On the expressiveness of the interval logic of Allen's relations over finite and discrete linear orders*

L. Aceto[1,2], D. Della Monica[1], A. Ingólfsdóttir[1], A. Montanari[3], and G. Sciavicco[4]

[1] *ICE-TCS, School of Computer Science, Reykjavik University, Iceland*
email: {luca | dariodm | annai}@ru.is
[2] *Gran Sasso Science Institute, INFN, L'Aquila, Italy*
[3] *Dept. of Mathematics and Computer Science, University of Udine, Italy*
email: angelo.montanari@uniud.it
[4] *Dept. of Information Engineering and Communications, University of Murcia, Spain*
email: guido@um.es

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Abstract

Interval temporal logics take time intervals, instead of time instants, as their primitive temporal entities. One of the most studied interval temporal logics is Halpern and Shoham's modal logic of time intervals HS, which associates a modal operator with each binary relation between intervals over a linear order (the so-called Allen's interval relations). A complete classification of all HS fragments with respect to their relative expressive power has been recently given for the classes of all linear orders and of all dense linear orders. The cases of discrete and finite linear orders turn out to be much more involved. In this paper, we make a significant step towards solving the classification problem over those classes of linear orders. First, we illustrate various non-trivial temporal properties that can be expressed by HS fragments when interpreted over finite and discrete linear orders; then, we provide a complete set of definabilities for the HS modalities corresponding to the Allen's relations *meets*, *later*, *begins*, *finishes*, and *during*, as well as the ones corresponding to their inverse relations. Given the results presented here, the only missing piece of the expressiveness puzzle is that of the definabilities for the modality corresponding to the Allen relation *overlaps* (those for the inverse relation *overlapped by* would immediately follow by symmetry).

1 Introduction

Interval reasoning naturally arises in various fields of computer science and artificial intelligence, ranging from hardware and real-time system verification to natural language

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processing, from constraint satisfaction to planning [4, 5, 14, 22, 23, 25]. Interval temporal logics make it possible to reason about interval structures over linearly ordered domains, where time intervals, rather than time instants, are the primitive ontological entities. The distinctive features of interval temporal logics turn out to be useful in various application domains [8, 11, 21, 22, 25]. For instance, they allow one to model *telic statements*, that is, statements that express goals or accomplishments, e.g., the statement: 'The airplane flew from Venice to Toronto' [21]. Moreover, when we restrict ourselves to discrete linear orders, such as, for instance, $\mathbb{N}$ or $\mathbb{Z}$, some interval temporal logics are expressive enough to constrain the length of intervals, thus allowing one to specify safety properties involving quantitative conditions [21]. This is the case, for instance, with the well-known 'gas-burner' example [25]. Temporal logics with interval-based semantics have also been proposed as suitable formalisms for the specification and verification of hardware [22] and of real-time systems [25].

The variety of binary relations between intervals in a linear order was first studied by Allen [4], who investigated their use in systems for time management and planning. In [16], Halpern and Shoham introduced and systematically analyzed the (full) logic of Allen's relations, called HS in this paper, that features one modality for each Allen relation. In particular, they showed that HS is highly undecidable over most classes of linear orders. This result motivated the search for (syntactic) HS fragments offering a good balance between expressiveness and decidability/complexity [6, 7, 9, 10, 12, 18, 20, 21]. A comparative analysis of the expressive power of HS fragments is far from being trivial, because some HS modalities are definable in terms of others, and thus syntactically different fragments may turn out to be equally expressive. Moreover, the definability of a specific modality in terms of other ones depends, in general, on the class of linear orders over which the logic is interpreted, and the classification of the relative expressive power of HS fragments with respect to a given class of linear orders cannot be directly transferred to another class. More precisely, while definabilities do transfer from a class $\mathcal{C}$ to all its proper sub-classes, there might be new definability relations that hold in some sub-class of $\mathcal{C}$, but not in $\mathcal{C}$ itself. Conversely, undefinabilities do transfer from a class to all its proper super-classes, but not vice versa. Proving a specific undefinability result amounts to providing a counterexample based on concrete linear orders from the considered class. As a matter of fact, different assumptions on the underlying linear orders give rise, in general, to different sets of definabilities [2, 13].

Contribution. Many classes of linear orders are of practical interest, including the class of all (resp., dense, discrete, finite) linear orders, as well as the particular linear order on $\mathbb{R}$ (resp., $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$). A precise characterization of the expressive power of all HS fragments with respect to the class of all linear orders and that of all dense linear orders has been given in [13] and [2], respectively. The classification of HS fragments over the classes of discrete and finite linear orders presents a number of convoluted technical difficulties. In [12], the authors focus on strongly discrete linear orders, by characterizing and classifying all *decidable* fragments of HS with respect to both complexity of the satisfiability problem and relative expressive power. In this paper, we make a significant step towards a complete classification of the expressiveness of all (*decidable* and *undecidable*) fragments of HS over finite and discrete linear orders, and in doing so we considerably extend the expressiveness results presented in [12]. As a matter of fact, given the present contributions, the only missing piece of the expressiveness puzzle is that of the definabilities for the modality corresponding to the Allen relation *overlaps* (those for the inverse relation *overlapped by* would immediately fol-

HS **modalities**	**Allen's relations**
$\langle A \rangle$	$[x,y]R_A[x',y'] \Leftrightarrow y = x'$
$\langle L \rangle$	$[x,y]R_L[x',y'] \Leftrightarrow y < x'$
$\langle B \rangle$	$[x,y]R_B[x',y'] \Leftrightarrow x = x', y' < y$
$\langle E \rangle$	$[x,y]R_E[x',y'] \Leftrightarrow y = y', x < x'$
$\langle D \rangle$	$[x,y]R_D[x',y'] \Leftrightarrow x < x', y' < y$
$\langle O \rangle$	$[x,y]R_O[x',y'] \Leftrightarrow x < x' < y < y'$

Graphical representation

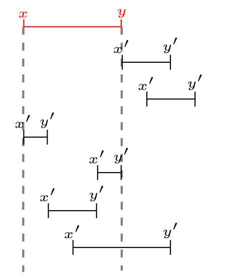

Figure 1: Allen's interval relations and the corresponding HS modalities.

low by symmetry).

Structure of the paper. In the next section, we introduce the logic HS. Then, in Section 3, we introduce the notion of definability of a modality in an HS fragment, and we present the main tool we use to prove our results. In order to provide the reader with an idea of the expressive power of HS modalities, we also illustrate some meaningful temporal properties, like counting and boundedness properties, which can be expressed in HS fragments when interpreted over discrete linear orders. Then, as a warm-up, in Section 4 we present a first, simple expressiveness result, by providing the complete set of definabilities for the HS modalities $\langle A \rangle$, $\langle L \rangle$, $\langle \overline{A} \rangle$, and $\langle \overline{L} \rangle$, corresponding to Allen's relations *meets* and *later*, and their inverses *met by* and *before*, respectively. Section 5 contains our main technical result, that is, a complete set of definabilities for the HS modalities $\langle D \rangle$, $\langle E \rangle$, $\langle B \rangle$, $\langle \overline{D} \rangle$, $\langle \overline{E} \rangle$, and $\langle \overline{B} \rangle$, corresponding to Allen's relations *during*, *finishes*, and *begins*, and their inverses *contains*, *finished by*, and *begun by*, respectively. The proofs of the results in this section are rather difficult and much more technically involved than the ones in Section 4. We conclude the paper with some final remarks.

2 Preliminaries

Let $\mathbb{D} = \langle D, < \rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[a, b]$, where $a, b \in D$ and $a \leq b$. An interval is called a *point interval* if $a = b$ and a *strict interval* if $a < b$. In this paper, we assume the *strict semantics*, that is, we exclude point intervals and only consider strict intervals. The adoption of the strict semantics, excluding point intervals, instead of the *non-strict semantics*, which includes them, conforms to the definition of interval adopted by Allen in [4], but differs from the one given by Halpern and Shoham in [16]. It has at least two strong motivations: first, a number of representation paradoxes arise when the non-strict semantics is adopted, due to the presence of point intervals, as pointed out in [4]; second, when point intervals are included, there seems to be no intuitive semantics for interval relations that makes them both pairwise disjoint and jointly exhaustive. If we exclude the identity relation, there are 12 different relations between two strict intervals in a linear order, often called *Allen's relations* [4]: the six relations R_A (adjacent to), R_L (later than), R_B (begins), R_E (ends), R_D (during), and R_O (overlaps), depicted in Figure 1, and their inverses, that is, $R_{\overline{X}} = (R_X)^{-1}$, for each $X \in \{A, L, B, E, D, O\}$.

We interpret interval structures as Kripke structures, with Allen's relations playing the role of the accessibility relations. Thus, we associate a modality $\langle X \rangle$ with each Allen relation R_X. For each $X \in \{A, L, B, E, D, O\}$, the *transpose* of modality $\langle X \rangle$ is modality $\langle \overline{X} \rangle$, corresponding to the inverse relation $R_{\overline{X}}$ of R_X. Halpern and Shoham's logic HS [16] is a multi-modal logic with formulae built from a finite, non-empty set $\mathcal{AP}$ of atomic propositions (also referred to as proposition letters), the propositional connectives $\vee$ and $\neg$, and a modality for each Allen relation. With every subset $\{R_{X_1}, \ldots, R_{X_k}\}$ of these relations, we associate the fragment $\mathsf{X_1X_2 \ldots X_k}$ of HS, whose formulae are defined by the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X_1 \rangle\varphi \mid \ldots \mid \langle X_k \rangle\varphi,$$

where $p \in \mathcal{AP}$. The other propositional connectives and constants (e.g., $\wedge$, $\rightarrow$, and $\top$), as well as the dual modalities (e.g., $[A]\varphi \equiv \neg\langle A \rangle\neg\varphi$), can be derived in the standard way. We define the *modal depth* of a formula as the largest nesting of modal operators in it. For a fragment $\mathcal{F} = \mathsf{X_1X_2 \ldots X_k}$ and a modality $\langle X \rangle$, we write $\langle X \rangle \in \mathcal{F}$ if $X \in \{X_1, \ldots, X_k\}$. Given two fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if $\langle X \rangle \in \mathcal{F}_1$ implies $\langle X \rangle \in \mathcal{F}_2$, for every modality $\langle X \rangle$. Finally, for a fragment $\mathcal{F} = \mathsf{X_1X_2 \ldots X_k}$ and a formula φ, we write $\varphi \in \mathcal{F}$ or, equivalently, we say that φ is an $\mathcal{F}$-formula, meaning that φ belongs to the language of $\mathcal{F}$.

The (strict) semantics of HS is given in terms of *interval models* $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$, where $\mathbb{D}$ is a linear order, $\mathbb{I}(\mathbb{D})$ is the set of all (strict) intervals over $\mathbb{D}$, and V is a *valuation function* $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$, which assigns to each atomic proposition $p \in \mathcal{AP}$ the set of intervals $V(p)$ on which p holds. The *truth* of a formula on a given interval $[x, y]$ in an interval model M is defined by structural induction on formulae as follows:

- $M, [x, y] \Vdash p$ if and only if $[x, y] \in V(p)$, for each $p \in \mathcal{AP}$;
- $M, [x, y] \Vdash \neg\psi$ if and only if it is not the case that $M, [x, y] \Vdash \psi$;
- $M, [x, y] \Vdash \varphi \vee \psi$ if and only if $M, [x, y] \Vdash \varphi$ or $M, [x, y] \Vdash \psi$;
- $M, [x, y] \Vdash \langle X \rangle\psi$ if and only if there exists $[x', y']$ such that $[x, y]R_X[x', y']$ and $M, [x', y'] \Vdash \psi$, for each modality $\langle X \rangle$.

Formulae of HS can be interpreted over a class of interval models (built on a given class of linear orders). Among others, we mention the following classes of (interval models built on important classes of) linear orders: *(i)* the class of *all* linear orders Lin; *(ii)* the class of (all) *dense* linear orders Den, that is, those in which for every pair of distinct points there exists at least one point in between them (e.g., $\mathbb{Q}$ and $\mathbb{R}$); *(iii)* the class of (all) *discrete* linear orders Dis, that is, those in which every element, apart from the greatest element, if it exists, has an immediate successor, and every element, other than the least element, if it exists, has an immediate predecessor (e.g., $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z} + \mathbb{Z}$); *(iv)* the class of (all) *finite* linear orders Fin, that is, those having only finitely many points. A formula ϕ of HS is *valid* over a class $\mathcal{C}$ of linear orders, denoted by $\Vdash_{\mathcal{C}} \phi$, if it is true on every interval in every interval model belonging to $\mathcal{C}$. Two formulae ϕ and ψ are *equivalent* relative to the class $\mathcal{C}$ of linear orders, denoted by $\phi \equiv_{\mathcal{C}} \psi$, if $\Vdash_{\mathcal{C}} \phi \leftrightarrow \psi$.

3 Definability and expressivenesss

Definition 1 (Definability) *A modality $\langle X \rangle$ of* HS *is* definable *in an* HS *fragment $\mathcal{F}$ relative to a class $\mathcal{C}$ of linear orders, denoted $\langle X \rangle \lhd_{\mathcal{C}} \mathcal{F}$, if $\langle X \rangle p \equiv_{\mathcal{C}} \psi$ for some $\mathcal{F}$-formula ψ over the atomic proposition p, for any $p \in \mathcal{AP}$. Then, the equivalence $\langle X \rangle p \equiv_{\mathcal{C}} \psi$ is called a* definability equation *for $\langle X \rangle$ in $\mathcal{F}$ relative to $\mathcal{C}$. We write $\langle X \rangle \ntriangleleft_{\mathcal{C}} \mathcal{F}$ if it is not the case that $\langle X \rangle \lhd_{\mathcal{C}} \mathcal{F}$.*

As we have already noted, smaller classes of linear orders inherit the definabilities holding for larger classes: if $\mathcal{C}_1$ and $\mathcal{C}_2$ are classes of linear orders such that $\mathcal{C}_1 \subset \mathcal{C}_2$, then all definabilities holding for $\mathcal{C}_2$ are also valid for $\mathcal{C}_1$. However, more definabilities can possibly hold for $\mathcal{C}_1$. On the other hand, undefinability results for $\mathcal{C}_1$ hold also for $\mathcal{C}_2$. In the rest of the paper, we omit the class of linear orders when it is clear from the context (e.g., we will simply write $\langle X\rangle p \equiv \psi$ and $\langle X\rangle \lhd \mathcal{F}$ for $\langle X\rangle p \equiv_{\mathcal{C}} \psi$ and $\langle X\rangle \lhd_{\mathcal{C}} \mathcal{F}$, respectively).

It is known from [16] that, when the strict semantics is assumed, all HS modalities are definable in the fragment containing modalities $\langle A\rangle$, $\langle B\rangle$, and $\langle E\rangle$, and their transposes $\langle\overline{A}\rangle$, $\langle\overline{B}\rangle$, and $\langle\overline{E}\rangle$, while in the non-strict semantics, the four modalities $\langle B\rangle$, $\langle E\rangle$, $\langle\overline{B}\rangle$, and $\langle\overline{E}\rangle$ suffice, as shown in [24]. Given two HS fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we say that $\mathcal{F}_2$ is *at least as expressive as* $\mathcal{F}_1$, denoted $\mathcal{F}_1 \preceq \mathcal{F}_2$, if each operator $\langle X\rangle \in \mathcal{F}_1$ is definable in $\mathcal{F}_2$, and that $\mathcal{F}_1$ is *strictly less expressive* than $\mathcal{F}_2$, denoted $\mathcal{F}_1 \prec \mathcal{F}_2$, if $\mathcal{F}_1 \preceq \mathcal{F}_2$ holds but $\mathcal{F}_2 \preceq \mathcal{F}_1$ does not. The notions of *expressively equivalent* fragments and *expressively incomparable* fragments can be defined likewise.

Definition 2 (Optimal definability) *A definability $\langle X\rangle \lhd \mathcal{F}$ is* optimal *if $\langle X\rangle \ntriangleleft \mathcal{F}'$ for each fragment $\mathcal{F}'$ such that $\mathcal{F}' \prec \mathcal{F}$.*

3.1 Proof techniques to disprove definability

In order to show non-definability of a given modality in a certain fragment, we use the standard notion of *N-bisimulation* [15, 17, 19], suitably adapted to our setting.

Definition 3 *Let $\mathcal{F}$ be an* HS*-fragment. An $\mathcal{F}_N$-bisimulation between two models $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}'), V'\rangle$ over a set of proposition letters $\mathcal{AP}$ is a sequence of N relations $Z_N, \ldots, Z_1 \subseteq \mathbb{I}(\mathbb{D})\times\mathbb{I}(\mathbb{D}')$ such that: (i) for every $([x,y],[x',y']) \in Z_h$, with $N \geq h \geq 1$, $M,[x,y] \Vdash p$ if and only if $M',[x',y'] \Vdash p$, for all $p \in \mathcal{AP}$* (local condition)*; (ii) for every $([x,y],[x',y']) \in Z_h$, with $N \geq h > 1$, if $[x,y]R_X[v,w]$ for some $[v,w] \in \mathbb{I}(\mathbb{D})$ and some $\langle X\rangle \in \mathcal{F}$, then there exists $([v,w],[v',w']) \in Z_{h-1}$ such that $[x',y']R_X[v',w']$* (forward condition)*; (iii) for every $([x,y],[x',y']) \in Z_h$, with $N \geq h > 1$, if $[x',y']R_X[v',w']$ for some $[v',w'] \in \mathbb{I}(\mathbb{D}')$ and some $\langle X\rangle \in \mathcal{F}$, then there exists $([v,w],[v',w']) \in Z_{h-1}$ such that $[x,y]R_X[v,w]$* (backward condition).

Given an $\mathcal{F}_N$-bisimulation, the truth of $\mathcal{F}$-formulae of modal depth at most $h-1$ is invariant for pairs of intervals belonging to Z_h, with $N \geq h \geq 1$ (see, e.g., [15]). Thus, to prove that a modality $\langle X\rangle$ is not definable in $\mathcal{F}$, it suffices to provide, for every natural number N, a pair of models M and M', and an $\mathcal{F}_N$-bisimulation between them for which there exists a pair $([x,y],[x',y']) \in Z_N$ such that $M,[x,y] \Vdash \langle X\rangle p$ and $M',[x',y'] \Vdash \neg\langle X\rangle p$, for some $p \in \mathcal{AP}$ (in this case, we say that the $\mathcal{F}_N$-bisimulation *violates* $\langle X\rangle$). To convince oneself that this is enough to ensure that $\langle X\rangle$ is not definable by any $\mathcal{F}$-formula of any modal depth, assume, towards a contradiction, that ϕ is an $\mathcal{F}$-formula of modal depth n such that $\langle X\rangle p \equiv \phi$. Since, for each N, there is an $\mathcal{F}_N$-bisimulation that violates $\langle X\rangle$, there exists, in particular, one such bisimulation for $N = n+1$. Let $([x,y],[x',y']) \in Z_N$ be the pair of intervals that *violates* $\langle X\rangle$, that is, $M,[x,y] \Vdash \langle X\rangle p$ and $M',[x',y'] \Vdash \neg\langle X\rangle p$. Then, the truth value of ϕ over $[x,y]$ (in M) and $[x',y']$ (in M') is the same, and this is in contradiction with the fact that $M,[x,y] \Vdash \langle X\rangle p$ and $M',[x',y'] \Vdash \neg\langle X\rangle p$. A result obtained following this argument applies to all classes of linear orders that contain (as their elements) both structures on which M and M' are based. Notice that, in some cases, it is convenient to define $\mathcal{F}_N$-bisimulations between a model M and itself.

