numerical semigroups with exactly n star operations.

2 Background and notation

Like [18], the notation and the terminology of this paper follow [4]; for further informations about numerical semigroups, the reader may consult [16].

A *numerical semigroup* is a subset $S \subseteq \mathbb{N}$ such that $0 \in S$, $a + b \in S$ for every $a, b \in S$ and such that $\mathbb{N} \setminus S$ is finite. If $a_1, \dots, a_n$ are natural numbers, $\langle a_1, \dots, a_n \rangle$ denotes the semigroup generated by $a_1, \dots, a_n$, i.e., the set $\{\lambda_1 a_1 + \cdots + \lambda_n a_n : \lambda_i \in \mathbb{N}\}$.

A *fractional ideal* (or simply an *ideal*) of S is a nonempty subset $I \subseteq S$ such that $i + s \in S$ for every $i \in I$, $s \in S$, and such that $d + I \subseteq S$ for some $d \in \mathbb{Z}$. We denote by $\mathcal{F}(S)$ the set of fractional ideals of S, and by $\mathcal{F}_0(S)$ the set of fractional ideals contained between S and $\mathbb{N}$ or, equivalently, the set of fractional ideals whose minimal element is 0. Note that, if I is an ideal, $I$is bounded below and $I - \min(I) \in \mathcal{F}_0(S)$. The intersection of a family of ideals, and the union of a finite family of ideals, is an ideal. If I, J are ideals of S, then $(I - J) := \{x \in \mathbb{Z} : x + J \subseteq I\}$ is an ideal; moreover, if $I, J \in \mathcal{F}_0(S)$ then $(I - J) \subseteq \mathbb{N}$.

The *Frobenius number* $g(S)$ of a numerical semigroup S is the biggest element of $\mathbb{Z} \setminus S$, while the *degree of singularity* $\delta(S)$ is the cardinality of $\mathbb{N} \setminus S$. The *multiplicity* $\mu(S)$ is the smallest positive integer in S.

A *star operation* on S is a map $* : \mathcal{F}(S) \longrightarrow \mathcal{F}(S)$, $I \mapsto I^*$, such that, for any $I, J \in \mathcal{F}(S)$, $a \in \mathbb{Z}$, the following properties hold:

(a) $I \subseteq I^*$;
(b) if $I \subseteq J$, then $I^* \subseteq J^*$;
(c) $(I^*)^* = I^*$;
(d) $a + I^* = (a + I)^*$;
(e) $S^* = S$.

An ideal I such that $I = I^*$ is said to be $*$-closed. The set of $*$-closed ideals is denoted by $\mathcal{F}^*(S)$; $*$ is uniquely determined by $\mathcal{F}^*(S)$, and even by $\mathcal{F}^*(S) \cap \mathcal{F}_0(S)$. The set of star operation on S is denoted by $\mathrm{Star}(S)$.

Star(S) has a natural ordering, where $*_1 \leq *_2$ if and only if $I^{*_1} \subseteq I^{*_2}$ for every ideal I or, equivalently, if and only if $\mathcal{F}^{*_1} \supseteq \mathcal{F}^{*_2}$. With this ordering, its minimum is the identity star operation (usually denoted by d), while the maximum is the star operation $I \mapsto (S - (S - I))$, usually denoted by v. Ideals that are v-closed are commonly said to be *divisorial*. We denote by $\mathcal{G}_0(S)$ the set of nondivisorial ideals I such that $\min I = 0$, that is, $\mathcal{G}_0(S) := \mathcal{F}_0(S) \setminus \mathcal{F}^v(S)$.

3 Ordering and antichains

Every set Δ of ideals of S defines a star operation $*_\Delta$ such that, for every ideal J of S,

$$J^{*_\Delta} := J^v \cap \bigcap_{I \in \Delta} (I - (I - J)) = J^v \cap \bigcap_{I \in \Delta} \bigcap_{\alpha \in (I-J)} (-\alpha + I). \qquad (1)$$

(For the equivalence of the two representations, see [18, Proposition 3.6].) Equivalently, $*_\Delta$ can be defined as the biggest star operation $*$ such that every element of Δ is $*$-closed. We call $*_\Delta$ the star operation *generated* by Δ. Denoting $*_{\{I\}}$ as $*_I$, we see that $*_\Delta = \inf_{I \in \Delta} *_I$. It is rapidly seen that $*_I = *_{a+I}$ for every ideal I and every integer a, so that we can always suppose $\Delta \subseteq \mathcal{F}_0(S)$, or even $\Delta \subseteq \mathcal{G}_0(S)$, since $*_I = v$ when I is divisorial.

A major problem is to find conditions under which two different sets of ideals generate different star operations. In general, it is possible that $*_\Delta = *_\Lambda$ while $\Delta \neq \Lambda$: the simplest example is maybe the case $\Lambda = \Delta \setminus \{J\}$, where J is a divisorial ideal. The non-unicity persists even if we discard divisorial ideals: in fact, whenever J is $*_I$-closed, both $\{I\}$ and $\{I, J\}$ define the same star operation.

Definition 3.1 Let S be a numerical semigroup and let $I, J \in \mathcal{G}_0(S)$. We say that I is $*$-*minor* than J, and we write $I \leq_* J$, if $*_I \geq *_J$ or, equivalently, if I is $*_J$-closed.

By [18, Theorem 3.8], if $I, J \in \mathcal{G}_0(S)$ and $I \neq J$ then $*_I \neq *_J$. In particular, $\leq_*$ is antisymmetric, and so it is an order on $\mathcal{G}_0(S)$.

By [18, Corollary 4.5], $(\mathcal{G}_0, \leq_*)$ has a maximum, $M_g := \{x \in \mathbb{N} : g - x \notin S\}$, but it has not (in general) a minimum, since the biggest star operation is v, and we are considering only operations generated by non-divisorial ideals. However, since the set $\mathcal{G}_0$ is finite, there are always minimal elements: more precisely, I is a minimal element if and only if $\mathcal{F}^{*_I} = \mathcal{F}^v \cup \{n + I : n \in \mathbb{Z}\}$. For example, if $S = \{0, \mu, \ldots\}$, then every ideal in the form $I = \{0, a, \ldots\}$ (with $1 < a < \mu$) is a minimal element of $(\mathcal{G}_0, \leq_*)$.

If a star operation $*$ closes an ideal I, then each ideal $*$-minor than I is $*$-closed. It follows that the set $\mathcal{A}(*) := \max_*(\mathcal{F}^* \cap \mathcal{G}_0)$ is uniquely determined by $*$ (where $\max_*$ denotes the maximum with respect to the $\leq_*$-ordering). The set $\mathcal{A}(*)$ is an example of antichain:

Definition 3.2 Let $(\mathcal{P}, \leq)$ be a partially ordered set. An *antichain* of $\mathcal{P}$ is a set $\Delta \subseteq \mathcal{P}$ such that no two members of Δ are comparable.

Let $\Omega(\mathcal{P})$ be the set of antichains of $\mathcal{P}$. By the previous observations, we have an injective map $\mathcal{A} : \mathrm{Star}(S) \longrightarrow \Omega(\mathcal{G}_0(S))$, given by $* \mapsto \mathcal{A}(*)$; conversely, (1) defines a map $* : \Omega(\mathcal{G}_0(S)) \longrightarrow \mathrm{Star}(S)$ which sends Δ into $*_\Delta$. It is clear that $*_{\mathcal{A}(*_\Delta)} = *_\Delta$

for every $\Delta \subseteq \mathcal{G}_0(S)$; therefore, $* \circ \mathcal{A}$ is the identity on $\mathrm{Star}(S)$, and $*$ is a surjective map. We shall show in Corollary 6.5 that, when $\mu = 3$, $\mathcal{A}$ and $*$ are bijective.

4 The graphical representation

The remainder of this article will deal excusively with semigroups of multiplicity 3. The following trivial observation is the basis of all our method.

Proposition 4.1 *Let S be a numerical semigroup of multiplicity 3, and I a fractional ideal of S. Then, there are uniquely determined $a, b, c \in \mathbb{Z}$ such that $I = (3a + 1 + 3\mathbb{N}) \cup (3b + 2 + 3\mathbb{N}) \cup (3c + 3\mathbb{N})$. If $I \in \mathcal{F}_0(S)$, then $c = 0$.*

Proof Since I is a fractional ideal of S, I is bounded below. Let a', b', c' be the minimal elements of I congruent (respectively) to 1, 2 and 0 modulo 3: defining a, b, c as the integers such that $a' = 3a + 1$, $b' = 3b + 2$ and $c' = 3c$ we obtain what we need, since $3 \in S$ implies that if $x \in I$ then also $x + 3 \in I$. If moreover $I \in \mathcal{F}_0(S)$, then $0 \in I$, so that $c \leq 0$, but $I \subseteq \mathbb{N}$, and thus $c \geq 0$. □

In particular, the above proposition applies when $I = S$: in this case, we use α and β instead of a and b, that is, we shall put $S = (3\alpha + 1 + 3\mathbb{N}) \cup (3\beta + 2 + 3\mathbb{N}) \cup 3\mathbb{N}$. In particular, we have $S = \langle 3, 3\alpha + 1, 3\beta + 2\rangle$.

