A Decidable Spatial Generalization of Metric Interval Temporal Logic

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Abstract

Temporal reasoning plays an important role in artificial intelligence. Temporal logics provide a natural framework for its formalization and implementation. A standard way of enhancing the expressive power of temporal logics is to replace their unidimensional domain by a multidimensional one. In particular, such a dimensional increase can be exploited to obtain spatial counterparts of temporal logics. Unfortunately, it often involves a blow up in complexity, possibly losing decidability. In this paper, we propose a spatial generalization of the decidable metric interval temporal logic RPNL+INT, *called Directional Area Calculus (DAC). DAC features two modalities, that respectively capture (possibly empty) rectangles to the north and to the east of the current one, and metric operators, to constrain the size of the current rectangle. We prove the decidability of the satisfiability problem for DAC, when interpreted over frames built on natural numbers, and we analyze its complexity. In addition, we consider a weakened version of DAC, called WDAC, which is expressive enough to capture meaningful qualitative and quantitative spatial properties and computationally better.*

1. Introduction

The transfer of formalisms, techniques, and results from the temporal context to the spatial one is quite common in computer science and artificial intelligence. However, it (almost) never comes for free: it involves a blow up in complexity, that can possibly yield undecidability. In this paper, we study a spatial generalization of the decidable metric interval temporal logic RPNL+INT [7]. The main goal of spatial formal systems is to capture common-sense knowledge about space and to provide a calculus of spatial information. Applications of spatial calculi include, for instance, spatial databases management, geographical information systems, image processing, and autonomous agents. Depending on the considered class of spatial relations, we can distinguish between *topological* and *directional* spatial reasoning. While topological relations between pairs of spatial objects (viewed as sets of points) are preserved under translation, scaling, and rotation, directional relations depend on the relative spatial position of the objects. A comprehensive and sufficiently up-to-date survey, which covers topological, directional, and combined constraint systems and relations, can be found in [9].

Deductive systems for reasoning about topological relations have been proposed in various papers, including Bennett's work [4, 5], later extended by Bennett et al. [6], Nutt's systems for generalized topological relations [21], the modal logic systems for a number of mathematical theories of space described in [1], the logic of connectedness constraints developed by Kontchakov et al. [15], and Lutz and Wolter's modal logic of topological relations [17]. Directional relations have been mainly dealt with following both the algebraic approach or the modal logic one. As for the first one, the most important contributions are those by Güsgen [13] and by Mukerjee and Joe [20], that introduce Rectangle Algebra (RA), later extended by Balbiani et al. in [2, 3]. As for the second one, we mention Venema's Compass Logic [23], whose undecidability has been shown by Marx and Reynolds in [18], Spatial Propositional Neighborhood Logic (SpPNL for short) by Morales et al. [19], that generalizes the logic of temporal neighborhood [12] to the two-dimensional space, and the fragment of SpPNL called Weak Spatial Propositional Neighborhood Logic (WSpPNL), presented in [8]. As for the *quantitative* level, the literature is very scarce. Condotta [10] presents a generalization of RA with the integration of quantitative constraints, for which there exist tractable fragments. Dutta [11] proposes an integrated framework for representing induced spatial constraints between a set of landmarks given imprecise, incomplete, and possibly conflicting quantitative and qualitative information about them, using fuzzy logic. Finally, Sheremet, Tishkovsky, Wolter and Zakharyaschev [22] propose a logic for reasoning about metric spaces with the induced topologies, which combines the qualitative interior and closure operators with quantitative operators "somewhere in the sphere of radius r" including or excluding the boundary; similar and related work can be also found in [14, 16].

In this paper, we present the Directional Area Calculus (DAC), that can be viewed as a two-dimensional variant of RPNL+INT [7]. DAC allows one to reason with basic shapes, such as lines, points, and rectangles, directional relations, and (a weak form of) areas. It features two modal operators (*somewhere in the north* and *somewhere to the east*). We show that DAC preserves the decidability of the satisfiability problem, and, moreover, it allows one to express meaningful spatial expressions despite its simplicity. DAC is interpreted over frames built over the set of natural numbers or prefixes of them, and, by means of special *atomic propositions* of the type $l_h = k$ and $l_v = k$, one can constraint the length of the horizontal (resp., vertical) projections of the considered objects; thus, combining these two features, it is possible to express statements such as *the area of the current object is less than 4 square meters*. Moreover, we study a proper fragment of DAC, denoted by WDAC (Weak DAC), which is expressive enough to capture meaningful qualitative and quantitative spatial properties and computationally better.

The paper is organized as follows. In Section 2, we present syntax and semantics of DAC and WDAC. In Section 3, we briefly discuss the expressive power of DAC; then, in Section 4, we prove that it is decidable. Next, in Section 5, we introduce WDAC, we show that it is strictly less expressive than DAC, and we provide a more efficient decision procedure for it.

