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RADICALS OF PRINCIPAL IDEALS AND THE CLASS GROUP OF A DEDEKIND DOMAIN

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For a Dedekind domain D, let $\mathcal{P}(D)$ be the set of ideals of D that are the radical of a principal ideal. We show that, if D and D' are Dedekind domains and there is an order isomorphism between $\mathcal{P}(D)$ and $\mathcal{P}(D')$, then the rank of the class groups of D and D' is the same.

1. Introduction

The class group $\mathrm{Cl}(D)$ of a Dedekind domain D is defined as the quotient between the group of the nonzero fractional ideals of D and the subgroup of the principal ideals of D. Since $\mathrm{Cl}(D)$ is trivial if and only if D is a principal ideal domain (equivalently, if and only if it is a unique factorization domain), the class group can be seen as a way to measure how much unique factorization fails in D. For this reason, the study of the class group is an important part of the study of Dedekind domains.

It is a nonobvious fact that the class group of D actually depends only on the multiplicative structure of D, or, from another point of view, depends only on the set of nonzero principal ideals of D. Indeed, the class group of $D^\bullet := D \setminus \{0\}$ as a monoid (where the operation is the product) is isomorphic to the class group of D as a Dedekind domain (see Chapter 2, in particular Section 2.10, of [Geroldinger and Halter-Koch 2006]), and thus, if D and D' are Dedekind domains whose sets of principal ideals are isomorphic (as monoids), then the class groups of D and D' are isomorphic too.

In this paper, we show that the rank of $\mathrm{Cl}(D)$ can be recovered by considering only the set $\mathcal{P}(D)$ of the ideals that are the *radical* of a principal ideal; that is, we show that if $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are isomorphic as partially ordered sets, then the ranks of $\mathrm{Cl}(D)$ and $\mathrm{Cl}(D')$ are equal. The proof of this result can be divided into two steps.

In Section 3, we show that an order isomorphism between $\mathcal{P}(D)$ and $\mathcal{P}(D')$ can always be extended to an isomorphism between the sets $\mathrm{Rad}(D)$ and $\mathrm{Rad}(D')$ of all radical ideals of D (Theorem 3.6): this is accomplished by considering these

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sets as (noncancellative) semigroups and characterizing coprimality in D through a version of coprimality in $\mathcal{P}(D)$ (Proposition 3.3).

In Section 4, we link the structure of $\mathcal{P}(D)$ and $\mathrm{Rad}(D)$ with the structure of the tensor product $\mathrm{Cl}(D)\otimes\mathbb{Q}$ as an ordered topological space; in particular, we interpret the set of inverses of a set $\Delta\subseteq\mathrm{Max}(D)$ with respect to $\mathcal{P}(D)$ (see Definition 4.1) as the negative cone generated by the images of Δ in $\mathrm{Cl}(D)\otimes\mathbb{Q}$ (Proposition 4.2) and use this connection to calculate the rank of $\mathrm{Cl}(D)$ as a function of some particular partitions of an "inverse basis" of $\mathrm{Max}(D)$ (Propositions 4.9 and 4.10). As this construction is invariant with respect to isomorphism, we get the main theorem (Theorem 4.11).

In Section 5, we give three examples, showing that some natural extensions of the main result do not hold.

2. Notation and preliminaries

Throughout the paper, D will denote a *Dedekind domain*, that is, a one-dimensional integrally closed Noetherian integral domain; equivalently, a one-dimensional Noetherian domain such that D_P is a discrete valuation ring for all maximal ideals P. For general properties about Dedekind domains, the reader may consult, for example, [Bourbaki 1989, Chapter 7, §2], [Atiyah and Macdonald 1969, Chapter 9] or [Neukirch 1999, Chapter 1].

We use $D^\bullet$ to indicate the set $D\setminus\{0\}$. We denote by $\mathrm{Max}(D)$ the set of maximal ideals of D. If I is an ideal of D, we set

$$V(I):=\{P\in\mathrm{Spec}(D)\mid I\subseteq D\}.$$

If $I=xD$ is a principal ideal, we write $V(x)$ for $V(xD)$. If $I\neq(0)$, the set $V(I)$ is always a finite subset of $\mathrm{Max}(D)$. We denote by $\mathrm{rad}(I)$ the radical of the ideal I, and we say that I is a *radical ideal* (or simply that I is *radical*) if $I=\mathrm{rad}(I)$.

Every nonzero proper ideal I of D can be written uniquely as $P_1^{e_1}\cdots P_n^{e_n}=P_1^{e_1}\cap\cdots\cap P_n^{e_n}$, where $P_1,\dots,P_n$ are distinct maximal ideals and $e_1,\dots,e_n\geq 1$. In particular, in this case we have $V(I)=\{P_1,\dots,P_n\}$, and $\mathrm{rad}(I)=P_1\cdots P_n$. An ideal is radical if and only if $e_1=\cdots=e_n=1$. If P is a maximal ideal, the *P-adic valuation* of an element x is the exponent of P in the factorization of xD; we denote it by $v_P(x)$. (If $x\notin P$, i.e., if P does not appear in the factorization, then $v_P(x)=0$.)

If $P_1,\dots,P_k$ are distinct maximal ideals and $e_1,\dots,e_k\in\mathbb{N}$, then by the approximation theorem for Dedekind domains (see, e.g., [Bourbaki 1989, Chapter VII, §2, Proposition 2]) there is an element $x\in D$ such that $v_{P_i}(x)=e_i$ for $i=1,\dots,k$.

A *fractional ideal* of D is a D-submodule I of the quotient field K of D such that $xI\subseteq D$ (and thus, xI is an ideal of D) for some $x\in D^\bullet$. The set $\mathcal{F}(D)$ of nonzero fractional ideals of D is a group under multiplication; the inverse of an ideal I is $I^{-1} := (D : I) := \{x \in K \mid xI \subseteq D\}$. A nonzero fractional ideal I can be written uniquely as $P_1^{e_1} \cdots P_n^{e_n}$, where $P_1, \dots, P_n$ are distinct maximal ideals and $e_1, \dots, e_n \in \mathbb{Z} \setminus \{0\}$ (with the empty product being equal to D). Thus, $\mathcal{F}(D)$ is isomorphic to the free abelian group over $\mathrm{Max}(D)$. The quotient between this group and its subgroup formed by the principal fractional ideals is called the *class group* of D, and is denoted by $\mathrm{Cl}(D)$.