Let $I \in \mathcal{F}_0(S)$. If $I = (3a+1+3\mathbb{N}) \cup (3b+2+3\mathbb{N}) \cup 3\mathbb{N}$, then we set $[a, b] := I$. We note that $\mathbb{N} = [0, 0]$ and $S = [\alpha, \beta]$.

Proposition 4.2 *Let $S = \langle 3, 3\alpha+1, 3\beta+2\rangle$ be a numerical semigroup of multiplicity 3, and suppose that $\alpha \leq \beta$.*

(a) If $I = [a, b] \in \mathcal{F}_0(S)$, then $0 \leq a \leq \alpha$, $0 \leq b \leq \beta$ and $-\alpha \leq b - a \leq \alpha$.
(b) Conversely, if a, b are integers, $0 \leq a \leq \alpha$, $0 \leq b \leq \beta$ and $b - a \leq \alpha$, then $I = [a, b]$ for some $I \in \mathcal{F}_0(S)$.

Proof (a) Suppose $I = [a, b]$. Since $I \subseteq \mathbb{N}$, $a, b \geq 0$ and, since $S \subseteq I$, we have $3\alpha + 1, 3\beta + 2 \in I$, and thus $a \leq \alpha$, $b \leq \beta$. In particular, $b - a \geq 0 - \alpha = -\alpha$. If $b - a > \alpha$, then

$$3a + 1 + 3\alpha + 1 = 3(a + \alpha) + 2 < 3(a + b - a) + 2 < 3b + 2$$

and thus $3a+1+3\alpha+1 \notin I$, while we should have $3a+1+3\alpha+1 \in 3a+1+S \subseteq I + S \subseteq I$. Hence $b - a \leq \alpha$.

(b) Let $I := (3a + 1 + 3\mathbb{N}) \cup (3b + 2 + 3\mathbb{N}) \cup \mathbb{N}$; we have to prove that I is indeed an ideal, and to do this it is enough to show that $I + 3$, $I + 3\alpha + 1$ and $I + 3\beta + 2$ belong to I. Clearly $I + 3 \subseteq I$; for $3\alpha + 1$, note that

$$3b + 2 + 3\mathbb{N} + 3\alpha + 1 = 3(b + \alpha + 1) + 3\mathbb{N} \subseteq S$$

since $b + \alpha + 1 \geq \alpha + 1 \geq 0$, while $3\alpha + 1 + 3\mathbb{N} \subseteq I$ since $a \geq \alpha$. Moreover,

$$3a + 1 + 3\mathbb{N} + 3\alpha + 1 = 3(a + \alpha) + 2 + 3\mathbb{N} \subseteq I$$

since $a + \alpha \geq a + b - a = b$. Analogously, $3a + 1 + 3\mathbb{N} + 3\beta + 2 \subseteq I$ and $3\mathbb{N} + 3\beta + 2 \subseteq I$, while

$$3b + 2 + 3\mathbb{N} + 3\beta + 2 = 3(b + \beta + 1) + 1 + 3\mathbb{N} \subseteq I$$

since $b + \beta + 1 \geq \beta \geq \alpha \geq a$.

□

Simmetrically, we have:

Proposition 4.3 *Let $S = \langle 3, 3\alpha+1, 3\beta+2\rangle$ be a numerical semigroup of multiplicity 3, and suppose that $\alpha \geq \beta$.*

(1) If $I = [a, b] \in \mathcal{F}_0(S)$, then $0 \leq a \leq \alpha$, $0 \leq b \leq \beta$ and $-\beta \leq a - b \leq \beta + 1$.
(2) Conversely, if a, b are integers, $0 \leq a \leq \alpha$, $0 \leq b \leq \beta$ and $a - b \leq \beta + 1$, then $I = [a, b]$ for some $I \in \mathcal{F}_0(S)$.

Proof It is enough to repeat the proof of Proposition 4.2. □

Suppose S is a numerical semigroup of multiplicity 3. If $I = [a, b] \in \mathcal{F}_0(S)$, then we can represent I by the point (a, b) in the lattice $\mathbb{Z} \times \mathbb{Z}$ of the integral points of the plane, and Propositions 4.2 and 4.3 determines the image of $\mathcal{F}_0(S)$: firstly, the bounds $0 \leq a \leq \alpha$ and $0 \leq b \leq \beta$ shows that it will be contained in the rectangle whose vertices are $[0, 0]$, $[0, \beta]$, $[\alpha, 0]$ and $[\alpha, \beta]$. Moreover, since each "ascending" diagonal (i.e., each diagonal going from the lower left to the upper right of the rectangle) is characterized by the quantity $b - a$, we see that if $\alpha \leq \beta$ then the image of $\mathcal{F}_0(S)$ will lack the upper left corner of the rectangle (the points with $b - a > \alpha$) while if $\alpha \geq \beta$ then we have to "cut" the lower right corner. In the case $\alpha = \beta$, $\mathcal{F}_0(S)$ will be represented by the whole rectangle (that will, indeed, be a square). Thus, $\mathcal{F}_0(S)$ will be represented by a polygon vaguely similar to a trapezoid, like the one showed in Fig. 1; we shall often identificate an ideal with the point corresponding to it in this graphical representation.

Proposition 4.4 *Let S be a numerical semigroup of multiplicity 3 and let $[a, b]$, $[a', b']$ be ideals in $\mathcal{F}_0(S)$. Then:*

(a) $[a, b] \subseteq [a', b']$ if and only if $a \geq a'$ and $b \geq b'$;
(b) $[a, b] \cap [a', b'] = [\max\{a, a'\}, \max\{b, b'\}]$;
(c) $[a, b] \cup [a', b'] = [\min\{a, a'\}, \min\{b, b'\}]$.

Proof Straightforward. □

Definition 4.5 Let $S = \langle 3, 3\alpha + 1, 3\beta + 2\rangle$.

- Σ^0 is the ascending diagonal that contains $S = [\alpha, \beta]$, i.e., the diagonal such that $b - a = \beta - \alpha$.
- $\Sigma^+ := \{[a, b] \in \mathcal{F}_0(S) : b - a > \beta - \alpha\}$.
- $\Sigma^- := \{[a, b] \in \mathcal{F}_0(S) : b - a < \beta - \alpha\}$.

The notation Σ^+ and Σ^- is chosen to highlight the position of the two sets in the graphical representation.

Lemma 4.6 *Let S be a numerical semigroup of multiplicity 3. The sets Σ^+, Σ^-, Σ^0, $\Sigma^+ \cup \Sigma^0$ and $\Sigma^- \cup \Sigma^0$ are closed by intersections.*

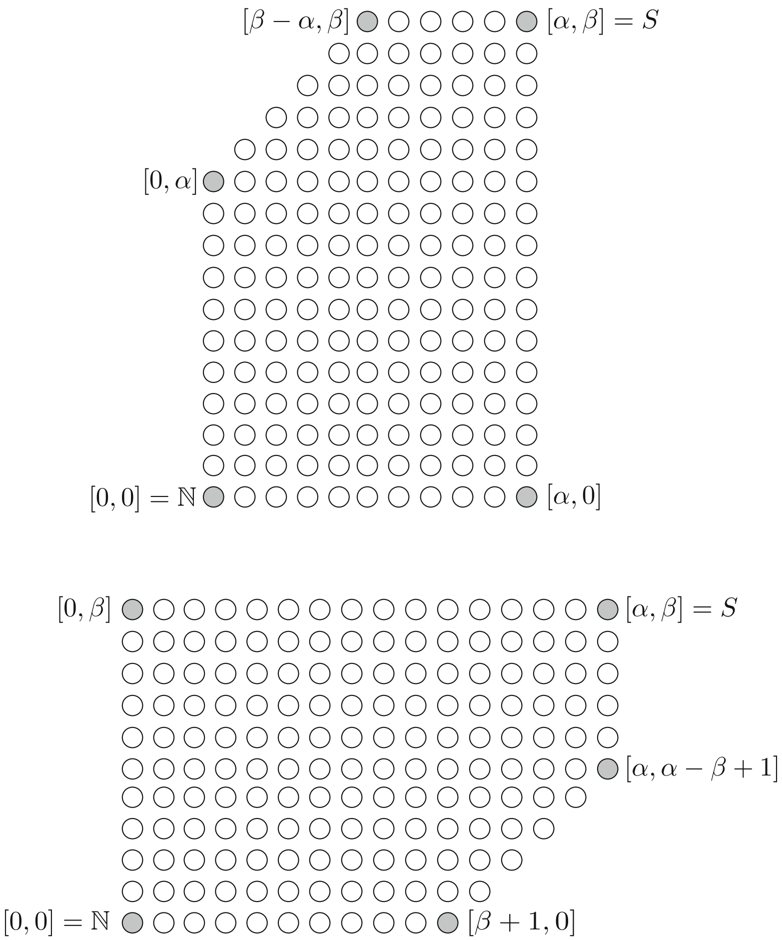


Fig. 1 Graphical representation of the ideals of a semigroup of multiplicity 3: above, the case $\alpha \leq \beta$; below, the case $\alpha \geq \beta$

Proof Σ^0 is linearly ordered, so this case is trivial.