2. Directional Area Calculi (DAC and WDAC)

The language of the Directional Area Calculus (DAC) and of Weak Directional Area Calculus (WDAC) consists of a set of propositional variables $\mathcal{AP}$, the logical connectives $\neg$ and $\vee$, and the modalities $\Diamond_e, \Diamond_n$, plus an infinite set of special atomic propositions of the type $l_h = k$ and $l_v = k$, with $k \in \mathbb{N}$. Let $p \in \mathcal{AP}$. Well-formed formulas, denoted by $\varphi, \psi, \ldots$, are recursively defined as follows:

$$\varphi ::= l_h = k \mid l_v = k \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond_e\varphi \mid \Diamond_n\varphi.$$

The other logical connectives, as well as the logical constants $\top$ and $\bot$ and universal modalities $\Box_e$ and $\Box_n$, can be defined in the usual way.

Let $\mathbb{D}_h = \langle D_h, < \rangle$ and $\mathbb{D}_v = \langle D_v, < \rangle$, where D_h (resp., D_v) is (a prefix of) the set of natural numbers $\mathbb{N}$ and $<$ is the usual linear order. Elements of $\mathbb{D}_h$ (resp., $\mathbb{D}_v$) will be denoted by $h_a, h_b, \ldots$ (resp., $v_a, v_b, \ldots$). A *spatial frame* is a structure $\mathbb{F} = \mathbb{D}_h \times \mathbb{D}_v$. The set of *objects* (rectangles, lines, and points) is the set $\mathbb{O}(\mathbb{F}) = \{\langle (h_a, v_b), (h_c, v_d)\rangle \mid h_a \leq h_c,\, v_b \leq v_d,\, h_a, h_c \in D_h,\, v_b, v_d \in D_v\}$. The semantics of DAC is given in terms of *spatial models* $M = \langle (\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{V}\rangle$, where $\mathbb{F}$ is a spatial frame, $\mathbb{O}(\mathbb{F})$ is the set of relevant objects, and $\mathcal{V} : \mathbb{O}(\mathbb{F}) \to 2^{\mathcal{AP}}$ is a *spatial valuation function*. The pair $(\mathbb{F}, \mathbb{O}(\mathbb{F}))$ is called *spatial structure*. Given a model M and an object $o = \langle (h_a, v_b), (h_c, v_d)\rangle$, the *truth* relation for DAC-formulas (resp., WDAC-formulas) is defined as follows:

- $M, o \Vdash l_h = k$ (resp., $l_v = k$) iff $(h_c - h_a) = k$ (resp., $(v_d - v_b = k)$);
- $M, o \Vdash p$ iff $p \in \mathcal{V}(o)$, for any $p \in \mathcal{AP}$;
- $M, o \Vdash \neg\phi$ iff $M, \langle (h_a, v_b), (h_c, v_d)\rangle \not\Vdash \phi$;
- $M, o \Vdash \phi \vee \psi$ iff $M, \langle (h_a, v_b), (h_c, v_d)\rangle \Vdash \phi$ or $M, \langle (h_a, v_b), (h_c, v_d)\rangle \Vdash \psi$;
- $M, o \Vdash \Diamond_e\psi$ iff there exist $h_e \in D_h$ (resp., $h_e, h_f \in D_h$) such that $h_c \leq h_e$ and there exist $v_g, v_i \in D_v$, such that $v_g \leq v_i$ and $M, \langle (h_c, v_g), (h_e, v_i)\rangle \Vdash \psi$ (resp., $M, \langle (h_e, v_g), (h_f, v_i)\rangle \Vdash \psi$);
- $M, o \Vdash \Diamond_n\psi$ iff there exist $v_e \in D_v$ (resp., $v_e, v_f \in D_v$) such that $v_d \leq v_e$ and there exists $h_g, h_i \in D_h$, such that $h_g \leq h_i$ and $M, \langle (h_g, v_d), (h_i, v_e)\rangle \Vdash \psi$ (resp., $M, \langle (h_g, v_e), (h_i, v_f)\rangle \Vdash \psi$).

Length constraints of the type $l_h > k$ or $l_h < k$ can be easily defined in terms of $l_h = k$, and similarly for the vertical ones.

Proposition 1 *The satisfiability problem for DAC and WDAC can be reduced to the satisfiability problem over an* initial *object* $\langle (0, 0), (h_0, v_0)\rangle$.

As we will show, WDAC is a proper fragment of DAC. The reason why we will consider both logics is that, even though both of them are decidable, we will provide a decision procedure for WDAC whose complexity is exponentially lower than that for DAC. In both cases, optimality is an open issue.

3. Expressive Power of DAC

As mentioned in [19], one of the possible measures of the expressive power of a directional-based spatial logic for rectangles is the comparison with Rectangle Algebra (RA) [20]. In RA, one considers a finite set of objects (rectangles) $O_1, \ldots O_n$, and a set of constraints between pairs of objects. Each constraint is a pair of Allen's Interval Algebra relations that capture the relationships between the projections on the x- and the y-axis of the objects. As an example, $O_1(b, d)O_2$ means that $before$ (resp., $during$) is the interval relation between the x-projections (resp., y-projections) of O_1 and O_2. In general, given an *algebraic constraint network*, the main problem is to establish whether the network is consistent, that is, if all constraints can be jointly