For a set S, we denote by $\mathfrak{P}_{\mathrm{fin}}(S)$ the set of all finite and nonempty subsets of S.

3. The two semilattices $\mathcal{P}(D)$ and $\mathrm{Rad}(D)$

Let $(X, \leq)$ be a *meet-semilattice*, that is, a partially ordered set where every pair of elements has an infimum. Then, the operation $x \wedge y$ associating to x and y their infimum is associative, commutative and idempotent, and it has a unit if and only if X has a maximum. The order of X can also be recovered from the operation: $x \geq y$ if and only if x divides y in $(X, \wedge)$, that is, if and only if there is a $z \in X$ such that $y = x \wedge z$. A *join-semilattice* is defined in the same way, but using the supremum instead of the infimum.

Let D be a Dedekind domain. We will be interested in two structures of this kind.

The first one is the semilattice $\mathrm{Rad}(D)$ of all nonzero radical ideals of D. In this case, the order $\leq$ is the usual containment order, while the product is equal to

$$I \wedge J := I \cap J = \mathrm{rad}(IJ).$$

The second one is the semilattice $\mathcal{P}(D)$ of the ideals of D that are radical of a nonzero, *principal* ideal of D. This is a subsemilattice of $\mathrm{Rad}(D)$, since

$$\mathrm{rad}(aD) \wedge \mathrm{rad}(bD) = \mathrm{rad}(abD),$$

i.e., the product of two elements of $\mathcal{P}(D)$ remains inside $\mathcal{P}(D)$.

A nonzero radical ideal I is characterized by the finite set $V(I)$. Hence, the map from $\mathrm{Rad}(D)$ to $\mathfrak{P}_{\mathrm{fin}}(\mathrm{Max}(D))$ sending I to $V(I)$ is an order-reversing isomorphism of partially ordered sets, which becomes an order-reversing isomorphism of semilattices if the operation on the power set is the union. We denote by $\mathcal{V}(D)$ the image of $\mathcal{P}(D)$ under this isomorphism; that is, $\mathcal{V}(D) := \{V(x) \mid x \in D^\bullet\}$. The inverse of this map is the one sending a set Z to the intersection of the prime ideals contained in Z.

Those semilattices have neither an absorbing element (which would be the zero ideal) nor a unit (which should be D itself).

Lemma 3.1. *Let $X, Y \in \mathfrak{P}_{\mathrm{fin}}(\mathrm{Max}(D))$ (respectively, $X, Y \in \mathcal{V}(D)$). Then, $X \mid Y$ in $\mathfrak{P}_{\mathrm{fin}}(\mathrm{Max}(D))$ (respectively, $X \mid Y$ in $\mathcal{V}(D)$) if and only if $X \subseteq Y$.*

Proof. If $X \mid Y$, then $Y = X \cup Z$ for some $Z \in \mathcal{V}(D)$, and thus $X \subseteq Y$. If $X \subseteq Y$, then $Y = Y \cup X$ and thus $X \mid Y$. (This works both in $\mathfrak{P}_{\mathrm{fin}}(\mathrm{Max}(D))$ and in $\mathcal{V}(D)$.) $\square$

Definition 3.2. Let M be a commutative semigroup. We say that $a_1, \dots, a_n \in M$ are *product-coprime* if, whenever there are $b_1, \dots, b_n \in M$ such that $a_1 b_1 = a_2 b_2 = \cdots = a_n b_n$, then for every j the element a_j divides $\prod_{i \neq j} b_i$.

Viewing $\mathcal{V}(D)$ as a semigroup, we can characterize when some elements are product-coprime.

Proposition 3.3. *Let D be a Dedekind domain, and let $a_1, \dots, a_n \in D^\bullet$. Then, $a_1, \dots, a_n$ are coprime in D if and only if $V(a_1), \dots, V(a_n)$ are product-coprime in $\mathcal{V}(D)$.*

Proof. Suppose that $a_1, \dots, a_n$ are coprime, and let $X \in \mathcal{V}(D)$ be such that $X = V(a_1) \cup B_1 = \cdots = V(a_n) \cup B_n$ for some $B_1, \dots, B_n \in \mathcal{V}(D)$. By symmetry, it is enough to prove that $V(a_1)$ divides $B_2 \cup \cdots \cup B_n$ in $\mathcal{V}(D)$, i.e., that $V(a_1) \subseteq B_2 \cup \cdots \cup B_n$. Take any prime ideal $P \in V(a_1)$: since $a_1, \dots, a_n$ are coprime there is a j such that $P \notin V(a_j)$. However, $P \in V(a_j) \cup B_j$, and thus $P \in B_j$. Therefore, $V(a_i) \subseteq B_2 \cup \cdots \cup B_n$, as claimed.

Conversely, suppose $V(a_1), \dots, V(a_n)$ are product-coprime, and suppose that $a_1, \dots, a_n$ are not coprime. Then, there is a prime ideal P containing all a_i; passing to powers, without loss of generality we can suppose that the P-adic valuation of the a_i is the same, say $v_P(a_i) = t$ for every i. By prime avoidance, there is a $b_1 \in D \setminus P$ such that $v_Q(b_1) \geq v_Q(a_i)$ for all $i > 1$ and all $Q \neq P$. Let $x := a_1 b_1$. By construction, $a_i \mid x$ for each i, and thus we can find $b_2, \dots, b_n \in D$ such that $x = a_i b_i$. Therefore, $V(x) = V(a_i) \cup V(b_i)$ for every i; by hypothesis, it follows that $V(a_1)$ divides $V(b_2) \cup \cdots \cup V(b_n)$, i.e., that $V(a_i) \subseteq V(b_2) \cup \cdots \cup V(b_n)$. However, $v_P(x) = v_P(a_1) + v_P(b_1) = t$, and thus $v_P(b_i) = 0$ for every i; in particular, $P \notin V(b_i)$ for every i. This is a contradiction, and thus $a_1, \dots, a_n$ are coprime. $\square$

Definition 3.4. Let M be a commutative semigroup. We say that $I \subsetneq M$ is *product-proper* if no finite subset of I is product-coprime. We denote the set of maximal product-proper subsets of M by $\mathfrak{M}(M)$.