Let $[a,b], [a',b'] \in \Sigma^+$, and suppose without loss of generality $a \leq a', b \geq b'$ (if $b \leq b'$, then $[a,b] \supseteq [a',b']$). Then $[a,b] \cap [a',b'] = [a,b']$, and $b'-a \geq b'-a' > \beta-\alpha$, and thus $[a,b'] \in \Sigma^+$.

For Σ^-, in the same way, if $[a,b] \cap [a',b'] = [a,b']$, then $b'-a \leq b-a < \beta-\alpha$ and $[a,b'] \in \Sigma^-$.

If $[a,b] \in \Sigma^+$ and $[a',b'] \in \Sigma^0$, then $b' = a'+\beta-\alpha$ and $b > a+\beta-\alpha$; hence $\min\{b,b'\} \geq \min\{a,a'\}+\beta-\alpha$ and $[a,b] \cap [a',b'] \in \Sigma^+ \cap \Sigma^0$.

Analogously, if $[a,b] \in \Sigma^-$ and $[a',b'] \in \Sigma^0$, then $\min\{b,b'\} \leq \min\{a,a'\}+\beta-\alpha$ and $[a,b] \cap [a',b'] \in \Sigma^- \cap \Sigma^0$. $\square$

5 Shifting ideals

Definition 5.1 If $I \in \mathcal{F}_0(S)$ and $k \in I$, the *k-shift* of I, denoted by $\rho_k(I)$, is the ideal $(I - k) \cap \mathbb{N}$.

It is clear that, if $\rho_k(I)$ is defined, then it is contained in $\mathcal{F}_0(S)$, since 0 belongs to $\rho_k(I)$. Since $3k \in S \subseteq I$ for every $k \in \mathbb{N}$, the $3k$-shift (and in particular the 3-shift) is always defined.

It is straightforward to see that, if $a, a+b \in I$, then $\rho_b(\rho_a(I)) = \rho_{a+b}(I)$. Therefore, applying repeatedly the 3-shift, we can always write $\rho_k(I)$ as $\rho_r \circ \rho_3^q(I)$, where $r \in \{0, 1, 2\}$ is congruent to k modulo 3. Hence, the study of the shifts reduces to the study of ρ_1, ρ_2 and ρ_3.

Lemma 5.2 *Let S be a numerical semigroup of multiplicity 3 and let $I = [a, b]$ be an ideal in $\mathcal{F}_0(S)$.*

(a) $\rho_3(I) = [\max\{0, a-1\}, \max\{0, b-1\}]$*; in particular, if $a, b > 0$, then* $\rho_3(I) = [a-1, b-1]$.
(b) $\rho_1(I)$ *is defined if and only if $a = 0$, and in this case* $\rho_1(I) = [b, 0]$.
(c) $\rho_2(I)$ *is defined if and only if $b = 0$, and in this case* $\rho_2(I) = [0, a-1]$.

In terms of the graphical representation, this means that ρ_1 and ρ_2 swap the x-axis $\{[a, 0] : 0 \le a \le \min\{\alpha, \beta + 1\}\}$ and the y-axis $\{[0, b] : 0 \le b \le \min\{\alpha, \beta\}\}$. On the other hand, ρ_3 moves the ideals one step closer to the origin (Fig. 2).

Proof Write $I = 3\mathbb{N} \cup (3a + 1 + 3\mathbb{N}) \cup (3b + 2 + 3\mathbb{N})$. Then,

- $I - 3 = (-3 + 3\mathbb{N}) \cup (3(a-1) + 1 + 3\mathbb{N}) \cup (3(b-1) + 2 + 3\mathbb{N})$,
- $I - 1 = 3a\mathbb{N} \cup (3b + 1 + 3\mathbb{N}) \cup (2 + 3\mathbb{N})$,
- $I - 2 = 3b\mathbb{N} \cup (1 + 3\mathbb{N}) \cup (3(a-1) + 2 + 3\mathbb{N})$.

If $\rho_1(I)$ (respectively, $\rho_2(I)$) is defined, then we must have $0 \in 3a\mathbb{N}$, and thus $a = 0$ (resp., $0 \in 3b\mathbb{N}$, and thus $b = 0$). The lemma now follows from the definition of $[x, y]$. □

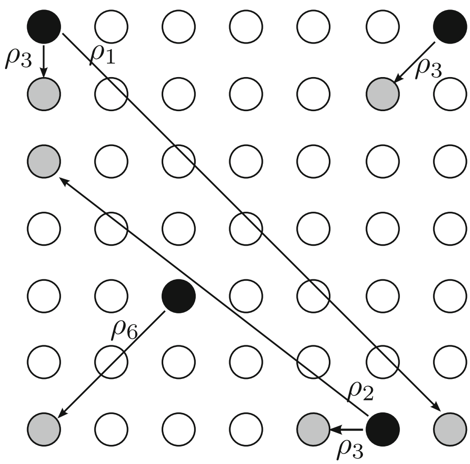


Fig. 2 Action of the shifts

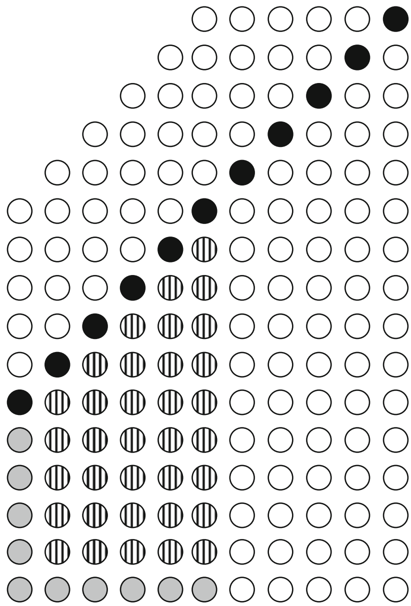


Fig. 3 Divisorial and nondivisorial ideals. *Black circles* represent ideals of Σ^0, *gray circles* other ideals in the form $\rho_x(S)$, *striped circles* are intersections of *black* and *gray* ideals. *White circles* represent non-divisorial ideals

6 Principal star operations

Lemma 6.1 *Let S be a numerical semigroup of multiplicity 3 and $\Delta \subseteq \mathcal{F}_0(S)$. Then $\Delta + \mathbb{Z} := \{d + I : d \in \mathbb{Z}, I \in \Delta\}$ is the set of closed ideals of a star operations if and only if $S \in \Delta$, Δ is closed by intersections and $\rho_k(I) \in \Delta$ whenever $I \in \Delta$ and $\rho_k(I)$ is defined.*

Proof It is merely a restatement of [18, Lemma 3.3]. □

We state separetely a corollary to underline a property which we will use many times:

Corollary 6.2 *Let S be a numerical semigroup of multiplicity 3, $I \in \mathcal{F}_0(S)$, $k \in I$ and $* \in \mathrm{Star}(S)$. If I is $*$-closed, so is $\rho_k(I)$.*

Proposition 6.3 *Let $S = \langle 3, 3\alpha+1, 3\beta+2\rangle$ be a numerical semigroup of multiplicity 3. Then:*

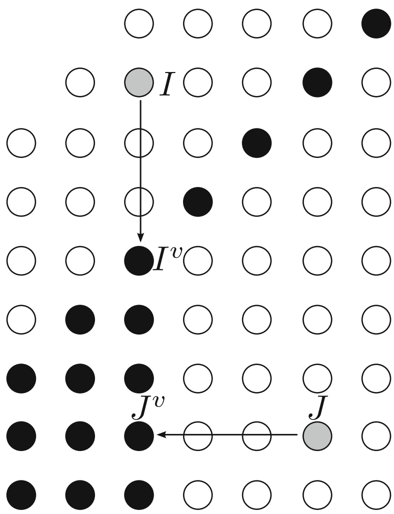
(a) if $\alpha \leq \beta$, then $\mathcal{F}^v(S) \cap \mathcal{F}_0(S) = \Sigma^0 \cup \{[a, b] \in \Sigma^- : a \leq \beta - \alpha\}$;
(b) if $\alpha \geq \beta$, then $\mathcal{F}^v(S) \cap \mathcal{F}_0(S) = \Sigma^0 \cup \{[a, b] \in \Sigma^+ : b \leq \alpha - \beta - 1\}$.

