On the Complexity of Fragments of the Modal Logic of Allen's Relations over Dense Structures

D. Bresolin[1], D. Della Monica[2], A. Montanari[3], P. Sala[4], and G. Sciavicco[5]

[1] Department of Computer Science and Engineering
University of Bologna (Italy) (davide.bresolin@unibo.it)
[2] ICE-TCS, School of Computer Science
Reykjavik University (Iceland) (dariodm@ru.is)
[3] Department of Mathematics and Computer Science
University of Udine (Italy) (angelo.montanari@uniud.it)
[4] Department of Computer Science
University of Verona (Italy) (pietro.sala@univr.it)
[5] Department of Information, Engineering and Communications
University of Murcia (Spain) (guido@um.es)

Abstract. Interval temporal logics provide a natural framework for temporal reasoning about interval structures over linearly ordered domains, where intervals are taken as the primitive ontological entities. Their computational behaviour and expressive power mainly depend on two parameters: the set of modalities they feature and the linear orders over which they are interpreted. In this paper, we consider all fragments of Halpern and Shoham's interval temporal logic HS with a decidable satisfiability problem over the class of all dense linear orders, and we provide a complete classification of them in terms of their complexity and expressiveness by solving the last two open cases.

1 Introduction

Most temporal logics proposed in the literature assume a point-based structure of time. They have been successfully applied in a variety of fields, ranging from the specification and verification of communication protocols to temporal data mining. However, a number of relevant application domains, such as, for instance, those of planning and synthesis of controllers, are often characterized by advanced features like durative actions (and their temporal relationships), accomplishments, and temporal aggregations, which are neglected or dealt with in an unsatisfactory way by point-based formalisms. The distinctive features of interval temporal logics turn out to be useful in these domains. As an example, they allow one to model telic statements [17], that is, statements that express goals or accomplishments, like the statement: "The airplane flew from Venice to Toronto" (see [8, Sect. II.B]). Temporal logics with interval-based semantics have also been proposed as suitable formalisms for the specification and verification of hardware [14] and of real-time systems [9]. Finally, successful implementations of interval-based systems can be found in the areas of learning (the adaptive learning system TERENCE [10], that provides a support to poor comprehenders and their educators, is based on the so-called Allen's interval algebra [3]) and real-time data systems (the algorithm RISMA [12], for performance and behaviour analysis of real-time data systems, is based on Halpern and Shoham's modal logic of Allen's relations [11]).

The variety of binary relations between intervals in a linear order was first studied by Allen [3], who investigated their use in systems for time management and planning. In [11], Halpern and Shoham introduced and systematically analyzed the (full) modal logic of Allen's relations (HS for short), that features one modality for each Allen relation. In particular, they showed that HS is highly undecidable over most classes of linear orders. This result motivated the search for (syntactic) fragments of HS offering a good balance between expressiveness and computational complexity. During the last decade, a systematic analysis has been carried out to characterize the complexity of the satisfiability problem for HS fragments [4, 5, 15], as well as their relative expressive power [1, 2, 5]. Such an analysis pointed out that such characterizations also depend on the class of linearly ordered set over which formulae are interpreted.

This paper aims at completing the classification of decidable HS fragments with respect to both their complexity and expressiveness, relative to the class of (all) dense linear orders. For our purposes, the class of dense linear orders and the linear order of the rational numbers $\mathbb{Q}$ are indistinguishable. Thus, all the results presented here directly apply to $\mathbb{Q}$ as well. The paper is organized as follows. In Section 2, we introduce syntax and semantics of (fragments of) HS. Next, in Section 3 we summarize known results about dense linear orders. In Section 4 and Section 5, we solve the last two open problems, thus completing the picture for the class of dense linear structures. It is worth mentioning that an analogous classification has been provided in [5] for the class of finite linear orders, the class of discrete linear orders, the linear order of the natural numbers $\mathbb{N}$, and the linear order of the integers $\mathbb{Z}$.

2 The Modal Logic of Allen's Relations

Let us consider a linearly ordered set $\mathbb{D} = \langle D, < \rangle$, where D is an element domain and $<$ is a total ordering on it. An *interval* over $\mathbb{D}$ is an ordered pair $[x, y]$, where $x, y \in D$ and $x \leq y$. An interval is called a *point interval* if $x = y$ and a *strict interval* if $x < y$. In this paper, we assume the *strict semantics*, that is, we exclude point intervals and only consider strict intervals. The adoption of the strict semantics, excluding point intervals, instead of the *non-strict semantics*, which includes them, conforms to the definition of interval adopted by Allen in [3], but differs from the one given by Halpern and Shoham in [11]. It has at least two strong motivations: first, a number of representation paradoxes arise when the non-strict semantics is adopted, due to the presence of point intervals, as pointed out in [3]; second, when point intervals are included there seems to be no intuitive semantics for interval relations that makes them both pairwise disjoint and jointly exhaustive. If we exclude the identity relation, there

HS **modalities**	**Allen's relations**	**Graphical representation**
$\langle A \rangle$	$[x,y]R_A[x',y'] \Leftrightarrow y = x'$	
$\langle L \rangle$	$[x,y]R_L[x',y'] \Leftrightarrow y < x'$	
$\langle B \rangle$	$[x,y]R_B[x',y'] \Leftrightarrow x = x', y' < y$	
$\langle E \rangle$	$[x,y]R_E[x',y'] \Leftrightarrow y = y', x < x'$	
$\langle D \rangle$	$[x,y]R_D[x',y'] \Leftrightarrow x < x', y' < y$	
$\langle O \rangle$	$[x,y]R_O[x',y'] \Leftrightarrow x < x' < y < y'$	

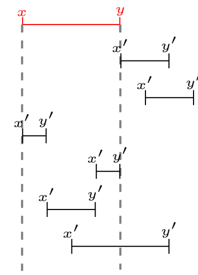

Fig. 1. Allen's interval relations and the corresponding HS modalities.

are 12 different relations between two strict intervals in a linear order, often called *Allen's relations* [3]: the six relations R_A (*meets* or *adjacent*), R_L (*after* or *later*), R_B (*starts* or *begins*), R_E (*finishes* or *ends*), R_D (*during*), and R_O (*overlaps*), depicted in Fig. 1, and their inverses, that is, $R_{\overline{X}} = (R_X)^{-1}$, for each $X \in \{A, L, B, E, D, O\}$.

We interpret interval structures as Kripke structures, with Allen's relations playing the role of the accessibility relations. Thus, we associate a modality $\langle X \rangle$ with each Allen relation R_X. For each $X \in \{A, L, B, E, D, O\}$, the *transpose* of modality $\langle X \rangle$ is modality $\langle \overline{X} \rangle$, corresponding to the inverse relation $R_{\overline{X}}$ of R_X. Halpern and Shoham's logic HS [11] is a multi-modal logic with formulae built from a finite, non-empty set $\mathcal{AP}$ of atomic propositions (also referred to as proposition letters), the propositional connectives $\vee$ and $\neg$, and a modality for each Allen relation. With every subset $\{R_{X_1}, \ldots, R_{X_k}\}$ of these relations, we associate the fragment $\mathsf{X_1X_2\ldots X_k}$ of HS, whose formulae are defined by the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X_1 \rangle\varphi \mid \ldots \mid \langle X_k \rangle\varphi,$$

where $p \in \mathcal{AP}$. The other propositional connectives and constants (e.g., $\wedge$, $\rightarrow$, and $\top$), as well as the dual modalities (e.g., $[A]\varphi \equiv \neg\langle A \rangle\neg\varphi$), can be derived in the standard way.