Proposition 3.5. *Let D be a Dedekind domain. The maps*

$$\nu \colon \operatorname{Max}(D) \to \mathfrak{M}(\mathcal{V}(D)), \quad P \mapsto \{V(x) \mid x \in P\}$$

and

$$\theta \colon \mathfrak{M}(\mathcal{V}(D)) \to \operatorname{Max}(D), \quad \mathcal{Y} \mapsto \{x \in D \mid V(x) \in \mathcal{Y}\}$$

are bijections, inverses of each other.

Proof. We first show that ν and θ are well-defined.

If P is a maximal ideal of D, then $P \in X$ for every $X \in \nu(P)$; thus, if $V(a) \in \nu(P)$, then $a \in P$ and $\nu(P)$ is product-proper. If $\nu(P) \subsetneq \mathcal{Y} \subseteq \mathcal{V}(D)$, take $Y \in \mathcal{Y} \setminus \nu(P)$: then, $Y = V(b)$ for some $b \notin P$. If $Y = \{Q_1, \dots, Q_k\}$, by prime avoidance we can find $a \in P \setminus (Q_1 \cup \cdots \cup Q_k)$; then, a and b are coprime, and thus $V(a)$ and $V(b)$ are product-coprime. Hence, $\nu(P)$ is a maximal product-proper subset of $\mathcal{V}(D)$.

Conversely, let $\mathcal{Y} \in \mathfrak{M}(\mathcal{V}(D))$. If $\theta(\mathcal{Y})$ is contained in some prime ideal P, then $\mathcal{Y} \subseteq \nu(P)$, and thus we must have $\mathcal{Y} = \nu(P)$; in particular, $\theta(\mathcal{Y}) = P \in \mathrm{Max}(D)$. If $\theta(\mathcal{Y})$ is not contained in any prime ideal, let $V(a) = \{Q_1, \ldots, Q_k\} \in \mathcal{Y}$. Since $\theta(\mathcal{Y}) \nsubseteq Q_i$, for every i we can find $b_i \notin Q_i$ such that $V(a_i) \in \mathcal{Y}$; then, $a, b_1, \ldots, b_n$ are coprime, and thus $V(a), V(b_1), \ldots, V(b_n)$ are a product-coprime subset of $\mathcal{Y}$, a contradiction. Hence, $\mathcal{Y} = \nu(P)$.

The fact that they are inverses of each other follows similarly. □

Theorem 3.6. *Let D, D' be Dedekind domains. If there is an order isomorphism $\psi : \mathcal{P}(D) \to \mathcal{P}(D)$, then there is an order isomorphism $\Psi : \mathrm{Rad}(D) \to \mathrm{Rad}(D')$ extending ψ.*

Proof. The statement is equivalent to saying that any isomorphism $\phi : \mathcal{V}(D) \to \mathcal{V}(D')$ can be extended to an isomorphism $\Phi : \mathfrak{P}_{\mathrm{fin}}(\mathrm{Max}(D)) \to \mathfrak{P}_{\mathrm{fin}}(\mathrm{Max}(D'))$. For simplicity, set $\mathfrak{P} := \mathfrak{P}_{\mathrm{fin}}(\mathrm{Max}(D))$ and $\mathfrak{P}' := \mathfrak{P}_{\mathrm{fin}}(\mathrm{Max}(D'))$.

If ϕ is an isomorphism, then it sends product-proper sets into product-proper sets, and thus ϕ induces a bijective map $\eta_1 : \mathfrak{M}(\mathcal{V}(D)) \to \mathfrak{M}(\mathcal{V}(D'))$. Using the map θ of Proposition 3.5, η_1 induces a bijection $\eta : \mathrm{Max}(D) \to \mathrm{Max}(D')$ such that the diagram

$$\begin{tikzcd} \mathfrak{M}(\mathcal{V}(D)) \arrow[r, "\theta"] \arrow[d, "\eta_1"] & \mathrm{Max}(D) \arrow[d, "\eta"] \\ \mathfrak{M}(\mathcal{V}(D')) \arrow[r, "\theta'"] & \mathrm{Max}(D) \end{tikzcd}$$

commutes (explicitly, $\eta = \theta' \circ \eta_1 \circ \theta^{-1}$). In particular, η induces an order isomorphism Φ between $\mathfrak{P}$ and $\mathfrak{P}'$, sending $X \subseteq \mathrm{Max}(D)$ to $\eta(X) \subseteq \mathrm{Max}(D')$. To conclude the proof, we need to show that Φ extends ϕ.

Let $X = \{P_1, \ldots, P_k\} \in \mathcal{V}(D)$. Then, by definition,

$$\Phi(X) = \eta(X) = \{\eta(P_1), \ldots, \eta(P_k)\}.$$

The maximal product-proper subsets of $\mathcal{V}(D)$ containing X are $\mathcal{Y}_i := \nu(P_i)$, for $i = 1, \ldots, k$; since ϕ is an isomorphism, the maximal product-proper subsets of $\mathcal{V}(D')$ containing $\phi(X)$ are the sets $\phi(\mathcal{Y}_i)$. By construction, $\phi(\mathcal{Y}_i) = \eta_1(\mathcal{Y}_i)$; however, $\theta'(\eta_1(\mathcal{Y}_i)) = \eta(P_i)$, and thus $\eta(X) = \{\phi(\mathcal{Y}_1), \ldots, \phi(\mathcal{Y}_k)\} = \phi(X)$. Thus, Φ extends ϕ, as claimed. □

The following corollary was obtained, with a more *ad hoc* reasoning, in the proof of [Spirito 2020, Theorem 2.6].