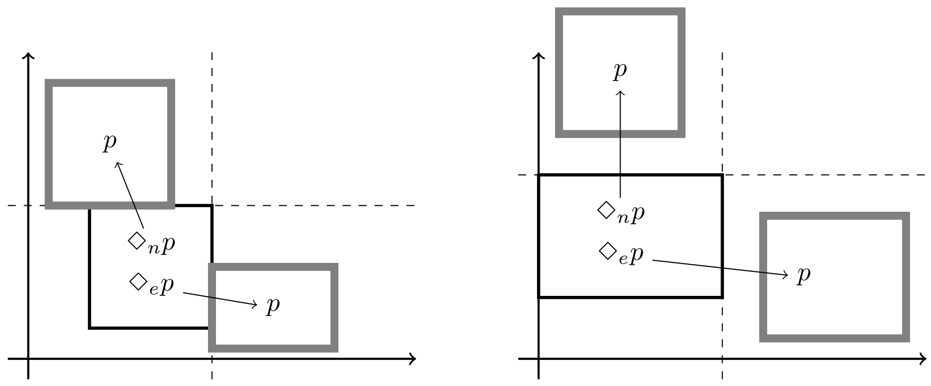

Figure 1. Intuitive semantics of DAC (left) and WDAC (right).

satisfied. In [19], it has been shown that SpPNL is powerful enough to express and to check the consistency of an RA-constraint network, while in [8] it has been proved that the weak, decidable version of that logic (WSpPNL) is powerful enough to do the same. Here, we consider the problem of checking the consistency of an *augmented interval and rectangle network* [10], which is, somehow, the metric version of the consistency problem for a RA-network. An augmented network is basically a RA-network paired with (at most two) set(s) of point-based constraints of the type $O_i^{X^+} - O_j^{X^-} = k$. Such a point-based constraint allows one to relate the endpoints of the various objects; thus, for example, one can force the object O_1 and the object O_2 to be 3 units distant along the x-axis, with O_2 after O_1, by means of $O_2^{X^-} - O_1^{X^+} = 3$. Moreover, with an augmented network, one can constraint the horizontal and/or the vertical length of the various objects, by means of constraints between endpoints of the same object. It is possible to show that such a metric network can be expressed in DAC as follows. First, we define, as in [8], a *weak universal operator* $\Box_u$, that guarantees that p is true "almost everywhere" in a model M, that is, over every object $\langle(h_a, v_b),(h_c, v_d)\rangle$ such that $h_a \neq 0$ or $v_b \neq 0$. Then, we define *weak nominals* (that is, formulas which are true "almost only" on the current object). Finally, given an augmented network with objects $O_1, \ldots, O_n$, we introduce a propositional variable for every object and we force it to be a weak nominal. As for metric constraints, we simply translate them using the metric features of DAC. As an example, the above constraint $O_2^{X^-} - O_1^{X^+} = 3$ can be encoded by the formula:

$$\Box_u(p_{O_1} \rightarrow \Diamond_e(l_h = 3 \wedge \Diamond_e p_{O_2})),$$

where p_{O_1} and p_{O_2} are the nominals corresponding to O_1 and O_2, respectively. In such a way, we are able to represent the network as a conjunction of DAC-formulas which is satisfiable if and only if the network is consistent.

Moreover, using DAC one is able to express very natural spatial statements. As an example, one can define the shortcut:

$$(Area = k) = (l_h = 1 \wedge l_v = k) \vee (l_h = 2 \wedge l_v = \frac{k}{2}) \vee \ldots,$$

by using all possible combinations of horizontal and vertical constraints that give the intended result. In a similar way, one can define $Area > k$ and $Area < k$. Then, it is simple to express the constraint: '*The area of the current object is less then 4 square meters*', by means of the formula: $(Area < 4)$.

Similarly, we can state that *If the area of the current object is greater than 6 square meters, then there exists a line of length 12 to the north of it with the property q, and a point with the property p to the east of it*, by using:

$$\begin{aligned}(Area > 6) \quad \rightarrow \quad & \Diamond_n(l_v = 0 \wedge l_h = 12 \wedge q) \\ & \wedge \Diamond_e(l_h = 0 \wedge l_v = 0 \wedge p).\end{aligned}$$

4. DAC: Decidability and Complexity

4.1. Basic Notions

Let φ be a DAC-formula to be checked for satisfiability and let $\mathcal{AP}$ be the set of its propositional variables. We define the notions of *closure*, *spatial requests*, *atom*, and *fulfilling labeled spatial structure* as follows.

Definition 1 *The* closure $\mathrm{CL}(\varphi)$ *of φ is the set of all subformulas of φ and of their negations (we identify $\neg\neg\psi$ with ψ). The set of* horizontal *(resp.,* vertical*)* spatial requests *of φ is the set $\mathrm{HF}(\varphi)$ (resp., $\mathrm{VF}(\varphi)$) of all horizontal (resp., vertical) spatial formulas in $\mathrm{CL}(\varphi)$, that is, $\mathrm{HF}(\varphi) = \{\Diamond_e\psi, \Box_e\psi \in \mathrm{CL}(\varphi)\}$ (resp., $\mathrm{VF}(\varphi) = \{\Diamond_n\psi, \Box_n\psi \in \mathrm{CL}(\varphi)\}$).*