The (strict) semantics of HS is given in terms of *interval models* $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$, where $\mathbb{D}$ is a linear order, $\mathbb{I}(\mathbb{D})$ is the set of all (strict) intervals over $\mathbb{D}$, and V is a *valuation function* $V : \mathcal{AP} \rightarrow 2^{\mathbb{I}(\mathbb{D})}$, which assigns to each atomic proposition $p \in \mathcal{AP}$ the set of intervals $V(p)$ on which p holds. The *truth* of a formula on a given interval $[x, y]$ in an interval model M is defined by structural induction on formulae as follows:

- $M, [x, y] \Vdash p$ if and only if $[x, y] \in V(p)$, for each $p \in \mathcal{AP}$;
- $M, [x, y] \Vdash \neg\psi$ if and only if it is not the case that $M, [x, y] \Vdash \psi$;
- $M, [x, y] \Vdash \varphi \vee \psi$ if and only if $M, [x, y] \Vdash \varphi$ or $M, [x, y] \Vdash \psi$;
- $M, [x, y] \Vdash \langle X \rangle\psi$ if and only if there exists $[x', y']$ such that $[x, y]R_X[x', y']$ and $M, [x', y'] \Vdash \psi$, for each modality $\langle X \rangle$.

Formulae of HS can be interpreted over a given class of interval models; we identify the class of interval models over linear orders in $\mathcal{C}$ with the class $\mathcal{C}$ itself.

Thus, we will use, for example, the expression 'formulae of HS are interpreted over the class $\mathcal{C}$ of linear orders' instead of the extended one 'formulae of HS are interpreted over the class of interval models over linear orders in $\mathcal{C}$'. Among others, we mention the following important classes of linear orders: *(i)* the class of *all* linear orders Lin; *(ii)* the class of all *dense* linear orders Den, that is, those in which for every pair of different points there exists at least one point in between them; *(iii)* the class of all *weakly discrete* linear orders WDis, that is, those in which every element, apart from the greatest one, if it exists, has an immediate successor, and every element, other than the least one, if it exists, has an immediate predecessor; *(iv)* the class of all *strongly discrete* linear orders Dis, that is, those in which for every pair of different points there are only finitely many points in between them; *(v)* the class of all *finite* linear orders Fin, that is, those having only finitely many points; *(vi)* the singleton classes consisting of the standard linear orders over $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$. The *mirror image* (or, simply, *mirror*) of a fragment $\mathcal{F}$ is obtained by simultaneously substituting $\langle A\rangle$ with $\langle\overline{A}\rangle$, $\langle B\rangle$ with $\langle E\rangle$, $\langle\overline{B}\rangle$ with $\langle\overline{E}\rangle$, $\langle O\rangle$ with $\langle\overline{O}\rangle$, $\langle L\rangle$ with $\langle\overline{L}\rangle$, and the other way around. When interpreted over left/right symmetric classes of structures (i.e., classes $\mathcal{C}$ such that if $\mathcal{C}$ contains a linear order $\mathbb{D} = \langle D, \prec\rangle$, then it also contains a linear order isomorphic to its dual linear order $\mathbb{D}^d = \langle D, \succ\rangle$, where $\succ$ is the inverse of $\prec$), such as Den, all computational properties of a fragment are preserved for its mirror one; thanks to this observation, we can safely deal with only one fragment for each pair of mirror fragments.

3 Known and Unknown Results

It has been proved in [1] that there are precisely nine different optimal definabilities that hold among HS modalities in the dense case; as a consequence, only 966 HS fragments are expressively different (out of 4096 different subsets of 12 modalities). Of those, 146 are decidable, thanks to the following results:

Undecidability: we know by [4] that every fragment containing (as definable) O, AD, or $\mathsf{A\overline{D}}$ is undecidable;

Non-primitive recursive: the decidability of $\mathsf{A\overline{A}B\overline{B}}$ has been proved in [13], where it has also been shown that every fragment containing $\mathsf{A\overline{A}B}$ or $\mathsf{A\overline{A}\overline{B}}$ is non-primitive recursive;

ExpSpace-completeness: as a consequence of the results presented in [8], we know that $\mathsf{AB\overline{B}\overline{L}}$ is in ExpSpace, and every fragment containing AB or $\mathsf{A\overline{B}}$ is ExpSpace-hard (in particular, the hardness result given in [8] for $\mathsf{AB\overline{B}}$ can be suitably rephrased to deal with the smaller fragments AB and $\mathsf{A\overline{B}}$);

NExpTime-completeness: it has been proved in [7] that $\mathsf{A\overline{A}}$ is in NExpTime, and both A and $\mathsf{\overline{A}}$ are NExpTime-hard;

PSpace-completeness: every sub-fragment of $\mathsf{B\overline{B}D\overline{D}L\overline{L}}$ that contains (as definable) D or $\mathsf{\overline{D}}$ is shown to be PSpace-complete in [6, 15].

The purpose of this paper is to fill in the few gaps still uncovered by this collection of results. Here, we shall prove that: *(i)* $\mathsf{B\overline{B}L\overline{L}}$ and all its fragments are

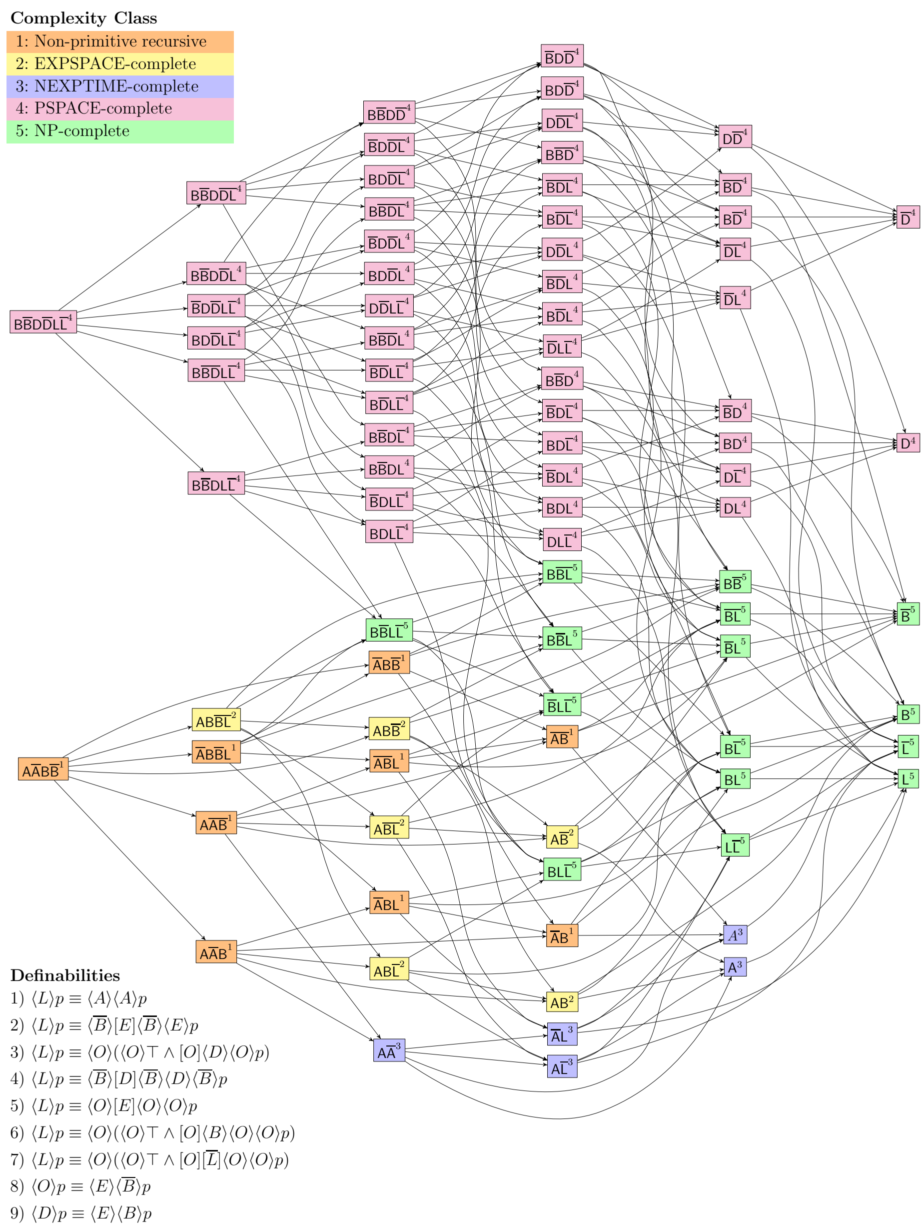

Fig. 2. Decidable fragments of HS in the dense case and their relative expressive power.