Corollary 3.7. *Let D, D' be Dedekind domains such that $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are order-isomorphic. Then, $\mathrm{Cl}(D)$ is torsion if and only if $\mathrm{Cl}(D')$ is torsion.*

Proof. The class group of D is torsion if and only if every prime ideal has a principal power [Gilmer and Ohm 1964, Theorem 3.1], and thus if and only if $\mathcal{P}(D) = \mathrm{Rad}(D)$.

If $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are isomorphic, then by Theorem 3.6 there is an isomorphism $\Phi : \mathrm{Rad}(D) \to \mathrm{Rad}(D')$ sending $\mathcal{P}(D)$ to $\mathcal{P}(D')$; hence, $\mathrm{Rad}(D) = \mathcal{P}(D)$ if and only if $\mathrm{Rad}(D') = \mathcal{P}(D')$. Therefore, $\mathrm{Cl}(D)$ is torsion if and only if $\mathrm{Cl}(D')$ is torsion. □

Remark 3.8. Let $\mathrm{Princ}(D)$ be the set of principal ideals of D and $\mathcal{I}(D)$ be the set of all ideals of D.

The method used in this section can also be applied to prove the analogous result for ideals that are not necessarily radical, i.e., to prove that an isomorphism $\phi : \mathrm{Princ}(D) \to \mathrm{Princ}(D')$ can be extended to an isomorphism $\Phi : \mathcal{I}(D) \to \mathcal{I}(D')$.

The most obvious analogue of Proposition 3.3 does not hold, since the ideals $(a_1), \dots, (a_n)$ may be product-coprime in $\mathrm{Princ}(D)$ without $a_1, \dots, a_n$ being coprime (for example, take $a_1 = y$, $a_2 = y^2$ and $a_3 = y^3$, where y is a prime element of D). However, this can be repaired: $a_1, \dots, a_n \in D^\bullet$ are coprime if and only if the ideals $(a_1)^{k_1}, \dots, (a_n)^{k_n}$ are product-coprime in $\mathrm{Princ}(D)$ for every $k_1, \dots, k_n \in \mathbb{N}$. The proof is essentially analogous to the one given for Proposition 3.3.

Proposition 3.5 carries over without significant changes: the maximal product-proper subsets of $\mathrm{Princ}(D)$ are in bijective correspondence with the maximal ideals of D. Theorem 3.6 carries over as well: the only difference is that, instead of the restricted power set $\mathfrak{P}_{\mathrm{fin}}(\mathrm{Max}(D))$, it is necessary to use the free abelian group generated by $\mathrm{Max}(D)$.

In particular, this result directly implies that if $\mathrm{Princ}(D)$ and $\mathrm{Princ}(D')$ are isomorphic as partially ordered sets, then the class groups $\mathrm{Cl}(D)$ and $\mathrm{Cl}(D')$ are isomorphic as groups, since the class group depends exactly on which ideals are principal. This result is also a consequence of the theory of monoid factorization (see [Geroldinger and Halter-Koch 2006]), of which this reasoning can be seen as a more direct (but less general) version.

4. Calculating the rank

The *rank* $\mathrm{rk}\, G$ of an abelian group G is the dimension of the tensor product $G \otimes \mathbb{Q}$ as a vector space over $\mathbb{Q}$. In particular, the rank of G is 0 if and only if G is a torsion group; therefore, Corollary 3.7 can be rephrased by saying that, if $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are order-isomorphic, then the rank of $\mathrm{Cl}(D)$ is 0 if and only if the rank of $\mathrm{Cl}(D')$ is 0. In this section, we want to generalize this result by showing that $\mathrm{rk}\,\mathrm{Cl}(D)$ is actually determined by $\mathcal{P}(D)$ in every case.

Let D be a Dedekind domain. If $\mathcal{I}(D)$ is the set of proper ideals of D, then the quotient from $\mathcal{F}(D)$ to $\mathrm{Cl}(D)$ restricts to a map $\pi : \mathcal{I}(D) \to \mathrm{Cl}(D)$, which is a monoid homomorphism (i.e., $\pi(IJ) = \pi(I) \cdot \pi(J)$). Moreover, π is surjective since the class of I coincide with the class of dI for every $d \in D^\bullet$.

There is also a natural map $\psi_0 : \mathrm{Cl}(D) \to \mathrm{Cl}(D) \otimes \mathbb{Q}$, $g \mapsto g \otimes 1$, from the class group to the $\mathbb{Q}$-vector space $\mathrm{Cl}(D) \otimes \mathbb{Q}$; the map ψ_0 is a group homomorphism, and its kernel is the torsion subgroup T of $\mathrm{Cl}(D)$. By construction, the image $\mathcal{C}$ of ψ_0 spans $\mathrm{Cl}(D) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space.

Thus, we have a chain of maps

$$\mathcal{I}(D) \xrightarrow{\pi} \mathrm{Cl}(D) \xrightarrow{\psi_0} \mathrm{Cl}(D) \otimes \mathbb{Q};$$

we denote by ψ the composition $\psi_0 \circ \pi$.

Definition 4.1. Let $\Delta \subseteq \mathrm{Max}(D)$. A maximal ideal Q is an *almost inverse* of Δ if there is a set $\{P_1, \ldots, P_k\} \subseteq \Delta$ (not necessarily nonempty) such that $Q \wedge P_1 \wedge \cdots \wedge P_n$ belongs to $\mathcal{P}(D)$. We denote the set of almost inverses of Δ as $\mathrm{Inv}(\Delta)$.

Our aim is to characterize $\mathrm{Inv}(\Delta)$ in terms of the map ψ; to do so, we use the terminology of *ordered topological spaces* (we refer the reader to [Davis 1954]). Given a $\mathbb{Q}$-vector space V and a set $S \subseteq V$, the *positive cone* spanned by S is

$$\mathrm{pos}(S) := \left\{ \sum_{i=1}^{k} \lambda_i \boldsymbol{v}_i \mid \lambda_i \in \mathbb{Q}^{\geq 0}, \boldsymbol{v}_i \in S \right\};$$

if $C = \mathrm{pos}(S)$, we say that C is *positively spanned* by S. Symmetrically, the *negative cone* is $\mathrm{neg}(S) := -\mathrm{pos}(S)$.

