The dark side of Interval Temporal Logic: sharpening the undecidability border

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***Abstract*—Unlike the Moon, the dark side of interval temporal logics is the one we usually see: their ubiquitous undecidability. Identifying *minimal* undecidable interval logics is thus a natural and important issue in the research agenda in the area. The decidability status of a logic often depends on the class of models (in our case, the class of interval structures) in which it is interpreted. In this paper, we have identified several new minimal undecidable logics amongst the fragments of Halpern-Shoham logic HS, including the logic of the *overlaps* relation, over the classes of all and finite linear orders, as well as the logic of the *meet* and *subinterval* relations, over the class of dense linear orders. Together with previous undecidability results, this work contributes to delineate the border of the dark side of interval temporal logics quite sharply.**

***Keywords*-temporal logic; interval logic; undecidability.**

I. Introduction

Temporal reasoning plays a major role in computer science. In the most standard approach, the basic temporal entities are time points and temporal domains are represented as ordered structures of time points. The interval reasoning approach adopts another, arguably more natural, perspective on time, according to which the primitive ontological entities are time intervals instead of time points.

The tasks of representing and reasoning about time intervals arises naturally in various fields of computer science, artificial intelligence, and temporal databases, such as theories of action and change, natural language processing, and constraint satisfaction problems. Temporal logics with interval-based semantics have also been proposed as a useful formalism for the specification and verification of hardware [1] and of real-time systems [2].

Interval temporal logics feature modal operators that correspond to (binary) relations between intervals usually known as Allen's relations [3]. In [4], Halpern and Shoham introduce a modal logic for reasoning about interval structures (HS), with a modal operator for each Allen's relation. HS is undecidable under very weak assumptions on the class of interval structures [4]. In particular, undecidability holds for any class of interval structures over linear orders that contains at least one linear order with an infinite ascending (or descending) sequence of points, thus including the natural time flows $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$. For a long time, such a sweeping undecidability result have discouraged attempts for practical applications and further research on interval logics. A renewed interest in the area has been recently stimulated by the discovery of some interesting decidable fragments of HS [5], [6], [7], [8]. Gradually, the quest for expressive decidable fragments of HS has become one of the main points of the current research agenda for interval temporal logic. In this context, many fragments of HS have already been shown to be undecidable [9], [10], [11], [12].

In this paper, we contribute to delineate the boundary between decidable and undecidable HS fragments by establishing new undecidability results. In particular, we exhibit the first known case of a single-modality HS fragment which is undecidable in the class of *all* linear orders, as well as in the class of all *finite* linear orders, strengthening previous results [10], [11]. Moreover, most undecidability proofs for interval logics hinge on the existence of a linear ordering with an infinite sequence of points; here we show how to relax such an assumption. For space reasons, the details of proofs are mostly omitted; they can be found in [13], together with a complete picture of the state-of-the-art on the classification of HS fragments w.r.t. decidability of satisfiability. The web page http://itl.dimi.uniud.it/content/logic-hs also provides a collection of online tools that enable one to verify the status (decidable/undecidable/unknown) of any fragment of HS w.r.t. the satisfiability problem, over various classes of linear orders (all, dense, discrete, and finite).

II. Preliminaries

Let $\mathbb{D} = \langle D, < \rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[a, b]$, where $a, b \in D$ and $a \leq b$.

Intervals of the form $[a, a]$ are called *point intervals*; if these are excluded, the resulting semantics is called *strict interval semantics* (*non-strict* otherwise). Our results hold in either semantics. There are 12 different non-trivial relations (excluding the equality) between two intervals in a linear order, often called *Allen's relations* [3]: the six relations depicted in Table I and their inverses. One can naturally associate a modal operator $\langle X \rangle$ with each Allen's relation R_X. For each operator $\langle X \rangle$, we denote by $\langle \overline{X} \rangle$ its *transpose*, corresponding to the inverse relation.

Halpern and Shoham's logic HS is a multi-modal logic with formulae built over a set $\mathcal{AP}$ of propositional letters, the propositional connectives $\vee$ and $\neg$, and a set of modal unary operators associated with all Allen's relations. For each subset $\{R_{X_1}, \ldots, R_{X_k}\}$ of these relations, we define the HS fragment $\mathsf{X_1 X_2 \ldots X_k}$, whose formulae are defined by the grammar:

$$\varphi ::= p \mid \pi \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X_1 \rangle\varphi \mid \ldots \mid \langle X_k \rangle\varphi,$$

where π is a modal constant, true precisely at point intervals. We omit π when it is definable in the language or when the strict semantics is adopted. The other propositional connectives, like $\wedge$ and $\rightarrow$, and the dual modal operators $[X]$ are defined as usual, e.g., $[X]\varphi \equiv \neg\langle X \rangle\neg\varphi$.

Let $\mathbb{I}(\mathbb{D})$ be the set of all intervals over $\mathbb{D}$. The semantics of an interval-based temporal logic is given in terms of *interval models* $M = \langle \mathbb{D}, V \rangle$, where $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$ is the *valuation function* that assigns to every $p \in \mathcal{AP}$ the set of intervals $V(p)$ over which it holds. The *truth of a formula over a given interval* $[a, b]$ in a model M is defined by structural induction on formulae:

- $M, [a, b] \Vdash \pi$ iff $a = b$;
- $M, [a, b] \Vdash p$ iff $[a, b] \in V(p)$, for all $p \in \mathcal{AP}$;
- $M, [a, b] \Vdash \neg\psi$ iff it is not the case that $M, [a, b] \Vdash \psi$;
- $M, [a, b] \Vdash \varphi \vee \psi$ iff $M, [a, b] \Vdash \varphi$ or $M, [a, b] \Vdash \psi$;
- $M, [a, b] \Vdash \langle X_i \rangle\psi$ iff there exists an interval $[c, d]$ such that $[a, b]\ R_{X_i}\ [c, d]$, and $M, [c, d] \Vdash \psi$,

Satisfiability is defined as usual.

The notion of sub-interval (*contains*) can be declined into two variants, namely, *proper* sub-interval ($[a, b]$ is a proper sub-interval of $[c, d]$ if $c \leq a$, $b \leq d$, and $[a, b] \neq [c, d]$), and *strict* sub-interval (when both $c < a$ and $b < d$). Both variants will play a central role in our technical results; notice that by sub-interval we usually mean the proper one.

$\langle A \rangle$	$[a, b] R_A [c, d] \Leftrightarrow b = c$
$\langle L \rangle$	$[a, b] R_L [c, d] \Leftrightarrow b < c$
$\langle B \rangle$	$[a, b] R_B [c, d] \Leftrightarrow a = c, d < b$
$\langle E \rangle$	$[a, b] R_E [c, d] \Leftrightarrow b = d, a < c$
$\langle D \rangle$	$[a, b] R_D [c, d] \Leftrightarrow a < c, d < b$
$\langle O \rangle$	$[a, b] R_O [c, d] \Leftrightarrow a < c < b < d$

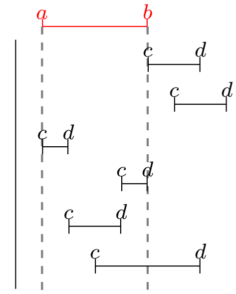

Table I
ALLEN'S INTERVAL RELATIONS AND THE CORRESPONDING HS MODALITIES.