Definition 2 *A φ-*atom *is a set $A \subseteq \mathrm{CL}(\varphi)$ such that i) for every $\psi \in \mathrm{CL}(\varphi)$, $\psi \in A$ iff $\neg\psi \notin A$, and ii) for every $\psi_1 \vee \psi_2 \in \mathrm{CL}(\varphi)$, $\psi_1 \vee \psi_2 \in A$ iff $\psi_1 \in A$ or $\psi_2 \in A$.*

We denote the set of all φ-atoms by A_φ. Let $|\varphi|$ (the size of φ) be the number of symbols of φ. By induction on the structure of φ, we can easily prove that for every formula φ, $|\operatorname{CL}(\varphi)|$ is linear and $|A_\varphi|$ is at most exponential in $|\varphi|$. Atoms are connected by the binary relations R^h_φ (resp., R^v_φ) over $A_\varphi \times A_\varphi$ such that, for every pair of atoms $(A, A') \in A_\varphi \times A_\varphi$, $A\ R^h_\varphi\ A'$ (resp., $A\ R^v_\varphi\ A'$) if and only if, for every $\Box_e\psi \in \operatorname{CL}(\varphi)$ (resp., $\Box_n\psi \in \operatorname{CL}(\varphi)$), if $\Box_e\psi \in A$ (resp., $\Box_n\psi \in A$), then $\psi \in A'$. We now introduce a suitable labeling of spatial structures based on φ-atoms.

Definition 3 *A φ-labeled spatial structure (*LSS *for short) is a pair* $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$*, where* $(\mathbb{F}, \mathbb{O}(\mathbb{F}))$ *is a spatial structure and* $\mathcal{L} : \mathbb{O}(\mathbb{F}) \to A_\varphi$ *is a* labeling function *such that, for every pair of objects* $\langle(h_a, v_b), (h_c, v_d)\rangle$ *and* $\langle(h_c, v_e), (h_f, v_g)\rangle$*,* $\mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)\ R^h_\varphi\ \mathcal{L}(\langle(h_c, v_e), (h_f, v_g)\rangle)$*, and for every pair of objects* $\langle(h_a, v_b), (h_c, v_d)\rangle$ *and* $\langle(h_e, v_d), (h_f, v_g)\rangle$*,* $\mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)\ R^v_\varphi\ \mathcal{L}(\langle(h_e, v_d), (h_f, v_g)\rangle)$*.*

An LSS $\mathbf{L}$ *is said to be:*

- horizontally *(resp.,* vertically*)* fulfilling *if for every formula of the type $\Diamond_e\psi$ (resp., $\Diamond_n\psi$) in $\operatorname{CL}(\varphi)$ and every object $\langle(h_a, v_b), (h_c, v_d)\rangle$, if $\Diamond_e\psi \in \mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)$ (resp., $\Diamond_n\psi \in \mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)$), then there exists an object $\langle(h_c, v_e), (h_f, v_g)\rangle$ (resp., $\langle(h_e, v_d), (h_f, v_g)\rangle$) such that ψ belongs to $\mathcal{L}(\langle(h_c, v_e), (h_f, v_g)\rangle)$ (resp., $\mathcal{L}(\langle(h_e, v_d), (h_f, v_g)\rangle)$);*
- length fulfilling *if and only if for every length constraint $l_h = k \in \operatorname{CL}(\varphi)$ (resp., $l_v = k \in \operatorname{CL}(\varphi)$) and every object $\langle(h_a, v_b), (h_c, v_d)\rangle \in \mathbb{O}(\mathbb{F})$, $l_h = k \in \mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)$ (resp., $l_v = k \in \mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)$) iff $(h_c - h_a) = k$ (resp., $(v_d - v_b = k)$);*
- fulfilling *if and only if it is horizontally, vertically, and length fulfilling.*

It is quite straightforward to prove that a formula φ is satisfiable if and only if there exists a fulfilling LSS such that φ belongs to the labeling of some initial object $\langle(0, 0), (h_0, v_0)\rangle$. This allows us to reduce the satisfiability problem for φ to the problem of finding a fulfilling LSS with an initial object labeled by φ. From now on, we say that a fulfilling LSS $\mathbf{L}$ *satisfies* φ if and only if $\varphi \in \mathcal{L}(\langle(0, 0), (h_0, v_0)\rangle)$ for some $h_0, v_0 \geq 0$.

4.2. The Elimination Lemma

Since fulfilling LSSs satisfying φ may be arbitrarily large or even infinite, we must find a way to finitely establish their existence. In the following, we will show how the techniques developed in [7] for the metric temporal logic RPNL+INT can be exploited to prove the decidability of DAC. We first give a bound on the size of finite fulfilling LSSs and then we show that in the infinite case we can safely restrict ourselves to infinite fulfilling LSSs with a finite bounded representation. To prove these results, we take advantage of the following two fundamental properties of LSSs: i) the labelings of all objects that share the rightmost horizontal (resp., topmost vertical) coordinate must agree on horizontal (resp., vertical) spatial formulas, that is, for every $\psi \in HF(\varphi)$ (resp., $\psi \in VF(\varphi)$), $\psi \in \mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)$ if and only if $\psi \in \mathcal{L}(\langle(h_e, v_f), (h_c, v_g)\rangle)$ (resp., $\psi \in \mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)$ if and only if $\psi \in \mathcal{L}(\langle(h_e, v_f), (h_g, v_d)\rangle)$); ii) $\frac{|\operatorname{HF}(\varphi)|}{2}$ different objects of the type $\langle(h_c, v_e), (h_f, v_g)\rangle$ are sufficient to fulfill the existential horizontal formulas belonging to the labeling of an object $\langle(h_a, v_b), (h_c, v_d)\rangle$ (and symmetrically for the vertical axis).