Proof We will prove only the case $\alpha \leq \beta$ (pictured in Fig. 3); the proof for $\alpha \geq \beta$ is entirely analogous.

Let Δ be the set on the right hand side. We will show that Δ verifies the hypotheses of Lemma 6.1 (so that $\Delta = \mathcal{F}^*(S) \cap \mathcal{F}_0(S)$ for some star operation $*$), and that each $I \in \Delta$ is divisorial: since $v \geq *$ for every $* \in \mathrm{Star}(S)$, the claim will follow.

If $[a, b] \in \Sigma^0$, then $[a, b] = [\alpha-k, \beta-k] = \rho_{3k}(S)$ for some $k \in \mathbb{N}$, so that $[a, b]$ is divisorial. In particular, $[0, \beta-\alpha] \in \mathcal{F}^v(S)$. Therefore, $[0, \beta-\alpha-x] = \rho_{3x}([0, \beta-\alpha])$

Fig. 4 Divisorial closure of ideals



is divisorial for every $x \geq 0$, and so is $[\beta - \alpha - x, 0] = \rho_1([0, \beta - \alpha - x])$. Let $[a, b] \in \Sigma^-$ such that $a \leq \beta - \alpha$. If $b \leq \beta - \alpha$, then $[a, b] = [a, 0] \cap [0, b]$ is the intersection of two divisorial ideals; if $b > \beta - \alpha$, then $[a, b] = [a, 0] \cap [b - (\beta - \alpha), b]$, and the latter is divisorial since it belongs to Σ^0. Hence $\mathcal{F}^v \subseteq \Delta$.

Let now $[a, b], [a', b'] \in \Delta$; if they are both in Σ^0 they are comparable, and thus the intersection is in Δ. If $[a, b] \in \Sigma^-$, then by Lemma 4.6 its intersection with $[a', b']$ is in $\Sigma^- \cup \Sigma^0$; moreover, $\min\{a, a'\} \leq a \leq \beta - \alpha$, and thus $[a, b] \cap [a', b'] \in \Delta$.

It is clear that $\rho_3(I) \in \Delta$ whenever $I \in \Delta$, since $\rho_3([a, b]) \in \Sigma^0$ if $[a, b] \in \Sigma^0$ and $a > 0$, while $\rho_3([0, \beta - \alpha]) = [0, \beta - \alpha - 1] \in \Delta$; if $[a, b] \in \Delta \setminus \Sigma^0$, then $\rho_3([a, b]) = [\max\{a - 1, 0\}, \max\{b - 1, 0\}]$, and $\max\{a - 1, 0\} \leq a$, so that $\rho_3([a, b]) \in \Delta$.

By the discussion in Sect. 5, we only need to show that $\rho_1([0, c]), \rho_2([c, 0]) \in \Delta$ if $[0, c]$ or $[c, 0]$ are in Δ. However, excluding the case $c = 0$ (which is trivial), we have $\rho_1([0, c]) = [c, 0]$ and $\rho_2([c, 0]) = [0, c - 1]$, and since $c \leq \beta - \alpha$ we have $[c, 0], [0, c - 1] \in \Delta$. □

Lemma 6.4 *Let S be a semigroup of multiplicity 3, and let $I \in \mathcal{F}(S)$. Then, the set of ideals between I and I^v is linearly ordered.*

Proof If $[a, b] \in \Sigma^0$, then it is divisorial.

Suppose $[a, b] \in \Sigma^+$. Then, $\rho_{3(\alpha - a)}([\alpha, \beta]) = [a, \min\{\beta - \alpha + a, 0\}]$. However, $\beta - \alpha + a \leq b - a + a = b$, and thus $[a, b] \subseteq [a, b'] = \rho_{3(\alpha - a)}(S)$. However, the ideals between $[a, b]$ and $[a, b']$ are linearly ordered, and $\rho_{3x}(S)$ is always divisorial (by Corollary 6.2); hence $[a, b]^v \subseteq [a, b']$ and the ideals between $[a, b]$ and $[a, b]^v$ are linearly ordered (Fig. 4).

If $[a, b] \in \Sigma^-$, then in the same way $[a, b]^v \subseteq \rho_{3(\beta - b)}([\alpha, \beta]) = [a', b]$ for some $a' \leq a$, and the claim follows. □

Corollary 6.5 *Let S be a semigroup of multiplicity 3. Then, the maps $\mathcal{A}$ and $*$ (defined at the end of Sect. 3) are bijections, and $|\mathrm{Star}(S)|$ is equal to the number of antichains of $(\mathcal{G}_0(S), \leq_*)$.*

Proof We need to show that, given two antichains $\Delta \neq \Lambda$ of $\mathcal{G}_0(S)$, we have $*_\Delta \neq *_\Lambda$. Suppose not, and suppose (without loss of generality) that there exists an $I \in \Delta \setminus \Lambda$. Then, $I = I^{*_\Delta} = I^{*_\Lambda} = \bigcap_{L\in\Lambda} I^{*_L}$. Since $I \subseteq I^* \subseteq I^v$ for every $* \in \mathrm{Star}(S)$, and the set of ideals between I and I^v is linearly ordered, there is an $J \in \Lambda$ such that $I^{*_J} = I$; it follows that $I \leq_* J$. Analogously, since $J = J^{*_\Lambda} = J^{*_\Delta}$, there is a $I' \in \Delta$ such that $J \leq_* I'$. Since Δ is an antichain in the $*$-order, it follows that $I = I' = J$, and thus $I \in \Lambda$, against the hypothesis. Therefore, $*_\Delta \neq *_\Lambda$. □

Corollary 6.6 *Let S be a semigroup of multiplicity 3 and let $I, J \in \mathcal{F}_0(S) \cap \mathcal{F}^*(S)$ for some $* \in \mathrm{Star}(S)$. Then, $I \cup J$ is $*$-closed.*

Proof Let $I = [a, b]$ and $J = [a', b']$. Without loss of generality, we can suppose $a < a'$ and $b > b'$ (if $b \leq b'$, then $I \supseteq J$ and $I \cup J = I$). Then, $I \cup J = [a, b']$.

Suppose $I \cup J \in \Sigma^+$. Then, since $a - b < a - b'$, it follows that $I \in \Sigma^+$. Hence, $[a, b'] = \rho_{3(b-b')}(I) \cap I^v$, and thus $[a, b'] \in \Sigma^+$. Analogously, if $I \cup J \in \Sigma^-$, then $J \in \Sigma^-$ and $[a, b'] = \rho_{3(a'-a)}(J) \cap J^v$. In both cases, $I \cup J$ is $*_I$- or $*_J$-closed, and in particular, since $* \leq *_I \wedge *_J$, it is $*$-closed. □

Note that the hypothesis $I, J \in \mathcal{F}_0(S)$ is necessary: for example, if $S = \langle 3, 5, 7\rangle$, $I = S$, $J = 4 + \mathbb{N}$, then both I and J are divisorial, but $I \cup J = S \cup \{4\}$ while $(I \cup J)^v = (S - M) = S \cup \{2, 4\}$.

Lemma 6.7 *Let S be a numerical semigroup of multiplicity 3, and let $I, J \in \mathcal{F}(S)$ such that J is $*_I$-closed. There are $\gamma_0, \gamma_1, \gamma_2 \in \mathbb{N}$, $\gamma_i \equiv i \bmod 3$, such that $J^{*_I} = J^v \cap (-\gamma_0 + I) \cap (-\gamma_1 + I) \cap (-\gamma_2 + I)$. In particular, if $I, J \in \mathcal{F}_0(S)$, then there are γ_i such that $J^{*_I} = J^v \cap \rho_{\gamma_0}(I) \cap \rho_{\gamma_1}(I) \cap \rho_{\gamma_2}(I)$.*

Proof Since J is $*_I$-closed, using (1) we have $J = J^v \cap \bigcap_{\gamma\in(I-J)} -\gamma + I$; separing the γ according to their residue class modulo 3 we have

$$J = J^v \cap \bigcap_{\gamma\in\Gamma_0} (-\gamma + I) \cap \bigcap_{\gamma\in\Gamma_1} (-\gamma + I) \cap \bigcap_{\gamma\in\Gamma_2} (-\gamma + I),$$

where $\Gamma_i := (I - J) \cap (i + 3\mathbb{Z})$; since $(I - J) \subseteq \mathbb{N}$, each Γ_i has a minimum. However, if $\gamma, \delta \in \Gamma_i$, then either $-\gamma + I \subseteq -\delta + I$ or $-\delta + I \subseteq -\gamma + I$; therefore it is enough to take $\gamma_i := \min \Gamma_i$.