III. A SHORT SUMMARY OF UNDECIDABILITY RESULTS

In this section, we first summarize the main undecidability results for fragments of HS. Then, we state the main results of this paper (Theorem III.1), which extend the previous ones under two different aspects: (i) we prove a number of new undecidability results for proper sub-fragments of logics that were already known to be undecidable, and (ii) we show how to adapt various existing undecidability proofs to a more general class of linear orders. The first undecidability result, for full HS, was obtained by Halpern and Shoham [4]. Since then, several other results have been published, starting from Lodaya [14], that proved the undecidability of the fragment BE, when interpreted over dense linear orders, or, alternatively, over $\langle \omega, < \rangle$, where infinite intervals are allowed. In [9], Bresolin at al. proved the undecidability of a number of interesting fragments, such as $\mathsf{AD^*E^*}$, $\mathsf{AD^*\overline{O}}$, $\mathsf{\overline{A}D^*B^*}$, $\mathsf{\overline{A}D^*O}$, $\mathsf{B\overline{E}}$, $\mathsf{\overline{B}E}$, and $\mathsf{\overline{B}\overline{E}}$, where, for each $\mathsf{X} \in \{\mathsf{A}, \mathsf{L}, \mathsf{B}, \mathsf{E}, \mathsf{D}, \mathsf{O}\}$, $\mathsf{X^*}$ denotes either X or $\mathsf{\overline{X}}$. In [10], the undecidability of all (HS-)extensions of the fragment O (and thus of $\mathsf{\overline{O}}$), except for those with the modalities $\langle L \rangle$ and $\langle \overline{L} \rangle$, has been proved when interpreted in any class of linear orders with at least one infinite ascending (or descending) sequence. In [11], the one-modality fragment O alone has been proved undecidable, but assuming discreteness. Recently, Marcinkowski et al. have first shown the undecidability of $\mathsf{B^*D^*}$ on finite and discrete linear orders [15], and, then, strengthened that result to the one-modality fragments D and $\mathsf{\overline{D}}$ [12].

Here, we first extend and complete the results from [10], [11] by providing an undecidability proof that assumes neither discreteness nor the presence of an infinite sequence. Second, we strengthen the undecidability results given in [9] by (i) proving that the logics $\mathsf{B^*E^*}$ are undecidable over the class of finite linear orders, and (ii) by showing that the weak fragments $\mathsf{A^*D^*}$ are undecidable with respect to all relevant classes of linear orders. As a consequence, we obtain a very sharp characterization of the decidability/undecidability border for the family of HS-fragments, as the undecidability for the mentioned logics holds over the class of all finite linear orders as well as over the classical orders based on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

Theorem III.1. *The satisfiability problem for the* HS *fragments* O, $\mathsf{\overline{O}}$, $\mathsf{A^*D^*}$, $\mathsf{B^*E^*}$ *is undecidable in any class of linear orders that contains, for each $n > 0$, at least one linear order with length greater than n.*

Due to space constraints, we only detail the case of O. First, we show how to relax the discreteness hypothesis;

then, we provide the changes needed to relax the hypothesis about the existence of at least one infinite sequence in the model. We refer the reader to [13] for full details.

IV. Undecidability of O

A. Intuition

As in [10], [11], our undecidability proof is based on a reduction from the so-called Octant Tiling Problem (OTP). This is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile the second octant of the integer plane $\mathcal{O} = \{(i,j) : i,j \in \mathbb{N} \wedge 0 \leq i \leq j\}$. For every tile type $t_i \in \mathcal{T}$, let $right(t_i)$, $left(t_i)$, $up(t_i)$, and $down(t_i)$ be the colors of the corresponding sides of t_i. To solve the problem, one must find a function $f : \mathcal{O} \rightarrow \mathcal{T}$ such that $right(f(n,m)) = left(f(n+1,m))$ and $up(f(n,m)) = down(f(n,m+1))$. By exploiting an argument similar to the one used in [16] to prove the undecidability of the Quadrant Tiling Problem, it can be shown that the Octant Tiling Problem is undecidable too. Given an instance OTP($\mathcal{T}$), where $\mathcal{T}$ is a finite set of tiles types, we build an O-formula $\Phi_{\mathcal{T}}$ in such a way that $\Phi_{\mathcal{T}}$ is satisfiable if and only if $\mathcal{T}$ tiles $\mathcal{O}$. The proof is similar to that of other undecidability results for HS fragments, but not readily derivable from those. It is based on the undecidability proof of O in the class of discrete linear orders [11]. The essential difference here concerns dropping the discreteness assumption, which turns out to be rather non-trivial, and it leads to a very general and elegant proof of the undecidability of O, structured as follows. First, we focus on the (sub)set $\mathcal{G}_{[a,b]}$ of all and only those intervals that are reachable in the language of O from a given starting interval $[a,b]$, by defining a suitable *global operator* $[G]$. Then, we set the tiling framework by forcing the existence of a unique infinite chain of u-intervals (i.e., intervals satisfying a designated proposition u) on the underlying linear ordering; the elements of such u-*chain* will be used as cells to arrange the tiling, and we will define in the language a derived modality to capture exactly the next u-interval from the current one. Third, we encode the octant by means of a unique infinite sequence of ld-intervals (ld-*chain*), each one of them representing a row of the octant. An ld-interval is composed by a sequence of u-intervals; each u-interval is used either to represent a part of the plane or to separate two consecutive rows; in the former case it is labelled with tile, in the latter case it is labelled with $*$. Fourth, by setting suitable propositions, we encode the *above-* and *right-neighbor* relations, which connect each tile in a row of the octant with, respectively, the one immediately above it and the one immediately at its right, if any. The encoding of such relations must be done in respect of the *commutativity property* (Def. IV.1 below). Throughout, if two tiles t_1 and t_2 are connected by the above-neighbor (resp., right-neighbor) relation, we say that t_1 is *above-connected* (resp., *right-connected*) to t_2, and similarly for tile-intervals (when they encode tiles of the octant that are above- or right-connected, respectively).

Definition IV.1 (commutativity property)**.** Given two tile-intervals $[c,d]$ and $[e,f]$, if there exists a tile-interval $[d_1,e_1]$, such that $[c,d]$ is right-connected to $[d_1,e_1]$ and $[d_1,e_1]$ is above-connected to $[e,f]$, then there exists also a tile-interval $[d_2,e_2]$ such that $[c,d]$ is above-connected to $[d_2,e_2]$ and $[d_2,e_2]$ is right-connected to $[e,f]$.