NP-complete (observe that every fragment is NP-hard, given that it is at least as expressive as propositional logic), and *(ii)* all the fragments that contain $\overline{\mathsf{AB}}$ or $\overline{\mathsf{A}}\mathsf{B}$ are non-primitive recursive. All the aforementioned results allow us to draw a picture that encompasses all HS fragments, ordered according to their relative expressive power and grouped by computational complexity. We show here such a picture (see Fig. 2), limited to all and only decidable HS fragments (for the sake of readability, we omit fragments that are expressively equivalent or mirror image of another fragment featured in the picture). In Fig. 2 we also show the nine definabilities that hold among HS modalities over dense linear orders.

4 NP-Complete Fragments

In this section we prove that the fragment $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ is NP-Complete. The proof consists of two main steps. First, we provide a pseudo-model property for formulas of $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$, that is there exists a finitely representable structure that satisfies a given set of constraints if and only if the $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ formula of input is satisfiable. Second we prove that each satisfiable formula admits a pseudo model of size at most $P(|\varphi|)$ where P is some polynomial, together with the fact that the constraints on the structure may be checked in non deterministic polynomial time the fact that our problem belongs to NP immediately follows.

We now introduce basic terminology and notation that are common in the temporal logic setting. The *closure* of a formula φ is defined as the set $\mathcal{C}l(\varphi)$ of all sub-formulas of φ and all their negations (we identify $\neg\neg\psi$ with ψ, $\neg\langle B\rangle\psi$ with $[B]\neg\psi$, etc.). For a technical reason that will be clear soon, we also introduce the *extended closure* of φ, denoted $\mathcal{ECl}(\varphi)$, that extends $\mathcal{C}l(\varphi)$ by adding all formulas of the form $\langle \mathsf{R}\rangle\psi$ and $[R]\psi$, with $R \in \{B, \bar{B}, L, \bar{L}\}$ and $\psi \in \mathcal{C}l(\varphi)$.

Now we introduce the concept of atom by means of a maximal "locally consistent" subset of $\mathcal{ECl}(\varphi)$. Formally, we call *φ-atom* any set $F \subseteq \mathcal{ECl}(\varphi)$ such that (i) $\psi \in F$ iff $\neg\psi \notin F$, for all $\psi \in \mathcal{ECl}(\varphi)$, (ii) $\psi \in F$ iff $\psi_1 \in F$ or $\psi_2 \in F$, for all $\psi = \psi_1 \vee \psi_2 \in \mathcal{ECl}(\varphi)$, We denote by *atoms*$(\varphi)$ the set of all φ-atoms.

Given an atom F and a relation $R \in \{B, \bar{B}, L, \bar{L},\}$, we let $\mathcal{R}eq_R(F)$ be the set of *requests of F for the interval relation R*, namely, the formulas $\psi \in \mathcal{C}l(\varphi)$ such that $\langle \mathsf{R}\rangle\psi \in F$. Similarly, we let $\mathcal{O}bs(F)$ be the set of *observables of F*, namely, the formulas $\psi \in F \cap \mathcal{C}l(\varphi)$ – intuitively, the observables of F are those formulas $\psi \in F$ that fulfil requests of the form $\langle \mathsf{R}\rangle\psi$ from other atoms. An atom F is said to be B-reflexive iff for each $[B]\psi \in F$ (resp. $[\bar{B}]\psi \in F$) we have $\psi \in F$. Given an given a linear order $\mathbb{D}$ a **labelled interval structure**, **LIS** for short, over it is a tuple $\mathbf{L} = (\mathbb{D}, \mathcal{L})$ where $\mathcal{L} : \mathbb{I}(\mathbb{D}) \to atoms(\varphi)$ is a function mapping each interval on $\mathbb{D}$ in an atom with the following properties:

1. for every pair of intervals $[x, y], [x', y'] \in \mathbb{I}(\mathbb{D})$ and for each $* \in \{B, \overline{B}, L, \overline{L}\}$ we have that if $[x, y] * [x', y']$ then $\mathcal{O}bs(\mathcal{L}([x', y'])) \subseteq \mathcal{R}eq_*([x, y])$;
2. for every interval $[x, y] \in \mathbb{I}(\mathbb{D})$, for each $* \in \{B, \overline{B}, L, \overline{L}\}$ and for each $\psi \in \mathcal{R}eq_*([x, y])$ we have that there exists $[x', y'] \in \mathbb{I}(\mathbb{D})$ with $[x, y] * [x', y']$ and $\psi \in \mathcal{O}bs(\mathcal{L}([x', y']))$;

3. there exists an interval $[x, y] \in \mathbb{I}(\mathbb{D})$ with $\varphi \in \mathcal{L}([x, y])$.

The following theorem states that **LIS** are sufficient witnesses for satisfiability.

Theorem 1. *Given an* $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ *formula* φ*,* φ *is satisfiable over* $\mathbb{Q}$ *if and only if there exists a* **LIS** $\mathbf{L} = (\mathbb{Q}, \mathcal{L})$ *for it.*

We omit the proof of Theorem 1 since it is straightforward and consists of simple adaptation of proofs presented in [7, 15].

Definition 1. *Given an* $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ *formula* φ *an* $L\overline{L}$ *sequence is a sequence of pairs in* $\sigma_{L\bar{L}} = (L_0, \bar{L}_0, k_0), \ldots, (L_n, \bar{L}_n, k_n)$ *with* $L_i, \bar{L}_i \subseteq \mathcal{C}l(\varphi)$ *and* $k_i \in \{cluster, point\}$ *for each* $1 \leq i \leq n$ *such that:*

- $k_0 = k_n = cluster$*, for each* $0 < i < n$ *if* $k_i = point$ *then* $k_{i-1} = k_{i+1} = cluster$*;*
- *for each* $0 \leq i < m$ *we have* $L_i \supseteq L_{i+1}$ *and* $\bar{L}_i \subseteq \bar{L}_{i+1}$ *with* $* \in \{B, L\}$*;*
- *for each* $0 \leq i < m$ *there exists* $*$ *in* $\{L, \bar{L}\}$ *for which* $*_i \neq *_{i+1}$*;*

Definition 2. *Given an* $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ *formula* φ *a set* $\bar{L} \subseteq \mathcal{C}l(\varphi)$ *an* $B\overline{B}$ *sequence over* $\bar{L}$ *is a sequence of tuples* $\sigma_{B\bar{B}} = (B_0, \bar{B}_0, L_0, \bar{L}_0, \Psi_0, k_0), \ldots, (B_m, L_m, \bar{L}_m, \Psi_m, k_m)$ *with* $B_i, \bar{L}_i \subseteq \mathcal{C}l(\varphi)$*,* $\Psi_i \subseteq \mathcal{C}l(\varphi)$ *and* $k_i \in \{cluster, point\}$ *for each* $1 \leq i \leq m$ *such that:*