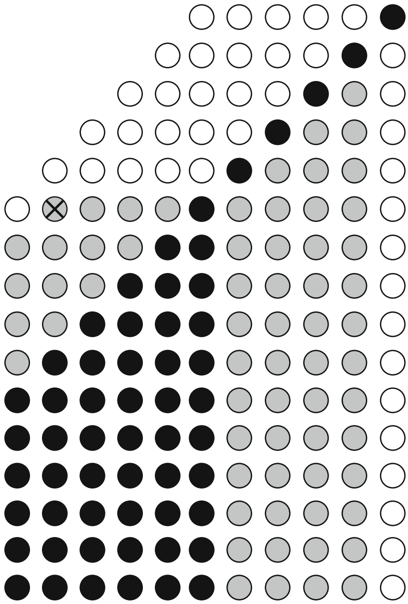
For the "in particular" statement, note that both J and J^v are contained in $\mathbb{N}$, so that the intersection does not change substituing $-\gamma_i + I$ with $-\gamma_i + I \cap \mathbb{N} = \rho_{\gamma_i}(I)$. □

We proceed to determine explicitly the set of ideals closed by a principal star operation (Fig. 5).

Proposition 6.8 *Let S be a numerical semigroup of multiplicity 3, and let $I = [a, b]$ be an ideal.*

- *If $[a, b] \in \Sigma^+$, then $\mathcal{F}^{*_I} \cap \Sigma^+ = \{[c, d] : d \leq b,\ d - c \leq b - a\}$.*

Fig. 5 The set of divisorial ideals (in *black*) and of non-divisorial $*_I$-closed ideals (in *gray*), where I is the marked ideal



- *If* $[a,b] \in \Sigma^-$, *then* $\mathcal{F}^{*_I} \cap \Sigma^- = \{[c,d] : c \le a,\ d-c \ge b-a\}$.

Proof Suppose $[a,b] \in \Sigma^+$, and let $[c,d] \in \Sigma^+$ such that $d \le b$ and $d-c \le b-a$. Then, $\rho_{3(b-d)}([a,b]) = [a-(b-d), b-(b-d)] = [a-b+d, d]$ is $*_{[a,b]}$-closed; moreover, $a-b+d \ge c-d+d = c$, and thus $[c,d] = [a-b+d,d] \cap [c,c']$, where $c'-c = \beta-\alpha$ (i.e., $c' = c+\beta-\alpha$), so that $[c,c'] \in \Sigma^0$ is divisorial, and $[c,d]$ is $*_{[a,b]}$-closed.

Conversely, let $\Delta := (\mathcal{F}^{*_I} \cap \Sigma^+) \setminus \{[c,d] : d \le b,\ d-c \le b-a\}$ and suppose $\Delta \ne \emptyset$. Note that, by Proposition 6.3, $\mathcal{F}^v(R) \cap \Delta = \emptyset$. Let B be the maximum b' such that $[a',b'] \in \Delta$ for some a', and let A be the minimum a' such that $[a',B] \in \Delta$. Let $J := [A,B]$.

By Lemma 6.7, $J = J^v \cap I_0 \cap I_1 \cap I_2$, where $I_i := \rho_{\gamma_i}(I) = [a_i,b_i]$. Since $J^v = [A,b'']$ for some $b'' < B$, at least one of the b_i must be equal to B. We have $I_i \in \Sigma^+$: indeed, if $I \in \Sigma^0$ it is divisorial, while if $I_i \in \Sigma^-$ then $L := [B-\beta+\alpha, B] \in \Sigma^0$ is divisorial and is contained between J and I_i: in both cases, $J^v \subseteq I_i$, so that $J^v \subseteq [A,b''] \cap [a_i,B] = [A,B] = J$, and J is divisorial, against $J \in \Delta$. Since $J \subseteq [a_i,B]$, we have $a_i \le A$. Suppose $a_i < A$: then, by definition of A, $I_i \notin \Delta$. However, I_i is $*_I$-closed: hence, $B \le b$ and $B-a_i \le b-a$. But $B-a_i \ge B-A$, so that $B-A \le b-a$; this would imply $J \notin \Delta$, against its definition. Therefore $a_i = A$, and $J = I_i$. However:

(1) if $i=0$, then $b_i \le b$, and $b_i - a_i = b-a$;
(2) if $i=1$, then $I_i \in \Sigma^-$;
(3) if $i=2$, then $[a_i,b_i] = [0,0]$ (since $J \in \Sigma^+$).

Therefore, $\Delta = \emptyset$.

If $[a, b] \in \Sigma^-$, we can use the same method reversing the rôle of a and b: we choose first A as the maximum a' such that $[a', b'] \in \Delta$ for some b', and then B as the minimum b' such that $[A, b'] \in \Delta$. It follows as above that $[a_i, b_i] = [A, B]$ for some i, and $I_i \in \Sigma^-$; moreover, if $i = 0$ then $[a_i, b_i] \notin \Delta$, if $i = 1$ then $[a_i, b_i] = [0, 0]$ and if $i = 2$ then $[a_i, b_i] \in \Sigma^+$. None of this cases is acceptable, and $\Delta = \emptyset$. □

Proposition 6.9 *Let S be a numerical semigroup of multiplicity 3, and let $I = [a, b]$ be an ideal.*

- *If $[a, b] \in \Sigma^+$, then $\mathcal{F}^{*_I} \cap \Sigma^- = \mathcal{F}^{*_{[b-a,0]}} \cap \Sigma^-$.*
- *If $[a, b] \in \Sigma^-$, then $\mathcal{F}^{*_I} \cap \Sigma^+ = \mathcal{F}^{*_{[0,b-a-1]}} \cap \Sigma^+$*

In particular, both depends only on $b - a$.

Proof Suppose $[a, b] \in \Sigma^+$. Since $[a, b]$ is closed, so is $[0, b - a]$, and thus also $[b - a, 0] = \rho_1([0, b - a])$ is closed. Hence $\mathcal{F}^{*_{[b-a,0]}} \cap \Sigma^- \subseteq \mathcal{F}^{*_I} \cap \Sigma^-$.

Let $\Delta := (\mathcal{F}^{*_I} \cap \Sigma^-) \setminus \mathcal{F}^{*_{[b-a,0]}}$ and suppose it is nonempty; as in the proof of the previous proposition, let A be the maximum a' such that $[a', b'] \in \Delta$ for some b' and let B be the minimum b' such that $[A, b'] \in \Delta$. Observe that $A > b-a$ since $[a', 0]$ is $*_{[b-a,0]}$-closed for every $a' \leq b-a$. Then $J := [A, B] \in \Delta$, and $J = \rho_\gamma(I)$ for some γ such that $\rho_\gamma(I) \in \Sigma^-$, and the unique possibility is $\gamma \equiv 1 \bmod 3$; let $\gamma = 3k + 1$. Then $\rho_{3k}([a, b]) = [0, c]$ for some $c \leq b - a$, and thus $\rho_\gamma(I) = [c - 1, 0]$, with $c - 1 \leq b - a$, which is impossible.

The case $[a, b] \in \Sigma^-$ is treated in the same manner. □

7 The number of star operations

Let $S = \langle 3, 3\alpha + 1, 3\beta + 2\rangle$ be a numerical semigroup, and suppose that $\alpha \leq \beta$; let k be an integer such that $\beta - \alpha \leq k < \alpha$. We define:

- $\mathcal{L}_k^+ := \{[k, \beta], [k - 1, \beta - 1], \ldots, [0, \beta - k]\}$;
- $\mathcal{L}_k^- := \{[\beta - k, 0], [\beta - k, 1], \ldots, [\beta - k, 2\beta - \alpha - k - 1]\}$;
- $\mathcal{L}_k := \mathcal{L}_k^+ \cup \mathcal{L}_k^-$.