Proposition 4.2. *Let* $\Delta \subseteq \mathrm{Max}(D)$. *Then*, $\mathrm{Inv}(\Delta) = \psi^{-1}(\mathrm{neg}(\psi(\Delta)))$.

Proof. Let $Q \in \mathrm{Inv}(\Delta)$, and let $P_1, \ldots, P_n \in \Delta$ be such that $L := Q \wedge P_1 \wedge \cdots \wedge P_n \in \mathcal{P}(D)$. Then, there is a principal ideal $I = aD$ with radical L; thus, there are positive integers $e, f_1, \ldots, f_n > 0$ such that $I = Q^e P_1^{f_1} \cdots P_n^{f_n}$ (this holds also if $Q = P_i$ for some i). Since I is principal, $\psi(I) = \mathbf{0}$; hence,

$$\mathbf{0} = \psi(I) = \psi(Q^e P_1^{f_1} \cdots P_n^{f_n}) = e\psi(Q) + \sum_{i=1}^{n} f_i \psi(P_i).$$

Solving in $\psi(Q)$, we see that $\psi(Q) = \sum_i -\frac{f_i}{e} \psi(P_i) \in \mathrm{neg}(\psi(\Delta))$, as claimed.

Conversely, suppose that $\psi(Q) \in \mathrm{neg}(\psi(\Delta))$. Then, either $\psi(Q) = \mathbf{0}$ (in which case $Q \in \mathrm{Inv}(\Delta)$ by taking no $P \in \Delta$ in the definition) or we can find $P_1, \ldots, P_n \in \Delta$ and negative rational numbers $q_1, \ldots, q_n$ such that $\psi(Q) = \sum_i q_i \psi(P_i)$. By multiplying for the minimum common multiple of the denominators of the q_i, we obtain a relation $e\psi(Q) + \sum_i f_i \psi(P_i) = \mathbf{0}$, with $e, f_i \in \mathbb{N}^+$. If $I := Q^e P_1^{f_1} \cdots P_n^{f_n}$, it follows that $\pi(I)$ is torsion in the class group, i.e., there is an $n > 0$ such that I^n is principal; thus $\mathrm{rad}(I^n) = \mathrm{rad}(I) = Q \wedge P_1 \wedge \cdots \wedge P_n \in \mathcal{P}(D)$, as claimed. $\square$

Corollary 4.3. *Let* $\Delta \subseteq \mathrm{Max}(D)$. *Then*, $\mathrm{Inv}(\Delta) = \mathrm{Max}(D)$ *if and only if* $\psi(\Delta)$ *positively spans* $\mathrm{Cl}(D) \otimes \mathbb{Q}$.

Proof. Suppose $\mathrm{Inv}(\Delta)=\mathrm{Max}(D)$, and let $\boldsymbol{q}\in \mathrm{Cl}(D)\otimes\mathbb{Q}$. Since the image $\mathcal{C}$ of ψ generates $\mathrm{Cl}(D)\otimes\mathbb{Q}$ as a $\mathbb{Q}$-vector space and is a subgroup, there is a $d\in\mathbb{N}^+$ such that $d\boldsymbol{q}\in\mathcal{C}$. Hence, $d\boldsymbol{q}=\psi(I)$ for some $I\in\mathcal{I}(D)$; factorize I as $P_1^{e_1}\cdots P_n^{e_n}$, with $P_i\in\mathrm{Max}(D)$ and $e_i>0$. By Proposition 4.2, we have

$$\psi(I)=\sum_i e_i\psi(P_i)\in\sum_i e_i\mathrm{neg}(\psi(\Delta))=\mathrm{neg}(\psi(\Delta)),$$

and thus also $\boldsymbol{q}=\frac{1}{d}\psi(I)\in\mathrm{neg}(\psi(\Delta))$. Hence, $\psi(\Delta)$ negatively spans $\mathrm{Cl}(D)\otimes\mathbb{Q}$, and thus it also positively spans $\mathrm{Cl}(D)\otimes\mathbb{Q}$.

Conversely, suppose $\psi(\Delta)$ positively spans $\mathrm{Cl}(D)\otimes\mathbb{Q}$; thus, it also negatively spans $\mathrm{Cl}(D)\otimes\mathbb{Q}$. Let $Q\in\mathrm{Max}(D)$: then, $\psi(Q)\in\mathrm{neg}(\psi(\Delta))$, so that $Q\in\mathrm{Inv}(\Delta)$ by Proposition 4.2. Hence, $\mathrm{Inv}(\Delta)=\mathrm{Max}(D)$. □

We can now characterize when the rank of $\mathrm{Cl}(D)$ is finite.

Proposition 4.4. *Let D be a Dedekind domain. Then,* $\mathrm{rk}\,\mathrm{Cl}(D)<\infty$ *if and only if there is a finite set* $\Delta\subseteq\mathrm{Max}(D)$ *such that* $\mathrm{Inv}(\Delta)=\mathrm{Max}(D)$.

Proof. Suppose first that $\mathrm{rk}\,\mathrm{Cl}(D)=n<\infty$. Then, $\mathrm{Inv}(\mathrm{Max}(D))=\mathrm{Max}(D)$, and thus $\psi(\mathrm{Max}(D))$ positively spans $\mathrm{Cl}(D)\otimes\mathbb{Q}$, by Corollary 4.3. Let $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_n\}$ be a basis of $\mathrm{Cl}(D)\otimes\mathbb{Q}$: then each $\boldsymbol{e}_i$ belongs to the positive cone spanned by a finite subset Λ_i of $\psi(\mathrm{Max}(D))$. Thus, the union Λ of the Λ_i is a finite set positively spanning $\mathrm{Cl}(D)\otimes\mathbb{Q}$, so the corresponding subset Δ of $\mathrm{Max}(D)$ is finite and $\mathrm{Inv}(\Delta)=\mathrm{Max}(D)$ by Corollary 4.3.