It is worth pointing out that the standard notion of $\mathcal{F}$-*bisimulation* can be recovered as a special case of $\mathcal{F}_N$-bisimulation. Formally, an $\mathcal{F}$-bisimulation can be thought of as an $\mathcal{F}_N$-bisimulation with $N = 2$ and $Z_1 = Z_2$. In the following, as is customary, we will treat $\mathcal{F}$-bisimulations as relations instead of sequences of two equal relations: if the sequence Z_2, Z_1 is an $\mathcal{F}$-bisimulation, with $Z_1 = Z_2 = Z$, then we will simply refer to it as to the relation Z. It is important to notice that showing that two intervals are related by an $\mathcal{F}$-bisimulation (i.e., they are $\mathcal{F}$-*bisimilar*) is stronger than showing that they are related by a relation Z_N, which belongs to a sequence $Z_N, \ldots, Z_1$ corresponding to an $\mathcal{F}_N$-bisimulation (i.e., the intervals are $\mathcal{F}_N$-*bisimilar*). Indeed, while in the latter case we are only guaranteed invariance of $\mathcal{F}$-formulae of modal depth at most $N - 1$, in the former case the truth of $\mathcal{F}$-formulae of any (possibly unbounded) modal depth is preserved. This means that undefinability results obtained using $\mathcal{F}$-bisimulations are not restricted to the finitary logics we consider in this paper, but also apply to extensions with infinite disjunctions and with fixed-point operators.

Since $\mathcal{F}$-bisimulations are notationally easier to deal with than $\mathcal{F}_N$-bisimulations, it is in principle more convenient to use the former, rather than the latter, when proving an undefinability result. However, while in few cases (see Section 4) a proof based on $\mathcal{F}$-bisimulations is possible, this is not generally the case, because some modalities that cannot be defined in fragments of HS can be expressed in their infinitary versions. In those cases (see Section 5), we resort to a proof via $\mathcal{F}_N$-bisimulations.

For a given modality $\langle X \rangle$ and a given class $\mathcal{C}$ of linear orders, we shall identify a set of definabilities for $\langle X \rangle$, and we shall prove its *soundness*, by shoving that each definability equation is valid in $\mathcal{C}$, and its *completeness*, by arguing that each definability is optimal and that there are no other optimal definabilities for $\langle X \rangle$ in $\mathcal{C}$. Completeness is proved by computing all maximal fragments $\mathcal{F}$ that cannot define $\langle X \rangle$ (in the attempt of defining $\langle X \rangle$ in $\mathcal{F}$, we can obviously make use of the set of known definabilities). For each modality, such fragments are listed in the last column of Figure 2. Depending on the number of known definabilities, such a task can be time-consuming and error-prone, so an automated procedure has been devised and implemented in [1] to serve the purpose. Then, for each such $\mathcal{F}$ and each $N \in \mathbb{N}$, we provide an $\mathcal{F}_N$-bisimulation that violates $\langle X \rangle$. Notice that all the classes of linear orders we consider in this paper are (left/right) *symmetric*, namely, if a class $\mathcal{C}$ contains a linear order $\mathbb{D} = \langle D, \prec \rangle$, then it also contains (a linear order isomorphic to) its dual linear order $\mathbb{D}^d = \langle D, \succ \rangle$, where $\succ$ is the inverse of $\prec$. This implies that the definabilities for $\langle \overline{L} \rangle$, $\langle \overline{A} \rangle$, $\langle B \rangle$, and $\langle \overline{B} \rangle$ can be immediately deduced (and shown to be sound and optimal) from those for $\langle L \rangle$, $\langle A \rangle$, $\langle E \rangle$, and $\langle \overline{E} \rangle$, respectively.

Figure 2 depicts the complete sets of optimal definabilities holding in Dis and Fin for the modalities $\langle L \rangle$, $\langle A \rangle$, $\langle D \rangle$, $\langle \overline{D} \rangle$, $\langle E \rangle$, and $\langle \overline{E} \rangle$ (recall that those for $\langle \overline{L} \rangle$, $\langle \overline{A} \rangle$, $\langle B \rangle$, and $\langle \overline{B} \rangle$ follow by symmetry). Section 4 and Section 5 are devoted to proving completeness of such sets. For all the modalities, but $\langle A \rangle$ and $\langle \overline{A} \rangle$, soundness is an immediate consequence of the corresponding soundness in Lin, shown in [13]. For lack of space, we omit the proofs of the soundness of the definabilities for $\langle A \rangle$ and $\langle \overline{A} \rangle$, which anyway are quite straightforward. Finally, while it is known from [16] that $\langle O \rangle \lhd \overline{\mathsf{B}}\mathsf{E}$ (resp., $\langle \overline{O} \rangle \lhd \mathsf{B}\overline{\mathsf{E}}$), it is still an open problem whether this is the only optimal definability for $\langle O \rangle$ (resp., $\langle \overline{O} \rangle$) in Dis and in Fin.

3.2 Expressing properties of a model in HS fragments

We give here a short account of meaningful temporal properties, such as counting and (un)boundedness ones, which can be expressed in HS fragments, when they are inter-

Modalities	Equations	Definabilities	Maximal fragments not defining it
$\langle L\rangle$	$\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$	$\langle L\rangle \lhd \mathsf{A}$	$\mathsf{BDO}\overline{\mathsf{ALBEDO}}$ $\mathsf{BEDO}\overline{\mathsf{ALEDO}}$
$\langle A\rangle$	$\langle A\rangle p \equiv \varphi(p) \vee \langle E\rangle\varphi(p)^*$	$\langle A\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$	$\mathsf{LBDO}\overline{\mathsf{ALBEDO}}$ $\mathsf{LBEDO}\overline{\mathsf{ALEDO}}$
	$^*\varphi(p) := [E]\bot \wedge \langle\overline{B}\rangle([E][E]\bot \wedge\langle E\rangle(p \vee \langle\overline{B}\rangle p))$		
$\langle D\rangle$	$\langle D\rangle p \equiv \langle B\rangle\langle E\rangle p$	$\langle D\rangle \lhd \mathsf{BE}$	$\mathsf{ALBO}\overline{\mathsf{ALBEDO}}$ $\mathsf{ALEO}\overline{\mathsf{ALBEDO}}$
$\langle\overline{D}\rangle$	$\langle\overline{D}\rangle p \equiv \langle\overline{B}\rangle\langle\overline{E}\rangle p$	$\langle D\rangle \lhd \overline{\mathsf{BE}}$	$\mathsf{ALBEDO}\overline{\mathsf{ALBO}}$ $\mathsf{ALBEDO}\overline{\mathsf{ALEO}}$
$\langle E\rangle$	no definabilities		$\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}$
$\langle\overline{E}\rangle$	no definabilities		$\mathsf{ALBEDO}\overline{\mathsf{ALBDO}}$

Figure 2: Optimal definabilities in Dis and Fin. The last column contains the maximal fragments not defining the modality under consideration.

preted over discrete linear orders. The outcomes of such an analysis are summarized in Figure 3 (other properties can obviously be expressed as Boolean combinations of those displayed). They demonstrate the expressiveness capabilities of HS modalities, which are of interest by themselves. As an example, the ability of constraining the length of intervals is a desirable feature of any formalism for representing and reasoning about temporal knowledge over a discrete domain. As a matter of fact, most HS fragments have many chances to succeed in practical applications, and thus it is definitely worth carrying out a taxonomic study of their expressiveness. As we alreay pointed out, such a study presents various intricacies. For instance, in some fragments, assuming the discreteness of the linear order suffices to constrain the lenght of intervals (this is the case with the fragment E); other fragments rely on additional assumptions (this is the case with the fragment DO, which requires the linear order to be right-unbounded). This gives evidence of how expressiveness results can be affected by the specific class of linear orders under consideration.

Counting properties. When the linear order is assumed to be discrete, some HS fragments are powerful enough to constrain (to some extent) the *length* of an interval, that is, the number of its points minus one. Let $\sim \in \{<, \leq, =, \geq, >\}$. For every $k \in \mathbb{N}$, we define $\ell_{\sim k}$ as a (pre-interpreted) atomic proposition which is true over all and only those intervals whose length is $\sim$-related to k. Moreover, for a modality $\langle X\rangle$, we denote by $\langle X\rangle^k\varphi$ the formula $\langle X\rangle \ldots \langle X\rangle\varphi$, with k occurrences of $\langle X\rangle$ before φ. Limiting ourselves to a few examples, we highlight here the ability of some of the HS modalities to express $\ell_{\sim k}$, for any k. It is well known that the fragments E and B can express $\ell_{\sim k}$, for every k and $\sim$ (see, e.g., [16]). As an example, the formulae $\langle E\rangle^k\top$ and $[E]^k\bot$ are equivalent to $\ell_{>k}$ and $\ell_{\leq k}$, respectively. The fragment D features limited counting properties, as, for every k, $\langle D\rangle^k\top \wedge [D]^{k+1}\bot$ is true over intervals whose length is either $2 \cdot k + 1$ or $2 \cdot (k + 1)$ (notice that, as a particular instance, $[D]\bot$ is true over intervals whose length is either 1 or 2). In a sense, it is not able to discriminate the parity of an interval. The counting capabilities of the fragment O are limited as well: it allows one to discriminate between *unit intervals* (intervals whose length is 1) and *non-unit intervals* (which are longer than 1), provided that the underlying linear order is right-unbounded, like $\mathbb{Z}$ or $\mathbb{N}$ ($\langle\overline{O}\rangle$ possesses the same capability, provided that the underlying linear order is left-unbounded, like $\mathbb{Z}$ or $\mathbb{Z}^-$). However, quite interestingly, by pairing $\langle D\rangle$ and $\langle O\rangle$, or, symmetrically, $\langle D\rangle$ and $\langle\overline{O}\rangle$, it is possible to express $\ell_{\sim k}$ for every k and $\sim$ over right-unbounded linear order (left-unbounded linear orders if

Counting properties		Right Unboundedness ($\exists_r$)	
$\ell_{>k}$	$\equiv \langle E\rangle^k\top$		$\langle\overline{B}\rangle\top$, $\langle A\rangle\top$
$\ell_{=k}$	$\equiv \langle E\rangle^{k-1}\top \wedge [E]^k\bot$	($\ddagger$)	$\langle O\rangle\top$, $[B]\langle L\rangle\top$
$\ell_{>2\cdot k}$	$\equiv \langle D\rangle^k\top$	($\S$)	$\langle\overline{D}\rangle\top$, $\langle\overline{E}\rangle\langle O\rangle\top$
$\ell_{\leq 2\cdot k}$	$\equiv [D]^k\bot$	($\ddagger,\S$)	$[\overline{O}]\langle L\rangle\top$
$\ell_{>1}$	$\equiv^\dagger \langle O\rangle\top$	($\flat$)	$[D]\langle L\rangle\top$
$\ell_{>2\cdot k+1}$	$\equiv^\dagger \langle D\rangle^k\langle O\rangle\top$		
$\ell_{=2\cdot(k+1)}$	$\equiv^\dagger \langle D\rangle^k\langle O\rangle\top \wedge [D]^{k+1}\bot$		

†: only on right-unbounded domains; ‡: only on intervals longer than 1;
§: only on left-unbounded domains; $^\flat$: only on intervals longer than 2.

Figure 3: Expressiveness of HS modalities over discrete linear orders.

$\langle O\rangle$ is replaced by $\langle\overline{O}\rangle$): it suffices to first use $\langle D\rangle$ to narrow the length down to k or $k+1$, and then $\langle O\rangle$ (or $\langle\overline{O}\rangle$) to discriminate the parity.

(Un)boundedness properties. Let us denote by $\exists_r$ (resp., $\exists_l$) a (pre-interpreted) atomic proposition that is true over all and only the intervals that have a point to their right (resp., left). Various combinations of HS operators can express $\exists_r$. Once again, while in some cases we need to assume only the discreteness of the underlying linear order, there are cases where the validity of the definability relies on additional assumptions. For example, to impose that the current interval has a point to the right within the fragment O, we can use $\langle O\rangle\top$ only on non-unit intervals (otherwise, $\langle O\rangle$ has no effect). Analogously, it is possible to express $\exists_l$, possibly under analogous assumptions.

4 The Easy Cases

In this section, we prove the completeness of the set of definabilities for the modalities $\langle L\rangle$, $\langle\overline{L}\rangle$, $\langle A\rangle$, and $\langle\overline{A}\rangle$, thus strengthening a similar result presented in [12, Theorem 1].

Theorem 1 *The sets of optimal definabilities for $\langle L\rangle$ and $\langle A\rangle$ (listed in Figure 2), as well as (by symmetry) those for $\langle\overline{L}\rangle$ and $\langle\overline{A}\rangle$, are complete for the classes* Dis *and* Fin.

Proof. The results for $\langle L\rangle$ (and, symmetrically, for $\langle\overline{L}\rangle$) immediately follows from [13], as the completeness proof for $\langle L\rangle$ presented there used a bisimulation between models based on finite linear orders. Notice that $\langle L\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$ holds in Dis and Fin, as it does in Lin. However, such a definability, which is optimal in Lin, is not optimal in Dis and Fin (and thus it is not listed in Figure 2), due to the fact that $\langle A\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$ (which is not a sound definability in Lin) holds over Dis. As a pleasing consequence, we can extend Venema's result from [24] concerning the expressive completeness of the fragment $\mathsf{BE}\overline{\mathsf{BE}}$ in the non-strict semantics to the strict one under the discreteness assumption.

According to Figure 2, $\langle A\rangle$ is definable in terms of $\overline{\mathsf{B}}\mathsf{E}$, implying that the maximal fragments not defining $\langle A\rangle$ are, as shown in the last column of Figure 2, $\mathsf{LBDO}\overline{\mathsf{ALBEDO}}$ and $\mathsf{LBEDO}\overline{\mathsf{ALEDO}}$. Thus, proving that $\langle A\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$ is the only optimal definability amounts to providing two bisimulations, namely an $\mathsf{LBDO}\overline{\mathsf{ALBEDO}}$- and an $\mathsf{LBEDO}\overline{\mathsf{ALEDO}}$-bisimulation that violate $\langle A\rangle$. As for the first one, we consider two models M and M', both based on the finite linear order $\{0,1,2\}$. We set $V(p)=\{[1,2]\}$, $V'(p)=\emptyset$, and $Z=\{([0,1],[0,1]),([0,2],[0,2])\}$. It is easy to verify that Z is an $\mathsf{LBDO}\overline{\mathsf{ALBEDO}}$-bisimulation that violates $\langle A\rangle$, as $M,[0,1] \Vdash \langle A\rangle p$ and $M',[0,1] \Vdash \neg\langle A\rangle p$. As for the second one, models and valuations are defined as before, but we take now $Z = \{([0,1],[0,1])\}$. Once again, it is easy to see that Z is an $\mathsf{LBEDO}\overline{\mathsf{ALEDO}}$-bisimulation that violates $\langle A\rangle$, as $M,[0,1] \Vdash \langle A\rangle p$ and $M',[0,1] \Vdash \neg\langle A\rangle p$. Since the result is based on a finite linear order, it holds for both Dis and Fin. ■

5 The hard cases

In this section, we provide the completeness result for the modalities $\langle D\rangle$ and $\langle\overline{D}\rangle$ (Theorem 2), as well as for $\langle E\rangle$, $\langle\overline{E}\rangle$, $\langle B\rangle$, and $\langle\overline{B}\rangle$ (Theorem 3).

A much more difficult and technically involved proof to deal with the modality $\langle D\rangle$ is sketched in [3], which is under submission. After the submission we devised a simpler proof, which we propose in the following. For the sake of completeness, we give the sketch of the original proof in Appendix A.

As a preliminary step, we introduce the notion of *equivalence up to*, denoted by $\simeq^g_h$, which is used in both proofs. It is a series of equivalence relations over $\mathbb{Z}$ up to a certain threshold, which is given by the value of a suitably defined distance function g on h.

Definition 4 ($\simeq^g_h$) *For any given function $g : \mathbb{D} \to \mathbb{N}$, where $\mathbb{D}$ can be any prefix of $\mathbb{N}$, that is, $\mathbb{D} = \{1,\ldots,N\}$, for some $N \in \mathbb{N}$, and for every $h \in \mathbb{D}$, we define the relation of* equivalence up to $g(h)$*, denoted $\simeq^g_h$, as follows. For every pair of integers $n_1, n_2 \in \mathbb{Z}$, $n_1 \simeq^g_h n_2$ if and only if one of the following holds:*

- *$n_1 = n_2$,*
- *$n_1, n_2 > g(h)$,*
- *$n_1, n_2 < -g(h)$.*

It is easy to see that the relation $\simeq^g_h$ is an equivalence, that is, it is reflexive, symmetric, and transitive. In particular, symmetry of $\simeq^g_h$ will be useful for our purposes. In addition, the following proposition can be easily verified.

Proposition 1 *If $g : \mathbb{D} \to \mathbb{N}$ is monotonically non-decreasing, then $n_1 \simeq^g_{h+1} n_2$ implies $n_1 \simeq^g_h n_2$, for each $n_1, n_2 \in \mathbb{Z}$ and each $h \in \mathbb{D}$.*

5.1 A simplified proof for $\langle D\rangle$

Theorem 2 *The sets of optimal definabilities for $\langle D\rangle$ and $\langle\overline{D}\rangle$ (listed in Figure 2) are complete for the classes* Dis *and* Fin.

In the following, we let $\mathbb{N}^{>c} = \{x \in \mathbb{N} \mid x > c\}$ and $\mathbb{Z}^{<-c} = \{x \in \mathbb{Z} \mid x < -c\}$, for each $c \geq 0$. Moreover, $\mathbb{N}^+$ and $\mathbb{Z}^-$ denote the sets $\mathbb{N}^{>0}$ and $\mathbb{Z}^{<-0}$, respectively.

First, we define the function $f : \{1,\ldots,N\} \to \mathbb{N}$ as:

$$f(h) = h + 1.$$

Let $\bar{\xi}$ be a bijection from $\mathbb{Z} \times \mathbb{N}^+$ to $\mathbb{N}^{>k}$ such that $\bar{\xi}(x,y) \geq x + k$ for each $(x,y) \in \mathbb{Z} \times \mathbb{N}^+$, and where $k = 2 \cdot f(N) + 4$. It is not difficult to convince oneself of the existence of such a function. For example, consider the classic enumeration of the plane $\mathbb{N}^+ \times \mathbb{N}^+$ (i.e., a bijection from $\mathbb{N}^+ \times \mathbb{N}^+$ to $\mathbb{N}^+$), shown in Figure 4.

			...			
5	11	...	...	...	...	
4	7	12	...	...	...	
3	4	8	13	...	...	...
2	2	5	9	14	...	
y 1	1	3	6	10	15	
	1	2	3	4	5	
	x					

Figure 4: An enumeration of the plane $\mathbb{N}^+ \times \mathbb{N}^+$.

It is clear that such enumeration defines a bijection $\mu : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ such that $\mu(x, y) \geq x$ for each $(x, y) \in \mathbb{N}^+ \times \mathbb{N}^+$. It is also easy to verify that the function $\bar{\mu} : \mathbb{Z} \setminus \{0\} \times \mathbb{N}^+ \to \mathbb{N}^+$, defined as

$$\bar{\mu}(x, y) = \begin{cases} 2 \cdot \mu(x, y) - 1 & \text{if } x > 0 \\ 2 \cdot \mu(-x, y) & \text{if } x < 0 \end{cases}$$

is a bijection from $\mathbb{Z} \setminus \{0\} \times \mathbb{N}^+$ to $\mathbb{N}^+$ such that $\bar{\mu}(x, y) \geq x$ for each $(x, y) \in \mathbb{Z} \setminus \{0\} \times \mathbb{N}^+$. Consequently, $\bar{\bar{\mu}} : \mathbb{Z} \setminus \{0\} \times \mathbb{N}^+ \to \mathbb{N}^{>k}$, defined as $\bar{\bar{\mu}}(x, y) = \bar{\mu}(x, y) + k$ is a bijection from $\mathbb{Z} \setminus \{0\} \times \mathbb{N}^+$ to $\mathbb{N}^{>k}$ such that $\bar{\bar{\mu}}(x, y) \geq x + k$ for each $(x, y) \in \mathbb{Z} \setminus \{0\} \times \mathbb{N}^+$. The existence of the desired bijection $\bar{\xi}$ immediately follows: it can be obtained following the same technique used for obtaining $\bar{\mu}$ from μ.