B. Technical details in the infinite case

Let $[a,b]$ be any interval of length at least 2 (i.e., such that there exists at least one point c with $a < c < b$). We define $\mathcal{G}_{[a,b]}$ as the set of all and only those intervals $[c,d]$ of length at least 2 such that $c > a$, $d > b$. Accordingly, the modality $[G]$, defined as $[G]p \equiv p \wedge [O]p \wedge [O][O]p$, refers to all and only intervals in $\mathcal{G}_{[a,b]}$. Because all formulae that we will use in the encoding will be prefixed with $\langle O \rangle$, $[O]$, or $[G]$, hereafter we only refer to intervals in $\mathcal{G}_{[a,b]}$; all others will be irrelevant.

Definition of the u-chain. The definition of the u-chain is the most difficult step in our construction, due to the extreme weakness of the language. This part represents the main difference with [11]: while there the definition of the u-chain hinges on the discreteness assumption, here we need to force its existence by means of a completely new approach. It involves three, related, aspects: (i) the existence of an infinite sequence of u-intervals $[b_0, b'_0], [b_1, b'_1], \ldots, [b_i, b'_i], \ldots$, with $b \leq b_0$ and $b'_i = b_{i+1}$ for each $i \in \mathbb{N}$; (ii) the existence of an interleaved auxiliary chain $[c_0, c'_0], [c_1, c'_1], \ldots, [c_i, c'_i], \ldots$, where $b_i < c_i < b'_i$, $b_{i+1} < c'_i < b'_{i+1}$, and $c'_i = c_{i+1}$ for each $i \in \mathbb{N}$, composed by k-intervals (each one of them overlapping exactly one u-chain), used to make it possible for us to reach the 'next' u-interval from the current one (see Fig. 1); (iii) guaranteeing that both chains are unique. This third aspect is the most difficult one. To obtain uniqueness, we show that under certain conditions the language of O can express properties of proper sub-intervals, which is quite surprising for a fragment so (apparently) weak. In particular, we show that whenever p is disjointly-bounded (see Def. IV.3 below), it is possible to express properties such as "for each interval $[a,b]$, if $[a,b]$ satisfies p then no proper sub-interval of $[a,b]$ satisfies p".

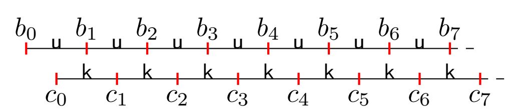

Figure 1. Encoding of the u-chain.

Let M be a model over the set $\mathcal{AP}$ of propositional letters, and let $[a,b]$ be our starting interval (which automatically defines the universe $\mathcal{G}_{[a,b]}$).

Definition IV.2. The propositions $p, q \in \mathcal{AP}$ are said to be *disjoint* if, for every pair of intervals $\langle [c,d],[e,f] \rangle$ such that $[c,d]$ satisfies p and $[e,f]$ satisfies q, either $d \le e$ or $f \le c$ (i.e., $[c,d] \cap [e,f] = \emptyset$). The proposition q is called *disjoint consequent* of p if p and q are disjoint and any p-interval is followed by a q-interval, that is, for each interval $[c,d] \in \mathcal{G}_{[a,b]}$ that satisfies p, there exists an interval $[e,f] \in \mathcal{G}_{[a,b]}$, with $e \ge d$, that satisfies q.

Definition IV.3. The proposition p is said to be *disjointly-bounded in* $\mathcal{G}_{[a,b]}$ (w.r.t. a disjoint consequent q) if: (i) $[a,b]$ neither satisfies p nor overlaps a p-interval, that is, p (possibly) holds only over intervals $[c,d]$, with $c \ge b$; (ii) p-intervals do not overlap each other, that is, there do not exist two intervals $[c,d]$ and $[e,f]$ satisfying p and such that $c < e < d < f$; (iii) p has a disjoint consequent q.

Now, whenever we can prove that a certain proposition p is is disjointly-bounded in $\mathcal{G}_{[a,b]}$ w.r.t. a disjoint consequent q, we may set an auxiliary proposition inside_p in such a way that it is true over all proper sub-intervals (in $\mathcal{G}_{[a,b]}$) of p-intervals; after that, by simply asserting that inside_p-intervals and p-intervals cannot overlap each other, we will be able to guarantee that p-intervals are never proper sub-intervals of other p-intervals. To define inside_p for the (disjointly bounded) letter p, we exploit the existence of its disjoint consequence q, plus an auxiliary proposition $\overrightarrow{p}$, which we make true over intervals starting inside a p-interval and ending outside it and inside a q-interval.

$$[G](p \rightarrow [O](\langle O \rangle q \rightarrow \overrightarrow{p})) \tag{1}$$
$$[G](\neg p \wedge [O](\langle O \rangle q \rightarrow \overrightarrow{p}) \rightarrow \mathsf{inside}_p) \tag{2}$$
$$[G]((\mathsf{inside}_p \rightarrow \neg \langle O \rangle p) \wedge (p \rightarrow \neg \langle O \rangle \mathsf{inside}_p)) \tag{3}$$

Lemma IV.4. *Let M be a model, $[a,b]$ be an interval over M, and $p, q \in \mathcal{AP}$ two propositions such that p is disjointly-bounded in $\mathcal{G}_{[a,b]}$ w.r.t. q. If $M,[a,b] \Vdash (1) \wedge (2) \wedge (3)$, then, in $\mathcal{G}_{[a,b]}$, there are no p-intervals properly contained in other p-intervals.*

Proof: Suppose, by contradiction, that there exist two intervals $[c,d]$ and $[e,f]$ (belonging to $\mathcal{G}_{[a,b]}$) satisfying p and such that $[e,f]$ is sub-interval of $[c,d]$. By definition of sub-interval, we have that $c < e$ or $f < d$. Without loss of generality, let us suppose that $c < e$ (the other case is analogous). Since $[e,f] \in \mathcal{G}_{[a,b]}$, then there exists a point in between e and f, say it e'. The interval $[c,e']$ is a sub-interval of $[c,d]$. Moreover, it cannot satisfy p, since it overlaps the p-interval $[e,f]$ (and p is a propositional letter disjointly-bounded in $\langle M,[a,b] \rangle$). By (1) and by the fact that q is a disjoint consequent of p, each interval starting in between c and d, and ending inside a q-interval, satisfies $\overrightarrow{p}$. Thus, $[c,e']$ satisfies $\neg p$ and $[O](\langle O \rangle q \rightarrow \overrightarrow{p})$. By (2), it must also satisfy inside_p. But this contradicts (3), hence the thesis. ■

From now on, given a proposition p disjointly-bounded w.r.t. to a disjoint consequent q, we use $\textit{non-sub}(p,q)$ to denote the formula $(1) \wedge (2) \wedge (3)$, and expressing the global property that no p-interval is sub-interval of another p-interval. By means of the following formulae, we force the letter u_1, u_2, k_1, and k_2 to be disjointly-bounded.