Definition 4 *Given an* LSS $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ *and* $h_c \in D_h$ *(resp.,* $v_d \in D_v$*), we denote by* $\operatorname{REQ}_h(h_c)$ *(resp.,* $\operatorname{REQ}_v(v_d)$*) the set of all and only the horizontal (resp., vertical) requests belonging to the labellings of the objects of the type* $\langle(h_a, v_b), (h_c, v_d)\rangle$*. The set* $\operatorname{REQ}_h(\varphi)$ *(resp.,* $\operatorname{REQ}_v(\varphi)$*) is the set of all possible sets of horizontal (resp., vertical) requests for the formula* φ*.*

In order to bound the size of finite LSSs that we must take into consideration when checking the satisfiability of a given formula φ, we determine the maximum number of times that any set in $\operatorname{REQ}_h(\varphi)$ (resp., $\operatorname{REQ}_v(\varphi)$) may appear in a given LSS.

Definition 5 *Given any* LSS $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$*, we say that a* horizontal *(resp.,* vertical*)* k-sequence in $\mathbf{L}$ *is a sequence of k consecutive points in D_h (resp., D_v). Given a horizontal sequence σ in $\mathbf{L}$, its* sequence of requests $\operatorname{REQ}_h(\sigma)$ *is defined as the sequence of horizontal requests at the points in σ, and similarly for the vertical component. We say that* $h \in D_h$ starts a horizontal k-sequence σ *if the horizontal requests at $h, \ldots, h + k - 1$ define an occurrence of $\operatorname{REQ}_h(\sigma)$, and similarly for the vertical component.*

Hereafter, let $m_h = \frac{|\operatorname{HF}(\varphi)|}{2}$, $m_v = \frac{|\operatorname{VF}(\varphi)|}{2}$, and $m = max\{m_h, m_v\}$, and let $k = max\{k', 1\}$, where k' is the the maximal constant that appears in length constraints occurring in φ.

Definition 6 *Given any* LSS $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$*, any sequence of horizontal requests* $\operatorname{REQ}_h(\sigma)$ *is said to be* abundant *in* $\mathbf{L}$ *if and only if it has at least* $k \cdot (m^2 + m) \cdot |\operatorname{REQ}_h(\varphi)|^2 + (m^2 + 3 \cdot m) \cdot |\operatorname{REQ}_h(\varphi)| + 1$ *distinct occurrences in* D_h*. The case of an abundant sequence of vertical requests is defined similarly.*

The above definition shows a quadratic increase in complexity from RPNL+INT: in the temporal case, a number of occurrences linear in m and $\mathrm{REQ}(\varphi)$ suffices to declare a sequence of requests as abundant. For any given horizontal k-sequence σ in $\mathbf{L}$, we will denote by h^σ_q the first point of the q-th occurrence of σ. Hereafter, whenever σ will be evident from the context, we will write h_q for h^σ_q. The next Lemma is analogous to Lemma 5.12 in [7]: in the spatial case we also need the existence of a certain number of occurrences of the sequence *before* a given point h_q to be able to reduce the size of the model.

Lemma 1 *Let $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ be an* LSS *and σ be a horizontal k-sequence in $\mathbf{L}$ such that $\mathrm{REQ}_h(\sigma)$ is abundant in $\mathbf{L}$. Then, there exists an index q such that:*

1. *for every pair $(\mathrm{REQ}_h(h), \mathrm{REQ}_h(h'))$ such that $h \in D^-_h = \{\overline{h} \mid h_q \leq \overline{h} < h_{q+1}\}$ and $h' - h \leq k$, there exist at least $m^2 + m$ distinct pairs of points h'', h''' in $D_h \setminus D^-_h$ such that $h''' - h'' = h' - h$ and $(\mathrm{REQ}_h(h''), \mathrm{REQ}_h(h'''))$ = $(\mathrm{REQ}_h(h), \mathrm{REQ}_h(h'))$;*
2. *for every element $\mathcal{R} \in \{REQ_h(h) \mid h \in D^-_h\}$, $\mathcal{R}$ occurs at least $m^2 + m$ times before h_q and at least $2 \cdot m$ times after h_{q+1}.*