In the following we will consider the function $\xi : \mathbb{Z} \times \mathbb{N}^+ \to \mathbb{Z}^{<-k}$, defined as: $\xi(x, y) = -\bar{\xi}(-x, y)$ for each $(x, y) \in \mathbb{Z} \times \mathbb{N}^+$. Clearly, ξ is a bijection such that $\xi(x, y) \leq x - k$ for each $(x, y) \in \mathbb{Z} \times \mathbb{N}^+$.

Now, we define the function $\eta : \mathbb{Z} \to \mathbb{N}^+$ as follows:

$$\eta(x) = \begin{cases} \bar{y} + \bar{x} - x & \text{if } x = \xi(\bar{x}, \bar{y}) \text{ for some } \bar{x}, \bar{y} \\ k - 2 & \text{otherwise} \end{cases}$$

Notice that if $x = \xi(\bar{x}, \bar{y})$, then $\eta(x) \geq k + 1$ holds, because $\xi(\bar{x}, \bar{y}) = x \leq \bar{x} - k$ and $\bar{y} \geq 1$. Thus, for each x, we have:

$$\eta(x) \geq k - 2. \tag{1}$$

Proposition 2 *There exist two integers x and $x + 1$ such that $\eta(x) \geq \eta(x + 1) + 3$.*

Proof. Consider some $x' = \xi(\bar{x}, \bar{y})$, for some $(\bar{x}, \bar{y}) \in \mathbb{Z} \times \mathbb{N}^+$. We have that $\eta(x') = \bar{y} + \bar{x} - x'$. Let y be the smallest integer such that $y > x'$ and $\eta(y) = k - 2$ (by the definitions of η and ξ such an integer exists and it is not greater than $-k$). Let

(a) $\delta(x, y) = y - x - \eta(x) > 0$.

(b) $\delta(w, z) = z - w - \eta(w) < 0$.

Figure 5: Function δ and valuation V.

$x = y - 1$. Since $\eta(x) \neq k - 2$, it obviously holds $\eta(x) = \bar{\bar{y}} + \bar{\bar{x}} - x$ for some $\bar{\bar{x}}$ and $\bar{\bar{y}}$ such that $\xi(\bar{\bar{x}}, \bar{\bar{y}}) = x$. The thesis follows from $\eta(x) \geq k + 1$ and $\eta(x + 1) = k - 2$. ■

Let $\delta(x, y) = y - x - \eta(x)$, for each interval $[x, y] \in \mathbb{I}(\mathbb{Z})$ (see Figure 5). The following lemma will be useful in the proof of Lemma 2.

Lemma 1 *The following statements hold.*

a) For each interval $[x, y]$ and each $i \in \mathbb{Z}$, with $-f(N) - 1 \leq i \leq f(N) + 1$, there exist x' and x'' such that $x - x' = |i|$ and $\delta(x'', x) = i$.

b) For each interval $[x, y]$ and each $i \in \mathbb{Z}$, with $-f(N) - 1 \leq i \leq f(N) + 1$, there exists $x' < x$ such that $\delta(x', y) = i$.

Proof.

a) The fact that for each interval $[x, y]$ and for each $i \in \mathbb{Z}$, with $-f(N) - 1 \leq i \leq f(N) + 1$, there exists x' such that $x - x' = |i|$ trivially holds. It suffices to take $x' = x - |i|$.

In order to show that for each interval $[x, y]$ and for each $i \in \mathbb{Z}$, with $-f(N) - 1 \leq i \leq f(N) + 1$, there exists x'' such that $\delta(x'', x) = i$, we proceed as follows. For any given interval $[x, y]$ and any given $-f(N) - 1 \leq i \leq f(N) + 1$, consider the interval $[x - f(N) - 2, x - i]$, and let $\bar{x} = x - f(N) - 2$ and $\bar{y} = x - i - \bar{x}$. Clearly, $(\bar{x}, \bar{y}) \in \mathbb{Z} \times \mathbb{N}^+$. We take $x'' = \xi(\bar{x}, \bar{y})$. From $\eta(x'') = \bar{y} + \bar{x} - x''$, it follows $\delta(x'', x) = i$, hence the thesis.

b) Analogously to what we have done above, for any given interval $[x, y]$ and any given $-f(N) - 1 \leq i \leq f(N) + 1$, we consider the interval $[\bar{x}, y - i]$, where $\bar{x} = \min\{y - f(N) - 2, x - 1\}$, and we let $\bar{y} = y - i - \bar{x}$. Clearly, $(\bar{x}, \bar{y}) \in \mathbb{Z} \times \mathbb{N}^+$. We take $x' = \xi(\bar{x}, \bar{y})$. Since $\delta(x', y) = i$, we have proven the thesis.

■

Now, we can define the model M as $M = \langle \mathbb{I}(\mathbb{Z}), V \rangle$, where the valuation V is as follows.

$$[x, y] \in V(p) \Leftrightarrow \delta(x, y) \geq 0.$$

Notice that the model M is parametric in N because k, used in the definitions of ξ and η, depends on N. Notice also that the length of p-intervals is at least $k - 2$. In Figure 5 both an interval $[x, y]$ satisfying p (thus $\delta(x, y) \geq 0$—Figure 5(a)) and an interval $[w, z]$ satisfying $\neg p$ (thus $\delta(w, z) < 0$—Figure 5(b)) are shown.

We introduce here a sequence of relations $Z_N, \ldots, Z_1$. In Lemma 2, we will show that it is an $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}_N$-bisimulation that violates $\langle D\rangle$. To this end, it is convenient to define the equivalence relations $\equiv_\ell^h$ and $\equiv_\delta^h$, for each $h \in \{1, \ldots, N\}$, as

$$\begin{array}{lll} [x,y] \equiv_\ell^h [w,z] & \text{if and only if} & y - x \simeq_h^f z - w \\ [x,y] \equiv_\delta^h [w,z] & \text{if and only if} & \delta(x,y) \simeq_h^f \delta(w,z). \end{array}$$

Intuitively, $\equiv_\ell^h$ relates pairs of intervals such that their lengths coincide or are both larger than $f(h)$. Analogously, $\equiv_\delta^h$ relates pairs of intervals $[x,y]$ and $[w,z]$ such that $\delta(x,y) = \delta(w,z)$ or $\min\{\delta(x,y), \delta(w,z)\} > f(h)$ or $\max\{\delta(x,y), \delta(w,z)\} < -f(h)$. Everything is set for the definition of the sequence of relations $\{Z_h\}_{1\leq h\leq N}$.

Definition 5 *For each $h \in \{1, \ldots, N\}$, the hth component Z_h of the sequence of relations $Z_N, \ldots, Z_1$ is defined as:*

$$[x,y]Z_h[w,z] \Leftrightarrow [x,y] \equiv_\ell^h [w,z] \textit{ and } [x,y] \equiv_\delta^h [w,z].$$

Since $\equiv_\ell^h$ and $\equiv_\delta^h$ are equivalence relations, so is Z_h, for each h. Moreover, for each h, the relations $\equiv_\ell^h$ and $\equiv_\delta^h$ induce the sets of equivalence classes

$$\{[i]_{\equiv_\ell^h} \mid i \in \{1, \ldots, f(h)\} \cup \{\infty\}\} \text{ and } \{[i]_{\equiv_\delta^h} \mid i \in \{-f(h), \ldots, f(h)\} \cup \{-\infty, +\infty\}\},$$

respectively, where,

$$[i]_{\equiv_\ell^h} = \begin{cases} \{[x,y] \mid y - x = i\} & \text{if } i \in \{1, \ldots, f(h)\} \\ \{[x,y] \mid y - x > f(h)\} & \text{if } i = \infty \end{cases}$$

and

$$[i]_{\equiv_\delta^h} = \begin{cases} \{[x,y] \mid \delta(x,y) = i\} & \text{if } i \in \{-f(h), \ldots, f(h)\} \\ \{[x,y] \mid \delta(x,y) < -f(h)\} & \text{if } i = -\infty \\ \{[x,y] \mid \delta(x,y) > f(h)\} & \text{if } i = +\infty. \end{cases}$$

Classes $[i]_{\equiv_\ell^h}$ and $[j]_{\equiv_\delta^h}$ are disjoint for each $i \in \{1, \ldots, f(h)\}$ and $j \in \{-f(h), \ldots, f(h)\}$. This is because $\delta(x,y) < -f(h)$ holds for each interval $[x,y]$ such that $y - x \leq f(h)$, as shown in the following:

$$\begin{array}{lll} \delta(x,y) & = y - x - \eta(x) & \\ & \leq f(h) - \eta(x) & \text{(by } y - x \leq f(h)) \\ & \leq f(h) - (k-2) & \text{(by Formula (1))} \\ & = f(h) - 2 \cdot f(N) - 2 & \text{(by } k = 2 \cdot f(N) + 4) \\ & \leq f(h) - 2 \cdot f(h) - 2 & \text{(by } f(N) \geq f(h)) \\ & < -f(h). & \end{array}$$

Let $[\cap]_{Z_h} = [\infty]_{\equiv_\ell^h} \cap [-\infty]_{\equiv_\delta^h}$ and let us rename $[i]_{\equiv_\ell^h}$ as $[\ell_i]_{Z_h}$ for each $i \in \{1, \ldots, f(h)\}$, and $[i]_{\equiv_\delta^h}$ as $[\delta_i]_{Z_h}$ for each $i \in \{-f(h), \ldots, f(h)\} \cup \{+\infty\}$. For each h, the set

$$\mathcal{E}_{Z_h} = \{[\sim]_{Z_h} \mid \sim \in \{\ell_1, \ldots, \ell_{f(h)}, \delta_{-f(h)}, \ldots, \delta_{f(h)}, \delta_{+\infty}, \cap\}\}$$

characterizes the equivalence classes of Z_h.

Lemma 2 *The sequence of relations $Z_N, \ldots, Z_1$ is an $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}_N$-bisimulation that violates $\langle D\rangle$.*

In order to make the proof of the lemma simpler, we introduce the following notation.

Definition 6 *For each modality $\langle X\rangle$, each interval $[x,y]$, and each $h \in \{2,\ldots,N\}$, and given a sequence of relations $Z_N,\ldots,Z_1$, we define the set $\rightsquigarrow^h_{X,[x,y]}$ as follows:*

$$\rightsquigarrow^h_{X,[x,y]}= \{[\sim]_{Z_{h-1}} \in \mathcal{E}_{Z_{h-1}} \mid [x,y]R_X[w,z] \textit{ for some } [w,z] \in [\sim]_{Z_{h-1}}\}.$$

Intuitively, $\rightsquigarrow^h_{X,[x,y]}$ contains the equivalence classes in $\mathcal{E}_{Z_{h-1}}$ that are reachable from the interval $[x,y]$ using the modality $\langle X\rangle$. We are now ready to prove Lemma 2.

Proof *(Lemma 2)***.** First, we show the existence of two intervals $[x,y]$ and $[w,z]$, with $([x,y],[w,z]) \in Z_N$, such that $M,[x,y] \Vdash \neg\langle D\rangle p$ and $M,[w,z] \Vdash \langle D\rangle p$. To this end, consider the intervals $[x,y]=[0,k-3]$ and the interval $[w,z]$, where $z = w+\eta(w)-1$ and w is such that $\eta(w) \geq \eta(w+1)+3$ (the existence of such w is guaranteed by Proposition 2). Now, since $\delta(x,y) = -1 = \delta(w,z)$, we have that $[x,y] \equiv^h_\delta [w,z]$ holds. To verify that also the condition $[x,y] \equiv^h_\ell [w,z]$ is fulfilled, it suffices to observe that $y-x = k-3 > f(N) \geq f(h)$ and $z-w = \eta(w)-1 \geq k-3 > f(N) \geq f(h)$ (recall that, by Formula (1), $\eta(w) \geq k-2$). Thus, we have that $([x,y],[w,z]) \in Z_N$.

Moreover, since $y-x < k-2$, it is clear, from the definition of V, that none of its sub-interval satisfies p (because p-intervals are long at least $k-2$), and thus $M,[x,y] \Vdash \neg\langle D\rangle p$ holds. Contrarily, $[w,z]$ is such that $M,[w,z] \Vdash \langle D\rangle p$ because the interval $[w+1,z-1]$ satisfies p. To see the latter, observe that $\delta(w+1,z-1) = z-1-w-1-\eta(w+1) = \eta(w)-\eta(w+1)-3 \geq 0$.

To complete the proof, we have to show that $Z_N,\ldots,Z_1$ is an $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}_N$-bisimulation. The local condition is trivially fulfilled, as $[x,w]Z_h[w,z]$ implies $[x,w] \equiv^h_\delta [w,z]$, which, in turn, implies $\delta(x,y) \geq 0$ if and only if $\delta(w,z) \geq 0$, and thus $M,[x,y] \Vdash p$ if and only if $M,[w,z] \Vdash p$.

We have to prove now that the forward and the backward conditions are fulfilled. According to the definabilities shown in Figure 2, the fragment $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}$ is equivalent to $\mathsf{ABO}\overline{\mathsf{ABE}}$, so we can restrict our attention to the operators featured by the latter fragment. Thus, we distinguish several cases, corresponding to the different modalities, and for each case we will show that the set of equivalence classes reachable from both $[x,y]$ and $[w,z]$ is the same. Formally, let $X \in \{A,B,O,\overline{A},\overline{B},\overline{E}\}$ and $([x,y],[w,z]) \in Z_h$ for some $h > 1$, we show that $\rightsquigarrow^h_{X,[x,y]}=\rightsquigarrow^h_{X,[w,z]}$. Both the forward and the backward conditions immediately follow.

- $X = A$. It holds $\rightsquigarrow^h_{X,[x',y']}= \mathcal{E}_{Z_{h-1}}$, for each $[x',y']$. Thus, it trivially holds $\rightsquigarrow^h_{X,[x,y]}=\rightsquigarrow^h_{X,[w,z]}$.
- $X = O$. Similarly to the previous case, we have that $\rightsquigarrow^h_{X,[x',y']}= \mathcal{E}_{Z_{h-1}} \setminus \{[\ell_1]_{Z_{h-1}}\}$ holds for each $[x',y']$. Thus, it trivially holds $\rightsquigarrow^h_{X,[x,y]}=\rightsquigarrow^h_{X,[w,z]}$.
- $X = B$. In this case, we observe that the value of $\rightsquigarrow^h_{X,[x',y']}$, for any given interval $[x',y']$, depends on the equivalence class $[x',y']$ belongs to. Precisely, it is as follows:
  - $\rightsquigarrow^h_{X,[x',y']}= \{[\ell_{i'}]_{Z_{h-1}} \mid i' < i\}$, if $([x,y],[w,z]) \in [\ell_i]_{Z_h}$, for some $i \in \{1,\ldots,f(h)\}$;
  - $\rightsquigarrow^h_{X,[x',y']}= \{[\ell_{i'}]_{Z_{h-1}} \mid i' < i\} \cup \{[\cap]_{Z_{h-1}}\}$, if $([x,y],[w,z]) \in [\cap]_{Z_h}$. Notice that the function f plays a role here, allowing us to "simulate" density: if an interval $[x,y]$ is h-long (i.e., its length is greater than $f(h)$), the interval $[x,y-1]$, despite being shorter than $[x,y]$, is $(h-1)$-long as well;

– $\rightsquigarrow^h_{X,[x',y']} = \{[\ell_{i'}]_{Z_{h-1}} \mid i' < i\} \cup \{[\cap]_{Z_{h-1}}\} \cup \{[\delta_{i'}]_{Z_{h-1}} \mid i' < i\}$, if $([x,y],[w,z]) \in [\delta_i]_{Z_h}$, for some $i \in \{-f(h), \ldots, f(h)\}$;
– $\rightsquigarrow^h_{X,[x',y']} = \mathcal{E}_{Z_{h-1}}$, if $([x,y],[w,z]) \in [\delta_\infty]_{Z_h}$ Again, we "simulate" density here in the same sense described before, to guarantee that, if $[x,y] \in [\delta_\infty]_{Z_h}$, then $[x, y-1] \in [\delta_\infty]_{Z_{h-1}}$.

Since $([x,y],[w,z]) \in Z_h$, they belong to the same equivalence class in $\mathcal{E}_{Z_h}$. Thus, $\rightsquigarrow^h_{X,[x,y]} = \rightsquigarrow^h_{X,[w,z]}$ holds.

- $X = \overline{B}$. In this case an argument very similar to the one for the previous case can be used to show that $\rightsquigarrow^h_{X,[x,y]} = \rightsquigarrow^h_{X,[w,z]}$ holds, so we omit the details.

- $X = \overline{A}$. It suffices to observe that from Lemma 1a it immediately follows that $\rightsquigarrow^h_{X,[x',y']} = \mathcal{E}_{Z_{h-1}}$, for each $[x',y']$. Thus, $\rightsquigarrow^h_{X,[x,y]} = \rightsquigarrow^h_{X,[w,z]}$ holds.

- $X = \overline{E}$: analogously to the previous case, by Lemma 1b, we have that $\rightsquigarrow^h_{X,[x',y']}$ contains the set $\{[\delta_i]_{Z_{h-1}} \mid i \in \{-f(h-1), \ldots, f(h-1)\} \cup \{+\infty\}\} \cup \{[\cap]_{Z_{h-1}}\}$. Moreover, if $([x,y],[w,z]) \in [\ell_i]_{Z_h}$, for some $i \in \{1, \ldots, f(h)\}$, then $\rightsquigarrow^h_{X,[x',y']}$ also includes the set of equivalence classes $\{[\ell_{i'}]_{Z_{h-1}} \mid i' > i\}$. Again, since $[x,y]$ and $[w,z]$ belong to the same equivalence class in $\mathcal{E}_{Z_h}$, it holds that $\rightsquigarrow^h_{X,[x,y]} = \rightsquigarrow^h_{X,[w,z]}$.

Since we have that $\rightsquigarrow^h_{X,[x,y]} = \rightsquigarrow^h_{X,[w,z]}$ holds in all the above cases, we can conclude that both the forward and the backward conditions are satisfied, hence the thesis. ■

The above proof makes use of a model based on the infinite set of integers $\mathbb{Z}$, and thus it proves the result for the class Dis. The whole construction can be adapted to deal with the class Fin as well, by using a finite, "large enough" portion of $\mathbb{Z}$, and then by taking special care of the intervals that are "close" to the borders.

Based on the observation that $\langle D \rangle$ and $\langle \overline{D} \rangle$ behave in a very similar way when interpreted over classes of finite linear orders, using the same idea it is possible to show that the result also holds for the modality $\langle \overline{D} \rangle$.

5.2 The cases $\langle E \rangle$, $\langle \overline{E} \rangle$, $\langle B \rangle$, and $\langle \overline{B} \rangle$

In what follow, we state the undefinability result for the remaining modalities, namely, $\langle E \rangle$, $\langle \overline{E} \rangle$, $\langle B \rangle$, and $\langle \overline{B} \rangle$. Because of the technical complexity of the proof, we only provide here a sketch that explains the main idea behind the proof, and refer the interested reader to Appendix B for the details.