$$\neg \mathsf{u} \wedge \neg \mathsf{k} \wedge [O](\neg \mathsf{u} \wedge \neg \mathsf{k}) \tag{4}$$
$$[G]((\mathsf{u} \leftrightarrow \mathsf{u}_1 \vee \mathsf{u}_2) \wedge (\mathsf{k} \leftrightarrow \mathsf{k}_1 \vee \mathsf{k}_2) \wedge (\mathsf{u}_1 \rightarrow \neg \mathsf{u}_2) \wedge (\mathsf{k}_1 \rightarrow \neg \mathsf{k}_2)) \tag{5}$$
$$[G]((\mathsf{u}_1 \rightarrow [O](\neg \mathsf{u} \wedge \neg \mathsf{k}_2)) \wedge (\mathsf{u}_2 \rightarrow [O](\neg \mathsf{u} \wedge \neg \mathsf{k}_1))) \tag{6}$$
$$[G]((\mathsf{k}_1 \rightarrow [O](\neg \mathsf{k} \wedge \neg \mathsf{u}_1)) \wedge (\mathsf{k}_2 \rightarrow [O](\neg \mathsf{k} \wedge \neg \mathsf{u}_2))) \tag{7}$$
$$[G]((\langle O \rangle \mathsf{u}_1 \rightarrow \neg \langle O \rangle \mathsf{u}_2) \wedge (\langle O \rangle \mathsf{k}_1 \rightarrow \neg \langle O \rangle \mathsf{k}_2)) \tag{8}$$
$$[G]((\mathsf{u}_1 \rightarrow \langle O \rangle \mathsf{k}_1) \wedge (\mathsf{k}_1 \rightarrow \langle O \rangle \mathsf{u}_2) \wedge (\mathsf{u}_2 \rightarrow \langle O \rangle \mathsf{k}_2) \wedge (\mathsf{k}_2 \rightarrow \langle O \rangle \mathsf{u}_1)) \tag{9}$$
$$(4) \wedge \ldots \wedge (9) \tag{10}$$

Lemma IV.5. *Let M be a model, and $[a,b]$ and interval over M such that $M,[a,b] \Vdash (10)$. Then u_1, u_2, k_1, and k_2 are disjointly-bounded.*

Finally, to build the u-chain, we state the following formulae, where first is used to identify the first interval of the chain.

$$\langle O \rangle \langle O \rangle (\mathsf{u}_1 \wedge \mathsf{first}) \tag{11}$$
$$[G](\mathsf{u} \vee \mathsf{k} \rightarrow [O]\neg \mathsf{first} \wedge [O][O]\neg \mathsf{first}) \tag{12}$$
$$[G]((\mathsf{first} \rightarrow \mathsf{u}_1) \wedge (\mathsf{first} \rightarrow [O][O]\neg \mathsf{first})) \tag{13}$$
$$\textit{non-sub}(\mathsf{u}_1,\mathsf{k}_1) \wedge \textit{non-sub}(\mathsf{u}_2,\mathsf{k}_2) \wedge \textit{non-sub}(\mathsf{k}_1,\mathsf{u}_2) \wedge \textit{non-sub}(\mathsf{k}_2,\mathsf{u}_1) \tag{14}$$
$$[G](\mathsf{u} \vee \mathsf{k} \rightarrow [O]\langle O \rangle (\mathsf{u} \vee \mathsf{k})) \tag{15}$$
$$(11) \wedge \ldots \wedge (15) \tag{16}$$

Lemma IV.6. *Let M be a model and $[a,b]$ an interval over M such that $M,[a,b] \Vdash (10) \wedge (16)$. Then:*

(a) there exists an infinite sequence of u*-intervals $[b_0,b'_0],[b_1,b'_1], \ldots, [b_i,b'_i], \ldots$, with $b \le b_0$, $b'_i = b_{i+1}$ for each $i \in \mathbb{N}$, and such that $M,[b_0,b'_0] \Vdash \mathsf{first}$,*

(b) there exists an infinite sequence of k*-intervals $[c_0,c'_0],[c_1,c'_1],\ldots,[c_i,c'_i],\ldots$ such that $b_i < c_i < b'_i$, $b_{i+1} < c'_i < b'_{i+1}$, and $c'_i = c_{i+1}$ for each $i \in \mathbb{N}$, and*

(c) every other interval $[c,d] \in \mathcal{G}_{[a,b]}$ satisfies neither of u*,* k*, or* first*, unless $c > b_i$ for every $i \in \mathbb{N}$.*

Within this framework, the operator $\langle X_{\mathsf{u}} \rangle$, used either to reach the first u-interval of the u-chain or to step from any given u-interval to the next one in the sequence, is definable:

$$\langle X_{\mathsf{u}} \rangle \varphi \equiv (\neg \mathsf{u} \wedge \langle O \rangle \langle O \rangle (\mathsf{first} \wedge \varphi)) \vee (\mathsf{u} \wedge \langle O \rangle (\mathsf{k} \wedge \langle O \rangle (\mathsf{u} \wedge \varphi)))$$

Definition of the ld-chain. In order to define the ld-chain, we make use of the following set of formulae:

$$\neg\mathsf{Id} \wedge \neg\langle O\rangle\mathsf{Id} \wedge [G](\mathsf{Id} \rightarrow \neg\langle O\rangle\mathsf{Id}) \quad (17)$$

$$\langle X_{\mathsf{u}}\rangle(* \wedge \langle X_{\mathsf{u}}\rangle(\mathsf{tile} \wedge \mathsf{Id} \wedge \langle X_{\mathsf{u}}\rangle* \wedge [G](* \rightarrow \langle X_{\mathsf{u}}\rangle(\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle\mathsf{tile})))) \quad (18)$$

$$[G]((\mathsf{u} \leftrightarrow * \vee \mathsf{tile}) \wedge (* \rightarrow \neg\mathsf{tile})) \quad (19)$$

$$[G](* \rightarrow \langle O\rangle(\mathsf{k} \wedge \langle O\rangle\mathsf{Id})) \quad (20)$$

$$[G](\mathsf{Id} \rightarrow \langle O\rangle(\mathsf{k} \wedge \langle O\rangle*)) \quad (21)$$

$$[G]((\mathsf{u} \rightarrow \neg\langle O\rangle\mathsf{Id}) \wedge (\mathsf{Id} \rightarrow \neg\langle O\rangle\mathsf{u})) \quad (22)$$

$$[G](\langle O\rangle* \rightarrow \neg\langle O\rangle\mathsf{Id}) \quad (23)$$

$$non\text{-}sub(\mathsf{Id}, \mathsf{k}) \quad (24)$$

$$(17) \wedge \ldots \wedge (24) \quad (25)$$

Lemma IV.7. *Let $M, [a, b] \Vdash (10) \wedge (16) \wedge (25)$ and let $b \leq b_1^0 < c_1^0 < b_1^1 < \ldots < b_1^{k_1-1} < c_1^{k_1-1} < b_1^{k_1} = b_2^0 < c_2^0 = c_1^{k_1} < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$ be the sequence of points, defined by Lemma IV.6, such that $[b_j^i, b_j^{i+1}]$ satisfies* u *and $[c_j^i, c_j^{i+1}]$ satisfies* k *for each $j \geq 1, 0 \leq i < k_j$. Then, for each $j \geq 1$, we have:*
(a) $M, [b_j^0, b_j^1] \Vdash *$;
(b) $M, [b_j^i, b_j^{i+1}] \Vdash$ tile for each $0 < i < k_j$;
(c) $M, [b_j^1, b_{j+1}^0] \Vdash$ Id;
(d) $k_1 = 2$, $k_l > 2$ for each $l > 1$;
and no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies $*$ (resp., tile, Id*), unless $c > b_j^i$ for each $i, j > 0$.*