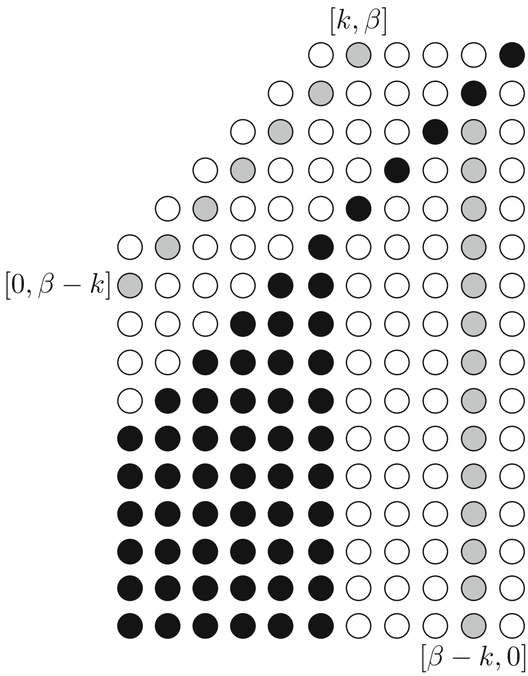
Equivalently, $\mathcal{L}_k^+$ is the set of ideals $[a, b]$ such that $b - a = \beta - k$, while $\mathcal{L}_k^-$ is the set of ideals $[a, b] \in \Sigma^-$ such that $a = \beta - k$. Note that, since $k < \alpha$, each element of $\mathcal{L}_k^+$ is in Σ^+ (Fig. 6).

Proposition 7.1 *Preserve the notation above. Then:*

(a) $\mathcal{L}_k \cap \mathcal{L}_j = \emptyset$ *if* $k \neq j$;
(b) $\bigcup_{k=\beta-\alpha}^{\alpha-1} \mathcal{L}_k = \mathcal{G}_0(S)$;
(c) $|\mathcal{L}_k| = 2\beta - \alpha + 1$;
(d) each $\mathcal{L}_k$ *is linearly ordered (in the* $*$*-order).*

Proof (a) Suppose $[a, b] \in \mathcal{L}_k \cap \mathcal{L}_j$. If $[a, b] \in \Sigma^+$, then $\beta - k = b - a = \beta - j$; if $[a, b] \in \Sigma^-$, then $\beta - k = a = \beta - j$. In both cases, $k = j$.

Fig. 6 A $\mathcal{L}_k$ *(gray circles)*



(b) Suppose $[a, b] \in \mathcal{L}_k$ for some k. If $[a, b] \in \Sigma^+$, then it is not divisorial by Proposition 6.3; if $[a, b] \in \Sigma^-$, then $a = \beta - k > \beta - \alpha$ and thus $[a, b] \neq [a, b]^v$, again by Proposition 6.3.
Conversely, suppose $[a, b] \neq [a, b]^v$. If $[a, b] \in \Sigma^+$, then $\beta - \alpha \leq b - a < \alpha$, and thus $[a, b] \in \mathcal{L}_{\beta-(b-a)}$; if $[a, b] \in \Sigma^-$, then by Proposition 6.3 we have $a > \beta - \alpha$, so that $\beta - a < \alpha$ and thus $[a, b] \in \mathcal{L}_{\beta-a}$.

(c) We have $|\mathcal{L}_k^+| = k + 1$ and $|\mathcal{L}_k^-| = 2\beta - \alpha - k$; since $\mathcal{L}_k^+$ and $\mathcal{L}_k^-$ are disjoint, $|\mathcal{L}_k| = 2\beta - \alpha + 1$.

(d) By Lemma 5.2, if $j \geq j'$ then $[k - j', \beta - j'] = \rho_{3(j-j')}([k - j, \beta - j])$, so that $\mathcal{L}_j^+$ is totally ordered, with minimum $[0, \beta - k]$; analogously, if $l \geq l'$, then $[a, l] = [a, l'] \cap [a, l]^v$ (see the proof of Lemma 6.4) and thus $[a, l] \leq_* [a, l']$, i.e., $\mathcal{L}_j^-$ is linearly ordered, with maximum $[\beta - k, 0]$. Moreover, $[\beta - k, 0] = \rho_1([0, \beta - k])$, and thus $\mathcal{L}_k$ is totally ordered. □

When $\alpha \geq \beta$, we can reason in a completely analogous way, but we have to reverse the rôle of Σ^+ and Σ^-: we choose an integer k such that $\alpha - \beta + 1 \leq k < \beta$, and define

- $\mathcal{L}_k^- := \{[\alpha, k], [\alpha - 1, k - 1], \ldots, [0, \alpha - k]\}$;
- $\mathcal{L}_k^+ := \{[0, \alpha - k - 1], [1, \alpha - k - 1], \ldots, [2\alpha - \beta - k - 2, \alpha - k - 1]\}$;
- $\mathcal{L}_k := \mathcal{L}_k^+ \cup \mathcal{L}_k^-$.

Then, the elements of $\mathcal{L}_k^-$ are in Σ^- and are characterized by $b - a$, while the elements of $\mathcal{L}_k^+$ are the ideals in Σ^+ with the same b. Proposition 7.1 becomes:

Proposition 7.2 *Preserve the notation above. Then:*

(a) $\mathcal{L}_k \cap \mathcal{L}_j = \emptyset$ *if* $k \neq j$*;*
(b) $\bigcup_{k=\alpha-\beta+1}^{\beta-1} \mathcal{L}_k = \mathcal{G}_0(S)$*;*
(c) $|\mathcal{L}_k| = 2\alpha - \beta$*;*
(d) each $\mathcal{L}_k$ *is linearly ordered (in the* $*$*-order).*

Corollary 7.3 *Let* $S = \langle 3, 3\alpha + 1, 3\beta + 2\rangle$ *be a numerical semigroup. Then,* $|\mathcal{G}_0(S)| = (2\alpha - \beta)(2\beta - \alpha + 1)$.

By a *rectangle* $a \times b$, indicated with $\mathcal{R}(a, b)$, we denote the cartesian product $\{1, \ldots, a\} \times \{1, \ldots, b\}$, endowed with the *reverse* product order (that is, $(x, y) \geq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$).

Theorem 7.4 *Let* $S = \langle 3, 3\alpha + 1, 3\beta + 2\rangle$ *be a numerical semigroup. Then,* $(\mathcal{G}_0(S), \leq_*)$ *is isomorphic (as an ordered set) to* $\mathcal{R}(2\alpha - \beta, 2\beta - \alpha + 1)$.

Proof Suppose $\alpha \leq \beta$, and let $I \in \mathcal{G}_0(S)$. If $I \in \mathcal{L}_k$, define $\psi_1(I) := k - (\beta - \alpha) + 1$. Moreover, if there are exactly $j - 1$ ideals in $\mathcal{L}_k$ strictly bigger (in the $*$-order) than I, then define $\psi_2(I) := j$. Explicitly, if $[a, b] \in \Sigma^+$ then $\psi_2([a, b]) = \beta - b + 1$, while if $[a, b] \in \Sigma^-$ then $\psi_2([a, b]) = k + 1 + b = \beta + 1 + b - a$ (using $a = \beta - k$). By Proposition 7.1, the map

$$\begin{aligned} \Psi : \mathcal{G}_0(S) &\longrightarrow \mathcal{R}(2\alpha - \beta, 2\beta - \alpha + 1) \\ [a, b] &\mapsto (\psi_1(I), \psi_2(I)) \end{aligned}$$

is a bijection.

For a partially ordered set $\mathcal{P}$, and a subset $\Delta \subseteq \mathcal{P}$, denote by $\overline{\Delta}$ the lower set of Δ: i.e., let $\overline{\Delta} := \{x \in \mathcal{P} : \exists y \in \Delta : x \leq y\}$. To show that Ψ is order-preserving, it is enough to show that $\Psi\left(\overline{\{I\}}\right) = \overline{\Psi(I)}$ for every ideal $I \in \mathcal{G}_0(S)$. Since $\overline{\{I\}} = \mathcal{G}_0(S) \cap \mathcal{F}^{*_I}$, we need to show that J is $*_I$-closed if and only if $\Psi(J) \leq \Psi(I)$.

Let $I = [a, b]$ and $J = [c, d]$ be ideals. If $I, J \in \Sigma^+$, then by Proposition 6.8 J is $*_I$-closed if and only if $d \leq b$ and $d - c \leq b - a$. We have $d \leq b$ if and only if $\psi_2(J) \geq \psi_2(I)$; on the other hand, $x - y = \beta - k$ if $[y, x] \in \mathcal{L}_k$, and thus $\psi_1([y, x]) = \beta - x + y$. Therefore, $d - c \leq b - a$ if and only if $\psi_1(J) \geq \psi_1(I)$. Hence (remember that the order on the rectangle is the reverse product order), $J \in \overline{\{I\}}$ if and only if $\Psi(J) \leq \Psi(I)$. On the other hand, if $I, J \in \Sigma^-$, then $J \in \overline{\{I\}}$ if and only if $c \leq a$ and $d - c \leq b - a$; the first condition if equivalent to the requirement that $\psi_1(J) \geq \psi_1(I)$, while the second is equivalent to $\psi_2(J) \geq \psi_2(I)$. Again, $J \in \overline{\{I\}}$ if and only if $\Psi(J) \leq \Psi(I)$.