Theorem 3 *There are no definabilities for $\langle E \rangle$ and $\langle \overline{E} \rangle$ (as shown in Figure 2), as well as for their transposes $\langle B \rangle$ and $\langle \overline{B} \rangle$, in the classes* Dis *and* Fin.

Proof *(sketch)***.** We only give the sketch of the proof for the operators $\langle E \rangle$ and $\langle \overline{E} \rangle$. The result for $\langle B \rangle$ and $\langle \overline{B} \rangle$ follows from a symmetric argument. According to Figure 2, there are no definabilities for $\langle E \rangle$ when the underlying structure is discrete, and therefore ALBDO$\overline{\text{ALBEDO}}$ is the only maximal fragment not defining it. This is also true on Lin and Den, but on Dis and Fin it is simply harder to prove. An indication of such a difficulty comes from the analysis of the proofs presented in [13], where the density of the models involved plays a major role. Similarly to the case of Theorem 2, $\langle E \rangle$ is definable in an infinitary extension of the language AB:

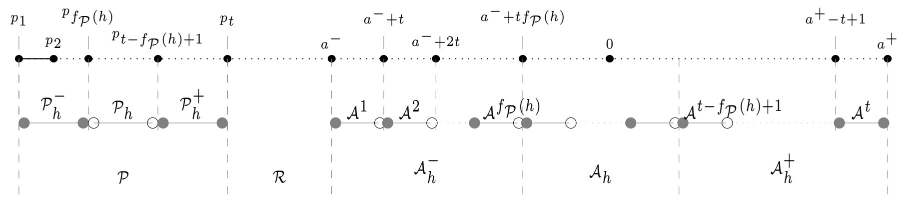

Figure 6: A graphical account of the $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}_N$-bisimulation that violates $\langle E\rangle$.

$$\langle E\rangle p \equiv \bigvee_{k\in\mathbb{N}}(\ell_{=k} \wedge \bigvee_{i<k}(\langle B\rangle(\ell_{=i} \wedge \langle A\rangle(\ell_{=k-i} \wedge p))),$$

since, as stated in Section 3.2, $\langle B\rangle$ can express $\ell_{=k}$, for every $k \in \mathbb{N}$. Thus, there exists no $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}$-bisimulation that violates $\langle E\rangle$, and we need to find an $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}_N$-bisimulation. Unlike the case of Theorem 2, the best way to sketch the construction is by explicitly giving models and relative valuation functions.

Let $\mathbb{D}$ be a finite domain, e.g., an arbitrarily large prefix of $\mathbb{N}$. We define a model M based on it and an $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}_N$-bisimulation between M and itself that violates $\langle E\rangle$. Given $N \in \mathbb{N}$, we make use of $h \leq N$ to refer to the hth component of the N-bisimulation, also called in the following the hth *step* of the N-bisimulation. Building the $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}_N$-bisimulation relies on a very technical construction that allows us to "simulate density" over discrete models up to a certain threshold. To this end, in analogy to what we did in the proof of Theorem 2, we will use monotonically increasing *threshold functions*, which are parametric in h and which characterize a notion of "long interval", relative to a generic *step* h of the N-bisimulation. Since such functions are monotonic, intervals that are "long" at the step h of the N-bisimulation always contain intervals that are still "long" at the step $h-1$, despite being obviously shorter of the the containing interval. We will also use suitably defined equivalences up to a threshold (given by the aforementioned threshold functions) to recognize when two intervals are "long enough" to be indistinguishable by modal formulae in the fragment $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}$ whose modal depth is less than $h \leq N$.

Now, we define the function $f(h) = h+1$, which will be used as threshold function, and the function $f_{\mathcal{P}}(h) = \sum_{i=1}^{h} f(i)$. Notice that both functions are monotonically increasing. Moreover, we let $t = 2(f_{\mathcal{P}}(1) + N + 4)$, $a^+ = \frac{t^2}{2} - 1$, and $a^- = -\frac{t^2}{2}$. Finally, we consider a partition of $\mathbb{D}$ as in Figure B.

Three subsets, from left to right, are clearly identified in Figure B:

$$\mathcal{P} = \{p_1, \ldots, p_t\}, \quad \mathcal{R} = \{x \in \mathbb{D} \mid p_t < x < a^-\}, \quad \mathcal{A} = \{x \in \mathbb{D} \mid a^- \leq x \leq a^+\},$$

where we let $p_t = a^- - t$ and, for each $i < t$, $p_i = p_{i+1} - 1$.

For each h, we define a further partition of the subsets $\mathcal{P}$ and $\mathcal{A}$, as follows:

$$\mathcal{P} = \bigcup \begin{cases} \mathcal{P}_h^- = \{x \mid p_1 \leq x \leq p_{f_{\mathcal{P}}(h)}\} \\ \mathcal{P}_h^+ = \{x \mid p_{t-f_{\mathcal{P}}(h)+1} \leq x \leq p_t\} \\ \mathcal{P}_h = \{x \mid p_{f_{\mathcal{P}}(h)} < x < p_{t-f_{\mathcal{P}}(h)+1}\}, \end{cases}$$

$$\mathcal{A}^i = \{x \in \mathbb{D} \mid a^- + (i-1)\cdot t \leq x < a^- + i\cdot t\},$$

$$\mathcal{A} = \bigcup \begin{cases} \mathcal{A}_h^- = \bigcup_{i=1}^{f_{\mathcal{P}}(h)} \mathcal{A}^i \\ \mathcal{A}_h^+ = \bigcup_{i=t-f_{\mathcal{P}}(h)+1}^{t} \mathcal{A}^i \\ \mathcal{A}_h = \mathcal{A} \setminus (A_h^- \cup A_h^+) = \bigcup_{i=f_{\mathcal{P}}(h)+1}^{t-f_{\mathcal{P}}(h)} \mathcal{A}^i. \end{cases}$$

Roughly speaking, we can say that stepping from $h+1$ to h, the sets $\mathcal{P}^-_{h+1}$, $\mathcal{P}^+_{h+1}$, $\mathcal{A}^-_{h+1}$, and $\mathcal{A}^+_{h+1}$ shrink, while the sets $\mathcal{P}_{h+1}$ and $\mathcal{A}_{h+1}$ expand. Now, let M be a model based on $\mathbb{D}$ described as above. We first define a function $\mathcal{V} : \mathcal{A} \to \mathcal{P}$, and then the valuation function V of M, which uses $\mathcal{V}$:

$$\begin{array}{rcl} \mathcal{V}(y) & = & \begin{cases} p_1 + i & \text{if } y = a^- + i \text{, for each } 0 \leq i < t \\ \mathcal{V}(y-t) & \text{if } a^- + t \leq y \leq a^+, \end{cases} \\ V(p) & = & \{[x,y] \mid y \in \mathcal{A} \text{ implies } x \leq \mathcal{V}(y)\}. \end{array}$$

In order to define an $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}_N$-bisimulation, we first define a sequence Z_N, $\ldots, Z_1$, which is common to both cases $\langle E \rangle$ and $\langle \overline{E} \rangle$, and then we show how to adjust it to obtain our results. To characterize the generic hth component Z_h of the sequence $Z_N, \ldots, Z_1$ we make use of an equivalence relation $\equiv_h$, parameterized by h, which is defined as follows. Let us denote by x (resp., w) the nth element of $\mathcal{A}_i$ (resp., the mth element of $\mathcal{A}_j$), that is, $x = a^i_n$ and $w = a^j_m$. Then, we have:

$$x \equiv_h w \text{ iff} \begin{cases} x = w \text{ or} \\ x, w \in \mathcal{P}_h \text{ or} \\ x, w \in \mathcal{A} \text{ and} \begin{cases} i = j \vee x, w \in \mathcal{A}_h, \text{ and} \\ m = n \vee f_{\mathcal{P}}(h) < m, n < t - f_{\mathcal{P}}(h) + 1. \end{cases} \end{cases}$$

As already pointed out, to define the desired N-bisimulation, we also need an equivalence up to a threshold. Such a relation, denoted $\simeq^f_h$, relates integers, which represent interval lengths, as follows: $a \simeq^f_h b$ if and only if either $a = b$ or both a and b are greater than the threshold $f(h)$. We can now define Z_h as follows: for each $1 \leq h \leq N$, $([x,y],[w,z]) \in Z_h$ if and only if: *(a)* $x \equiv_h w$ and $y \equiv_h z$, *(b)* $y - x \simeq^f_h z - w$, *(c)* if $x, w \in \mathcal{P}$ and $y, z \in \mathcal{A}$, then $\mathcal{V}(y) - x \simeq^f_h \mathcal{V}(z) - w$, and *(d)* if $x \in \mathcal{A}^i$ and $y \in \mathcal{A}^j$ for some $i, j \in \{1, \ldots, t\}$, then $w \in \mathcal{A}^k$ and $z \in \mathcal{A}^\ell$ for some $k, \ell \in \{1, \ldots, t\}$ such that $j - i \simeq^f_h \ell - k$. As a last step, we define a new sequence of relations $Z^E_N, \ldots, Z^E_1$ such that $Z^E_N \cup Z_N, \ldots, Z^E_1 \cup Z_1$ is an $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}_N$-bisimulation (the proof is technically involved, so details are omitted). Consider a point $a = a^i_m$ such that $i = m = \frac{t}{2}$, that is, a is the $\frac{t}{2}$th point of the $\frac{t}{2}$th sub-group of $\mathcal{A}$. It holds that $\mathcal{V}(a) = p_m = p_{\frac{t}{2}}$. Now, for each $1 \leq h \leq N$, let $Z^E_h = \{([\mathcal{V}(a) - (N-h+1), a], [\mathcal{V}(a) - (N-h), a])\}$. It is easy to see that $M, [\mathcal{V}(a) - 1, a] \Vdash \langle E \rangle p$, $M, [\mathcal{V}(a), a] \Vdash \neg\langle E \rangle p$, and $([\mathcal{V}(a) - 1, a], [\mathcal{V}(a), a]) \in Z^E_N$. Thus, $Z^E_N \cup Z_N, \ldots, Z^E_1 \cup Z_1$ is an $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}_N$-bisimulation that violates $\langle E \rangle$.

To deal with the modality $\langle \overline{E} \rangle$, a new sequence $Z^{\overline{E}}_N, \ldots, Z^{\overline{E}}_1$ can be defined, following a technique similar to the above-described one, so that $Z^{\overline{E}}_N \cup Z_N, \ldots, Z^{\overline{E}}_1 \cup Z_1$ is an $\mathsf{ALBEDO}\overline{\mathsf{ALBDO}}_N$-bisimulation that violates $\langle \overline{E} \rangle$. Once again, since the proof only uses a finite linear order, the result holds for both Dis and Fin. ■

6 Conclusions

In this paper we studied the expressiveness of fragments of the interval temporal logic HS interpreted over both discrete and finite linear orders. A complete classification of all such fragments with respect to their relative expressive power has been recently given for the classes of all linear orders and all dense linear orders. The cases of discrete and finite linear orders turn out to be much more involved. We illustrated here various non-trivial temporal properties that can be expressed when HS is interpreted over them, and we provided a complete set of definabilities for the modalities corresponding to the Allen's relations *meets*, *later*, *begins*, *finishes*, and *during*, plus their transposes.

We leave open the problem of identifying the complete set of definabilities for the modalities corresponding to the Allen relation *overlaps* and to its inverse *overlapped by*.

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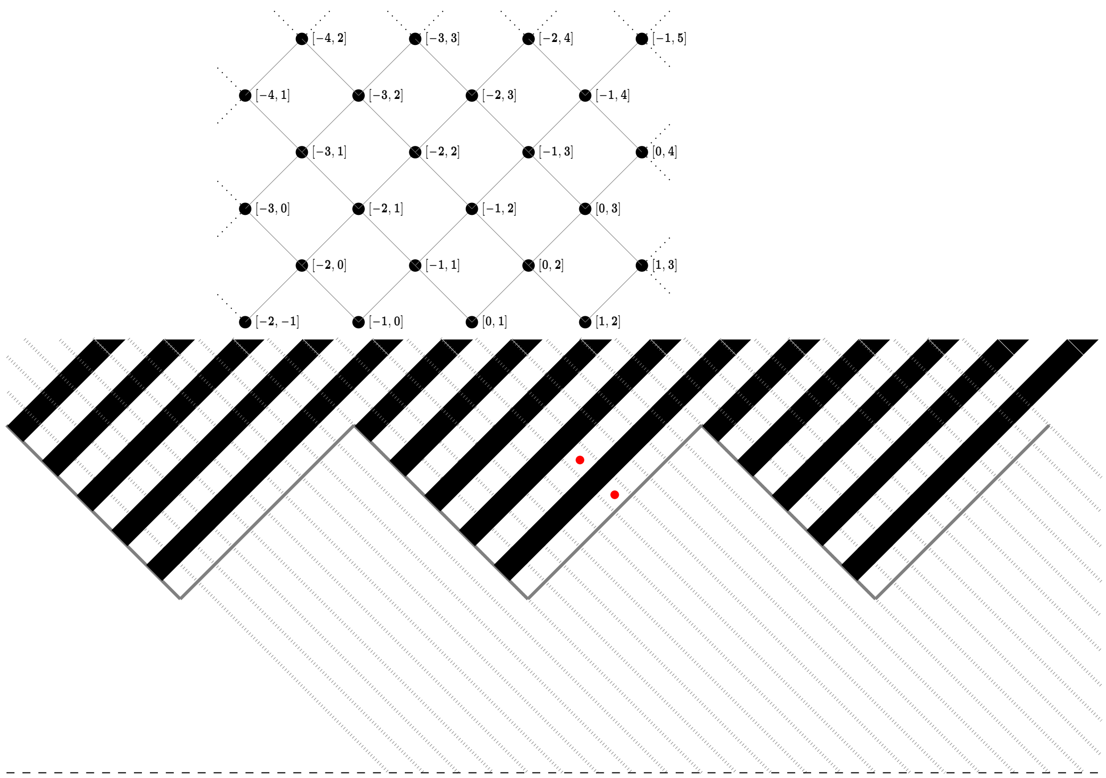

Figure 7: Grid-based interpretation of intervals (top) and a graphical account of the $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}_N$-bisimulation that violates $\langle D\rangle$ (bottom).

Appendix

In this appendix we first give, in Appendix A, the sketch of the original proof of Theorem 2, proposed in [3], which is under submission. A simplified, fully-detailed proof is provided in Section 5.1. Then, we provide, in Appendix B, full details of the proof of Theorem 3.

A Sketch of the (more complex) original proof of Theorem 2

Theorem 2 *The sets of optimal definabilities for $\langle D\rangle$ and $\langle\overline{D}\rangle$ (listed in Figure 2) are complete for the classes* Dis *and* Fin*.*

Proof *(sketch)***.** According to Figure 2, $\langle D\rangle$ is definable in terms of BE; thus there are two maximal fragments not defining it, namely, $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}$ and $\mathsf{ALEO}\overline{\mathsf{ALBEDO}}$. First, we observe that it is possible to define $\langle D\rangle$ in infinitary extensions of AB or $\overline{\mathsf{A}}\mathsf{E}$, using, respectively, the following formulae of unbounded modal depths:

$$\langle D\rangle p \equiv \begin{cases} \bigvee_{k\in\mathbb{N}}(\ell_{=k} \wedge \bigvee_{i<k-1}(\langle B\rangle(\ell_{=i} \wedge \langle A\rangle(\ell_{<k-i} \wedge p)))), \\ \bigvee_{k\in\mathbb{N}}(\ell_{=k} \wedge \bigvee_{i<k-1}(\langle E\rangle(\ell_{=i} \wedge \langle\overline{A}\rangle(\ell_{<k-i} \wedge p)))), \end{cases}$$

where length constraints of the form $\ell_{=k}$ and $\ell_{<k}$ can be expressed using either $\langle B \rangle$ or $\langle E \rangle$ (see Section 3.2). It immediately follows that there exists no $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}$-bisimulation (resp., $\mathsf{ALEO}\overline{\mathsf{ALBEDO}}$-bisimulation) that violates $\langle D \rangle$, and thus we have to resort to $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}_N$-bisimulations (resp., $\mathsf{ALEO}\overline{\mathsf{ALBEDO}}_N$-bisimulations). Besides, since the two fragments $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}$ and $\mathsf{ALEO}\overline{\mathsf{ALBEDO}}$ are symmetric, that is, they are indistinguishable over symmetric classes of linear orders, providing an $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}_N$-bisimulation that violates $\langle D \rangle$ suffices to prove the result.

For the purposes of the proof, it is convenient to introduce a new interpretation for intervals over grid-like structures (the so-called compass structures [24]), by exploiting the existence of a natural bijection between the intervals $[x, y]$ of an interval model and the points $p = (x, y)$, with $x < y$, of an $N \times N$ grid. A graphical account is given in Figure 7 (top), where the $N \times N$ grid has been rotated by a 45-degree angle clockwise, so that the bisector of the I and III quadrant is the base of the picture.

First, we define the model M, as depicted in Figure 7 (bottom), where intervals satisfying p are all and only the points belonging to the black areas. Thus, intervals satisfying p are grouped into stripes. The dotted lines in the picture are perpendicular to the stripes, more precisely, to (the ideal continuations of) their edges. Each dotted line intersects exactly one such continuation at the base of the picture (dashed line, representing the bisector of the I and III quadrant). Intersections of dotted lines with stripes give rise to small squares. Black (resp., white) squares only contains intervals satisfying p (resp., $\neg p$). Now, let us focus on the gray, zigzag solid line. If we ideally draw the straight lines continuing the segments making up such a zigzag line, their intersections shape bigger squares, each of them containing a (square) number of the above-mentioned small squares.

In order to define an $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}_N$-bisimulation, we focus on the generic hth element of the sequence, namely, the relation Z_h. The idea is to relate points that are either "far enough" from the elements of discontinuity of the model (stripes' edges, dotted lines, dashed line, and gray line) or at the same distance from them. The key element is the notion of "far enough", which can be formalized by means of monotonically increasing *distance functions* on h, representing the number of nested modalities that can still be used to build a formula that discriminates between the related intervals, before reaching the greatest allowed modal depth N. In other words, the notion of distance is induced by h through suitable distance functions, and the distance decreases as h does: in this way, if an interval i_1 is far from a significant element e of the model, according to the notion of distance induced by some h (i.e., i_1 is h-far from e), it is always possible to find another interval i_2, that is closer to e, but still far from e according to the "new" notion of distance induced by $h - 1$ (i.e., i_2 is $(h - 1)$-far from e).

Now, still at a very high level, by exploiting such a notion of "far enough", we can conclude that the two red circles in the two white stripes in the middle of the picture are Z_h-related, because, according to suitable distance functions, both of them are far from all the elements of discontinuity of the model, that is, the edges of their own small squares (both points are in the middle of a small square, with enough points in between them and the edges), as well as the ones of the big square. Moreover, the relative position of the two small squares in the big one is the same (up to a certain distance from the edges of the big square), with the exception of the position relative to the bottom-right edge of the big square: one of the circles is in the first small square, the other in the third one. This is not a problem, because distances in the bottom-right directions can be ignored as moving in that direction corresponds to using the modality $\langle E \rangle$, which does not belong to $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}$. Finally, from Figure 7, it

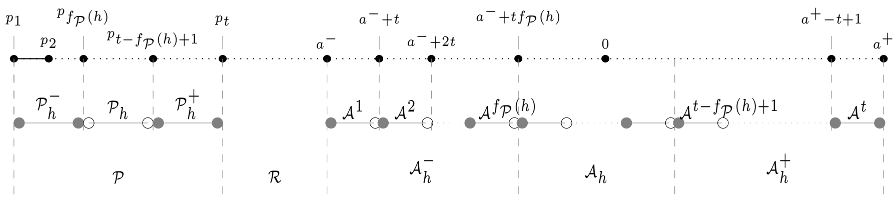

Figure 8: A graphical account of the $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}_N$-bisimulation that violates $\langle E\rangle$.

is clear that the lower circle does not "see" any interval satisfying p (black stripes) in the triangle underneath, and thus $\langle D\rangle p$ is false on it. On the contrary, the higher circle "sees" intervals satisfying p in the triangle underneath, which means that $\langle D\rangle p$ is true over it. Thus, we have an $\mathsf{ALBO}\overline{\mathsf{ALBEDO}}_N$-bisimulation that violates $\langle D\rangle$. A similar construction can be done to deal with the modality $\langle\overline{D}\rangle$, which somehow turns the picture upside-down, thus showing that the result holds also for $\langle\overline{D}\rangle$. ∎

B Proof of Theorem 3

In this section, we provide a fully-detailed proof of Theorem 3.