Above-neighbor relation. We proceed now with the encoding of the above-neighbor relation (Fig. 2), by means of which we connect each tile-interval with its vertical neighbor in the octant (e.g., t_2^2 with t_3^2 in Fig. 2). For technical reasons, we need to distinguish between *backward* and *forward* rows of $\mathcal{O}$ using the propositions bw and fw: we label each u-interval with bw (resp., fw) if it belongs to a backward (resp., forward) row (formulae (26)-(27)). Intuitively, the tiles belonging to forward rows of $\mathcal{O}$ are encoded in ascending order, while those belonging to backward rows are encoded in descending order (the tiling is encoded in a zig-zag manner). In particular, this means that the left-most tile-interval of a backward level encodes the last tile of that row (and not the first one) in $\mathcal{O}$. Let $\alpha, \beta \in \{\mathsf{bw}, \mathsf{fw}\}$, with $\alpha \neq \beta$:

$$\langle X_{\mathsf{u}}\rangle\mathsf{bw} \wedge [G]((\mathsf{u} \leftrightarrow \mathsf{bw} \vee \mathsf{fw}) \wedge (\mathsf{bw} \rightarrow \neg\mathsf{fw})) \quad (26)$$

$$[G]((\alpha \wedge \neg\langle X_{\mathsf{u}}\rangle* \rightarrow \langle X_{\mathsf{u}}\rangle\alpha) \wedge (\alpha \wedge \langle X_{\mathsf{u}}\rangle* \rightarrow \langle X_{\mathsf{u}}\rangle\beta)) \quad (27)$$

$$(26) \wedge (27) \quad (28)$$

Lemma IV.8. *If $M, [a, b] \Vdash (10) \wedge (16) \wedge (25) \wedge (28)$, then the sequence of points defined in Lemma IV.7 is such that $M, [b_j^i, b_j^{i+1}] \Vdash$ bw if and only if j is an odd number, and $M, [b_j^i, b_j^{i+1}] \Vdash$ fw if and only j is an even number. Furthermore, we have that no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies* bw *or* fw*, unless $c > b_j^i$ for each $i, j > 0$.*

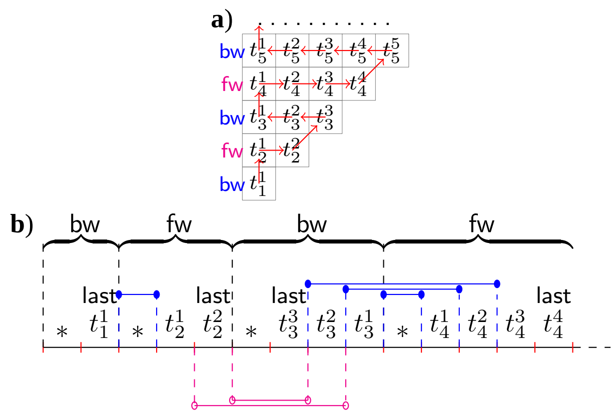

Figure 2. Encoding of the above-neighbor relation.

We make use of such an alternation between backward and forward rows to correctly encode the above-neighbor relation. We constrain each up_rel-interval starting from a backward (resp., forward) row not to overlap any other up_rel-interval starting from a backward (resp., forward) row. The structure of the encoding is shown in Fig. 2, where up_rel-intervals starting inside forward (resp., backward) rows are placed one inside the other. Consider, for instance, how the 3rd and 4th level of the octant are encoded in Fig. 2b. The 1st tile-interval of the 3rd level (t_3^3) is connected to the second from last tile-interval of the 4th level (t_4^3), the 2nd tile-interval of the 3rd level (t_3^2) is connected to the third from last tile-interval of the 4th level (t_4^2), and so on. Notice that, in forward (resp., backward) level, the last (resp., first) tile-interval has no tile-intervals above-connected to it, in order to constrain each level to have exactly one tile-interval more than the previous one (these tile-intervals are labeled with last).

Formally, we define the above-neighbor relation as follows. If $[b_j^i, b_j^{i+1}]$ is a tile-interval belonging to a forward (resp., backward) row, then we say that it is above-connected to the tile-interval $[b_{j+1}^{j+2-i}, b_{j+1}^{j+2-i+1}]$ (resp., $[b_{j+1}^{j+2-i-1}, b_{j+1}^{j+2-i}]$). To do so, we label with up_rel the interval $[c_j^i, c_{j+1}^{j+2-i}]$ (resp., $[c_j^i, c_{j+1}^{j+2-i-1}]$). Moreover, we distinguish between up_rel-intervals starting from forward and backward rows and, within each case, between those starting from odd and even tile-intervals. To this end, we use a new proposition, namely, $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{bw}}$ (resp., $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{bw}}$, $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{fw}}$, $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{fw}}$) to label up_rel-intervals starting from an odd tile-interval of a backward row (resp., even tile-interval/backward row, odd/forward, even/forward). To ease the reading of the formulae, we group $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{bw}}$ and $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{bw}}$ in $\mathsf{up\_rel}^{\mathsf{bw}}$ ($\mathsf{up\_rel}^{\mathsf{bw}} \leftrightarrow \mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{bw}} \oplus \mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{bw}}$), and similarly for $\mathsf{up\_rel}^{\mathsf{fw}}$. Finally, up_rel is exactly one among $\mathsf{up\_rel}^{\mathsf{bw}}$ and $\mathsf{up\_rel}^{\mathsf{fw}}$ ($\mathsf{up\_rel} \leftrightarrow \mathsf{up\_rel}^{\mathsf{bw}} \oplus \mathsf{up\_rel}^{\mathsf{fw}}$). In such a way, we encode the correspondence between tiles of consecutive rows of the plane induced by the above-neighbor relation.