Suppose $I \in \Sigma^+$ and $J \in \Sigma^-$. If J is $*_I$-closed, then by Proposition 6.9 it is $*_{[b-a,0]}$-closed, and, by the previous paragraph, this happens if and only if $\Psi(J) \leq \Psi([b - a, 0])$. However, $[b - a, 0]$ and I belong to the same $\mathcal{L}_k$ (since $[b - a, 0] = \rho_1\rho_{3(b-a)}([a, b])$), and thus $\Psi([b - a, 0]) \leq \Psi(I)$; hence $\Psi(J) \leq \Psi(I)$. Conversely, if $\Psi(J) \leq \Psi(I)$ then $J = [c, d]$ belongs to $\mathcal{L}_j$ for some $j \geq k$ (where $I = [a, b] \in \mathcal{L}_k$) and thus $c \leq a$, and J is $*_I$-closed (applying again Proposition 6.9). If $I \in \Sigma^-$ and $J \in \Sigma^+$, the same reasoning applies; therefore, in all cases, $J \in \overline{\{I\}}$ if and only if $\Psi(J) \leq \Psi(I)$, that is, if and only if $\Psi(J) \in \overline{\Psi(I)}$. Hence Ψ is an order isomorphism.

If $\alpha \geq \beta$, then we can apply the same method: we define a map

$$\begin{aligned}\Psi : &\mathcal{G}_0(S) &&\longrightarrow \mathcal{R}(2\beta-\alpha+1, 2\alpha-\beta)\\ &[a,b] &&\mapsto (\psi_1(I), \psi_2(I))\end{aligned}$$

where, if $I \in \mathcal{L}_k$, then $\psi_1(I) = k-(\alpha-\beta+1)+1$, and $\psi_2(I) = j$ if there are exactly $j-1$ elements of $\mathcal{L}_k$ $*$-bigger than I. Proposition 7.2 shows that Ψ is a bijection, and (as before) the use of Propositions 6.8 and 6.9 shows that it is an order isomorphism. Since $\mathcal{R}(2\beta-\alpha+1, 2\alpha-\beta) \simeq \mathcal{R}(2\alpha-\beta, 2\beta-\alpha+1)$, the theorem is proved. □

Lemma 7.5 *The number of antichains in* $\mathcal{R}(a, b)$ *is* $\binom{a+b}{a} = \binom{a+b}{b}$.

Proof Let $A := \{1, \dots, a\}$ and $B := \{1, \dots, b\}$.

For each antichain Δ, let $\overline{\Delta}$ be the lower set of Δ; clearly $\Delta = \max \overline{\Delta}$, so that the number of antichains is equal to that of the sets that are downward closed (i.e., sets Λ such that $\Lambda = \overline{\Lambda}$). When restriced to a single row $A \times \{c\}$, $\overline{\Delta}$ becomes a segment $\{a_c, \dots, a\} \times \{c\}$; moreover, if $d \leq c$, then $a_d \leq a_c$. Thus the number of antichains is equal to the number of sequences $\{1 \leq a_1 \leq \dots \leq a_b \leq a+1\}$ (where $a_i = a+1$ if and only if $(A \times \{i\}) \cap \overline{\Delta} = \emptyset$), that in turn is equal to the number of combinations with repetitions of b elements of $\{1, \dots, a+1\}$. This is equal to $\binom{a+1+b-1}{b} = \binom{a+b}{b} = \binom{a+b}{a}$. □

Theorem 7.6 *Let* $S = \langle 3, 3\alpha+1, 3\beta+2\rangle$ *be a numerical semigroup of multiplicity 3,* $g := g(S)$, $\delta := \delta(S)$. *Then,*

$$|\mathrm{Star}(S)| = \binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{\alpha+\beta+1}{2\beta-\alpha+1} = \binom{\delta+1}{g-\delta+2}.$$

Proof By Corollary 6.5, $|\mathrm{Star}(S)|$ is equal to the number of antichains of $\mathcal{G}_0(S)$, which is equal (by Theorem 7.4) to the number of antichains of $\mathcal{R}(2\alpha-\beta, 2\beta-\alpha+1)$. Lemma 7.5 now completes the reasoning.

To show the last equality, note that an element in $\mathbb{N} \setminus S$ can be written as $3a+1$ or $3b+2$, where $0 \leq a < \alpha$ or $0 \leq b < \beta$, and thus $\delta = \alpha + \beta$. On the other hand, if $\alpha > \beta$ then $g = 3\alpha - 2$, and thus $2\alpha - \beta = g - \delta + 2$, while if $\alpha \leq \beta$ then $g = 3\beta - 1$, and again $2\beta - \alpha + 1 = g - \delta + 2$. □

Remark 7.7 We can compare the explicit counting supplied by Theorem 7.6 with the three main estimates obtained in [18].

(1) The most general one (assuming only that S is not symmetric) is $|\mathrm{Star}(S)| \geq \left\lceil \frac{g}{2\mu} \right\rceil$. If $\alpha > \beta$, then (using the proof of Theorem 7.6) in the case of multiplicity 3 we can translate it as

$$|\mathrm{Star}(S)| \geq \left\lceil \frac{3\alpha-2}{6} \right\rceil \geq \frac{1}{2}\alpha - \frac{1}{3}.$$

Being linear, this estimate is very far from the actual numer of star operation, which grows as a binomial coefficient. This is especially evident when α is close to β: for example, if $\alpha = \beta$, then $|\mathrm{Star}(S)| = \binom{2\alpha+1}{\alpha}$ is asymptotic to $\frac{2}{\sqrt{\pi}} \cdot \frac{4^\alpha}{\sqrt{\alpha}}$. The same phenomenon happens, simmetrically, when $\beta \geq \alpha$ (but we will have a linear estimate in β instead of α).

(2) A second estimate, valid only in some cases, is $|\mathrm{Star}(S)| \geq 2^{\left\lceil \frac{\mu-1}{2} \right\rceil}$, which however does not distinguish between different semigroups of the same multiplicity.

(3) A third estimate is $|\mathrm{Star}(S)| \geq \delta + 1$, which is valid when S has an hole $a < \mu$ (an integer a is said to be an *hole* of S if $a, g - a \notin S$). When $g \equiv 1 \bmod 3$, the only possible hole smaller than μ is 2: in this case, the elements of $\mathbb{N} \setminus S$ are $\{1, 2, 4, 5, \ldots, 3(\beta-1)+1, 3(\beta-1)+2, g = 3\beta+1\}$, and thus $\delta = 2\beta+1$; hence, $|\mathrm{Star}(S)| = \binom{2\beta+2}{\beta+2}$, which is much bigger than $\delta + 1 = 2\beta + 2$. Analogously, when $g \equiv 2 \bmod 3$, the only possibile hole $a < \mu$ is $a = 1$: in this case, we obtain $\delta = 2\alpha$, $g = 3\alpha - 1$ and $|\mathrm{Star}(S)| = \binom{2\alpha+1}{\alpha+1}$, which is much bigger than $\delta + 1 = 2\alpha + 1$.

A numerical semigroup is called *pseudosymmetric* if g is even and $(S - M) = S \cup \{g, g/2\}$.

Proposition 7.8 *Let S be a numerical semigroup of multiplicity 3 such that $\mathcal{G}_0(S) \neq \emptyset$. Then, the following are equivalent:*

(i) S is pseudosymmetric;
(ii) $\alpha = 2\beta$ or $\beta = 2\alpha - 1$;
(iii) $(\mathcal{G}_0(S), \leq_*)$ is linearly ordered;
(iv) $\mathrm{Star}(S)$ is linearly ordered.
(v) every star operation on S is principal.

Proof (i $\iff$ ii) Let $a := 3\alpha + 1 - 3 = 3\alpha - 2$ and $b := 3\beta + 2 - 3 = 3\beta - 1$: then, $a, b \notin S$ but $a + 3, b + 3 \in S$. Hence, S is pseudosymmetric if and only if $a = 2b$ or $b = 2a$.
If $\alpha \geq \beta$, then $a \geq b$, and thus S is pseudosymmetric if and only if $3\alpha - 2 = 2(3\beta - 1)$, that is, if and only if $\alpha = 2\beta$. Analogously, if $\beta \geq \alpha$, S is pseudosymmetric if and only if $3\beta - 1 = 2(2\alpha - 2)$, that is, if and only if $\beta = 2\alpha + 1$.