Proof. (sketch) By Definition 6, there exist at least $k \cdot (m^2+m) \cdot |\mathrm{REQ}_h(\varphi)|^2 + (m^2+3\cdot m)| \cdot |\mathrm{REQ}_h(\varphi)| + 1$ points $h \in D_h$ such that h is the first element of a distinct occurrence of σ. For every index i, if there exists a pair $(\mathrm{REQ}_h(h), \mathrm{REQ}_h(h'))$, with $h_i \leq h < h' \leq h_{i+1} + k$, such that there exist no $m^2 + m$ distinct pairs of points h'', h''' in $D_h \setminus \{h \mid h_i \leq h < h_{i+1}\}$ with $h''' - h'' = h' - h$, then q cannot be equal to i. By an easy combinatorial argument, we can prove that there exist at most $k \cdot (m^2+m) \cdot |\mathrm{REQ}_h(\varphi)|^2$ such indexes, where $|\mathrm{REQ}_h(\varphi)|^2$ is the number of possible pairs $(\mathrm{REQ}_h(h), \mathrm{REQ}_h(h'))$, k is the number of possible values for $h' - h$, and, for any pair $(\mathrm{REQ}_h(h), \mathrm{REQ}_h(h'))$ and any distance $h' - h$, $m^2 + m$ is the greatest number of occurrences of a pair $(\mathrm{REQ}_h(h), \mathrm{REQ}_h(h'))$ that may lead to a violation of condition 1. Since σ is abundant in $\mathbf{L}$, we can conclude that there exist at least $(m^2 + 3 \cdot m)|\mathrm{REQ}_h(\varphi)| + 1$ indexes in D_h that satisfy condition 1. Let us now restrict our attention on these indexes. In the worst case, for at most $(m^2 + m) \cdot |\mathrm{REQ}_h(\varphi)|$ indexes i it may happen that there exist no $m^2 + m$ occurrences of $\mathcal{R}$ before h_i for some $\mathcal{R} \in \{REQ_h(h) \mid h_i \leq h < h_{i+1}\}$. Hence, there exist at least $2 \cdot m \cdot |\mathrm{REQ}_h(\varphi)| + 1$ indexes that satisfy both the above conditions. By applying the same argument, we can conclude that for at most $2\cdot m\cdot|\mathrm{REQ}_h(\varphi)|$ indexes i it may happen that there exist no $2\cdot m$ occurrences of $\mathcal{R}$ after h_i for some $\mathcal{R} \in \{REQ_h(h) \mid h_i \leq h < h_{i+1}\}$. This allows us to conclude that there exists at least one index i that satisfies the conditions of the lemma. □

Lemma 2 *(Horizontal Elimination Lemma) Let $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ be a fulfilling* LSS *that satisfies φ. Suppose that there exists an abundant k-sequence of horizontal requests $\mathrm{REQ}_h(\sigma)$ and let D^-_h be the set whose existence is guaranteed by Lemma 1. Then, there exists a fulfilling* LSS *$\overline{\mathbf{L}} = ((\overline{\mathbb{F}}, \mathbb{O}(\overline{\mathbb{F}})), \overline{\mathcal{L}})$ that satisfies φ, with $\overline{D}_h = D_h \setminus D^-_h$ and $\overline{D}_v = D_v$.*

Proof. Let us fix a fulfilling LSS $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ satisfying φ at some $\langle(0, 0), (h_0, v_0)\rangle$, an abundant k-sequence of horizontal requests $\mathrm{REQ}_h(\sigma)$ and let D^-_h be the set whose existence is guaranteed by Lemma 1. Now, let $D'_h = D_h \setminus D^-_h$ and, accordingly, the spatial frame $\mathbb{F}'_1$ and set of objects $\mathbb{O}(\mathbb{F}'_1)$, and $\mathcal{L}'_1 = \mathcal{L}|_{\mathbb{O}(\mathbb{F}'_1)}$ (the restriction of $\mathcal{L}$ to the objects in $\mathbb{O}(\mathbb{F}'_1)$). For the sake of readability, the points in D'_h will be denoted by the same numbers as in D_h. Now, $\mathcal{L}'_1$ is still a LSS, but not necessarily a fulfilling one, thus we have the problem of suitably re-defining the evaluation of objects in a way that preserves the spatial requests and still satisfying φ.

Fixing lengths. First of all, we have to change the labelling of those objects whose horizontal length has changed after the elimination, and it is less or equal to k in $\mathbb{O}(\mathbb{F}'_1)$. To this end, for all $h < h_q$, all $v_a, v_b \in D_v$, and all $0 \leq r \leq k$, we put $\mathcal{L}'_1(\langle(h, v_a), (h_{q+1} + r, v_b)\rangle) = \mathcal{L}(\langle(h, v_a), (h_q + r, v_b)\rangle)$. In this way, we have guaranteed that the objects whose horizontal length was changed have now a correct labelling in terms of all length constraints.

Fixing defects. After the above re-labelling, we can still have the following four types of defects:

1. there is a formula $\Diamond_e\psi \in \mathrm{REQ}_h(h_a)$, for some $h_a \in D_h$, that is not fulfilled anymore because of the elimination of some object $\langle(h_a,v_b),(h,v_c)\rangle$, where $h \in D^-_h$. Notice that for this to be the case, it must be that $(h - h_a) > k$. Since there are at least $2 \cdot m$ points $h_1, \dots, h_{2\cdot m}$ after h_{q+1}, for at least one of them we have that the label of the object $\langle(h_a,v_b),(h_i,v_c)\rangle$ satisfies neither vertical requests from $\mathrm{REQ}_v(v_b)$ nor horizontal requests from $\mathrm{REQ}_h(h_a)$, or it satisfies only requests that are satisfied elsewhere. So we put $\mathcal{L}'_1(\langle\langle(h_a,v_b),(h_i,v_c)\rangle\rangle)$ = $\mathcal{L}(\langle\langle(h_a,v_b),(h,v_c)\rangle\rangle)$, thus fixing the defect;
2. there is a formula $\Diamond_n\psi \in \mathrm{REQ}_v(v_a)$, for some $v_a \in D_v$, that is not fulfilled anymore because of the elimination of some object $\langle(h_b,v_a),(h,v_c)\rangle$, where $h \in D^-_h$. Again, for this to be the case, it must be that $(h - h_b) > k$. To fix this defect, we proceed exactly as in the previous case;
3. there is a formula $\Diamond_n\psi \in \mathrm{REQ}_v(v_a)$, for some $v_a \in D_v$, that is not fulfilled anymore because of the elimination of some object $\langle(h,v_a),(h_b,v_c)\rangle\rangle$, where $h, h_b \in D^-_h$ and $h_b - h \leq k$. Recall that, by hypothesis, there are at least $m^2 + m$ distinct pairs $(h_1, h'_1), \dots, (h_{m^2+m}, h'_{m^2+m})$ such that for all i we have $h_i, h'_i \in D_h\setminus D^-_h$ and $(h'_i-h_i) = (h_b-h)$. Let us consider the horizontal requests $\{\Diamond_e\tau_1, \dots, \Diamond_e\tau_q\} \subseteq$

$\mathrm{REQ}_h(h)$, where $q \leq m$. For each $\Diamond_e \tau_r$ we take an object of the type $\langle (h,v_{\tau_r}),(h_{\tau_r},v'_{\tau_r})\rangle$ containing τ_r in its labelling (in $\mathbf{L}$). Each v_{τ_r} has at most m vertical requests, which are satisfied, in the worst case, using objects with leftmost horizontal coordinate of the type h_i. Then, at most m^2 horizontal coordinates are needed to satisfy the vertical requests of the vertical coordinates of the type v_{τ_r}. Let us consider now the vertical coordinate v_a. Again, the vertical requests in $\mathrm{REQ}_v(v_a)$ different from $\Diamond_n \psi$ are at most $m-1$, so, there must be at least one horizontal coordinate h_i such that no object with h_i as leftmost horizontal coordinate satisfy any vertical request of v_a or of any of the v_{τ_r}. We can then put $\mathcal{L}'_1(\langle (h_i,v_a),(h'_i,v_c)\rangle) = \mathcal{L}(\langle (h,v_a),(h_b,v_c)\rangle)$, thus fixing the defect. However, in general, such a substitution can introduce a new defect, since there can be some $\Diamond_e \theta \in \mathrm{REQ}_h(h_i)$ which was satisfied by $\langle (h_i,v_a),(h'_i,v_c)\rangle$ and it is not satisfied anymore. Now, since $\mathrm{REQ}_h(h_i) = \mathrm{REQ}_h(h)$, $\theta = \tau_r$ for some r. We can fix this new defect by putting $\mathcal{L}'_1(\langle (h_i,v_{\tau_r}),(h'_i,v'_{\tau_r})\rangle) = \mathcal{L}(\langle (h,v_{\tau_r}),(h_{\tau_r},v'_{\tau_r})\rangle)$. By repeating this last substitution in a suitable way at most m times, we can fix all new defects that can be possibly introduced;

4. there is a formula $\Diamond_n \psi \in \mathrm{REQ}_v(v_a)$, for some $v_a \in D_v$, that is not fulfilled anymore because of the elimination of some object $\langle (h,v_a),(h_b,v_c)\rangle$, where $h \in D_h^-$, and $(h_b - h) > k$. To fix this defect, we proceed exactly as in case 3, but using only the $m^2 + m$ copies of h before h_q, and maintaining h_b as the rightmost horizontal coordinate.

In this way we can eliminate all defects; at the end of the process we obtain $\overline{\mathbf{L}}$ as claimed. $\Box$

Similarly, we have:

Lemma 3 *(Vertical Elimination Lemma) Let* $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ *be a fulfilling* LSS *that satisfies* φ. *Suppose that there exists an abundant* k*-sequence of vertical requests* $\mathrm{REQ}_v(\sigma)$ *and let* D_v^- *be the set whose existence is guaranteed by the (vertical version of) Lemma 1. Then, there exists a fulfilling* LSS $\overline{\mathbf{L}} = ((\overline{\mathbb{F}}, \mathbb{O}(\overline{\mathbb{F}})), \overline{\mathcal{L}})$ *that satisfies* φ, *with* $\overline{D}_v = D_v \setminus D_v^-$ *and* $\overline{D}_h = D_h$.

Lemma 2 and 3 are the spatial counterpart of the Elimination Lemma for RPNL+INT [7]. However, while in the temporal case we have to deal only with defects of type 1, the interaction between the two spatial operators of DAC adds two more types of defects.

4.3 Satisfiability for DAC

Thanks to the horizontal and vertical elimination lemmas above, we have that the following theorem holds.