Conversely, suppose there is a finite set $\Delta=\{P_1,\dots,P_k\}\subseteq\mathrm{Max}(D)$ such that $\mathrm{Inv}(\Delta)=\mathrm{Max}(D)$. For every $Q\in\mathrm{Max}(D)$, there are $i_1,\dots,i_r$ such that $Q\wedge P_{i_1}\wedge\cdots\wedge P_{i_r}\in\mathcal{P}(D)$; as in the proof of Proposition 4.2, it follows that there are $e,f_1,\dots,f_r>0$ such that $Q^eP_{i_1}^{f_1}\cdots P_{i_r}^{f_r}$ is principal. Thus, $\pi(Q)\otimes 1$ belongs to the $\mathbb{Q}$-vector subspace of $\mathrm{Cl}(D)\otimes\mathbb{Q}$ generated by $\pi(P_{i_1})\otimes 1,\dots,\pi(P_{i_r})\otimes 1$ (where $\pi:\mathcal{I}(D)\to\mathrm{Cl}(D)$ is the quotient map, as above). Since Q was arbitrary, the set $\{\pi(P_1)\otimes 1,\dots,\pi(P_k)\otimes 1\}$ is a basis of $\mathrm{Cl}(D)\otimes\mathbb{Q}$. In particular, we have $\mathrm{rk}\,\mathrm{Cl}(D)=\dim_{\mathbb{Q}}\mathrm{Cl}(D)\otimes\mathbb{Q}\le k<\infty$. □

We will also need a criterion to understand when $\mathrm{Inv}(\Delta)$ corresponds to a linear subspace.

Proposition 4.5. *Let* $\Delta\subseteq\mathrm{Max}(D)$. *Then,* $\mathrm{neg}(\psi(\Delta))$ *is a linear subspace of* $\mathrm{Cl}(D)\otimes\mathbb{Q}$ *if and only if* $\Delta\subseteq\mathrm{Inv}(\Delta)$.

Proof. Suppose $\mathrm{neg}(\psi(\Delta))$ is a linear subspace, and let $Q\in\Delta$. Then, there are $P_i\in\Delta$, $\lambda_i\in\mathbb{Q}^-$ such that $\psi(Q)=\sum_i\lambda_i\psi(P_i)$; multiplying by the minimum common multiple of the denominators we get an equality $e\psi(Q)+\sum_i f_i\psi(P_i)=\mathbf{0}$, where $e,f_i\in\mathbb{N}^+$. Let $I:=Q^eP_1^{f_1}\cdots P_n^{f_n}$: then, $\psi(I)=\mathbf{0}$, so that $\pi(I)$ is torsion in $\mathrm{Cl}(D)$, i.e., I^n is principal for some n. Thus, $Q\wedge P_1\wedge\cdots\wedge P_n\in\mathcal{P}(D)$, and $Q\in\mathrm{Inv}(\Delta)$.

Conversely, suppose $\Delta \subseteq \mathrm{Inv}(\Delta)$, and let $\boldsymbol{q}$ be an element of the linear subspace generated by $\psi(\Delta)$. Then, there are $P_i, Q_j \in \Delta$, $\theta_i \in \mathbb{Q}^+$ and $\mu_j \in \mathbb{Q}^-$ such that

$$\boldsymbol{q} = \sum_i \theta_i \psi(P_i) + \sum_i \mu_j \psi(Q_j).$$

By construction, each $\theta_i \psi(P_i)$ belongs to $\mathrm{pos}(\psi(\Delta))$. Furthermore, each $\psi(Q_j)$ is in $\mathrm{neg}(\psi(\Delta))$ by Proposition 4.2, and thus $\mu_j \psi(Q_j) \in \mathrm{pos}(\psi(\Delta))$ for every j. Therefore, $\boldsymbol{q} \in \mathrm{pos}(\psi(\Delta))$, so the positive cone of $\psi(\Delta)$ is a linear subspace and $\mathrm{neg}(\psi(\Delta)) = \mathrm{pos}(\psi(\Delta))$ is a subspace too. □

Proposition 4.4 can be interpreted by saying that $\mathrm{rk}\,\mathrm{Cl}(D)$ is finite if and only if $\mathrm{Max}(D)$ is "negatively generated" by a finite set. In the case of finite rank, we need a way to link the dimension of $\mathrm{Cl}(D) \otimes \mathbb{Q}$ with the cardinality of the sets spanning it as a positive cone; that is, we need to consider a notion analogue to the basis of a vector space.

Since we need only to consider the case of finite rank, from now on we suppose that $n := \mathrm{rk}\,\mathrm{Cl}(D) < \infty$ and we identify $\mathrm{Cl}(D) \otimes \mathbb{Q}$ with $\mathbb{Q}^n$.

Definition 4.6. A set $X \subseteq \mathbb{Q}^n$ is *positive basis* of $\mathbb{Q}^n$ if $\mathrm{pos}(X) = \mathbb{Q}^n$ and if $\mathrm{pos}(X \setminus \{x\}) \neq \mathbb{Q}^n$ for every $x \in X$.

Definition 4.7. A subset $\Delta \subseteq \mathrm{Max}(D)$ is an *inverse basis* of $\mathrm{Max}(D)$ if $\mathrm{Inv}(\Delta) = \mathrm{Max}(D)$ and $\mathrm{Inv}(\Delta') \neq \mathrm{Max}(D)$ for every $\Delta' \subsetneq \Delta$.

These two notions are naturally connected.