(ii $\iff$ iii) $\mathcal{G}_0(S)$ is linearly ordered if and only if $\mathcal{R}(2\alpha-\beta, 2\beta-\alpha+1)$ is linearly ordered; but this happens if and only if one of the sides of the rectangle has length 1, that is, if and only if $2\alpha - \beta = 1$ (i.e., $\beta = 2\alpha - 1$) or $2\beta - \alpha + 1 = 1$ (i.e., $\alpha = 2\beta$).

(iv $\Longrightarrow$ iii) is obvious.

(iii $\Longrightarrow$ iv,v) Let $*$ be a star operation. Then, $* = *_{I_1} \wedge \cdots \wedge *_{I_n}$ for some $I_1, \ldots, I_n$; since $\mathcal{G}_0(S)$ is linearly ordered, $* = *_{I_j}$ for some j. Hence each star operation is principal, and $\mathrm{Star}(S)$ is linearly ordered.

(v $\Longrightarrow$ ii) Suppose $\alpha \neq 2\beta$ and $\beta \neq 2\alpha - 1$. Then, the length of both sides of the rectangle $\mathcal{R}(2\alpha - \beta, \beta - 2\alpha + 1)$ is 2 or more; consider the set Δ composed by $(1, 2)$ and $(2, 1)$. Then, Δ is an antichain; therefore, so is $\Psi^{-1}(\Delta)$, where Ψ is the isomorphism defined in the proof of

Theorem 7.4. By hypothesis, $*_{\Psi^{-1}(\Delta)}$ is principal, i.e., $*_{\Psi^{-1}(\Delta)} = *_I$ for some $I \in \mathcal{G}_0(S)$; however, by Corollary 6.5, this would imply $\Psi^{-1}(\Delta) = \{I\}$, which is absurd. Hence S is pseudosymmetric. □

8 Quantitative estimates

Let $\xi_3(n)$ denote the number of numerical semigroups of multiplicity 3 with exactly n star operations.

Proposition 8.1 *If $n \equiv 0, 1 \bmod 3$, $n > 1$, then there is a unique pseudosymmetric semigroup of multiplicity 3 such that $|\mathrm{Star}(S)| = n$; if $n \equiv 2 \bmod 3$, there is no such S.*

Proof Let S be a pseudosymmetric semigroup of multiplicity 3.

If $\alpha \geq \beta$, then by Proposition 7.8 we have $\beta = 2\alpha - 1$; hence $|\mathrm{Star}(S)| = \binom{\alpha+\beta+1}{2\beta-\alpha+1} = \alpha + \beta + 1 = 3\beta + 1$; for each $n \equiv 1 \bmod 3$ there is a unique β and thus a unique pseudosymmetric semigroup.

Analogously, if $\beta \geq \alpha$, then $\alpha = 2\beta$, and $|\mathrm{Star}(S)| = \binom{\alpha+\beta+1}{2\beta-\alpha+1} = \alpha+\beta+1 = 3\alpha$, and every $n \equiv 0 \bmod 3$ can be (uniquely) obtained this way. □

Proposition 8.2 $\xi_3(n) = |\{\binom{a}{b} : \binom{a}{b} = n, a + b \equiv 1 \bmod 3\}|$.

Proof If $S = \langle 3, 3\alpha + 1, 3\beta + 2\rangle$, then $|\mathrm{Star}(S)| = \binom{\alpha+\beta+1}{2\alpha-\beta}$ and $\alpha + \beta + 1 + 2\alpha - \beta = 3\alpha + 1 \equiv 1 \bmod 3$; conversely, if $a + b \equiv 1 \bmod 3$, then the linear system

$$\begin{cases} \alpha + \beta + 1 = a \\ 2\alpha - \beta = b \end{cases}$$

has solutions $\alpha = \frac{a+b-1}{3}$, $\beta = \frac{2a-b-2}{3}$ which are integers if $a + b \equiv 1 \bmod 3$, and verify $\alpha \leq 2\beta + 1$ and $\beta \leq 2\alpha$. Hence to each semigroup we can attach a binomial coefficient and to each coefficient a semigroup, these maps are inverses and the two sets have the same cardinality. □

Thus, to find all numerical semigroups of multiplicity 3 with exactly n star operations, we only need to determine the binomial coefficients $\binom{a}{b}$ equal to n. Since $\binom{a}{b} \geq a$ if $\binom{a}{b} \neq 1$, this means that we only need to inspect the case $a \leq n$.

Removing the congruence condition, we get the function $\eta(n) := |\{\binom{a}{b} : \binom{a}{b} = n\}|$, that has been studied in [17] and [1]. It is straightforward to see that $\eta(n)$ is finite for every $n > 1$, and it is also quick to show (quantifying the previous reasoning) that $\eta(n) \leq 2 + 2\log_2 n$ [17]. A deeper analysis, using results about the distribution of the primes, proves that $\eta(n) = O(\log n/\log\log n)$ [1]; these results are however weaker than the expected, since in [17] it is conjectured that η is bounded for $n > 1$.

Clearly, $\xi_3(n) \leq \eta(n)$, and thus we get another proof (independent from [18]) that $\xi_3(n) < \infty$ for every $n > 1$. Note also that $\xi_3(1) = \infty$, because $|\mathrm{Star}(S)| = 1$ whenever $\alpha = 2\beta + 1$ or $\beta = 2\alpha$.

Proposition 8.3 *For every $n \in \mathbb{N}$, $\xi_3(n) \leq \frac{\eta(n)}{2}$.*

Proof If $n = 1$, then both sides of the equality are infinite; suppose $n > 1$. Then, $\eta(n) = \xi_3(n) + \xi_3^{(0)}(n) + \xi_3^{(2)}(n)$, where $\xi_3^{(i)}$ is the number of binomial coefficients $\binom{a}{b}$ such that $\binom{a}{b} = n$ and $a + b \equiv i \bmod 3$. We will show that $\xi_3(n) = \xi_3^{(2)}$, from which the claim follows.

Suppose $\binom{a}{b} = n$ and $a + b \equiv 1 \bmod 3$. Then also $\binom{a}{a-b} = n$, and $a + (a - b) = 2a - b \equiv 2a + 2b \bmod 3 \equiv 2 \bmod 3$. Therefore, $\xi_3(n) = \xi_3^{(2)}(n)$. □

Proposition 8.4 *Let* $Z(x) := \{n : 1 < n \leq x, \xi_3(n) > 1\}$.

(a) $|Z(x)| = O(\sqrt{x})$.
(b) There are an infinite number of integers n such that $\xi_3(n) = 0$.

Proof Following the proof of [1, Theorem 1], let $g(x) := \{n : 1 < n \leq x, \eta(n) > 2\}$. If $\xi_3(n) > 1$, then $\eta(n) \geq 2\xi_3(n) > 2$. Therefore, $Z(x) \leq g(x) = O(\sqrt{x})$, applying again the proof of [1, Theorem 1].

Take an $n \in \mathbb{N}$ such that $\eta(n) = 2$. Then, the only binomial coefficients such that $\binom{a}{b} = n$ are $\binom{n}{1}$ and $\binom{n}{n-1}$. It follows that $\xi_3(n) = 1$ if $n+1$ or $n+(n-1)$ are congruent to 1 modulo 3, i.e., if $n \equiv 0 \bmod 3$ or $n \equiv 1 \bmod 3$, while $\xi_3(n) = 0$ otherwise, i.e., if $n \equiv 2 \bmod 3$. (Compare Proposition 8.1.)

Suppose that $\xi_3(n) = 0$ only for $n \in \{n_1, \ldots, n_k\}$. For every $m \equiv 2 \bmod 3$ such that $m \neq n_i$ for every i, there is a binomial coefficient $\binom{a}{b}$ such that $\binom{a}{b} = m$ and $a + b \equiv 1 \bmod 3$. The last condition implies that $a - b \neq b$ (otherwise, $a + b = a - b + 2b = 3b \equiv 0 \bmod 3$); if $b = 1$ or $b = a - 1$, then $\binom{a}{b} = a = m$, and so $a + b \equiv m + 1 \equiv 0 \bmod 3$ or $a + b \equiv 2m - 1 \equiv 0 \bmod 3$, against the congruence condition. Therefore, $\binom{a}{b} = \binom{a}{a-b} = \binom{m}{1} = \binom{m}{m-1} = m$, and the four coefficients are different from each other, so that $\eta(m) \geq 4$. Thus, $g(x) \geq \frac{1}{3}x - k$, against the fact that $g(x) = O(\sqrt{x})$. Hence, $\xi_3(n) = 0$ infinitely often. □

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