- $k_0 = k_m = cluster$*, for each* $0 < i < m$ *if* $k_i = point$ *then* $k_{i-1} = k_{i+1} = cluster$*;*
- *for each* $0 \leq i < m$ *we have* $*_i \supseteq *_{i+1}$ *and* $\bar{*}_i \subseteq \bar{*}_{i+1}$ *with* $* \in \{B, L\}$*;*
- *for each* $0 \leq i < m$ *there exists* $*$ *in* $\{B, \bar{B}, L, \bar{L}\}$ *for which* $*_i \neq *_{i+1}$*;*
- *for each* $0 \leq i \leq n$ *if* $k_i = cluster$ *then for each* $\psi \in \Psi_i$ *there exists a* B*-reflexive atom* F *with* $\mathcal{R}eq(F) = B_i \cup \bar{B}_i \cup L_i \cup \bar{L}$ *and* $\psi \in F$*, if* $k_i = point$ *we have that* $\Psi_i \cup B_i \cup \bar{B}_i \cup L_i \cup \bar{L}$ *is an atom;*
- *for each* $0 \leq i < m$ *we have* $L_i \supseteq L_{i+1}$ *and* $\bar{L}_i \subseteq \bar{L}_{i+1}$ *and* $L_i \neq L_{i+1}$ *or* $\bar{L}_i \neq \bar{L}_{i+1}$*;*
- *for each* $0 \leq i \leq m$ *and for each* $\psi \in B_i$ *(resp.* $\psi \in \bar{B}_i$*) there exists* $0 \leq i' \leq i$ *(resp.* $i \leq i' \leq m$*) with* $\psi \in \Psi_{i'}$*, if* $k_i = point$ *we have* $i' \neq i$*;*
- *for each* $0 \leq i \leq m$ *we have* $\bigcup_{0 \leq i' < i} \Psi_{i'} \subseteq \bar{L}_i$ *if* $k_i = cluster$ *we have also* $\Psi_i \subseteq \bar{L}_i$*.*

We denote with $\Sigma_{B\bar{B}}$ the set of all the $B\bar{B}$ sequences over $\bar{L}$ for some $\bar{L} \subseteq \mathcal{C}l(\varphi)$. We extend the concept of observables to $B\overline{B}$ sequences. Given an $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ formula φ and a $B\overline{B}$ sequence $\sigma_{B\bar{B}} = (B_0, \bar{B}_0, L_0, \bar{L}_0, \Psi_0, k_0), \ldots, (B_m, L_m, \bar{L}_m, \Psi_m, k_m)$ and an index $0 \leq j \leq m$ we identify with $\mathcal{O}bs(\sigma_{B\bar{B}})|_{L_j, \bar{L}_j}$ the set $\bigcup_{0 \leq j'' \leq j'} \Psi''_j$ with $j' = j - 1$ if $k_j = point$ and $j' = j$ otherwise. We will use $\mathcal{O}bs(\sigma_{B\bar{B}})$ to denote $\mathcal{O}bs(\sigma_{B\bar{B}})|_{L_m, \bar{L}_m}$ (i.e.,when restrictions are not needed).

Definition 3. *Given an* $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ *formula* φ*, a set* $\bar{L} \subseteq \mathcal{C}l(\varphi)$*, an* $L\overline{L}$ *sequence* $\sigma_{L\bar{L}} = (L_0, \bar{L}_0, k_0), \ldots, (L_n, \bar{L}_n, k_n)$ *and a* $B\overline{B}$ *sequence* $\sigma_{B\bar{B}} = (B'_0, \bar{B}'_0, L'_0, \bar{L}'_0, \Psi'_0, k'_0), \ldots, (B'_m, L'_m, \bar{L}'_m, \Psi'_m, k'_m)$ *over* $\bar{L}$ *we say that* $\sigma_{B\bar{B}}$ agrees with $\sigma_{L\bar{L}}$ at point i *if and only if* $\bar{L} = \bar{L}_i$ *and there exists a strictly increasing function* $f : [i, n] \to [0, m]$ *such that (i)* $f(n) = m$ *and* $f(i) = 0$*; (ii) if* $k_i = point$ *we have* $L_i = L'_1$ *and* $L_i = L'_0$ *otherwise (iii) for each* $i < i' \leq n$ *we have* $L_{i'} = L'_{f(i')}$ *,* *for each $i \le i' \le n$ we have $\bar{L}_{i'} = \bar{L}'_{f(i')}$ and $k_{i'} = k'_{f(i')}$; (iv) for each $1 \le i' < n$ and for each pair of indexes $f(i') \le j, j' < f(i'+1)$ we have $L'_j = L'_{j'}$ and $\bar{L}'_j = \bar{L}'_{j'}$.*

Definition 4. *Given an $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ formula φ a pseudo model for it is a tuple $(\sigma_{L\bar{L}}, \Sigma_0, \ldots, \Sigma_n)$ where $\sigma_{L\bar{L}} = (L_0, \bar{L}_0, k_0), \ldots, (L_n, \bar{L}_n, k_n)$ is an $L\overline{L}$ sequence with $\varphi \in L_i$ for some $0 \le i \le n$, and for each $0 \le i \le n$ the following conditions hold:*

1. *$\Sigma_i = \{\sigma^{i,0}_{B\bar{B}}, \ldots, \sigma^{i,h_i}_{B\bar{B}}\}$ is a non-empty set of $B\overline{B}$ sequences and for each $0 \le j \le h_i$ we have that $\sigma^{i,j}_{B\bar{B}}$ agrees with $\sigma_{L\bar{L}}$ at point i and $|\Sigma_i| = 1$ if $k_i = point$;*
2. *for each $\psi \in L_i$ there exists $i \le i' \le n$ such that $\Sigma_{i'} = \{\sigma^{i',0}_{B\bar{B}}, \ldots, \sigma^{i',h_{i'}}_{B\bar{B}}\}$ and there exists $0 \le j \le h_{i'}$ for which $\psi \in \mathcal{O}bs(\sigma^{i',j}_{B\bar{B}})$, if $k_i = point$ we have $i < i'$;*
3. *for each $\psi \in \bar{L}_i$ there exists $0 \le i' \le i \le n$ such that $\Sigma_{i'} = \{\sigma^{i',0}_{B\bar{B}}, \ldots, \sigma^{i',h_{i'}}_{B\bar{B}}\}$ and there exists $0 \le j \le h_{i'}$ for which $\psi \in \mathcal{O}bs(\sigma^{i',j}_{B\bar{B}})|_{L_i,\bar{L}_i}$, if $k_i = point$ we have $i' < i$, if $k_{i'} = point$ and $k_i = cluster$ we have $\psi \in \mathcal{O}bs(\sigma^{i',j}_{B\bar{B}})|_{L_{i-1},\bar{L}_{i-1}}$.*

Theorem 2. *For every $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ formula φ we have that φ is satisfiable over the class of dense linear orders if and only if there exists a pseudo-model for it.*

Proof. Let $(\sigma_{L\bar{L}}, \Sigma_0, \ldots, \Sigma_n)$ be a pseudo model for φ with $\sigma_{L\bar{L}} = (L_0, \bar{L}_0, k_0), \ldots, (L_n, \bar{L}_n, k_n)$ and $\Sigma_i = \{\sigma^{i,0}_{B\bar{B}}, \ldots, \sigma^{i,h_i}_{B\bar{B}}\}$ for each $0 \le i \le n$. First we take a non-decreasing function $f_{sup} : \{0, \ldots, n-1\} \to \mathbb{R}$ such that for each $0 \le i \le n$ we have that $f_{sup}(i) = f_{sup}(i+1)$ if and only if $k_i = cluster$ and $k_{i+1} = point$ and $f_{sup}(i) \in \mathbb{Q}$ if and only if $k_i = point$ or $k_{i+1} = point$. Using f_{sup} we build a region function $f_{reg} : \mathbb{Q} \to \{0, \ldots, n\}$ such that for each i with $k_i = point$ we have $f_{reg}(q) = i$ for the rational q with $f_{sup}(i) = q$, if $q \notin img(f_{sup})$ we have $f_{reg}(q) = i$ such that $f_{sup}(i-1) < q < f_{sup}(i)$.