Proposition 4.8. *Let $\Delta \subseteq \mathrm{Max}(D)$. Then, Δ is an inverse basis of $\mathrm{Max}(D)$ if and only if $\psi(\Delta)$ is a positive basis of $\mathbb{Q}^n$.*

Proof. If Δ is an inverse basis, then $\psi(\Delta)$ positively spans $\mathbb{Q}^n$ by Corollary 4.3, while $\psi(\Delta')$ does not for every $\Delta' \subsetneq \Delta$ (again by the corollary). Hence, $\psi(\Delta)$ is a positive basis. The converse follows in the same way. □

Given a positive basis X of $\mathbb{Q}^n$, we call a partition $\{X_1, \dots, X_s\}$ of X a *weak Reay partition* if, for every j, the positive cone of $X_1 \cup \dots \cup X_i$ is a linear subspace of $\mathbb{Q}^n$. The following is a variant of [Reay 1965, Theorem 2]:

Proposition 4.9. *Let X be a positive basis of $\mathbb{Q}^n$. Then*:

(a) *every weak Reay partition of X has cardinality at most $|X| - n$*;

(b) *there is a weak Reay partition of X of cardinality $|X| - n$.*

Proof. Let $\{X_1, \dots, X_s\}$ be a weak Reay partition, and let V_i be the linear space spanned by $X_1, \dots, X_i$ (with $V_0 := (0)$). We claim that $\dim V_i - \dim V_{i-1} \leq |X_i| - 1$. Indeed, let $X_i := \{z_1, \dots, z_t\}$: then, $-z_t$ belongs to the positive cone generated by V_{i-1} and X_i, and thus we can write $-z_t = y + \sum_j \lambda_j z_j$ for some $y \in V_{i-1}$ and $\lambda_j \geq 0$. It follows that $-(1+\lambda_t) z_t = y + \lambda_1 z_1 + \dots + \lambda_{t-1} z_{t-1}$, and since $\lambda_t \neq -1$

we have that z_t is linearly dependent from $X_1 \cup \cdots \cup X_{i-1} \cup \{z_1, \ldots, z_{t-1}\}$. Hence, $\dim V_i \le \dim V_{i-1} + t - 1$, as claimed.

Therefore,

$$\begin{aligned} n = \dim \mathbb{Q}^n &= (\dim V_s - \dim V_{s-1}) + \cdots + \dim V_1 \\ &\le (|X_s| - 1) + \cdots + (|X_1| - 1) = |X| - s, \end{aligned}$$

and thus $s \le |X| - n$, and (a) is proved. (b) is a direct consequence of [Reay 1965, Theorem 2]. □

Similarly, if $\Delta \subseteq \mathrm{Max}(D)$ is an inverse basis of $\mathrm{Max}(D)$, we call a partition $\{\Delta_1, \ldots, \Delta_s\}$ a *weak Reay partition* if $\Delta_1 \cup \cdots \cup \Delta_i \subseteq \mathrm{Inv}(\Delta_1 \cup \cdots \cup \Delta_i)$ for every i.

Proposition 4.10. *Let $\Delta \subseteq \mathrm{Max}(D)$ be an inverse basis of $\mathrm{Max}(D)$, and let $\{\Delta_1, \ldots, \Delta_s\}$ be a partition of Δ. Then, $\{\Delta_1, \ldots, \Delta_s\}$ is a weak Reay partition of Δ if and only if $\{\psi(\Delta_1), \ldots, \psi(\Delta_s)\}$ is a weak Reay partition of $\psi(\Delta)$.*

Proof. By Proposition 4.5, $\Delta_1 \cup \cdots \cup \Delta_i \subseteq \mathrm{Inv}(\Delta_1 \cup \cdots \cup \Delta_i)$ if and only if the positive cone of $\psi(\Delta_1 \cup \cdots \cup \Delta_i) = \psi(\Delta_1) \cup \cdots \cup \psi(\Delta_i)$ is a linear subspace of $\mathbb{Q}^n$. The claim now follows from the definition. □

Theorem 4.11. *Let D and D' be Dedekind domains such that $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are isomorphic. Then, $\mathrm{rk}\,\mathrm{Cl}(D) = \mathrm{rk}\,\mathrm{Cl}(D')$.*

Proof. Let $\phi : \mathcal{P}(D) \to \mathcal{P}(D')$ be an isomorphism; by Theorem 3.6, we can find an isomorphism $\Phi : \mathrm{Rad}(D) \to \mathrm{Rad}(D')$ sending $\mathcal{P}(D)$ to $\mathcal{P}(D')$. In particular, $\Phi(\mathrm{Max}(D)) = \mathrm{Max}(D')$.

Since $\mathrm{Inv}(\Delta)$ is defined only through $\mathcal{P}(D)$ and $\mathrm{Rad}(D)$, Φ respects the inverse construction, in the sense that $\Phi(\mathrm{Inv}(\Delta)) = \mathrm{Inv}(\Phi(\Delta))$ for every $\Delta \subseteq \mathrm{Max}(D)$. In particular, $\mathrm{Inv}(\Delta) = \mathrm{Max}(D)$ if and only if $\mathrm{Inv}(\Phi(\Delta)) = \mathrm{Max}(D')$; by Proposition 4.4, it follows that $\mathrm{rk}\,\mathrm{Cl}(D) = \infty$ if and only if $\mathrm{rk}\,\mathrm{Cl}(D') = \infty$.

Suppose now that the two ranks are finite, say equal to n and n', respectively. Let $\Delta \subseteq \mathrm{Max}(D)$ be an inverse basis of $\mathrm{Max}(D)$. Let $\{\Delta_1, \ldots, \Delta_s\}$ be a weak Reay partition of Δ of maximum cardinality; by Propositions 4.9 and 4.10 , $s = |\Delta| - n$.

Every weak Reay partition of Δ gets mapped by Φ into a weak Reay partition of $\Delta' := \psi(\Delta)$, and conversely; therefore, the maximum cardinality of the weak Reay partitions of Δ' is again $|\Delta| - n$. However, applying Propositions 4.9 and 4.10 to Δ', we see that this quantity is $|\Delta'| - n'$; since $|\Delta| = |\Delta'|$, we get $n = n'$, as claimed. □

Corollary 4.12. *Let D, D' be Dedekind domains, and let $T(D)$ (respectively, $T(D')$) be the torsion subgroup of $\mathrm{Cl}(D)$ (respectively, $\mathrm{Cl}(D')$). If $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are isomorphic and if $\mathrm{Cl}(D)$ and $\mathrm{Cl}(D')$ are finitely generated, then $\mathrm{Cl}(D)/T(D) \simeq \mathrm{Cl}(D')/T(D')$.*

Proof. Since $\mathrm{Cl}(D)$ is finitely generated, it has finite rank n and $\mathrm{Cl}(D)/T(D) \simeq \mathbb{Z}^n$; analogously, $\mathrm{Cl}(D')/T(D') \simeq \mathbb{Z}^m$, where $m := \operatorname{rk}\mathrm{Cl}(D')$. By Theorem 4.11, $n = m$; and, in particular, $\mathrm{Cl}(D)/T(D) \simeq \mathrm{Cl}(D')/T(D')$. □

5. Counterexamples

In this section, we collect some examples showing that Theorem 4.11 is, in many ways, the best possible.