Let $\alpha, \beta \in \{\mathsf{bw}, \mathsf{fw}\}$ and $\gamma, \delta \in \{\mathsf{o}, \mathsf{e}\}$, with $\alpha \neq \beta$ and $\gamma \neq \delta$:

$$\neg \mathsf{up\_rel} \wedge \neg \langle O \rangle \mathsf{up\_rel} \tag{29}$$

$$[G]((\mathsf{up\_rel} \leftrightarrow \mathsf{up\_rel}^{\mathsf{bw}} \vee \mathsf{up\_rel}^{\mathsf{fw}}) \wedge (\mathsf{up\_rel}^{\alpha} \leftrightarrow \mathsf{up\_rel}^{\alpha}_{\mathsf{o}} \vee \mathsf{up\_rel}^{\alpha}_{\mathsf{e}})) \tag{30}$$

$$[G]((\mathsf{k} \vee * \rightarrow \neg \langle O \rangle \mathsf{up\_rel}) \wedge (\mathsf{up\_rel} \rightarrow \neg \langle O \rangle \mathsf{k})) \tag{31}$$

$$[G](\mathsf{u} \wedge \langle O \rangle \mathsf{up\_rel}^{\alpha}_{\gamma} \rightarrow \neg \langle O \rangle \mathsf{up\_rel}^{\alpha}_{\delta} \wedge \neg \langle O \rangle \mathsf{up\_rel}^{\beta}) \tag{32}$$

$$[G](\mathsf{up\_rel}^{\alpha} \rightarrow \neg \langle O \rangle \mathsf{up\_rel}^{\alpha}) \tag{33}$$

$$[G](\mathsf{up\_rel} \rightarrow \langle O \rangle \mathsf{ld}) \tag{34}$$

$$[G](\langle O \rangle \mathsf{up\_rel} \rightarrow \neg \langle O \rangle \mathsf{first}) \tag{35}$$

$$[G](\mathsf{up\_rel}^{\alpha}_{\gamma} \rightarrow \langle O \rangle (\mathsf{tile} \wedge \langle O \rangle \mathsf{up\_rel}^{\beta}_{\gamma})) \tag{36}$$

$$(29) \wedge \ldots \wedge (36) \tag{37}$$

Lemma IV.9. *If $M, [a, b] \Vdash (10) \wedge (16) \wedge (25) \wedge (28) \wedge (37)$, then the sequence of points defined in Lemma IV.7 is such that, for each $i \geq 0$, $j > 0$, the following properties hold:*

a) if $[c, d]$ satisfies up_rel*, then $c = c^i_j$ and $d = c^{i'}_{j'}$ for some $i, i', j, j' > 0$; that is, each* up_rel*-interval starts and ends inside a* tile*-interval. More precisely, it starts (resp., ends) at the same point in which a* k*-interval starts (resp., ends);*

b) $[c^i_j, c^{i'}_{j'}]$ satisfies up_rel *if and only if it satisfies exactly one between* $\mathsf{up\_rel}^{\mathsf{bw}}$ *and* $\mathsf{up\_rel}^{\mathsf{fw}}$ *and $[c^i_j, c^{i'}_{j'}]$ satisfies* $\mathsf{up\_rel}^{\mathsf{bw}}$ *(resp.,* $\mathsf{up\_rel}^{\mathsf{fw}}$*) if and only if it satisfies exactly one between* $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{o}}$ *and* $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}}$ *(resp., between* $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$ *and* $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$*);*

c) for each $\alpha, \beta \in \{\mathsf{bw}, \mathsf{fw}\}$ and $\gamma, \delta \in \{\mathsf{o}, \mathsf{e}\}$, if $[c^i_j, c^{i'}_{j'}]$ satisfies $\mathsf{up\_rel}^{\alpha}_{\gamma}$*, then there is no other interval starting at c^i_j satisfying* $\mathsf{up\_rel}^{\beta}_{\delta}$ *such that* $\mathsf{up\_rel}^{\alpha}_{\gamma} \neq \mathsf{up\_rel}^{\beta}_{\delta}$*;*

d) each $\mathsf{up\_rel}^{\mathsf{bw}}$*-interval (resp.,* $\mathsf{up\_rel}^{\mathsf{fw}}$*-interval) does not overlap any other* $\mathsf{up\_rel}^{\mathsf{bw}}$*-interval (resp.,* $\mathsf{up\_rel}^{\mathsf{fw}}$*-interval);*

e) if $[c^i_j, c^{i'}_{j'}]$ satisfies $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{o}}$ *(resp.,* $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}}$*,* $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$*,* $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$*), then there exists an* $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$*-interval (resp.,* $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{o}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}}$*-interval) starting at $c^{i'}_{j'}$.*

We constrain each tile-interval, apart from the last one of some level, to have a tile-interval above-connected to it. To this end, we label the last tile-interval of every row with the new proposition last (formulae (43)-(45)). Then we force all tile-intervals not labelled with last to have a tile-interval above-connected to them (formulae (46)-(49)):

$$[G](\mathsf{tile} \rightarrow \langle O \rangle \mathsf{up\_rel}) \tag{38}$$

$$[G](\alpha \rightarrow [O](\mathsf{up\_rel} \rightarrow \mathsf{up\_rel}^{\alpha})) \tag{39}$$

$$[G](\mathsf{up\_rel}^{\alpha} \rightarrow \langle O \rangle \beta) \tag{40}$$

$$[G](\langle O \rangle * \rightarrow \neg(\langle O \rangle \mathsf{up\_rel}^{\mathsf{bw}} \wedge \langle O \rangle \mathsf{up\_rel}^{\mathsf{fw}})) \tag{41}$$

$$[G](\mathsf{tile} \wedge \langle O \rangle \mathsf{up\_rel}^{\alpha}_{\gamma} \wedge \langle X_{\mathsf{u}} \rangle \mathsf{tile} \rightarrow \langle X_{\mathsf{u}} \rangle (\mathsf{tile} \wedge \langle O \rangle \mathsf{up\_rel}^{\alpha}_{\delta})) \tag{42}$$

$$[G](\mathsf{last} \rightarrow \mathsf{tile}) \tag{43}$$

$$[G]((* \wedge \mathsf{bw} \rightarrow \langle X_{\mathsf{u}} \rangle \mathsf{last}) \wedge (\mathsf{fw} \wedge \langle X_{\mathsf{u}} \rangle * \rightarrow \mathsf{last})) \tag{44}$$

$$[G]((\mathsf{last} \wedge \mathsf{fw} \rightarrow \langle X_{\mathsf{u}} \rangle *) \wedge (\mathsf{bw} \wedge \langle X_{\mathsf{u}} \rangle \mathsf{last} \rightarrow *)) \tag{45}$$

$$[G](* \wedge \mathsf{fw} \rightarrow \langle X_{\mathsf{u}} \rangle (\mathsf{tile} \wedge \langle O \rangle (\mathsf{up\_rel} \wedge \langle O \rangle (\mathsf{tile} \wedge \langle X_{\mathsf{u}} \rangle *)))) \tag{46}$$

$$[G](\mathsf{last} \wedge \mathsf{bw} \rightarrow \langle O \rangle (\mathsf{up\_rel} \wedge \langle O \rangle (\mathsf{tile} \wedge \langle X_{\mathsf{u}} \rangle (\mathsf{tile} \wedge \langle X_{\mathsf{u}} \rangle *)))) \tag{47}$$