The proof is done by building iteratively a model for φ as the limit of an infinite sequence of models $\mathbf{M}_0, \mathbf{M}_1, \ldots$. Each $\mathbf{M}_i$ is a partial interval model build on some finite subset $Q_i \subset \mathbb{Q}$ with $Q_i \subseteq Q_{i+1}$ for every $i \in \mathbb{N}$. We say partial model because, as we will see, the labeling for some intervals on Q_i may not be defined at step i. In order to provide a fairness condition that will guarantee that the final construction is a model for φ we define a partial order $\le$ over the intervals on $\bigcup_{i\in\mathbb{N}} Q_i$. Given a point $q \in Q_i$ we define its *birthdate* as $birth(q) = \min_{j\in\mathbb{N}} q \in Q_j$ Given two intervals $[q, q'], [\overline{q}, \overline{q}'] \in \bigcup_{i\in\mathbb{N}} Q_i$ we have that $[q, q'] \le [\overline{q}, \overline{q}']$ if and only if $\max(birth(q), birth(q')) \le \max(birth(\overline{q}), birth(\overline{q}'))$.

For each i we introduce an auxiliary function $f^i_{B\overline{B}} : Q_i \to \bigcup_{0\le i\le n} \Sigma_i$. For each step $i \in \mathbb{N}$ we guarantee the following invariant conditions:

1. for each $q \in Q_i$ we have that $f^i_{B\overline{B}}(q) \in \Sigma_j$ iff either $q \in img(f_{sup})$ and $f_{sup}(j) = q$ or $q \notin img(f_{sup})$ and $f_{sup}(j-1) < q < f_{sup}(j)$;
2. for each labelled interval $[q, q']$ on Q_i let F be its label and $f^i_{\sigma_{B\bar{B}}}(q) = (B_0, \bar{B}_0, L_0, \bar{L}_0, \Psi_0, k_0), \ldots, (B_m, L_m, \bar{L}_m, \Psi_m, k_m)$ we have that $\mathcal{R}eq_{\bar{L}}(F) = \bar{L}_{f_{reg}(q)}$, $\mathcal{R}eq_L(\ F) = L_{f_{reg}(q')}$ and there exists $0 \le j \le m$ such that $\mathcal{R}eq_*(F) = *_j$ with $* \in \{B, \overline{B}, L\}$.

Moreover for each point q on Q_i let $f_{\sigma_{B\bar{B}}}(q) = (B_0, \bar{B}_0, L_0, \bar{L}_0, \Psi_0, k_0), \ldots, (B_m, L_m, \bar{L}_m, \Psi_m, k_m)$ we define a function $f^q_{\sigma_{B\bar{B}}} : \{q' > q : q \in Q_i\} \rightarrow [0...m]$ which is defined for $q' \in Q_i$ iff $[q, q']$ has been labelled up to step i. For every $q' \in Q_i$ for which the function $f^q_{\sigma_{B\bar{B}}}$ is defined we have that $f^q_{\sigma_{B\bar{B}}}$ satisfies: (i) $\mathcal{R}eq_*(\mathcal{L}([q, q'])) = *_{f^q_{\sigma_{B\bar{B}}}(q')}$ for each $* \in \{B, \overline{B}L\}$ and $\mathcal{O}bs(\mathcal{L}([q, q'])) \subseteq \Psi_{f^q_{\sigma_{B\bar{B}}}(q')}$ (ii) for each $q'' > q'$ in Q_i for which $f^q_{\sigma_{B\bar{B}}}(q'')$ is defined we have $f^q_{\sigma_{B\bar{B}}}(q') \leq f^q_{\sigma_{B\bar{B}}}(q'')$. Now we proceed describing the initial step 0 and then how the iterations are done at the generic step i. From Definition 4 we have that there exists an index i for which $\varphi \in L_i$ and for condition 2 of the same definition we have that there exists $i \leq i'$ for which $\Sigma_{i'} = \{\sigma^{i',0}_{B\bar{B}}, \ldots, \sigma^{i',h_{i'}}_{B\bar{B}}\}$ and there exists $0 \leq j \leq h_{i'}$ for which $\varphi \in \mathcal{O}bs(\sigma^{i',j}_{B\bar{B}})$ two cases may arise:

1. $k_{i'} = point$ then we have that $|\Sigma_{i'}| = 1$ then we put $f^0_{BB}(q) = \sigma^{i',j}_{B\bar{B}}$ and we take $q = f_{sup}(i')$;
2. $k_{i'} = cluster$ then we take a point $q \in \mathbb{Q}$ such that $f_{reg}(q) = i'$ then we put $f^0_{BB}(q) = \sigma^{i',j}_{B\bar{B}}$.

Let $\sigma^{i',j}_{B\bar{B}} = (B^{i',j}_0, \bar{B}^{i',j}_0, L^{i',j}_0, \bar{L}^{i',j}_0, \Psi^{i',j}_0, k^{i',j}_0), \ldots, (B^{i',j}_{m(i',j)}, L^{i',j}_{m(i',j)}, \bar{L}^{i',j}_{m(i',j)}, \Psi^{i',j}_{m(i',j)}, k^{i',j}_{m(i',j)})$ since $\varphi \in \mathcal{O}bs(\sigma^{i',j}_{B\bar{B}})$ we have that there exists $0 \leq j' \leq m(i', j)$ for which with $\varphi \in \Psi_{i',j'}$ then since we have that $\sigma^{i',j}_{B\bar{B}})$ agrees with $\sigma_{L\bar{L}}$ at i' then there exists i'' for which $(L^{i',j}_{j'}, \bar{L}^{i',j}_{j'}) = (L_{i''}, \bar{L}_{i''})$ again we have two cases:

1. $k_{i''} = point$ then we have that $|\Sigma_{i''}| = 1$ then we put $f^0_{BB}(q') = \sigma^{i''}_{B\bar{B}}$ where $\sigma^{i''}_{B\bar{B}}$ is the sole element in $\Sigma_{i''}$ and we take $q = f_{sup}(i'')$;
2. $k_{i''} = cluster$ then we take a point $q' \in \mathbb{Q}$ such that $f_{reg}(q') = i''$ then we put $f^0_{BB}(q') = \sigma^{i''}_{B\bar{B}}$ for some $\sigma^{i''}_{B\bar{B}}$ in $\Sigma_{i''}$.

If the first case arises we label the interval $[q, q']$ with the atom $\bar{L}_{i'} \cup \Psi_{i',j'} \cup B_{i',j'} \cup \overline{B}_{i',j'} \cup L_{i',j'}$, if the second one does we have by Definition 2 that there exists an atom F with $\mathcal{R}eq(F) = \bar{L}_{i'} \cup B_{i',j'} \cup \overline{B}_{i',j'} \cup L_{i',j'}$ and $\varphi \in A$ in such a case we use A for labeling the interval $[q, q']$. At the generic step i the procedure executes the first among the following operations that is active:

1. suppose that there exists an interval $[q, q']$ in Q_i for which the labeling has not yet been defined. Let $f^i_{\sigma_{B\bar{B}}}(q) = (B_0, \bar{B}_0, L_0, \bar{L}_0, \Psi_0, k_0), \ldots, (B_m, L_m, \bar{L}_m, \Psi_m, k_m)$, $f_{reg}(q) = L$ and $f_{reg}(q') = L'$ for some $\overline{L}, L' \subseteq \mathcal{C}l(\varphi)$ From the agreement property we have that there exists an index j and an atom F such that $\mathcal{R}eq_L(F) = L_j = L'$, $\mathcal{R}eq_{\overline{L}}(F) = \overline{L}$ and $\mathcal{R}eq_*(F) = *_j$ with $* \in \{B, \overline{B}\}$ then we put F as the label for the interval $[q, q']$;
2. let $[q, q'] \in Q_i$ be an interval such that q' is the successor of q in Q_i and for all the intervals $[\overline{q}, \overline{q}']$ we have that $[q, q'] \leq [\overline{q}, \overline{q}']$ or there exists $\overline{q} < \overline{q}'' < \overline{q}'$ in Q_i. Three cases may arise (i) $q' = \max Q_i$ is such a case we take a new point q'' from $\mathbb{Q}$ with $q'' > q'$ and we define $Q_{i+1} = Q_i \cup \{q''\}$ and $f_{\sigma_{B\bar{B}}}(q'') = \sigma_{B\bar{B}}$ for some $\sigma_{B\bar{B}} \in \Sigma_n$ (ii)$q = \min Q_i$ is such a case we take a new point q'' from $\mathbb{Q}$ with $q'' < q'$ and we define $Q_{i+1} = Q_i \cup \{q''\}$ and $f_{\sigma_{B\bar{B}}}(q'') = \sigma_{B\bar{B}}$ for some $\sigma_{B\bar{B}} \in \Sigma_0$ (iii) we take $q < q'' < q'$ and we put $f_{\sigma_{B\bar{B}}}(q'') = \sigma_{B\bar{B}}$ for some $\sigma_{B\bar{B}} \in \Sigma_{f_{reg}(q'')}$.