Theorem 3 *There are no definabilities for $\langle E\rangle$ and $\langle\overline{E}\rangle$ (as shown in Fig. 2), as well as for their transposes $\langle B\rangle$ and $\langle\overline{B}\rangle$, in the classes* Dis *and* Fin.

B.1 The auxiliary functions f and $f_{\mathcal{P}}$

- The function $f : \{1, \ldots, N\} \to \mathbb{N}$ is defined as:

  $$f(h) = N + 2 - h, \text{ for each } h \in \{1, \ldots, N\}$$

  Notice that $f(h-1) - f(h) = 1$ for each $h > 1$.

  If the length of an interval $[x, y]$ is greater than $f(h)$, then we say that $[x, y]$ is *h-long*; otherwise, we say that $[x, y]$ is *h-short*.

  The function f is used for conditions (b), (c), and (d) of the definition of Z_h.

- The function $f_{\mathcal{P}} : \{1, \ldots, N\} \to \mathbb{N}$ is defined as:

  $$f_{\mathcal{P}}(h) = \sum_{i=h}^{N} f(i), \text{ for each } h \in \{1, \ldots, N\}$$

  Notice that $f_{\mathcal{P}}(h-1) - f_{\mathcal{P}}(h) = f(h-1) > 0$ for each $h > 1$, and that $f_{\mathcal{P}}(h) \geq f(h)$ for each h; indeed, $f_{\mathcal{P}}(h) > f(h)$ if $h < N$ and $f_{\mathcal{P}}(h) = f(h)$ otherwise.

  Moreover, for each $h > 1$, the following properties of $f_{\mathcal{P}}$ hold:

  - $f_{\mathcal{P}}(h-1) > f(h)$;

- $[x, y]$ is $(h-1)$-long (and thus h-long as well), for each $y > f_{\mathcal{P}}(h-1)$ and $x \leq f_{\mathcal{P}}(h)$;
- for each $y > f_{\mathcal{P}}(h-1)$, there exists $x > f_{\mathcal{P}}(h)$ such that $[x, y]$ is h-long.

The function $f_{\mathcal{P}}$ is used, in combination with the function f, for condition (c) of the definition of Z_h.

B.2 Partitioning D

Let t be defined as $t = 2 \cdot (f_{\mathcal{P}}(1) + N + 4)$. We partition D as follows (see Figure B):

- $\mathcal{A} = \{x \in D \mid a^- \leq x \leq a^+\}$, where $a^- = -\frac{t^2}{2}$ and $a^+ = \frac{t^2}{2} - 1$. Thus, $|\mathcal{A}| = t^2$,
- $\mathcal{P} = \{p_1, \ldots, p_t\}$ such that $p_i = p_{i+1} - 1$, for each $1 \leq i < t$, and $p_t \ll a^-$, e.g., $a^- - p_t = t$,
- $\mathcal{R} = \{x \in D \mid p_t < x < a^-\}$,
- $\mathcal{R}^- = \{x \in D \mid x < p_1\}$,
- $\mathcal{R}^+ = \{x \in D \mid x > a^+\}$.

Now, for each $h \in \{1, \ldots, N\}$, we define new partitions for both $\mathcal{P}$ and $\mathcal{A}$. The sets $\mathcal{P}_h^-$, $\mathcal{P}_h$, and $\mathcal{P}_h^+$, defined below, define a partition of $\mathcal{P}$:

- $\mathcal{P}_h^- = \{x \in \mathcal{P} \mid p_1 \leq x \leq p_{f_{\mathcal{P}}(h)}\}$
- $\mathcal{P}_h = \{x \in \mathcal{P} \mid p_{f_{\mathcal{P}}(h)} < x < p_{t-f_{\mathcal{P}}(h)+1}\}$
- $\mathcal{P}_h^+ = \{x \in \mathcal{P} \mid p_{t-f_{\mathcal{P}}(h)+1} \leq x \leq p_t\}$

The sets $\mathcal{A}^i$, for $1 \leq i \leq t$, defined below, define a partition of $\mathcal{A}$:

- $\mathcal{A}^i = \{x \in \mathcal{A} \mid a^- + (i-1) \cdot t \leq x < a^- + i \cdot t\}$

For each $1 \leq i \leq t$, we denote by $a_1^i, a_2^i, \cdots, a_t^i$ (with $a_{j+1}^i = a_j^i + 1$ for every $1 \leq j < t$) the elements of $\mathcal{A}^i$. Thus, $a_1^1 = a^-$ and $a_t^t = a^+$.

Finally, the sets $\mathcal{A}_h^-$, $\mathcal{A}_h$, and $\mathcal{A}_h^+$, defined below, define a partition of $\mathcal{A}$:

- $\mathcal{A}_h^- = \bigcup_{i=1}^{f_{\mathcal{P}}(h)} \mathcal{A}^i$
- $\mathcal{A}_h^+ = \bigcup_{i=t-f_{\mathcal{P}}(h)+1}^{t} \mathcal{A}^i$
- $\mathcal{A}_h = \mathcal{A} \setminus (A_h^- \cup A_h^+) = \bigcup_{i=f_{\mathcal{P}}(h)+1}^{t-f_{\mathcal{P}}(h)} \mathcal{A}^i$

Notice that, when stepping from $h-1$ to h, the sets A_{h-1}^- and A_{h-1}^+ shrink, while A_{h-1} grows. Precisely, $|\mathcal{A}_{h-1}^-| - |\mathcal{A}_h^-| = |\mathcal{A}_{h-1}^+| - |\mathcal{A}_h^+| = t \cdot (f_{\mathcal{P}}(h-1) - f_{\mathcal{P}}(h)) = t \cdot f(h-1)$, and $|\mathcal{A}_h| - |\mathcal{A}_{h-1}| = 2 \cdot (|\mathcal{A}_{h-1}^-| - |\mathcal{A}_h^-|) = 2 \cdot t \cdot f(h-1)$, for each $h > 1$.

B.3 The valuation function V

$V(p) = \{[x, y] \mid y \in \mathcal{A} \text{ implies } x \leq \mathcal{V}(y)\}$, where $\mathcal{V} : \mathcal{A} \to \mathcal{P}$ is defined as follows:

$$\mathcal{V}(y) = \begin{cases} p_1 + i & \text{if } y = -a + i, \text{for each } 0 \leq i < t \\ \mathcal{V}(y - t) & \text{if } -a + t \leq y \leq a \end{cases}$$

Let $y = a^i_m$ be a point in $\mathcal{A}$ (i.e., y is the mth point in $\mathcal{A}^i$). The following properties of $\mathcal{V}$ hold:

- $\mathcal{V}(y) = p_m$;
- if $m \leq f_\mathcal{P}(h)$ (resp., $m \geq t - f_\mathcal{P}(h) + 1$, $f_\mathcal{P}(h) < m < t - f_\mathcal{P}(h) + 1$), then $\mathcal{V}(y) \in \mathcal{P}^-_h$ (resp., $\mathcal{V}(y) \in \mathcal{P}^+_h$, $\mathcal{V}(y) \in \mathcal{P}_h$);
- for each $x \in \mathcal{P}^-_h$ (resp., $x \in \mathcal{P}^+_h$), if $f_\mathcal{P}(h-1) < m < t - f_\mathcal{P}(h-1) + 1$, then $[x, \mathcal{V}(y)]$ is h-long (resp., $[\mathcal{V}(y), x]$ is h-long);
- for each $x = a^j_n$ for some $n, j \in \{1, \ldots, t\}$ (i.e., x is the nth point in $\mathcal{A}^j$), if $m = n$, then $\mathcal{V}(x) = \mathcal{V}(y)$.
- for each $x \in \mathcal{A}^{i'}$, for some $i' \leq f_\mathcal{P}(h)$ (resp., $i' \geq t - f_\mathcal{P}(h) + 1$), i.e., $x \in \mathcal{A}^-_h$ (resp., $x \in \mathcal{A}^+_h$), if $f_\mathcal{P}(h-1) < i < t - f_\mathcal{P}(h-1) + 1$, i.e., $y \in \mathcal{A}_{h-1}$, then $i - i' > f(h)$ (resp., $i' - i > f(h)$);

B.4 The N-bisimulation relation

First, we define the equivalences $\equiv_h$ $(1 \leq h \leq N)$ between points as follows: $x \equiv_h w$ if and only if the following conditions hold:

(i) one of the following holds:

- $x = w$,
- $x \neq w$ and $x, w \in \mathcal{P}_h$,
- $x \neq w$ and $x, w \in \mathcal{A}$;

(ii) if $x, w \in \mathcal{A}$, then the following properties hold (we denote x by a^i_m and w by a^j_n, i.e., x is the ith element in $\mathcal{A}^i$ and w is the nth element in $\mathcal{A}^j$):

- either $i = j$ or $x, w \in \mathcal{A}_h$ (i.e., $f_\mathcal{P}(h) < min\{i, j\} < max\{i, j\} < t - f_\mathcal{P}(h) + 1$),
- either $m = n$ or $f_\mathcal{P}(h) < m < t - f_\mathcal{P}(h) + 1$ and $f_\mathcal{P}(h) < n < t - f_\mathcal{P}(h) + 1$.

For each $h \in \{1, \ldots, N\}$, Z_h is defined as follows. $[x, y] Z_h [w, z]$ if and only if all of the following hold:

(a) **endpoints $\equiv_h$-related**: $x \equiv_h w$ and $y \equiv_h z$,

(b) **same length up to $f(h)$**: $y - x \simeq^f_h z - w$,

(c) **same distance between the left endpoint and the value of $\mathcal{V}$ over the right endpoint up to $f(h)$**: if $x, w \in \mathcal{P}$ and $y, z \in \mathcal{A}$, then $\mathcal{V}(y) - x \simeq^f_h \mathcal{V}(z) - w$,

(d) **same number of sub-groups $\mathcal{A}^i$ up to $f(h)$**: if $x \in \mathcal{A}^i$ and $y \in \mathcal{A}^j$ for some $i, j \in \{1, \ldots, t\}$, then $w \in \mathcal{A}^k$ and $z \in \mathcal{A}^\ell$ for some $k, \ell \in \{1, \ldots, t\}$ such that $j - i \simeq^f_h \ell - k$.

Proposition 3 *The relations Z_h ($1 \leq h \leq N$) are symmetric.*

Notice that, for each pair $([x, y], [w, z]) \in Z_h$ ($1 \leq h \leq N$), at least one condition among *(c)* and *(d)* is vacuously satisfied (i.e., the premise is false).

Lemma 3 (forward condition wrt $\langle B \rangle$) *Let $[x, y]Z_{h-1}[w, z]$ for some $2 \leq h \leq N$ and let $[x, y]B[x, y']$. Then, there exists an interval $[w, z']$ such that $[w, z]B[w, z']$ and $[x, y']Z_h[w, z']$.*

Proof. First, notice that, whenever $[x, y] = [w, z]$, the property is trivially verified. Thus, in the rest of the proof, we will assume $[x, y] \neq [w, z]$. We proceed by cases.

- If $x = w$, then it must be $y \neq z$. Thus, both $[x, y]$ and $[w, z]$ are $(h-1)$-long. We can safely assume $y' \geq z$ because if it was $y' < z$, then we would select $z' = y'$ and we are done. As an immediate consequence of $y' \geq z$ and $y' < y$, we have that $z < y$ holds. Moreover, $[x, y']$ is h-long, because it is not shorter than $[w, z]$, which, in turn, is $(h-1)$-long, and thus also h-long. Now, since $y \neq z$, either $y, z \in \mathcal{P}_{h-1}$ or $y, z \in \mathcal{A}$.

  - If $y, z \in \mathcal{P}_{h-1}$, then $y' \in \mathcal{P}_h$ (because $z \leq y' < y$). We select $z' = z - 1$, and $[w, z']$ is such that $[w, z]B[w, z']$ and $([x, y'], [w, z']) \in Z_h$.
  - If $y, z \in \mathcal{A}$, then $y' \in \mathcal{A}$, too (due to $z \leq y' < y$). Let us denote y (resp., z, y') by a^i_m (resp., a^j_n, $a^{i'}_{m'}$). We distinguish the following cases.
    * If $i \neq j$ (notice that the only possibility is that $j < i$, because $z < y$), then $y, z \in \mathcal{A}_{h-1}$ (i.e., $f_{\mathcal{P}}(h-1) < j < i < t - f_{\mathcal{P}}(h-1) + 1$). From $z \leq y' < y$, it follows $j \leq i' \leq i$, which implies $y' \in \mathcal{A}_h$. Moreover, by the condition *(d)* of the definition of Z_h, if $x(= w) \in \mathcal{A}^k$, for some k, then $i - k > f(h-1)$ and $j - k > f(h-1)$. We choose z' in $\mathcal{A}^{j-1}$ such that $z' - a^{j-1}_1 = y' - a^{i'}_1$, and $[w, z']$ is such that $[w, z]B[w, z']$ and $([x, y'], [w, z']) \in Z_h$.
    * If $i = j$, then it must be $n < m$ (because $z < y$), and thus $f_{\mathcal{P}}(h-1) < n < m < t - f_{\mathcal{P}}(h-1) + 1$. Since $z \leq y' < y$, $i' = i$ and $n \leq m' < m$ hold. By the condition *(c)* of the definition of Z_h and by the fact that $x = w$ and $m \neq n$ hold, if $x, w \in \mathcal{P}$, then $|\mathcal{V}(y) - x| > f(h-1)$, $|\mathcal{V}(z) - w| > f(h-1)$, and $\mathcal{V}(y) > x$ iff $\mathcal{V}(z) > w$. Now, if $\mathcal{V}(y) < x$, then $|\mathcal{V}(y') - x| > |\mathcal{V}(y) - x| > f(h-1) > f(h)$; if $\mathcal{V}(y) > x$, then $|\mathcal{V}(y') - x| > |\mathcal{V}(z) - w| > f(h-1) > f(h)$. Then, $|\mathcal{V}(y') - x| > f(h)$ holds. We select $z' = z - 1$ and $[w, z']$ is such that $[w, z]B[w, z']$ and $([x, y'], [w, z']) \in Z_h$. Notice, in particular, that, if $x, w \in \mathcal{P}$, then $|\mathcal{V}(z') - w| > f(h)$ and $\mathcal{V}(y') > x$ iff $\mathcal{V}(z') > w$.

- If $x \neq w$, then either $x, w \in \mathcal{P}_{h-1}$ or $x, w \in \mathcal{A}$.

  - If $x, w \in \mathcal{P}_{h-1}$, then we distinguish the following cases.
    * If $y' \notin \mathcal{A} \cup \mathcal{P}_h$, then we select $z' = y'$ and $[w, z']$ is such that $[w, z]B[w, z']$ and $([x, y'], [w, z']) \in Z_h$.

* If $[x, y']$ is h-short (thus $y' \in \mathcal{P}_h$) then we select $z' = w + (y' - x)$. Notice that $z' \in \mathcal{P}_h$ as well, and $[w, z']$ is such that $[w, z]B[w, z']$ and $([x, y'], [w, z']) \in Z_h$.
* If $y' \in \mathcal{P}_h$ and $[x, y']$ is h-long, then we select $z' = min\{z, t - f_{\mathcal{P}}(h) + 1\} - 1$. Notice that z' is such that $z' < z$, $z' \in \mathcal{P}_h$, and $[w, z']$ is h-long. Thus, $[w, z']$ is such that $[w, z]B[w, z']$ and $([x, y'], [w, z']) \in Z_h$.
* If $y' \in \mathcal{A}$ (thus $[x, y']$ is h-long), let us denote y' by a^i_m. We select $z' = a^j_n \in \mathcal{A}$ such that $z' < z$ and according to the following:
  - if $m \leq f_{\mathcal{P}}(h)$ or $m \geq t - f_{\mathcal{P}}(h) + 1$, and $y' \notin A_h$ (i.e., $y' \in A^-_h \cup A^+_h$), then $n = m$ and $j = i$;
  - if $m \leq f_{\mathcal{P}}(h)$ or $m \geq t - f_{\mathcal{P}}(h) + 1$, and $y' \in A_h$, then $n = m$ and $f_{\mathcal{P}}(h) < j < t - f_{\mathcal{P}}(h) + 1$;
  - if $f_{\mathcal{P}}(h) < m < t - f_{\mathcal{P}}(h) + 1$ and $y' \notin A_h$ (i.e., $y' \in A^-_h \cup A^+_h$), then $f_{\mathcal{P}}(h) < n < t - f_{\mathcal{P}}(h) + 1$ and $j = i$;
  - if $f_{\mathcal{P}}(h) < m < t - f_{\mathcal{P}}(h) + 1$ and $y' \in A_h$, then $f_{\mathcal{P}}(h) < n < t - f_{\mathcal{P}}(h) + 1$ and $f_{\mathcal{P}}(h) < j < t - f_{\mathcal{P}}(h) + 1$.

  It is not difficult to verify that such a z' exists and that $\mathcal{V}(y') - x \simeq^f_h \mathcal{V}(z') - w$. To convince oneself about the latter, observe that, if $m \leq f_{\mathcal{P}}(h)$ (resp., $m \geq t - f_{\mathcal{P}}(h) + 1$), then $\mathcal{V}(y') = \mathcal{V}(z') \in \mathcal{P}^-_h$ (resp., $\mathcal{V}(y') = \mathcal{V}(z') \in \mathcal{P}^+_h$) holds; since, $x, w \in \mathcal{P}_{h-1}$, it follows $x - \mathcal{V}(y') > f(h)$ and $w - \mathcal{V}(z') > f(h)$ (resp., $\mathcal{V}(y') - x > f(h)$ and $\mathcal{V}(z') - w > f(h)$). Thus, $[w, z']$ is such that $[w, z]B[w, z']$ and $([x, y'], [w, z']) \in Z_h$.

– If $x, w \in \mathcal{A}$, let us denote x (resp., w) by a^i_m (resp., a^j_n). If $y' \notin \mathcal{A}$ (i.e., $y' \in \mathcal{R}^+$), then we select $z' = y'$, and $[w, z']$ is such that $[w, z]B[w, z']$ and $([x, y'], [w, z']) \in Z_h$. Similarly, if $[x, y']$ is h-short, then we choose z' such that $z' = w + (y' - x)$, and $[w, z']$ is such that $[w, z]B[w, z']$ and $([x, y'], [w, z']) \in Z_h$. Thus, we assume in this context that y', which we denote by $a^{i'}_{m'}$ is such that $y' \in \mathcal{A}$ and $[x, y']$ is h-long. We distinguish the following cases.