$$[G](\mathsf{k} \wedge \langle O \rangle (\mathsf{tile} \wedge \langle O \rangle \mathsf{up\_rel}^{\alpha}_{\gamma}) \rightarrow [O]((\langle O \rangle \mathsf{up\_rel}^{\alpha}_{\gamma} \wedge \langle O \rangle (\mathsf{k} \wedge \langle O \rangle (\mathsf{tile} \wedge \langle O \rangle \mathsf{up\_rel}^{\beta}_{\delta} \wedge \neg \mathsf{last})) \rightarrow \langle O \rangle \mathsf{up\_rel}^{\alpha}_{\delta})) \tag{48}$$

$$[G](\mathsf{up\_rel} \rightarrow \neg \langle O \rangle \mathsf{last}) \tag{49}$$

$$(38) \wedge \ldots \wedge (49) \tag{50}$$

Lemma IV.10. *If $M, [a, b] \Vdash (10) \wedge (16) \wedge (25) \wedge (28) \wedge (37) \wedge (50)$, then the sequence of points defined in Lemma IV.7 is such that the following properties hold:*

a) for each up_rel*-interval $[c^i_j, c^{i'}_{j'}]$, connecting the* tile*-interval $[b^i_j, b^{i+1}_j]$ to the* tile*-interval $[b^{i'}_{j'}, b^{i'+1}_{j'}]$, if $[c^i_j, c^{i'}_{j'}]$ satisfies* $\mathsf{up\_rel}^{\mathsf{bw}}$ *(resp.,* $\mathsf{up\_rel}^{\mathsf{fw}}$*), then $[b^i_j, b^{i+1}_j]$ satisfies* bw *(resp.,* fw*) and $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ satisfies* fw *(resp.,* bw*);*

b) (strict alternation property) for each tile*-interval $[b^i_j, b^{i+1}_j]$, with $i < k_j - 1$, such that there exists an* $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{o}}$*-interval (resp.,* $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$*-interval) starting at c^i_j, there exists an* $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}}$*-interval (resp.,* $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{o}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$*-interval) starting at c^{i+1}_j;*

c) for every tile*-interval $[b^i_j, b^{i+1}_j]$ satisfying* last*, there is no* up_rel*-interval ending at c^i_j;*

d) for each up_rel*-interval $[c^i_j, c^{i'}_{j'}]$, with $0 < i < k_j$, we have that $j' = j + 1$.*

Lemma IV.11. *Each* tile*-interval $[b^i_j, b^{i+1}_j]$ is above-connected to exactly one* tile*-interval and, if it does not satisfy* last*, then there exists exactly one* tile*-interval which is above-connected to it.*

Right-neighbor relation. The right-neighbor relation connects each tile with its horizontal neighbor in the octant, if any (e.g., t^2_3 with t^3_3 in Fig. 2). Again, in order to encode the right-neighbor relation, we must distinguish between forward and backward levels: a tile-interval belonging to a forward (resp., backward) level is right-connected to the tile-interval immediately to the right (resp., left), if any. For example, in Fig. 2b, the 2nd tile-interval of the 4th level (t^2_4) is right-connected to the tile-interval immediately to the right (t^3_4), since the 4th level is a forward one, while the 2nd tile-interval of the 3rd level (t^2_3) is right-connected to the tile-interval immediately to the left (t^3_3), since the 3rd level is a backward one. Therefore, we define the right-neighbor relation as follows: if $[b^i_j, b^{i+1}_j]$ is a tile-interval belonging to a forward (resp., backward) ld-interval, with $i \neq k_j - 1$

(resp., $i \neq 1$), then we say that it is right-connected to the tile-interval $[b_j^{i+1}, b_j^{i+2}]$ (resp., $[b_j^{i-1}, b_j^{i}]$).

Lemma IV.12 (Commutativity property)**.** *If* $M, [a,b] \Vdash (10) \wedge (16) \wedge (25) \wedge (28) \wedge (37) \wedge (50)$, *then the commutativity property holds over the sequence defined in Lemma IV.7.*

Tiling the plane. The following formulae constrain each tile-interval (and no other interval) to be tiled by exactly one tile (formula (51)) and constrain the tiles that are right- or above-connected to respect the color constraints (from (52) to (54)):

$$[G]((\bigvee_{i=1}^{k} \mathtt{t_i} \leftrightarrow \mathsf{tile}) \wedge (\bigwedge_{i,j=1, i\neq j}^{k} \neg(\mathtt{t_i} \wedge \mathtt{t_j})) \quad (51)$$

$$[G](\mathsf{tile} \rightarrow \bigvee_{up(t_i)=down(t_j)} (\mathtt{t_i} \wedge \langle O\rangle(\mathsf{up\_rel} \wedge \langle O\rangle \mathtt{t_j}))) \quad (52)$$

$$[G](\mathsf{tile} \wedge \mathsf{fw} \wedge \langle X_{\mathsf{u}}\rangle \mathsf{tile} \rightarrow \bigvee_{right(t_i)=left(t_j)} (\mathtt{t_i} \wedge \langle X_{\mathsf{u}}\rangle \mathtt{t_j})) \quad (53)$$

$$[G](\mathsf{tile} \wedge \mathsf{bw} \wedge \langle X_{\mathsf{u}}\rangle \mathsf{tile} \rightarrow \bigvee_{left(t_i)=right(t_j)} (\mathtt{t_i} \wedge \langle X_{\mathsf{u}}\rangle \mathtt{t_j})) \quad (54)$$

$$(51) \wedge \ldots \wedge (54) \quad (55)$$

Given the set of tile types $\mathcal{T} = \{t_1, t_2, \ldots, t_k\}$, let $\Phi_{\mathcal{T}}$ be the formula $(10) \wedge (16) \wedge (25) \wedge (28) \wedge (37) \wedge (50) \wedge (55)$.

Lemma IV.13. *The formula* $\Phi_{\mathcal{T}}$ *is satisfiable if and only if* $\mathcal{T}$ *can tile the second octant* $\mathcal{O}$.

C. Extending undecidability to finite linear orders

In this section, we show how to adapt the construction of the previous section in order to encode the Finite Tiling Problem. This provides us with an undecidability proof for the fragment O that works in any class of *strongly discrete* linear orders – that is, linear orders satisfying the property that every interval contains only finitely many points – that contains arbitrarily (finitely) long orders. In particular, this allow us to conclude that O is undecidable when interpreted in the class of all finite linear orders.

The Finite Tiling Problem is formally defined as the problem of establishing if there exists two natural number k and l such that a finite set of of tile types $\mathcal{T}$, containing two distinguished tile types t_0 and t_f, can tile the $\{0, \ldots, k\} \times \{0, \ldots, l\}$ finite plane, under the restriction that $f(0,0) = t_0$ and $f(k,l) = t_f$. This problem has been first introduced and shown to be undecidable in [17].