3. if there exists an interval $[q, q']$ on Q_i such that there exists a requests $\psi \in \mathcal{R}eq_B(\mathcal{L}([q, q']))$ and does not exist a point $q'' \in Q_i$ with $\psi \in \mathcal{O}bs(\mathcal{L}([q, q'']))$. Let $f_{\sigma_{B\bar{B}}}(q) = (B_0, \bar{B}_0, L_0, \bar{L}_0, \Psi_0, k_0), \ldots, (B_m, L_m, \bar{L}_m, \Psi_m, k_m)$ from the second invariant condition we have that there exists $0 \leq j \leq m$ for which $\mathcal{R}eq_*(\mathcal{L}([q, q'])) = *_j$ for each $* \in \{B, \overline{B}, L, \overline{L}\}$ then by Definition 2 we have that there exists $j' \leq j$ for which $\psi \in \Psi_{j'}$. Let $q \leq \overline{q} < \overline{q}' \leq q'$ be a pair of consecutive points such that $\mathcal{R}eq_*(\mathcal{L}([q, \overline{q}])) \subseteq *_j \subseteq \mathcal{R}eq_*(\mathcal{L}([q, \overline{q}']))$ (if q' is the immediate successor and $* \in \{B, \overline{B}, L\}$ of q in Q_i we simply take $\overline{q} = q$ and $\overline{q}' = q'$) then we take a point $\overline{q} < q'' < \overline{q}'$ in $\mathbb{Q} \setminus Q_i$ and we define the labeling $\mathcal{L}([q, q''])$ such that $\psi \in \mathcal{L}([q, q''])$ and $\mathcal{R}eq_*(\mathcal{L}([q, q''])) = B_j, \overline{B}_j, L_j, \overline{L}_{f_{reg}(q)}$. Moreover we define $f^q_{\sigma_{B\bar{B}}}(q'') = j$ $f_{\sigma_{B\bar{B}}}(q'') = \sigma'_{B\bar{B}}$ for some $\sigma_{B\bar{B}}$ in $\Sigma_{i'}$ where $(L_{i'}, \overline{L}_{i'}) = (L_j, \overline{L}_j)$;
4. if there exists an interval $[q, q']$ on Q_i such that there exists a requests $\psi \in \mathcal{R}eq_{\overline{B}}(\mathcal{L}([q, q']))$ and does not exist a point $q'' \in Q_i$ with $\psi \in \mathcal{O}bs(\mathcal{L}([q, q'']))$. In such a case we operate in a very symmetric way with respect to case 3;
5. if there exists an interval $[q, q']$ on Q_i such that there exists a requests $\psi \in \mathcal{R}eq_L(\mathcal{L}([q, q']))$ and does not exist a point $q'' \in Q_i$ with $q' < q''$ and $\psi \in \mathcal{O}bs(f_{\sigma_{B\bar{B}}}(q''))$. Let $F_{reg}(q') = i'$ by definition 4 there exists $i'' \geq i'$ and $\sigma_{B\bar{B}} \in \Sigma_{i''}$ with $\psi \in \mathcal{O}bs(\sigma_{B\bar{B}})$. Two cases may arise if $k_{i''} = \mathit{point}$ then we define $Q_{i+1} = Q_i \cup \{f_{sup}(i'')\}$ and $f_{\sigma_{B\bar{B}}}(f_{sup}(i'')) = \sigma_{B\bar{B}}$. If $k_{i''} = \mathit{cluster}$ we choose a point $q' < q'' < f_{sup}(i'')$ and we define $Q_{i+1} = Q_i \cup \{q''\}$ and $f_{\sigma_{B\bar{B}}}(q'') = \sigma_{B\bar{B}}$;
6. if there exists an interval $[q, q']$ on Q_i such that there exists a requests $\psi \in \mathcal{R}eq_{\overline{L}}(\mathcal{L}([q, q']))$ and does not exist a point $q'' \in Q_i$ with $q'' < q$ and $\psi \in \mathcal{R}eq_B(\mathcal{L}([q'', q]))$. Let $F_{reg}(q) = i'$ by definition 4 there exists $i'' \leq i'$ and $\sigma_{B\bar{B}} \in \Sigma_{i''}$ with $\psi \in \mathcal{O}bs(\sigma_{B\bar{B}})|_{L_i, \overline{L}_i}$. Let $\sigma_{B\bar{B}} = (B_0, \bar{B}_0, L_0, \bar{L}_0, \Psi_0, k_0), \ldots, (B_m, L_m, \bar{L}_m, \Psi_m, k_m)$ two cases may arise if $k_{i''} = \mathit{point}$ then we define $Q_{i+1} = Q_i \cup \{f_{sup}(i'')\}$ and $f_{\sigma_{B\bar{B}}}(f_{sup}(i'')) = \sigma_{B\bar{B}}$. If $k_{i''} = \mathit{cluster}$ we choose a point $q'' < q'' < f_{sup}(i'')$ and we define $Q_{i+1} = Q_i \cup \{q''\}$ and $f_{\sigma_{B\bar{B}}}(q'') = \sigma_{B\bar{B}}$. From definition 4 we have that there exists j such that $\psi \in B_j$ $\bar{L}_j = \mathcal{R}eq_{\bar{L}}(\mathcal{L}([q, q']))$ $(L_j, \bar{L}) = f_{reg}(q')$ for some $\bar{L}$ and $\mathcal{L}([q'', q]) = F$ s.t. $\mathcal{R}eq_*(F) = *_j$, for $* \in \{B, \overline{B}, L\}$ and $\mathcal{R}eq_{\bar{L}}(F) = \bar{L}_{i''}$.

Definition 5. *Given an $\mathsf{B\overline{B}L\overline{L}}$ formula φ and a pseudo model $P = (\sigma_{L\bar{L}}, \Sigma_0, \ldots, \Sigma_n)$ for it we say that P is minimal if and only if for each $0 \leq i \leq n$ and each $\sigma_{B\bar{B}} \in \Sigma_i$ the structure $P' = (\sigma_{L\bar{L}}, \Sigma_0, \ldots, \Sigma_i \setminus \{\sigma_{B\bar{B}}\}, \ldots, \Sigma_n)$ is not a pseudo model for φ.*

Theorem 3. *Given an $\mathsf{B\overline{B}L\overline{L}}$ formula φ and a minimal pseudo model $P = (\sigma_{L\bar{L}}, \Sigma_0, \ldots, \Sigma_n)$ for it then we have that (i) $n \leq 2 \cdot |\varphi|$, (ii) $|\bigcup_{0 \leq i \leq n} \Sigma_i| \leq 4 \cdot |\varphi|$, (iii) for each $0 \leq i \leq n$ and each $\sigma_{B\bar{B}} = \{\sigma^{i,0}_{B\bar{B}}, \ldots, \sigma^{i,h_i}_{B\bar{B}}\}$ in Σ_i we have $h_i \leq 4 \cdot |\varphi|$.*

Corollary 1. *The satisfiability problem for the logic $\mathsf{B\overline{B}L\overline{L}}$ interpreted over the class of dense linear orders is in NP.*

5 Non-Primitive Recursive Fragments

As we have mentioned, the last piece needed to complete the picture in Fig. 2 concerns the non-primitive recursive fragments. In [13] the non-primitive recursiveness of $\mathsf{A\overline{A}B}$ and $\mathsf{A\overline{AB}}$ has been proved. We shall prove here that, in actuality, every fragment that contains $\mathsf{\overline{AB}}$ or $\mathsf{\overline{A}B}$ is non-primitive recursive.