Example 5.1. It is not possible to improve the conclusion of Theorem 4.11 from "$\operatorname{rk}\mathrm{Cl}(D) = \operatorname{rk}\mathrm{Cl}(D')$" to "$\mathrm{Cl}(D) \simeq \mathrm{Cl}(D')$". Indeed, if $\operatorname{rk}\mathrm{Cl}(D) = 0$ (i.e., if $\mathrm{Cl}(D)$ is torsion) then $\mathcal{P}(D) = \mathrm{Rad}(D)$; and thus, whenever $\operatorname{rk}\mathrm{Cl}(D) = \operatorname{rk}\mathrm{Cl}(D') = 0$, the posets $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are isomorphic.

For the next examples, we need to use a construction of Claborn [1968].

Let $G := \sum_i x_i\mathbb{Z}$ be the free abelian group on the countable set $\{x_i\}_{i\in\mathbb{N}}$. Let I be a subset of G satisfying the following two properties:

- All coefficients of the elements of I (with respect to the x_i) are nonnegative.
- For every finite set $x_{i_1}, \dots, x_{i_k}$ and every $n_1, \dots, n_k \in \mathbb{N}$, there is an element y of I such that the component of y relative to x_{i_t} is n_t.

Then, [Claborn 1968, Theorem 2.1] says that there is an integral domain D with countably many maximal ideals $\{P_i\}_{i\in\mathbb{N}}$ such that the map sending the ideal $P_1^{n_1} \cdots P_k^{n_k}$ to $n_1x_1 + \cdots + n_kx_k$ sends principal ideals to elements of the subgroup H generated by I. In particular, $\mathrm{Cl}(D) \simeq G/H$.

Example 5.2. Corollary 4.12 does not hold without the hypothesis that $\mathrm{Cl}(D)$ and $\mathrm{Cl}(D')$ are finitely generated.

For example, let H_1 be the subgroup of G generated by $x_n + x_{n+1}$, as n ranges in $\mathbb{N}$, and I_1 be the subset of the elements of H_1 having all coefficients nonnegative. Then, I_1 satisfies the above conditions. The corresponding domain D_1 has a class group isomorphic to $\mathbb{Z}$, and its prime ideals are concentrated in two classes: if n is even, P_n is equivalent to P_0; if n is odd, P_n is equivalent to P_1; furthermore, P_0P_1 is principal. (This is exactly Example 3-2 of [Claborn 1968].) In particular, $\mathcal{P}(D_1)$ is equal to the members of $\mathrm{Rad}(D_1)$ that are contained both in some P_n with n even and in some P_m with m odd.

Let now H_2 be the subgroup of G generated by $x_n + 2x_{n+1}$, as n ranges in $\mathbb{N}$, and let I_2 be the subset of the elements of H_2 having all coefficients nonnegative. Then, I_2 also satisfies the condition above. Let D_2 be the corresponding Dedekind domain. Then, $\mathrm{Cl}(D_2)$ is isomorphic to the quotient G/H_2, which is isomorphic to the subgroup $\mathbb{Z}(2^\infty)$ of $\mathbb{Q}$ generated by $1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \dots$ (that is, to the Prüfer 2-group): this can be seen by noting that the map

$$G \to \mathbb{Q}, \quad P_n \mapsto (-1)^n \frac{1}{2^n}$$

is a group homomorphism with kernel H_2 and range $\mathbb{Z}(2^\infty)$. In this isomorphism, the prime ideals Q_n, with n even, are mapped to positive elements of $\mathbb{Z}(2^\infty)$, while the prime ideals Q_m, with m odd, are mapped to the negative elements. Hence, $\mathcal{P}(D_2)$ is equal to the member of $\mathrm{Rad}(D_2)$ that are contained in both an "even" and an "odd" prime.

Therefore, the map $\mathrm{Rad}(D_1)\to\mathrm{Rad}(D_2)$ sending $P_{i_1}\cap\cdots\cap P_{i_k}$ to $Q_{i_1}\cap\cdots\cap Q_{i_k}$ is an isomorphism sending $\mathcal{P}(D_1)$ to $\mathcal{P}(D_2)$. However, the class groups of D_1 and D_2 are both torsion-free (i.e., $T(D_1)=T(D_2)=0$), but not isomorphic.

Example 5.3. The converse of Theorem 4.11 does not hold; that is, it is possible that $\mathrm{rk}\,\mathrm{Cl}(D)=\mathrm{rk}\,\mathrm{Cl}(D')$ even if $\mathcal{P}(D)$ and $\mathcal{P}(D')$ are not isomorphic.

Take H_1 and D_1 as in the previous example.

Take H_3 to be the subgroup of G generated by x_0 and by x_n+x_{n+1} for $n>0$, and let I_3 be the subset of the elements of H_3 having all coefficients nonnegative. Then, I_3 satisfies Claborn's conditions, and the corresponding domain D_3 satisfies $\mathrm{Cl}(D_3)\simeq\mathbb{Z}$ (in particular, $\mathrm{rk}\,\mathrm{Cl}(D_3)=1$), so $\mathrm{Cl}(D_1)$ and $\mathrm{Cl}(D_3)$ are isomorphic.

However, D_3 has a principal maximal ideal (the one corresponding to x_0), while D_1 does not. Therefore, there is no isomorphism $\mathrm{Rad}(D_1)\to\mathrm{Rad}(D_3)$ sending $\mathcal{P}(D_1)$ to $\mathcal{P}(D_3)$; by Theorem 3.6, it follows that $\mathcal{P}(D_1)$ and $\mathcal{P}(D_3)$ cannot be isomorphic.

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