Definition of the u**-chain.** The main difference from the reduction of the octant tiling problem described in the previous section is the finiteness of the rectangular area. This requires the existence of an arbitrarily long, but not infinite, u-chain. Hence, we introduce an auxiliary propositions $\mathsf{last_u}$ to denote the last u-interval of the (finite) u-chain. The properties of $\mathsf{last_u}$ are defined as follows.

$$\langle O\rangle\langle O\rangle \mathsf{last_u} \quad (56)$$

$$[G](\mathsf{last_u} \rightarrow * \wedge [O](\neg \mathsf{u} \wedge \neg \mathsf{k}) \wedge [O][O](\neg \mathsf{u} \wedge \neg \mathsf{k})) \quad (57)$$

Now, we analyze the formulae used in the previous section, showing only those that need to be changed for the finite case. Formula (9) is replaced by (58) in order to guarantee the existence of the u- and k-chains.

$$[G]((\mathsf{u_1} \wedge \neg \mathsf{last_u} \rightarrow \langle O\rangle \mathsf{k_1}) \wedge (\mathsf{k_1} \rightarrow \langle O\rangle \mathsf{u_2}) \wedge (\mathsf{u_2} \wedge \neg \mathsf{last_u} \rightarrow \langle O\rangle \mathsf{k_2}) \wedge (\mathsf{k_2} \rightarrow \langle O\rangle \mathsf{u_1})) \quad (58)$$

Since $\mathsf{u_1}$- and $\mathsf{u_2}$-intervals (resp., $\mathsf{k_1}$- and $\mathsf{k_2}$-intervals) do not infinitely alternate with each other in the finite case, we introduce the new proposition cons, and we force it to be a disjoint consequent of u and k. In this way, we can force $\mathsf{u_1}$, $\mathsf{u_2}$, $\mathsf{k_1}$, and $\mathsf{k_2}$ to be disjointly-bounded.

$$\neg \mathsf{cons} \wedge [O]\neg \mathsf{cons} \wedge [G](\mathsf{u} \wedge \mathsf{k} \rightarrow \langle O\rangle\langle O\rangle \mathsf{cons}) \quad (59)$$

$$[G](\langle O\rangle \mathsf{u} \vee \langle O\rangle \mathsf{k} \rightarrow \neg\langle O\rangle \mathsf{cons}) \quad (60)$$

$$[G]((\mathsf{u} \vee \mathsf{k} \rightarrow \neg\langle O\rangle \mathsf{cons}) \wedge (\mathsf{cons} \rightarrow [O](\neg \mathsf{u} \wedge \neg \mathsf{k}))) \quad (61)$$

Finally, we replace (14) and (15) with (62) and (63).

$$\textit{non-sub}(\mathsf{u_1}, \mathsf{cons}) \wedge \textit{non-sub}(\mathsf{u_2}, \mathsf{cons}) \wedge \textit{non-sub}(\mathsf{k_1}, \mathsf{cons}) \wedge \textit{non-sub}(\mathsf{k_2}, \mathsf{cons}) \quad (62)$$

$$[G](\mathsf{u} \vee \mathsf{k} \rightarrow [O](\langle O\rangle\langle O\rangle \mathsf{last_u} \rightarrow \langle O\rangle(\mathsf{u} \vee \mathsf{k}))) \quad (63)$$

Notice that formulae (56), ..., (63) guarantees the existence of the u-chain also when interpreted over arbitrary linear orders, but that the strong discreteness assumption is crucial to guarantee the finiteness of the chain. As a counterexample, consider the model over $\mathbb{Q}$ depicted in Figure 3, where $\mathsf{u_1}$ holds over every interval $[2-\frac{1}{2^n}, 2-\frac{1}{2^{n+1}}]$ such that n is even, $\mathsf{u_2}$ holds over every interval $[2-\frac{1}{2^n}, 2-\frac{1}{2^{n+1}}]$ such that n is odd, the sequence of $\mathsf{k_1}$- and $\mathsf{k_2}$-intervals are defined consistently, and $\mathsf{last_u}$ holds over the interval $[2, 2+\frac{1}{2}]$. Such a model satisfy formulae (56), ..., (63), but contains an infinite u-chain.

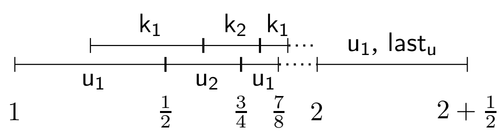

Figure 3. Infinite u-chain counterexample.

Definition of the Id**-chain.** To guarantee that Id is a disjointly-bounded proposition, we exploit the fact that, by definition, cons is also a disjoint consequent of Id. Moreover, as for the u-chain, we have to make sure that the chain is finite: to this end, we introduce the proposition $\mathsf{last_{Id}}$ to denoting the last Id-interval of the (finite) Id-chain.

$$[G]((\mathsf{last_{Id}} \rightarrow \mathsf{Id}) \wedge (\mathsf{Id} \wedge \langle O\rangle(\mathsf{k} \wedge \langle O\rangle \mathsf{last_u}) \rightarrow \mathsf{last_{Id}})) \quad (64)$$

Finally, we redefine formulae (18) and (20) as follows.

$$\langle X_{\mathsf{u}}\rangle * \wedge [G](* \to \langle X_{\mathsf{u}}\rangle \mathsf{tile}) \tag{65}$$

$$[G](* \wedge \neg \mathsf{last_u} \to \langle O\rangle(\mathsf{k} \wedge \langle O\rangle \mathsf{Id})) \tag{66}$$

Above-neighbor relation. In the finite case, every row has exactly the same number of tiles; therefore, the formulae (43), (44), (45), (47), and (49) can be dismissed. Formulae (36), (38), and (48) are replaced by the following ones.

$$[G](\mathsf{up\_rel}^{\alpha}_{\gamma} \to (\langle O\rangle \mathsf{tile} \wedge (\langle O\rangle\langle O\rangle(* \wedge \neg \mathsf{last_u}) \to \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\beta}_{\gamma})))) \tag{67}$$

$$[G](\mathsf{tile} \wedge \langle O\rangle\langle O\rangle(* \wedge \neg \mathsf{last_u}) \to \langle O\rangle \mathsf{up\_rel}) \tag{68}$$

$$[G](\mathsf{k} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\gamma}) \to [O](\langle O\rangle \mathsf{up\_rel}^{\alpha}_{\gamma} \wedge \langle O\rangle(\mathsf{k} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\beta}_{\delta})) \to \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\delta})) \tag{69}$$

To complete the construction is sufficient to add the constraints on the first and last tile of the plane. Therefore, undecidability of O is proven also for finite linear orders.

V. Conclusions

In this paper, we filled in many gaps in the characterization of HS fragments with respect to decidability/undecidability. More precisely, we proved that O, $\overline{\mathsf{O}}$, $\mathsf{B}^*\mathsf{E}^*$, and $\mathsf{A}^*\mathsf{D}^*$ are undecidable when interpreted in any significant class of linear orders. However, this is not the end of the story, because the status of some meaningful fragments is still unknown. As an example, the (un)decidability of D and $\overline{\mathsf{D}}$ over the class of all linear orders cannot be trivially derived from known results about finite, discrete, and dense ones [12], [18].

Acknowledgements

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