Lossy counter machines are a variant of Minsky counter automata where transitions may non-deterministically decrease the values of counters. A comprehensive survey on faulty machines and on the relevant complexity, decidability, and undecidability results can be found in [16]. Formally, a *counter automaton* is a tuple $\mathcal{A} = (Q, q_0, C, \Delta)$, where Q is a finite set of control *states*, $q_0 \in Q$ is the initial state, $C = \{c_1, \ldots, c_k\}$ is the set of *counters*, whose values range over $\mathbb{N}$, and Δ is a *transition relation*. The relation Δ is a subset of $Q \times L \times Q$, where L is the *instruction set* $L = \{inc, dec, ifz\} \times \{1, \ldots, k\}$. A *configuration* of $\mathcal{A}$ is a pair $(q, \bar{v})$, where $q \in Q$ and $\bar{v}$ is the vector of counter values. A *run* of a Minsky (i.e., with no error) counter automaton is a finite or infinite sequence of configurations such that, for every pair of consecutive configurations $(q, \bar{v}), (q', \bar{v}')$, a *transition* $(q, \bar{v}) \xrightarrow{l} (q', \bar{v}')$ has been taken (for some $(q, l, q') \in \Delta$). The value of $\bar{v}'$ is obtained from the value of $\bar{v}$ by performing instruction l, where $l = (dec, i)$ requires $v_i > 0$ and $l = (ifz, i)$ requires $v_i = 0$. In lossy machines, which is the type in which we are interested, once a faulty transition has been taken, counter values may have been decreased nondeterministically before or after the execution of the exact transition by an arbitrary natural number. We use the notation $(q, \bar{v}) \xrightarrow{l}_{\dagger} (q', \bar{v}')$ to denote that there exist $\bar{v}_{\dagger}, \bar{v}'_{\dagger}$ such that $\bar{v} \geq \bar{v}_{\dagger}$, $(q, \bar{v}_{\dagger}) \xrightarrow{l} (q', \bar{v}'_{\dagger})$, and $\bar{v}'_{\dagger} \geq \bar{v}'$, where the ordering $\leq$ is defined component-wise in the obvious way. We are interested here in the *non-termination problem* for lossy machines, defined as the problem of deciding whether $\mathcal{A}$ has at least one infinite run starting with the *initial configuration* $(q_0, \bar{0})$. This problem is non-primitive recursive [16].

Lemma 1. *There exists a reduction from the non-termination problem for lossy counter machines to the satisfiability problem for* $\mathsf{\overline{A}B}$ *over the class of all dense linear orders.*

Proof. Let $\mathcal{A} = (Q, q_0, C, \Delta)$ be a lossy counter machine. We write an $\mathsf{\overline{A}B}$-formula $\varphi_{\mathcal{A}}$ which is satisfiable over a dense linear order if and only if $\mathcal{A}$ has at least one infinite run starting with the initial configuration. The computation is encoded left-to-right over a dense domain $\mathbb{D}$, by choosing an evaluation interval $[x, y]$ that works as the "last" one, and taking into account that, given any $x_0 < x$, there are infinitely many intervals between x_0 and x. We shall make use of the propositional letters u (*units*), q_i (*states*, where i ranges from 0 to $|Q|$), *conf* (*configurations*), c_i (*counters' instances*, where i ranges from 1 to $|C|$), and *corr*, $corr_i$ (*corresponds*; i ranges from 1 to $|C|$). Counters' instances, or simply *counters*, allow us to encode the counters of $\mathcal{A}$: given a configuration where the value of the i-th counter is n, the corresponding *conf*-interval will contain precisely n c_i-intervals. (By *p-interval* we denote those intervals that satisfy p, for every propositional letter p.) Additional propositional letters will be used in the reduction for technical reasons.

Let $[G]$ (*universal modality*) be the following shortcut:

$$[G]\varphi = \varphi \wedge [B]\varphi \wedge [\overline{A}]\varphi \wedge [\overline{A}][\overline{A}]\varphi.$$

The first step in our construction consists in *discretizing* the domain, making use of a propositional variable u. In doing so, we also set the first configuration:

$$\varphi_{u-chain} = \begin{cases} \langle\overline{A}\rangle\langle\overline{A}\rangle(u \wedge conf \wedge start \wedge q_0) \wedge [\overline{A}](\langle\overline{A}\rangle u \rightarrow \langle B\rangle u) \\ [G](u \rightarrow [B]\neg u) \wedge [G](u \rightarrow [B]u_b) \wedge [G](u \rightarrow [\overline{A}]\neg u_b) \\ [G](start \rightarrow u) \wedge [G](start \rightarrow [\overline{A}](\neg u \wedge [\overline{A}]\neg u)) \end{cases}$$

Consider an interval $[x, y]$ over which the formula of our reduction is evaluated. The sense of the above formula $\varphi_{u-chain}$ is to generate an infinite discrete chain $x_0, x_1, \ldots$ such that $x_0 < x_1 < \ldots < x < y$, and that each $[x_k, x_{k+1}]$ is labeled by u. With the above formulae we also guarantee that *start* is unique and no u-interval overlaps a u-interval in the chain.

With the next formulae we make sure that there is a infinite sequence of configurations. The first one (*start*) coincides with the unit $[x_0, x_1]$, and contains the starting state q_0 only. This is consistent with our requirement that all counters start with the value 0. Moreover, we guarantee that configurations' endpoints coincide with endpoints of elements of the u-chain, that every configuration contains a state, and that *start* is unique. In our reduction, the state is placed on the last unit of every configuration.

$$\varphi_{conf-chain} = \begin{cases} [G](conf \rightarrow (u \vee \langle B\rangle u)) \wedge [G](\langle\overline{A}\rangle conf \rightarrow \langle\overline{A}\rangle u) \\ [\overline{A}](\langle\overline{A}\rangle conf \rightarrow \langle B\rangle conf) \wedge [G](conf \rightarrow [B]conf_b \wedge [B]\neg conf) \\ [G](conf \rightarrow [\overline{A}]\neg conf_b) \wedge [G](\langle\overline{A}\rangle conf \leftrightarrow \langle\overline{A}\rangle(\bigvee_{i=0,\ldots,|Q|} q_i)) \end{cases}$$

Notice that states (q_i-intervals) occur exactly as last u-intervals of configurations. Since configurations do not overlap, this implies that each configuration contains exactly one state.