  * If $i' - i = 0$, we select $z' = a^{j'}_{n'}$ such that $z' < z$, $j' = j$, $[w, z']$ is h-long, and either $n' = m'$ (if $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h) + 1$) or $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$ (if $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h) + 1$). The interval $[w, z']$ is such that $[w, z]B[w, z']$ and $([x, y'], [w, z']) \in Z_h$.
  * If $0 < i' - i \leq f(h)$, then, by the properties of $f_{\mathcal{P}}$, it is guaranteed the existence of a point $z' = a^{j'}_{n'} < z$, with $j' = j + (i' - i)$ and either $n' = m'$ (if $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h) + 1$) or $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$ (if $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h) + 1$). The interval $[w, z']$ is such that $[w, z]B[w, z']$ and $([x, y'], [w, z']) \in Z_h$.
  * If $i' - i > f(h)$, then we distinguish four cases.
    - If either $i' \leq f_{\mathcal{P}}(h)$ or $i' \geq t - f_{\mathcal{P}}(h) + 1$, and either $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h) + 1$, then we select $z' = y'$.
    - If $f_{\mathcal{P}}(h) < i' < t - f_{\mathcal{P}}(h) + 1$ and either $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h) + 1$, then we select $z' = a^{j'}_{n'}$ such that $n' = m'$ and $j' = min\{k, t - f_{\mathcal{P}}(h) + 1\} - 1$, where k is t if $z \notin \mathcal{A}$, otherwise it is the index such that $z \in \mathcal{A}^k$.

- $\cdot$ If either $i' \leq f_{\mathcal{P}}(h)$ or $i' \geq t - f_{\mathcal{P}}(h) + 1$, and $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h) + 1$, then we select $z' = a^{j'}_{n'} < z$ such that $j' = i'$ and $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$.
- $\cdot$ If $f_{\mathcal{P}}(h) < i' < t - f_{\mathcal{P}}(h) + 1$ and $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h) + 1$, then we select $z' = a^{j'}_{n'}$ such that $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$ and $j' = min\{k, t - f_{\mathcal{P}}(h) + 1\} - 1$, where k is t if $z \notin \mathcal{A}$, otherwise it is the index such that $z \in \mathcal{A}^k$.

In all the above cases $[w, z']$ is such that $[w, z]B[w, z']$ and $([x, y'], [w, z']) \in Z_h$.

■

Lemma 4 (forward condition wrt $\langle\overline{B}\rangle$) *Let $[x, y]Z_{h-1}[w, z]$ for some $2 \leq h \leq N$ and let $[x, y]\bar{B}[x, y']$. Then, there exists an interval $[w, z']$ such that $[w, z]\bar{B}[w, z']$ and $[x, y']Z_h[w, z']$.*

Proof. The proof of this lemma proceeds likewise to the one of Lemma 3. As usual, we can safely assume $[x, y] \neq [w, z]$, as the property is trivially verified when $[x, y] = [w, z]$ holds. We proceed by cases.

- If $x = w$, then it must be $y \neq z$. Thus, both $[x, y]$ and $[w, z]$ are $(h-1)$-long. We can safely assume $y < y' \leq z$ because if it was $y' > z$, then we would select $z' = y'$ and we are done. Moreover, $[x, y']$ is h-long, because it is longer than $[x, y]$. Now, since $y \neq z$, either $y, z \in \mathcal{P}_{h-1}$ or $y, z \in \mathcal{A}$.
  - If $y, z \in \mathcal{P}_{h-1}$, then $y' \in \mathcal{P}_h$ (because $y < y' \leq z$). We select $z' = z + 1$, and $[w, z']$ is such that $[w, z]\bar{B}[w, z']$ and $([x, y'], [w, z']) \in Z_h$.
  - If $y, z \in \mathcal{A}$, then $y' \in \mathcal{A}$, too (due to $y < y' \leq z$). Let us denote y (resp., z, y') by a^i_m (resp., a^j_n, $a^{i'}_{m'}$). We distinguish the following cases.
    - $*$ If $i \neq j$ (notice that the only possibility is that $i < z$, because $y < z$), then $y, z \in \mathcal{A}_{h-1}$ (i.e., $f_{\mathcal{P}}(h-1) < i < j < t - f_{\mathcal{P}}(h-1) + 1$). From $y < y' \leq z$, it follows $i \leq i' \leq j$, which implies $y' \in \mathcal{A}_h$. Moreover, by the condition *(d)* of the definition of Z_h, if $x(= w) \in \mathcal{A}^k$, for some k, then $i - k > f(h-1)$ and $j - k > f(h-1)$. We choose z' in $\mathcal{A}^{j+1}$ such that $z' - a^{j+1}_1 = y' - a^{i'}_1$, and $[w, z']$ is such that $[w, z]\bar{B}[w, z']$ and $([x, y'], [w, z']) \in Z_h$.
    - $*$ If $i = j$, then it must be $m < n$ (because $y < z$), and thus $f_{\mathcal{P}}(h-1) < m < n < t - f_{\mathcal{P}}(h-1) + 1$. Since $y < y' \leq z$, $i' = i$ and $m < m' \leq n$ hold. By the condition *(c)* of the definition of Z_h and by the fact that $x = w$ and $m \neq n$ hold, if $x, w \in \mathcal{P}$, then $|\mathcal{V}(y) - x| > f(h-1)$, $|\mathcal{V}(z) - w| > f(h-1)$, and $\mathcal{V}(y) > x$ iff $\mathcal{V}(z) > w$. Now, if $\mathcal{V}(y) < x$, then $|\mathcal{V}(y') - x| > |\mathcal{V}(z) - w| > f(h-1) > f(h)$; if $\mathcal{V}(y) > x$, then $|\mathcal{V}(y') - x| > |\mathcal{V}(y) - x| > f(h-1) > f(h)$. Then, $|\mathcal{V}(y') - x| > f(h)$ holds. We select $z' = z + 1$ and $[w, z']$ is such that $[w, z]\bar{B}[w, z']$ and $([x, y'], [w, z']) \in Z_h$. Notice, in particular, that, if $x, w \in \mathcal{P}$, then $|\mathcal{V}(z') - w| > f(h)$ and $\mathcal{V}(y') > x$ iff $\mathcal{V}(z') > w$.
- If $x \neq w$, then either $x, w \in \mathcal{P}_{h-1}$ or $x, w \in \mathcal{A}$.
  - If $x, w \in \mathcal{P}_{h-1}$, then we distinguish the following cases.

* If $y' \notin \mathcal{A} \cup \mathcal{P}_h$, then we select $z' = y'$ and $[w, z']$ is such that $[w, z]\bar{B}[w, z']$ and $([x, y'], [w, z']) \in Z_h$.
* If $[x, y']$ is h-short (thus $y' \in \mathcal{P}_h$) then we select $z' = w + (y' - x)$. Notice that $z' \in \mathcal{P}_h$ as well, and $[w, z']$ is such that $[w, z]\bar{B}[w, z']$ and $([x, y'], [w, z']) \in Z_h$.
* If $y' \in \mathcal{P}_h$ and $[x, y']$ is h-long, then it must be $y < y' < p_{t-f_{\mathcal{P}}(h)+1}$, which, by $y \equiv_h z$, implies $z < p_{t-f_{\mathcal{P}(h)}+1} - 1$. Then, we select $z' = z+1$. Notice that z' is such that $z' > z$, $z' \in \mathcal{P}_h$, and $[w, z']$ is h-long. Thus, $[w, z']$ is such that $[w, z]\bar{B}[w, z']$ and $([x, y'], [w, z']) \in Z_h$.
* If $y' \in \mathcal{A}$ (thus $[x, y']$ is h-long), let us denote y' by a^i_m. We select $z' = a^j_n \in \mathcal{A}$ such that $z' > z$ and according to the following:
  · if $m \leq f_{\mathcal{P}}(h)$ or $m \geq t - f_{\mathcal{P}}(h) + 1$, and $y' \notin A_h$ (i.e., $y' \in A^-_h \cup A^+_h$), then $n = m$ and $j = i$;
  · if $m \leq f_{\mathcal{P}}(h)$ or $m \geq t - f_{\mathcal{P}}(h) + 1$, and $y' \in A_h$, then $n = m$ and $f_{\mathcal{P}}(h) < j < t - f_{\mathcal{P}}(h) + 1$;
  · if $f_{\mathcal{P}}(h) < m < t - f_{\mathcal{P}}(h) + 1$ and $y' \notin A_h$ (i.e., $y' \in A^-_h \cup A^+_h$), then $f_{\mathcal{P}}(h) < n < t - f_{\mathcal{P}}(h) + 1$ and $j = i$;
  · if $f_{\mathcal{P}}(h) < m < t - f_{\mathcal{P}}(h) + 1$ and $y' \in A_h$, then $f_{\mathcal{P}}(h) < n < t - f_{\mathcal{P}}(h) + 1$ and $f_{\mathcal{P}}(h) < j < t - f_{\mathcal{P}}(h) + 1$.

  It is not difficult to verify that such a z' exists and that $\mathcal{V}(y') - x \simeq^f_h \mathcal{V}(z') - w$. To convince oneself about the latter, observe that, if $m \leq f_{\mathcal{P}}(h)$ (resp., $m \geq t - f_{\mathcal{P}}(h) + 1$), then $\mathcal{V}(y') = \mathcal{V}(z') \in \mathcal{P}^-_h$ (resp., $\mathcal{V}(y') = \mathcal{V}(z') \in \mathcal{P}^+_h$) holds; since, $x, w \in \mathcal{P}_{h-1}$, it follows $x - \mathcal{V}(y') > f(h)$ and $w - \mathcal{V}(z') > f(h)$ (resp., $\mathcal{V}(y') - x > f(h)$ and $\mathcal{V}(z') - w > f(h)$). Thus, $[w, z']$ is such that $[w, z]\bar{B}[w, z']$ and $([x, y'], [w, z']) \in Z_h$.

– If $x, w \in \mathcal{A}$, let us denote x (resp., w) by a^i_m (resp., a^j_n). If $y' \notin \mathcal{A}$ (i.e., $y' \in \mathcal{R}^+$), then we select $z' = y'$, and $[w, z']$ is such that $[w, z]\bar{B}[w, z']$ and $([x, y'], [w, z']) \in Z_h$. Similarly, if $[x, y']$ is h-short, then we choose z' such that $z' = w + (y' - x)$, and $[w, z']$ is such that $[w, z]\bar{B}[w, z']$ and $([x, y'], [w, z']) \in Z_h$. Thus, we assume in this context that y', which we denote by $a^{i'}_{m'}$ is such that $y' \in \mathcal{A}$ and $[x, y']$ is h-long. We distinguish the following cases.

  * If $i' - i = 0$, we select $z' = a^{j'}_{n'}$ such that $j' = j$, $[w, z']$ is h-long, and either $n' = m'$ (if $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h) + 1$) or $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$ (if $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h) + 1$).
  * If $0 < i' - i \leq f(h)$, then, by the properties of $f_{\mathcal{P}}$, it is guaranteed the existence of a point $z' = a^{j'}_{n'} > z$, with $j' = j + (i' - i)$ and either $n' = m'$ (if $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h) + 1$) or $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$ (if $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h) + 1$). The interval $[w, z']$ is such that $[w, z]\bar{B}[w, z']$ and $([x, y'], [w, z']) \in Z_h$.
  * If $i' - i > f(h)$, then we distinguish four cases.
    · If either $i' \leq f_{\mathcal{P}}(h)$ or $i' \geq t - f_{\mathcal{P}}(h) + 1$, and either $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h) + 1$, then we select $z' = y'$.
    · If $f_{\mathcal{P}}(h) < i' < t - f_{\mathcal{P}}(h) + 1$ and either $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h) + 1$, then we select $z' = a^{j'}_{n'}$ such that $n' = m'$ and $j' = max\{k, f_{\mathcal{P}}(h)\} + 1$, where k is the index such that $z \in \mathcal{A}^k$.

- · If either $i' \le f_{\mathcal{P}}(h)$ or $i' \ge t - f_{\mathcal{P}}(h) + 1$, and $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h) + 1$, then we select $z' = a_{n'}^{j'}$ such that $j' = i'$ and $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$.
- · If $f_{\mathcal{P}}(h) < i' < t - f_{\mathcal{P}}(h) + 1$ and $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h) + 1$, then we select $z' = a_{n'}^{j'}$ such that $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$ and $j' = max\{k, f_{\mathcal{P}}(h)\} + 1$, where k is the index such that $z \in \mathcal{A}^k$.

In all the above cases $[w, z']$ is such that $[w, z]\bar{B}[w, z']$ and $([x, y'], [w, z']) \in Z_h$.

■

Lemma 5 (forward condition wrt $\langle E \rangle$) *Let $[x, y]Z_{h-1}[w, z]$ for some $2 \le h \le N$ and let $[x, y]E[x', y]$. Then, there exists an interval $[w', z]$ such that $[w, z]E[w', z]$ and $[x', y]Z_h[w', z]$.*

Proof. As we did above, we will assume $[x, y] \neq [w, z]$. We proceed by cases.

- If $y = z$, then it must be $x \neq w$. Thus, both $[x, y]$ and $[w, z]$ are $(h-1)$-long. As usual, we can assume $x < x' \le w$ because if it was $x' > w$, then we would select $w' = x'$ and we are done. Moreover, $[x', y]$ is h-long, because it is not shorter than $[w, z]$, which, in turn, is $(h-1)$-long, and thus also h-long. Now, since $x \neq w$, either $x, w \in \mathcal{P}_{h-1}$ or $x, w \in \mathcal{A}$.
  - If $x, w \in \mathcal{P}_{h-1}$, then $x' \in \mathcal{P}_h$ (because $x < x' \le w$). Notice that, if $y(= z) \in \mathcal{A}$, then, by condition *(c)* of definition of Z_h, and since $x \neq w$ and $y = z$ hold, it must be $|\mathcal{V}(y) - x| > f(h-1)$, $|\mathcal{V}(z) - w| > f(h-1)$, and $\mathcal{V}(y) > x$ iff $\mathcal{V}(z) > w$. Now, if $\mathcal{V}(y) < x$, then $|\mathcal{V}(y) - x'| > |\mathcal{V}(y) - x| > f(h-1) > f(h)$; if $\mathcal{V}(y) > x$, then $|\mathcal{V}(y) - x'| > |\mathcal{V}(z) - w| > f(h-1) > f(h)$. Then, $|\mathcal{V}(y) - x'| > f(h)$ holds. We select $w' = w + 1$, and $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$. Notice, in particular, that, if $y(= z) \in \mathcal{A}$, then $|\mathcal{V}(z) - w'| > f(h)$ and $\mathcal{V}(y) > x'$ iff $\mathcal{V}(z) > w'$.
  - if $x, w \in \mathcal{A}$, then $x' \in \mathcal{A}$, too (due to $x < x' \le w$). Let us denote x (resp., w, x') by a_m^i (resp., a_n^j, $a_{m'}^{i'}$). We distinguish the following cases.
    - ∗ If $i \neq j$ (notice that the only possibility is that $i < j$, because $x < w$), then $x, w \in \mathcal{A}_{h-1}$ (i.e., $f_{\mathcal{P}}(h-1) < i < j < t - f_{\mathcal{P}}(h-1) + 1$). From $x < x' \le w$, it follows $i \le i' \le j$, which implies $x' \in \mathcal{A}_h$. Moreover, by the condition *(d)* of the definition of Z_h, if $y(= z) \in \mathcal{A}^k$, for some k, then $k - i > f(h-1)$ and $k - j > f(h-1)$. We choose w' in $\mathcal{A}^{j+1}$ such that $w' - a_1^{j+1} = x' - a_1^{i'}$, and $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$.
    - ∗ If $i = j$, then it must be $m < n$ (because $x < w$), and thus $f_{\mathcal{P}}(h-1) < m < n < t - f_{\mathcal{P}}(h-1) + 1$. Since $x < x' \le w$, $i' = i$ and $m < m' \le n$ hold. We select $w' = w + 1$ and $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$.
- If $y \neq z$, then either $y, z \in \mathcal{P}_{h-1}$ or $y, z \in \mathcal{A}$.
  - If $y, z \in \mathcal{P}_{h-1}$, then we distinguish the following cases.

- * If $x' \leq p_{f_{\mathcal{P}}(h)}$ (i.e., $x' \notin \mathcal{P}_h$), then we select $w' = x'$, and $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$.
  * If $[x', y]$ is h-long and $x' \in \mathcal{P}_h$, then we select $w' = max\{p_{f_{\mathcal{P}}(h)}, w\} + 1$. Notice that it holds $w' > p_{f_{\mathcal{P}}(h)}$, which implies $w' \in \mathcal{P}_h$. Thus, $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$.
  * If $[x', y]$ is h-short (and thus, by properties of $f_{\mathcal{P}}(h)$, $x' \in \mathcal{P}_h$), then we select $w' = z - (y - x')$, and $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$. Notice that, again by properties of $f_{\mathcal{P}}(h)$, $w' \in \mathcal{P}_h$, too.
- If $y, z \in \mathcal{A}$, let us denote y (resp., z) by a^i_m (resp., a^j_n). We distinguish the following cases.
  * If $x' \notin \mathcal{A} \cup \mathcal{P}$, then we select $w' = x'$, and $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$.
  * If $x' \in \mathcal{P}$, then we distinguish the following cases.
    · If $m = n$, then $\mathcal{V}(y) = \mathcal{V}(z)$. Thus, we can assume $x < x' \leq w$ (indeed, if it was $x' > w$, then we would select $w' = x'$ and we are done). Notice that $x \neq w$ implies $x, w \in \mathcal{P}_{h-1}$ or $x, w \in \mathcal{A}$. Since $x' \in \mathcal{P}$ and $x < x'$, it must be $x, w \in \mathcal{P}_{h-1}$. Moreover, from $x < x' \leq w$, it follows $x' \in \mathcal{P}_{h-1} \subseteq \mathcal{P}_h$. We select $w' = w + 1$ and $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$. Notice that, to guarantee condition *(c)* of the definition of Z_h, we can use the same argument used before (case $y = z$, with $x, w \in \mathcal{P}_{h-1}$, of this lemma).
    · If $m \neq n$, then it must be $f_{\mathcal{P}}(h-1) < min\{m, n\} < max\{m, n\} < t - f_{\mathcal{P}}(h-1) + 1$, and thus $\mathcal{V}(y), \mathcal{V}(z) \in \mathcal{P}_{h-1}$. We distinguish the following cases.
      • If $x' \in \mathcal{P}^-_h \cup P^+_h$, then we select $w' = x'$, and $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$. Notice that, from $\mathcal{V}(y), \mathcal{V}(z) \in \mathcal{P}_{h-1}$, condition *(c)* of the definition of Z_h immediately follows.
      • If $x' \in \mathcal{P}_h$ and $\mathcal{V}(y) - x' > f(h)$, then $\mathcal{V}(y) - x > f(h) + 1$ and $\mathcal{V}(z) - w > f(h) + 1$. We select $w' = max\{w, p_{f_{\mathcal{P}}(h)}\} + 1$ and $\mathcal{V}(z) - w' > f(h)$ holds. Thus, $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$.
      • If $x' \in \mathcal{P}_h$ and $|\mathcal{V}(y) - x'| \leq f(h)$, then we select $w' = \mathcal{V}(z) - (\mathcal{V}(y) - x')$, and $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$. Notice that, since $|\mathcal{V}(y) - x'| \leq f(h)$ and $\mathcal{V}(z) \in \mathcal{P}_{h-1}$, it follows $w' \in \mathcal{P}_h$.
      • If $x' \in \mathcal{P}_h$ and $x' - \mathcal{V}(y) > f(h)$, then we select $w' = max\{w, \mathcal{V}(z) + f(h)\} + 1$. The interval $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$.
  * If $x' \in \mathcal{A}$ and $[x', y]$ is h-short, then we select $w' = z - (y - x')$, and $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$.
  * If $x' \in \mathcal{A}$ and $[x', y]$ is h-long, let us denote x' by $a^{i'}_{m'}$. We distinguish the following cases.
    · If $i - i' = 0$, we select $w' = a^{j'}_{n'}$ such that $w' > w$, $j' = j$, $[w', z]$ is h-long, and either $n' = m'$ (if $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq$

$t - f_{\mathcal{P}}(h) + 1$) or $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$ (if $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h) + 1$). The interval $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$.