Configurations also contain counters' instances c_i for each counter i whose value is greater than zero. Besides, a special placeholder c_i^+ or c_i^- may be placed in a configuration, in order to make it possible to deal with increment and decrement operations. States, counters' instances, and placeholders may only hold over units, which, in turn, all have to contain one of the above. A placeholder must be placed over the counter to which it refers. Moreover, counters and states are mutually incompatible, and there cannot be more than one per type on a given unit. These requirements are guaranteed by the following formula:

$$\varphi_{units} = \begin{cases} [G](\bigwedge_{i=0,\ldots,|Q|}(q_i \rightarrow u) \wedge \bigwedge_{i=1,\ldots,|C|}((c_i \vee c_i^+ \vee c_i^-) \rightarrow u)) \\ [G](u \rightarrow ((\bigvee_{i=0,\ldots,|Q|} q_i) \vee (\bigvee_{i=1,\ldots,|C|} c_i))) \\ [G] \bigwedge_{i=0,\ldots,|Q|}(q_i \rightarrow (\bigwedge_{j=i+1,\ldots,|Q|} \neg q_j)) \\ [G] \bigwedge_{i=0,\ldots,|Q|}(q_i \rightarrow (\bigwedge_{j=1,\ldots,|C|} \neg c_j)) \\ [G] \bigwedge_{i=1,\ldots,|C|}((c_i \rightarrow (\bigwedge_{j=i+1,\ldots,|C|} \neg c_j)) \wedge (c_i^- \rightarrow c_i) \wedge (c_i^+ \rightarrow c_i)) \end{cases}$$

Before we can actually encode the transition function Δ, we have to axiomatize the properties of $corr$ and $corr_i$ for each i. In a perfect (non-faulty) machine, when a counter is not modified by any operation from a configuration to the next one its value is preserved. Since we are encoding a lossy machine, it suffices to guarantee that no counter's value is ever incremented, except for the special case of an incrementing operation. To this end, we use the propositional letter $corr$ as a basis for correspondence, and the proposition $corr_i$ to identify the correspondence for the i-th counter:

$$\varphi_{corr} = \begin{cases} [G]\bigwedge_{i=1,\ldots,|C|}(((c_i \wedge \neg c_i^+) \rightarrow \langle\overline{A}\rangle corr_i) \wedge (c_i^+ \rightarrow \neg\langle\overline{A}\rangle corr_i)) \\ [G]\bigwedge_{i=1,\ldots,|C|}(corr_i \rightarrow corr) \\ [G]\bigwedge_{i=1,\ldots,|C|}(corr_i \rightarrow \langle\overline{A}\rangle(c_i \wedge \neg c_i^-)) \wedge [G](corr \rightarrow [B]corr_b) \\ [G](((\bigvee_{i=0,\ldots,|Q|} q_i) \wedge corr_b) \rightarrow corr_b{}^*) \\ [G]((\bigvee_{i=0,\ldots,|Q|} q_i) \rightarrow [\overline{A}](corr_b \rightarrow corr_b{}^*)) \\ [G](corr \rightarrow [B]\neg corr) \wedge [G](corr_b{}^* \rightarrow [B]\neg corr_b{}^*) \\ [G](\langle\overline{A}\rangle corr_b{}^* \rightarrow \langle\overline{A}\rangle u) \wedge [G](corr \rightarrow \langle B\rangle corr_b{}^*) \\ [G]((u \wedge \neg(\bigvee_{i=0,\ldots,|Q|} q_i)) \rightarrow [\overline{A}]\neg corr_b{}^*) \end{cases}$$

To finalize the reduction, we now take care of incrementing and decrementing operations, as well as of the zero test. For each $(q, l, q') \in \Delta$, let $conf_{(q,l,q')}$ be a special propositional letter holding on a configuration and carrying information on which transition produced that configuration. Clearly, every configuration but $start$ is the result of precisely one transition. Therefore, we have:

$$\varphi_{conf} = \begin{cases} [G]((conf \wedge \neg start) \leftrightarrow (\bigvee_{(q,l,q')\in\Delta} conf_{(q,l,q')})) \\ [G](\bigwedge_{(q,l,q')\in\Delta}(conf_{(q,l,q')} \rightarrow (\bigwedge_{(q'',l',q''')\neq(q,l,q')} \neg conf_{(q'',l',q''')}))) \end{cases}$$

We can now implement the actual transitions. To deal with the increment (resp., decrement) operation we make use of the symbol c_i^+ (resp., c_i^-), as follows:

$$\varphi_{inc} = \begin{cases} [G](\bigwedge_{(q,(inc,i),q')\in\Delta}(conf_{(q,(inc,i),q')} \rightarrow (\langle\overline{A}\rangle q \wedge \langle B\rangle c_{i,b}^+))) \\ [G](\bigwedge_{(q,(inc,i),q')\in\Delta}(\langle\overline{A}\rangle conf_{(q,(inc,i),q')} \rightarrow \langle\overline{A}\rangle q')) \\ [G](\bigwedge_{i=1,\ldots,|C|}(\langle\overline{A}\rangle c_{i,b}^+ \leftrightarrow \langle\overline{A}\rangle c_i^+)) \\ [G](\bigwedge_{i=1,\ldots,|C|}(c_{i,b}^+ \rightarrow (\langle\overline{A}\rangle conf \wedge [B]\neg conf))) \\ [G](\bigwedge_{i,j=1,\ldots,|C|}(c_{i,b}^+ \rightarrow [B]\neg c_{j,b}^+)) \\ [G](\bigwedge_{i=1,\ldots,|C|}((conf \wedge \langle B\rangle c_{i,b}^+) \rightarrow (\bigvee_{q,q'\in Q} conf_{(q,(inc,i),q')}))) \end{cases}$$

$$\varphi_{dec} = \begin{cases} [G](\bigwedge_{(q,(dec,i),q')\in\Delta}(conf_{(q,(dec,i),q')} \rightarrow (\langle\overline{A}\rangle q \wedge [\overline{A}](conf \rightarrow \langle B\rangle c_{i,b}^-)))) \\ [G](\bigwedge_{(q,(dec,i),q')\in\Delta}(\langle\overline{A}\rangle conf_{(q,(dec,i),q')} \rightarrow \langle\overline{A}\rangle q')) \\ [G](\bigwedge_{i=1,\ldots,|C|}(\langle\overline{A}\rangle c_{i,b}^- \leftrightarrow \langle\overline{A}\rangle c_i^-)) \\ [G](\bigwedge_{i=1,\ldots,|C|}(c_{i,b}^- \rightarrow (\langle\overline{A}\rangle conf \wedge [B]\neg conf))) \\ [G](\bigwedge_{i,j=1,\ldots,|C|}(c_{i,b}^- \rightarrow [B]\neg c_{j,b}^-)) \\ [G](\bigwedge_{i=1,\ldots,|C|}((conf \wedge \langle\overline{A}\rangle\langle B\rangle c_{i,b}^-) \rightarrow (\bigvee_{q,q'\in Q} conf_{(q,(dec,i),q')}))) \end{cases}$$

$$\varphi_{ifz} = \begin{cases} [G](\bigwedge_{(q,(ifz,i),q')\in\Delta}(conf_{(q,(ifz,i),q')} \rightarrow (\langle\overline{A}\rangle q \wedge [\overline{A}](conf \rightarrow [B]c_{i,b}^z)))) \\ [G](\bigwedge_{(q,(ifz,i),q')\in\Delta}(\langle\overline{A}\rangle conf_{(q,(ifz,i),q')} \rightarrow \langle\overline{A}\rangle q')) \\ [G](\bigwedge_{i=1,\ldots,|C|}((\langle\overline{A}\rangle c_i \rightarrow [\overline{A}]c_{i,b}^z) \wedge (\neg\langle\overline{A}\rangle c_i \rightarrow [\overline{A}]\neg c_{i,b}^z))) \end{cases}$$

The formula:

$$\varphi_{u-chain} \wedge \varphi_{conf-chain} \wedge \varphi_{units} \wedge \varphi_{corr} \wedge \varphi_{conf} \wedge \varphi_{inc} \wedge \varphi_{dec} \wedge \varphi_{ifz}$$

is satisfiable if and only if $\mathcal{A}$ has at least one infinite run. $\square$

Since it is possible to construct a similar reduction using the fragment $\overline{\mathsf{AB}}$, we can conclude the following theorem.

Theorem 4. *The complexity of the satisfiability problem for the fragments* $\overline{\mathsf{AB}}$ *and* $\overline{\mathsf{A}}\mathsf{B}$ *over the class of dense linear orders is non-primitive recursive.*

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