· If $0 < i - i' \leq f(h)$, then, by the properties of $f_{\mathcal{P}}$, it is guaranteed the existence of a point $w' = a^{j'}_{n'} > w$, with $j' = j - (i - i')$ and either $n' = m'$ (if $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h) + 1$) or $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$ (if $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h) + 1$). The interval $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$.

· If $i - i' > f(h)$, then we distinguish four cases.

- If either $i' \leq f_{\mathcal{P}}(h)$ or $i' \geq t - f_{\mathcal{P}}(h) + 1$, and either $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h) + 1$, then we select $w' = x'$.
- If $f_{\mathcal{P}}(h) < i' < t - f_{\mathcal{P}}(h) + 1$ and either $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h) + 1$, then we select $w' = a^{j'}_{n'}$ such that $n' = m'$ and $j' = max\{k, f_{\mathcal{P}}(h)\} + 1$, where k is 0 if $w \notin \mathcal{A}$, otherwise it is the index such that $w \in \mathcal{A}^k$.
- If either $i' \leq f_{\mathcal{P}}(h)$ or $i' \geq t - f_{\mathcal{P}}(h) + 1$, and $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h) + 1$, then we select $w' = a^{j'}_{n'} > w$ such that $j' = i'$ and $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$.
- If $f_{\mathcal{P}}(h) < i' < t - f_{\mathcal{P}}(h) + 1$ and $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h) + 1$, then we select $w' = a^{j'}_{n'}$ such that $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$ and $j' = max\{k, f_{\mathcal{P}}(h)\} + 1$, where k is 0 if $w \notin \mathcal{A}$, otherwise it is the index such that $w \in \mathcal{A}^k$.

In all the above cases $[w', z]$ is such that $[w, z]E[w', z]$ and $([x', y], [w', z]) \in Z_h$.

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Lemma 6 (forward condition wrt $\langle \overline{E} \rangle$) *Let $[x, y]Z_{h-1}[w, z]$ for some $2 \leq h \leq N$ and let $[x, y]\bar{E}[x', y]$. Then, there exists an interval $[w', z]$ such that $[w, z]\bar{E}[w', z]$ and $[x', y]Z_h[w', z]$.*

Proof. As we did above, we will assume $[x, y] \neq [w, z]$. We proceed by cases.

- If $y = z$, then it must be $x \neq w$. Thus, both $[x, y]$ and $[w, z]$ are $(h-1)$-long. As usual, we can assume $w \leq x' < x$ because if it was $x' < w$, then we would select $w' = x'$ and we are done. Moreover, $[x', y]$ is h-long, because it is longer than $[x, y]$, which, in turn, is $(h-1)$-long, and thus also h-long. Now, since $x \neq w$, either $x, w \in \mathcal{P}_{h-1}$ or $x, w \in \mathcal{A}$.

  – If $x, w \in \mathcal{P}_{h-1}$, then $x' \in \mathcal{P}_h$ (because $w \leq x' < x$). Notice that, if $y (= z) \in \mathcal{A}$, then, by condition *(c)* of definition of Z_h, and since $x \neq w$ and $y = z$ hold, it must be $|\mathcal{V}(y) - x| > f(h-1)$, $|\mathcal{V}(z) - w| > f(h-1)$, and $\mathcal{V}(y) > x$ iff $\mathcal{V}(z) > w$. Now, if $\mathcal{V}(y) < x$, then $|\mathcal{V}(y) - x'| > |\mathcal{V}(z) - w| > f(h-1) > f(h)$; if $\mathcal{V}(y) > x$, then $|\mathcal{V}(y) - x'| > |\mathcal{V}(y) - x| > f(h-1) > f(h)$. Then, $|\mathcal{V}(y) - x'| > f(h)$ holds. We select $w' = w - 1$, and $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$. Notice, in particular, that, if $y (= z) \in \mathcal{A}$, then $|\mathcal{V}(z) - w'| > f(h)$ and $\mathcal{V}(y) > x'$ iff $\mathcal{V}(z) > w'$.

- if $x, w \in \mathcal{A}$, then $x' \in \mathcal{A}$, too (due to $w \leq x' < x$). Let us denote x (resp., w, x') by a^i_m (resp., a^j_n, $a^{i'}_{m'}$). We distinguish the following cases.
  - If $i \neq j$ (notice that the only possibility is that $j < i$, because $w < x$), then $x, w \in \mathcal{A}_{h-1}$ (i.e., $f_{\mathcal{P}}(h-1) < j < i < t - f_{\mathcal{P}}(h-1)+1$). From $w \leq x' < x$, it follows $j \leq i' \leq i$, which implies $x' \in \mathcal{A}_h$. Moreover, by the condition *(d)* of the definition of Z_h, if $y(= z) \in \mathcal{A}^k$, for some k, then $k - i > f(h-1)$ and $k - j > f(h-1)$. We choose w' in $\mathcal{A}^{j-1}$ such that $w' - a^{j-1}_1 = x' - a^{i'}_1$, and $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$.
  - If $i = j$, then it must be $n < m$ (because $w < x$), and thus $f_{\mathcal{P}}(h-1) < n < m < t - f_{\mathcal{P}}(h-1) + 1$. Since $w \leq x' < x$, $i' = i$ and $n \leq m' < m$ hold. We select $w' = w - 1$ and $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$.

- If $y \neq z$, then either $y, z \in \mathcal{P}_{h-1}$ or $y, z \in \mathcal{A}$.
  - If $y, z \in \mathcal{P}_{h-1}$, then we distinguish the following cases.
    - If $x' \leq p_{f_{\mathcal{P}}(h)}$ (i.e., $x' \notin \mathcal{P}_h$), then we select $w' = x'$, and $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$.
    - If $[x', y]$ is h-long and $x' \in \mathcal{P}_h$, then it must be $p_{f_{\mathcal{P}}(h)} < x' < x$, which, by $x \equiv_h w$, implies $w > p_{f_{\mathcal{P}(h)}} + 1$. Then, we select $w' = w - 1$. Notice that w' is such that $w' < w$, $w' \in \mathcal{P}_h$, and $[w', z]$ is h-long. Thus, $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$.
    - If $[x', y]$ is h-short (and thus, by properties of $f_{\mathcal{P}}(h)$, $x' \in \mathcal{P}_h$), then we select $w' = z - (y - x')$, and $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$. Notice that, again by properties of $f_{\mathcal{P}}(h)$, $w' \in \mathcal{P}_h$, too.
  - If $y, z \in \mathcal{A}$, let us denote y (resp., z) by a^i_m (resp., a^j_n). We distinguish the following cases.
    - If $x' \notin \mathcal{A} \cup \mathcal{P}$, then we select $w' = x'$, and $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$.
    - If $x' \in \mathcal{P}$, then we distinguish the following cases.
      - If $m = n$, then $\mathcal{V}(y) = \mathcal{V}(z)$. Thus, we can assume $w \leq x' < x$ (indeed, if it was $x' < w$, then we would select $w' = x'$ and we are done). Notice that $x \neq w$ implies $x, w \in \mathcal{P}_{h-1}$ or $x, w \in \mathcal{A}$. Since $x' \in \mathcal{P}$ and $w \leq x'$, it must be $x, w \in \mathcal{P}_{h-1}$. Moreover, from $w \leq x' < x$, it follows $x' \in \mathcal{P}_{h-1} \subseteq \mathcal{P}_h$. We select $w' = w - 1$ and $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$. Notice that, to guarantee condition *(c)* of the definition of Z_h, we can use the same argument used before (case $y = z$, with $x, w \in \mathcal{P}_{h-1}$, of this lemma).
      - If $m \neq n$, then it must be $f_{\mathcal{P}}(h-1) < min\{m, n\} < max\{m, n\} < t - f_{\mathcal{P}}(h-1) + 1$, and thus $\mathcal{V}(y), \mathcal{V}(z) \in \mathcal{P}_{h-1}$. We distinguish the following cases.
        - If $x' \in \mathcal{P}^-_h \cup P^+_h$, then we select $w' = x'$, and $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$. Notice that, from $\mathcal{V}(y), \mathcal{V}(z) \in \mathcal{P}_{h-1}$, condition *(c)* of the definition of Z_h immediately follows.

- If $x' \in \mathcal{P}_h$ and $x' - \mathcal{V}(y) > f(h)$, then $x - \mathcal{V}(y) > f(h)+1$ and $w - \mathcal{V}(z) > f(h)+1$. We select $w' = min\{w, p_{t-f_{\mathcal{P}}(h)+1}\} - 1$ and $w' - \mathcal{V}(z) > f(h)$ holds. Thus, $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$.
- If $x' \in \mathcal{P}_h$ and $|\mathcal{V}(y) - x'| \leq f(h)$, then we select $w' = \mathcal{V}(z) - (\mathcal{V}(y) - x')$, and $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$. Notice that, since $|\mathcal{V}(y) - x'| \leq f(h)$ and $\mathcal{V}(z) \in \mathcal{P}_{h-1}$, it follows $w' \in \mathcal{P}_h$.
- If $x' \in \mathcal{P}_h$ and $\mathcal{V}(y) - x' > f(h)$, then we select $w' = min\{w, \mathcal{V}(z) - f(h)\} - 1$. The interval $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$.

$*$ If $x' \in \mathcal{A}$ and $[x', y]$ is h-short, then we select $w' = z - (y - x')$, and $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$.

$*$ If $x' \in \mathcal{A}$ and $[x', y]$ is h-long, then $x \in \mathcal{A}$, which implies $w \in \mathcal{A}$, too. Let us denote x' (resp., x, w) by $a^{i'}_{m'}$ (resp., a^k_ℓ, $a^{k'}_{\ell'}$). We distinguish the following cases.

$\cdot$ If $i - i' = 0$, we select $w' = a^{j'}_{n'}$ such that $w' < w$, $j' = j$, $[w', z]$ is h-long, and either $n' = m'$ (if $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h)+1$) or $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h)+1$ (if $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h)+1$). The interval $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$.

$\cdot$ If $0 < i - i' \leq f(h)$, then, by the properties of $f_{\mathcal{P}}$, it is guaranteed the existence of a point $w' = a^{j'}_{n'} < w$, with $j' = j - (i - i')$ and either $n' = m'$ (if $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h)+1$) or $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h)+1$ (if $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h)+1$). The interval $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$.

$\cdot$ If $i - i' > f(h)$, then we distinguish four cases.

- If either $i' \leq f_{\mathcal{P}}(h)$ or $i' \geq t - f_{\mathcal{P}}(h) + 1$, and either $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h) + 1$, then we select $w' = x'$.
- If $f_{\mathcal{P}}(h) < i' < t - f_{\mathcal{P}}(h) + 1$ and either $m' \leq f_{\mathcal{P}}(h)$ or $m' \geq t - f_{\mathcal{P}}(h)+1$, then we select $w' = a^{j'}_{n'}$ such that $n' = m'$ and $j' = min\{k - 1, f_{\mathcal{P}}(h) + 1\}$.
- If either $i' \leq f_{\mathcal{P}}(h)$ or $i' \geq t - f_{\mathcal{P}}(h) + 1$, and $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h) + 1$, then we select $w' = a^{j'}_{n'} < w$ such that $j' = i'$ and $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$.
- If $f_{\mathcal{P}}(h) < i' < t - f_{\mathcal{P}}(h)+1$ and $f_{\mathcal{P}}(h) < m' < t - f_{\mathcal{P}}(h)+1$, then we select $w' = a^{j'}_{n'}$ such that $f_{\mathcal{P}}(h) < n' < t - f_{\mathcal{P}}(h) + 1$ and $j' = min\{k - 1, f_{\mathcal{P}}(h) + 1\}$.

In all the above cases $[w', z]$ is such that $[w, z]\bar{E}[w', z]$ and $([x', y], [w', z]) \in Z_h$.

■

Lemma 7 *Let $[x, y]Z_h[w, z]$ for some $1 \leq h < N$ and let $[x, y]X[x', y']$ for some $X \in \{A, L, B, E, D, O, \bar{A}, \bar{L}, \bar{B}, \bar{E}, \bar{D}, \bar{O}\}$. Then, there exists an interval $[w', z']$ such that $[w, z]X[w', z']$ and $([x', y'], [w', z']) \in Z_{h+1} \cup Z_{h+2} \cup Z_{h+3} \cup Z_{h+4}$.*

Proof. If $X = B$ (resp., $X = E$, $X = \bar{B}$, $X = \bar{E}$), then the thesis immediately follows from Lemma 3 (resp., Lemma 5, Lemma 4, Lemma 6).

As for the other modalities, we use the known inter-definabilities as follows. Consider the case $X = D$. Let $[x, y]Z_h[w, z]$ for some $1 \leq h < N - 1$ and let $[x, y]D[x', y']$. We show that there exists an interval $[w', z']$ such that $[w, z]D[w', z']$ and $([x', y'], [w', z']) \in Z_{h+2}$. Consider the interval $[x', y]$: it holds $[x, y]E[x', y]$. By Lemma 5, there exists $[w'', z'']$ such that $([x', y], [w'', z'']) \in Z_{h+1}$, with $[w, z]E[w'', z'']$. By Lemma 3 (on $([x', y], [w'', z''])$), since $[x', y]B[x', y']$, there exists $[w', z']$ such that $([x', y'], [w', z']) \in Z_{h+2}$, with $[w'', z'']B[w', z']$. It is easy to see that $[w, z]D[w', z']$, hence the thesis.

A very similar argument can be used to proof that if $[x, y]Z_h[w, z]$ for some $1 \leq h < N-1$ and if $[x, y]X[x', y']$ for some $X \in \{\bar{D}, O, \bar{O}\}$, then there exists an interval $[w', z']$ such that $[w, z]X[w', z']$ and $([x', y'], [w', z']) \in Z_{h+2}$.

The cases where $X = A$ and $X = \bar{A}$ hinge on the same argument but they are a bit more involved. We give the details only the former case ($X = A$); the other case is symmetric. Let $[x, y]Z_h[w, z]$ for some $1 \leq h < N-1$ and let $[x, y]A[x', y']$. We show that there exists an interval $[w', z']$ such that $[w, z]A[w', z']$ and $([x', y'], [w', z']) \in Z_{h+4}$. Consider the interval $[y-1, y+1]$: it is such that either $[x, y]\bar{B}[x', y]$ (if $x = y-1$) or $[x, y]O[x', y]$ (if $x < y-1$). In the former case we have $([y-1, y+1], [z-1, z+1]) \in Z_{h+1}$ by Lemma 4; in the latter one we have $([y-1, y+1], [z-1, z+1]) \in Z_{h+2}$ by the case $X = O$ above. If $([y-1, y+1], [z-1, z+1]) \in Z_{h+1}$ (resp., $([y-1, y+1], [z-1, z+1]) \in Z_{h+2}$), then, by Lemma 5, we have $([y, y+1], [z, z+1]) \in Z_{h+2}$ (resp., $([y, y+1], [z, z+1]) \in Z_{h+3}$). Now, if it was the case that $[x', y'] = [y, y+1]$, then we are done as we $[z, z+1]$ is the interval we were looking for. If that was not the case, then it would be $[y, y+1]\bar{B}[x', y']$ and, by Lemma 4, there would exist $[w', z']$ such that $[z, z+1]\bar{b}[w', z']$ and $([y, y+1], [z, z+1]) \in Z_{h+3}$ (if $x = y-1$) or $([y, y+1], [z, z+1]) \in Z_{h+4}$ (if $x < y-1$). It is easy to see that $[w, z]A[w', z']$, hence the thesis.

Finally, the cases where $X = L$ and $X = \bar{L}$ can be dealt with by using the same argument and the inter-definabilities of $\langle L \rangle$ and $\langle \overline{L} \rangle$ in terms of $\langle A \rangle$ and $\langle \overline{A} \rangle$, respectively. ■

Let us recall that by a^i_m we denote the mth element of the set $\mathcal{A}^i$, for each i and m, and by p_m we denote the mth element of the set $\mathcal{P}$, for each m. Now, let $a = a^i_m$, with $i = m = \frac{t}{2}$, that is, a is the $\frac{t}{2}$th point in the $\frac{t}{2}$ sub-group $\mathcal{A}^{\frac{t}{2}}$ of $\mathcal{A}$. Roughly speaking, a is the central point in the central sub-group in $\mathcal{A}$. Consequently, $\mathcal{V}(a) = p_m = p_{\frac{t}{2}}$ is the central point in $\mathcal{P}$.

Now, we define the sequence of relations $Z^E_h = \{([\mathcal{V}(a)-h, a], [\mathcal{V}(a)-h+1, a])\}$, for $h \in \{1, \ldots, N\}$. It is worth pointing out that $([\mathcal{V}(a)-h, a], [\mathcal{V}(a)-h+1, a]) \notin Z_h$, for any h. This is because condition *(c)* of the definition of Z_h is not fulfilled by pairs in Z^E_h (on the other hand, conditions *(a)*, *(b)*, and *(d)* are verified). Notice also that, unlike Z_h, relations Z^E_h ($1 \leq h \leq N$) are not symmetric. It is easy to see that $([\mathcal{V}(a)-1, a], [\mathcal{V}(a), a]) \in Z^E_1$ is such that $M, [\mathcal{V}(a)-1, a] \Vdash \langle E \rangle p$, while $M, [\mathcal{V}(a), a] \Vdash \neg\langle E \rangle p$. In the following lemma, we use x as an abbreviation for $\mathcal{V}(a) - (h-1)$.

Lemma 8 *Let* $X \in \{A, B, D, O, \bar{A}, \bar{B}, \bar{E}\}$ *and* $([x, a], [x+1, a]) \in Z^E_{h-1}$ *for some* $h \in \{2, \ldots, N\}$. *If* $[x, a]X[x', y']$ *for some interval* $[x', y']$*, then there exists an interval* $[w', z']$ *such that* $[x+1, a]X[w', z']$ *and* $[x', y']Z_h \cup Z^E_h[w', z']$ *(forward condition). Moreover, if* $[x+1, a]X[w', z']$ *for some interval* $[w', z']$*, then there exists an interval* $[x', y']$ *such that* $[x, a]X[x', y']$ *and* $[x', y']Z_h \cup Z^E_h[w', z']$ *(backward condition).*

Proof. We first prove the forward condition and then the backward one.
Forward condition. Let $X \in \{A, B, D, O, \bar{A}, \bar{B}, \bar{E}\}$ and $([x, a], [x+1, a]) \in Z^E_{h-1}$, for some $h \in \{2, \ldots, N\}$. We assume that $[x, a]X[x', y']$ holds, and we show the existence of an interval $[w', z']$ such that $[x+1, a]X[w', z']$ and $[x', y']Z_h \cup Z^E_h[w', z']$. First, notice that both x and $x+1$ belong to $\mathcal{P}_{h-1}$ (and thus to $\mathcal{P}_h$ as well). The proof proceeds by cases, depending on the value of X.

- If $X = A$, then we select $z' = y'$, and $[a, z']$ is such that $[x+1, a]A[a, z']$ and $[a, y']Z_h[a, z']$.
- If $X = B$, then the proof proceeds exactly as the one for the case of $x \neq w$ and $x, w \in \mathcal{P}_{h-1}$ in Lemma 3 (condition *(c)* of the definition of Z_h, which discriminates against pairs in Z^E_h, is not used there).
- If $X = D$, then we distinguish two cases. If $x' > x+1$, then we select $w' = x'$ and $z' = y'$, and $[w', z']$ is such that $[x+1, a]D[w', z']$ and $[x', y']Z_h[w', z']$. If, on the other hand, $x' = x+1$, then the proof proceeds exactly as the one for the case of $x \neq w$ and $x, w \in \mathcal{P}_{h-1}$ in Lemma 3, with $x+1$ in x's stead, $x+2$ in w's stead, and a in both y's and z's stead (again, condition *(c)* of the definition of Z_h is not used there).
- If $X = O$, then we distinguish two cases. If $x' > x+1$, then we select $w' = x'$ and $z' = y'$, and $[w', z']$ is such that $[x+1, a]O[w', z']$ and $[x', y']Z_h[w', z']$. If, on the other hand, $x' = x+1$, then the proof proceeds exactly as the one for the case of $x \neq w$ and $x, w \in \mathcal{P}_{h-1}$ in Lemma 4, with $x+1$ in x's stead, $x+2$ in w's stead, and a in both y's and z's stead (again, condition *(c)* of the definition of Z_h is not used there — notice also that only the first and fourth sub-cases play a role here).
- If $X = \bar{A}$, then we distinguish two cases. If $[x', x]$ is h-long, then we select $w' = x'$, and $[w', x+1]$ is such that $[x+1, a]\bar{A}[w', x+1]$ and $[x', x]Z_h[w', x+1]$. Otherwise, if $[x', x]$ is h-short, then we select $w' = (x+1) - (x - x')$, and $[w', x+1]$ is such that $[x+1, a]\bar{A}[w', x+1]$ and $[x', x]Z_h[w', x+1]$.
- If $X = \bar{B}$, then the proof proceeds exactly as the one for the case of $x \neq w$ and $x, w \in \mathcal{P}_{h-1}$ in Lemma 4, (again, condition *(c)* of the definition of Z_h is not used there — notice also that only the first and fourth sub-cases play a role here).
- If $X = \bar{E}$, then we select $w' = x'$, and $[w', a]$ is such that $[x+1, a]\bar{E}[w', a]$ and $[x', a]Z_h[w', a]$.

Backward condition. Let $X \in \{A, B, D, O, \bar{A}, \bar{B}, \bar{E}\}$ and $([x, a], [x+1, a]) \in Z^E_{h-1}$, for some $h \in \{2, \ldots, N\}$. We assume that $[x+1, a]X[w', z']$ holds, and we show the existence of an interval $[x', y']$ such that $[x, a]X[x', y']$ and $[x', y']Z_h \cup Z^E_h[w', z']$. As we did above, we notice that both x and $x+1$ belong to $\mathcal{P}_{h-1}$ (and thus to $\mathcal{P}_h$ as well). The proof proceeds by cases, depending on the value of X.

- If $X = A$, then we select $y' = z'$, and $[a, y']$ is such that $[x, a]A[a, y']$ and $[a, y']Z_h[a, z']$.
- If $X = B$, then the proof proceeds exactly as the one for the case of $x \neq w$ and $x, w \in \mathcal{P}_{h-1}$ in Lemma 3 (again, condition *(c)* of the definition of Z_h is not used there).

- If $X = D$, then we select $x' = w'$ and $y' = z'$, and $[x', y']$ is such that $[x, a]D[x', y']$ and $[x', y']Z_h[w', z']$.
- If $X = O$, then we select $x' = w'$ and $y' = z'$, and $[x', y']$ is such that $[x, a]O[x', y']$ and $[x', y']Z_h[w', z']$.
- If $X = \bar{A}$, then we distinguish two cases. If w' is such that $[w', x]$ is h-long, then we select $x' = w'$, and $[x', x]$ is such that $[x, a]\bar{A}[x', x]$ and $[x', x]Z_h[w', x+1]$. Otherwise, if w' is such that $[w', x]$ is h-short, then we select $x' = x - ((x + 1) - w')$, and $[x', x]$ is such that $[x, a]\bar{A}[x', x]$ and $[x', x]Z_h[w', x+1]$.
- If $X = \bar{B}$, then the proof proceeds exactly as the one for the case of $x \neq w$ and $x, w \in \mathcal{P}_{h-1}$ in Lemma 4, (again, condition *(c)* of the definition of Z_h is not used there — notice also that only the first and fourth sub-cases play a role here).
- If $X = \bar{E}$, then we distinguish two cases. If $w' < x$, then we select $x' = w'$, and $[x', a]$ is such that $[x, a]\bar{E}[x', a]$ and $[x', a]Z_h[w', a]$. If, on the other hand, $w' = x$, then we select $x' = w' - 1$, and $[x', a]$ is such that $[x, a]\bar{E}[x', a]$ and $[x', a]Z_h^E[w', a]$.

■

Lemma 9 *For each $N \in \mathbb{N}$, the sequence of relations $Z_h^E \cup Z_h$, for $h \in \{1, \ldots, N\}$, define a N-bisimulation for* $\mathsf{ABDO\bar{A}\bar{B}\bar{E}}$.

Proof. First, we prove the local condition, then the forward and backward ones.

Local condition. First, we consider the pairs $([\mathcal{V}(a) - h, a], [\mathcal{V}(a) - h + 1, a]) \in Z_h^E$, with $h \in \{1, \ldots, N\}$. It is immediate to verify that, for every $h \in \{1, \ldots, N\}$, both $M, [\mathcal{V}(a) - h, a] \Vdash p$ and $M, [\mathcal{V}(a) - h + 1, a] \Vdash p$ hold, so the condition is respected with respect to the sets Z_h^E ($h \in \{1, \ldots, N\}$). Consider now the pair $([x, y], [w, z]) \in Z_h$, for some $h \in \{1, \ldots, N\}$. If $y \notin \mathcal{A}$, then, by definition of $\equiv_h$, $z \notin \mathcal{A}$, either. Thus, both $[x, y]$ and $[w, z]$ satisfy p. If $y \in \mathcal{A}$, then, by definition of $\equiv_h$, $z \in \mathcal{A}$, too. Moreover, by condition *(c)* of the definition of Z_h, either $x \leq \mathcal{V}(y)$ and $w \leq \mathcal{V}(z)$, that is, both $[x, y]$ and $[w, z]$ satisfy p, or $x > \mathcal{V}(y)$ and $w > \mathcal{V}(z)$, that is, both $[x, y]$ and $[w, z]$ satisfy $\neg p$.

Forward condition. It immediately follows from Lemmas 3, 5, 4, 6, and 7 if no interval pairs of the kind $([\mathcal{V}(a)-h, a], [\mathcal{V}(a)-h+1, a]) \in Z_h^E$ are involved; otherwise, it follows from Lemma 8.

Backward condition. Let $X \in \{A, B, D, O, \bar{A}, \bar{B}, \bar{E}\}$ and $([x, y], [w, z]) \in Z_{h-1}^E \cup Z_{h-1}$, for some $h \in \{2, \ldots, N\}$. We assume that $[w, z]X[w', z']$ holds for some interval $[w', z']$ and we show that there exists an interval $[x', y']$ such that $[x, y]X[x', y']$ and $([x', y'], [w', z']) \in Z_h^E \cup Z_h$ hold. In Lemma 8, we have already shown that this is the case if $([x, y], [w, z]) \in Z_{h-1}^E$. We have to consider now the case when $([x, y], [w, z]) \in Z_{h-1}$. To this end, we use the symmetry of Z_h (see Proposition 3). By symmetry of Z_{h-1}, we have that $([w, z], [x, y]) \in Z_{h-1}$; by forward condition, there exists an interval $[x', y']$ such that $[x, y]X[x', y']$ and $([w', z'], [x', y']) \in Z_h$; by symmetry of Z_h, $([x', y'], [w', z']) \in Z_h$. ■

Corollary 1 *There are no inter-definabilities for $\langle E \rangle$ in any class of discrete linear orders.*

Proof. It immediately follows from Lemma 9 and from the following facts: $([\mathcal{V}(a)-1,a],[\mathcal{V}(a),a]) \in Z_1^E$, $M,[\mathcal{V}(a)-1,a] \Vdash \langle E\rangle p$, and $M,[\mathcal{V}(a),a] \Vdash \neg\langle E\rangle p$. ■

Now, analogously to what we have done before to disprove the existence of inter-definabilities for $\langle E\rangle$, we define the sequence of relations $Z_h^{\bar{E}} = \{([\mathcal{V}(a)+h,a],[\mathcal{V}(a)+h+1,a])\}$, for $h \in \{1,\dots,N\}$. The following properties, similar to the ones stated for the above-defined sequence of relations Z_h^E $(1 \leq h \leq N)$, hold for the newly defined sequence $Z_h^{\bar{E}}$ $(1 \leq h \leq N)$. First, $([\mathcal{V}(a)+h,a],[\mathcal{V}(a)+h+1,a]) \notin Z_h$, for any h, because condition *(c)* of the definition of Z_h is not fulfilled by pairs in $Z_h^{\bar{E}}$ (the other conditions *(a)*, *(b)*, and *(d)* are verified). Second, relations $Z_h^{\bar{E}}$ $(1 \leq h \leq N)$ are not symmetric. Third, the pair $([\mathcal{V}(a)+1,a],[\mathcal{V}(a)+2,a]) \in Z_1^{\bar{E}}$ is such that $M,[\mathcal{V}(a)+1,a] \Vdash \neg\langle\overline{E}\rangle\neg p$, while $M,[\mathcal{V}(a)+2,a] \Vdash \langle\overline{E}\rangle\neg p$. In the following lemma, we use x as an abbreviation for $\mathcal{V}(a)+(h-1)$.

Lemma 10 *Let $X \in \{A,B,E,\bar{A},\bar{B},\bar{D},\bar{O}\}$ and $([x,a],[x+1,a]) \in Z_{h-1}^{\bar{E}}$ for some $h \in \{2,\dots,N\}$. If $[x,a]X[x',y']$ for some interval $[x',y']$, then there exists an interval $[w',z']$ such that $[x+1,a]X[w',z']$ and $[x',y']Z_h \cup Z_h^{\bar{E}}[w',z']$ (forward condition). Moreover, if $[x+1,a]X[w',z']$ for some interval $[w',z']$, then there exists an interval $[x',y']$ such that $[x,a]X[x',y']$ and $[x',y']Z_h \cup Z_h^{\bar{E}}[w',z']$ (backward condition).*

Proof. The proof of this lemma proceeds likewise to the one of Lemma 8.
Forward condition. Let $X \in \{A,B,E,\bar{A},\bar{B},\bar{D},\bar{O}\}$ and $([x,a],[x+1,a]) \in Z_{h-1}^{\bar{E}}$, for some $h \in \{2,\dots,N\}$. We assume that $[x,a]X[x',y']$ holds, and we show the existence of an interval $[w',z']$ such that $[x+1,a]X[w',z']$ and $[x',y']Z_h \cup Z_h^{\bar{E}}[w',z']$. First, notice that both x and $x+1$ belong to $\mathcal{P}_{h-1}$ (and thus to $\mathcal{P}_h$ as well). The proof proceeds by cases, depending on the value of X.

- If $X = A$, then we select $z' = y'$, and $[a,z']$ is such that $[x+1,a]A[a,z']$ and $[a,y']Z_h[a,z']$.
- If $X = B$, then the proof proceeds exactly as the one for the case of $x \neq w$ and $x,w \in \mathcal{P}_{h-1}$ in Lemma 3 (condition *(c)* of the definition of Z_h, which discriminates against pairs in $Z_h^{\bar{E}}$, is not used there).
- If $X = E$, then we distinguish two cases. If $x' > x+1$, then we select $w' = x'$, and $[w',a]$ is such that $[x+1,a]E[w',a]$ and $[x',a]Z_h[w',a]$. If, on the other hand, $x' = x+1$, then we select $w' = x'+1$, and $[w',a]$ is such that $[x+1,a]E[w',a]$ and $[x',a]Z_h^{\bar{E}}[w',a]$.
- If $X = \bar{A}$, then we distinguish two cases. If $[x',x]$ is h-long, then we select $w' = x'$, and $[w',x+1]$ is such that $[x+1,a]\bar{A}[w',x+1]$ and $[x',x]Z_h[w',x+1]$. Otherwise, if $[x',x]$ is h-short, then we select $w' = (x+1)-(x-x')$, and $[w',x+1]$ is such that $[x+1,a]\bar{A}[w',x+1]$ and $[x',x]Z_h[w',x+1]$.
- If $X = \bar{B}$, then the proof proceeds exactly as the one for the case of $x \neq w$ and $x,w \in \mathcal{P}_{h-1}$ in Lemma 4, (again, condition *(c)* of the definition of Z_h is not used there — notice also that only the first and fourth sub-cases play a role here).
- If $X = \bar{D}$, then we select $w' = x'$ and $z' = y'$, and $[w',z']$ is such that $[x+1,a]\bar{D}[w',z']$ and $[x',y']Z_h[w',z']$.
- If $X = \bar{O}$, then we distinguish two cases. If $y' > x+1$, then we select $w' = x'$ and $z' = y'$, and $[w',z']$ is such that $[x+1,a]\bar{O}[w',z']$ and $[x',y']Z_h[w',z']$. If, on the other hand, $y' = x+1$, then we distinguish two cases.

– If $[x', y']$ is h-long, then we select $z' = y' + 1$ and $w' = x'$, and $[w', z']$ is such that $[x + 1, a]\bar{O}[w', z']$ and $[x', y']Z_h[w', z']$.

– If $[x', y']$ is h-short, then we select $z' = y' + 1$ and $w' = z' - (y' - x')$, and $[w', z']$ is such that $[x + 1, a]\bar{O}[w', z']$ and $[x', y']Z_h[w', z']$.

Backward condition. Let $X \in \{A, B, E, \bar{A}, \bar{B}, \bar{D}, \bar{O}\}$ and $([x, a], [x+1, a]) \in Z^{\bar{E}}_{h-1}$, for some $h \in \{2, \dots, N\}$. We assume that $[x + 1, a]X[w', z']$ holds, and we show the existence of an interval $[x', y']$ such that $[x, a]X[x', y']$ and $[x', y']Z_h \cup Z^{\bar{E}}_h[w', z']$. As we did above, we notice that both x and $x + 1$ belong to $\mathcal{P}_{h-1}$ (and thus to $\mathcal{P}_h$ as well). The proof proceeds by cases, depending on the value of X.

- If $X = A$, then we select $y' = z'$, and $[a, y']$ is such that $[x, a]A[a, y']$ and $[a, y']Z_h[a, z']$.

- If $X = B$, then the proof proceeds exactly as the one for the case of $x \neq w$ and $x, w \in \mathcal{P}_{h-1}$ in Lemma 3 (again, condition *(c)* of the definition of Z_h is not used there).

- If $X = E$, then we select $x' = w'$, and $[x', a]$ is such that $[x, a]E[x', a]$ and $[x', a]Z_h[w', a]$.

- If $X = \bar{A}$, then we distinguish two cases. If w' is such that $[w', x]$ is h-long, then we select $x' = w'$, and $[x', x]$ is such that $[x, a]\bar{A}[x', x]$ and $[x', x]Z_h[w', x + 1]$. Otherwise, if w' is such that $[w', x]$ is h-short, then we select $x' = x - ((x + 1) - w')$, and $[x', x]$ is such that $[x, a]\bar{A}[x', x]$ and $[x', x]Z_h[w', x + 1]$.

- If $X = \bar{B}$, then the proof proceeds exactly as the one for the case of $x \neq w$ and $x, w \in \mathcal{P}_{h-1}$ in Lemma 4, (again, condition *(c)* of the definition of Z_h is not used there — notice also that only the first and fourth sub-cases play a role here).

- If $X = \bar{D}$, then we distinguish two cases. If $w' < x$, then we select $x' = w'$ and $y' = z'$, and $[x', y']$ is such that $[x, a]\bar{D}[x', y']$ and $[x', y']Z_h[w', z']$. If, on the other hand, $w' = x$, then the proof proceeds exactly as the one for the case of $x \neq w$ and $x, w \in \mathcal{P}_{h-1}$ in Lemma 4, with $x - 1$ in x's stead, x in w's stead, and a in both y's and z's stead (again, condition *(c)* of the definition of Z_h is not used there — notice also that only the first and fourth sub-cases play a role here).

- If $X = \bar{O}$, then we distinguish two cases. If $w' < x$, then we select $x' = w'$ and $y' = z'$, and $[x', y']$ is such that $[x, a]\bar{O}[x', y']$ and $[x', y']Z_h[w', z']$. If, on the other hand, $w' = x$, then the proof proceeds exactly as the one for the case of $x \neq w$ and $x, w \in \mathcal{P}_{h-1}$ in Lemma 3, with $x - 1$ in x's stead, x in w's stead, and a in both y's and z's stead (again, condition *(c)* of the definition of Z_h is not used there).

■

Lemma 11 *For each $N \in \mathbb{N}$, the sequence of relations $Z^{\bar{E}}_h \cup Z_h$, for $h \in \{1, \dots, N\}$, define a N-bisimulation for* $\mathsf{ABE\bar{A}\bar{B}\bar{D}\bar{O}}$.

Proof. First, we prove the local condition, then the forward and backward ones.

Local condition. We have already showed in Lemma 9 that the local condition is fulfilled as far as relations Z_h ($1 \leq h \leq N$) are concerned. Let us consider now pairs

$([\mathcal{V}(a)+h, a], [\mathcal{V}(a)+h+1, a]) \in Z_h^{\bar{E}}$, with $1 \leq h \leq N$. It is immediate to verify that, for every $h \in \{1, \ldots, N\}$, both $M, [\mathcal{V}(a)+h, a] \Vdash \neg p$ and $M, [\mathcal{V}(a)+h+1, a] \Vdash \neg p$ hold, hence the thesis.

Forward condition. It immediately follows from Lemmas 3, 5, 4, 6, and 7 if no interval pairs of the kind $([\mathcal{V}(a)+h, a], [\mathcal{V}(a)+h+1, a]) \in Z_h^{\bar{E}}$ are involved; otherwise, it follows from Lemma 10.

Backward condition. Let $X \in \{A, B, E, \bar{A}, \bar{B}, \bar{D}, \bar{O}\}$ and $([x, y], [w, z]) \in Z_{h-1}^{\bar{E}} \cup Z_{h-1}$, for some $h \in \{2, \ldots, N\}$. We assume that $[w, z]X[w', z']$ holds for some interval $[w', z']$ and we show that there exists an interval $[x', y']$ such that $[x, y]X[x', y']$ and $([x', y'], [w', z']) \in Z_h^{\bar{E}} \cup Z_h$ hold. In Lemma 10, we have already shown that this is the case if $([x, y], [w, z]) \in Z_{h-1}^{\bar{E}}$. The same argument based on the symmetry of Z_h $(1 \leq h \leq N)$ that we used in the proof of Lemma 9 applies here to show that, for each pair $([x, y], [w, z]) \in Z_{h-1}$ $(2 \leq h \leq N)$ and each $[w', z']$ such that $[w, z]X[w', z']$ for some $X \in \{A, B, E, \bar{A}, \bar{B}, \bar{D}, \bar{O}\}$, there exists an interval $[x', y']$ such that $[x, y]X[x', y']$ and $([x', y'], [w', z']) \in Z_h$. ■

Corollary 2 *There are no inter-definabilities for $\langle \overline{E} \rangle$ in any class of discrete linear orders.*

Proof. It immediately follows from Lemma 11 and from the following facts: $([\mathcal{V}(a)+1, a], [\mathcal{V}(a)+2, a]) \in Z_1^{\bar{E}}$, $M, [\mathcal{V}(a)+1, a] \Vdash \neg\langle \overline{E} \rangle \neg p$, and $M, [\mathcal{V}(a)+2, a] \Vdash \langle \overline{E} \rangle \neg p$. ■