Theorem 1 (Small Model Theorem) *If* φ *is any finitely satisfiable formula of DAC, then it is satisfiable in a finite* LSS $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ *such that* $|D_h| \leq (k \cdot (m^2+m) \cdot |\mathrm{REQ}_h(\varphi)|^2 + (m^2+3\cdot m) \cdot |\mathrm{REQ}_h(\varphi)|) \cdot |\mathrm{REQ}_h(\varphi)|^k + k - 1$, *and* $|D_v| \leq (k \cdot (m^2+m) \cdot |\mathrm{REQ}_v(\varphi)|^2 + (m^2 + 3\cdot m) \cdot |\mathrm{REQ}_v(\varphi)|) \cdot |\mathrm{REQ}_v(\varphi)|^k + k - 1$.

Corollary 1 *Finite satisfiability for DAC is decidable.*

Infinite structures can be dealt with in a similar way. First of all, we must distinguish among three types of infinite LSSs, depending on whether only one domain is infinite (and which one) or both. For each of these types, an appropriate representation can be obtained as follows.

Definition 7 *Any* LSS $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ *is* horizontally ultimately periodic, *with prefix* Pre_H, *period* $Per_H \geq 0$ *and threshold* k, *if and only if:*

1. *for all* h, h' *such that* $h' \geq h_{Pre_H}$ *and* $(h' - h) > k$, *and for all* v, v', $\mathcal{L}(\langle (h,v),(h',v')\rangle) = \mathcal{L}(\langle (h,v),(h'+Per_H,v')\rangle)$;
2. *for each object* $\mathcal{L}(\langle (h,v),(h',v')\rangle)$ *such that* $h \geq h_{Pre_H}$, $\mathcal{L}(\langle (h,v),(h',v')\rangle) = \mathcal{L}(\langle (h+Per_H,v),(h'+Per_H,v')\rangle)$.

The notion of vertically ultimately periodic LSS *can be defined in a similar way. Finally, a* LSS *is simply* ultimately periodic *if it is (i) both horizontally and vertically ultimately periodic, or (ii) horizontally ultimately periodic and vertically finite, or (iii) horizontally finite and vertically ultimately periodic.*

Note that every ultimately periodic LSS is finitely presentable.

Lemma 4 *Let* $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ *be an horizontally infinite, vertically finite* LSS *that satisfies* φ. *Then, there exists an ultimately periodic* LSS $\overline{\mathbf{L}}$ *that satisfies* φ.

An analogous of Lemma 4 can be stated for the vertical component, and, thus, any infinite LSS can be transformed in a ultimately periodic one.

Theorem 2 (Periodic Small Model Theorem) *Let* $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ *be any* LSS *that satisfies* φ. *Then, there exists an ultimately periodic* LSS $\overline{\mathbf{L}}$ *that satisfies* φ, *and such that length of the horizontal prefix and the horizontal period are bounded by* $(k \cdot (m^2+m) \cdot |\mathrm{REQ}_h(\varphi)|^2 + (m^2+3\cdot m) \cdot |\mathrm{REQ}(\varphi)|) \cdot |\mathrm{REQ}_h(\varphi)|^k + k - 1$, *and similarly for the vertical component.*

Once again, the spatial features of DAC causes a quadratic increase on the size of (prefixes and periods of) the model with respect to the metric temporal logic RPNL+INT [7].

Corollary 2 *The satisfiability problem for DAC is decidable.*

4.4 Complexity Issues

In [8] it has been shown that the non-metric version of DAC presents a NEXPTIME-complete satisfiability problem. This means that DAC is at least NEXPTIME-hard. To correctly state the complexity of the satisfiability problem for DAC, we have to consider three different cases, depending on the representation of length constraints. As a direct consequence of the theorems of the above section, a nondeterministic decision procedure that guesses an ultimately periodic model satisfying the formula φ can be easily built. Such a procedure works in NTIME($2^{|\varphi|\cdot k}$), and its exact complexity class depends on how the metric constants are encoded.

Theorem 3 *The satisfiability for DAC is:*

- *NEXPTIME-complete, if k is a constant;*
- *NEXPTIME-complete, if k is represented in unary;*
- *Between EXPSPACE and 2NEXPTIME, if k is represented in binary.*

NEXPTIME inclusion (cases 1 and 2) can be proved simply observing that $O(2^{|\varphi|\cdot k}) = O(2^{|\varphi|})$ if k is constant or represented in unary (respect to the length of the formula); NEXPTIME-hardness is a consequence of NEXPTIME-hardness for SpPNL [8]. In these cases there is no complexity increase with respect to the temporal counterpart RPNL+INT, that it is NEXPTIME-hard as well [7]. Conversely, when k is represented in binary (Case 3) we have that RPNL+INT is EXPSPACE-complete, and thus that DAC is at least EXPSPACE-hard. However, since $k = O(2^{|\varphi|})$, the non-deterministic procedure runs in time $O(2^{2^{|\varphi|}})$, giving us a 2NEXPTIME upper bound on the complexity. We do not know yet which is the exact complexity class for DAC in this case, and if the spatial generalization causes an increase on the complexity or not.

5. Weak Directional Area Calculus (WDAC)

In this section we discuss expressive power, decidability, and complexity of Weak DAC, comparing it with the full Directional Area Calculus.

First of all, formulas of WDAC can be translated to DAC-formulas by replacing any sub-formula of the form $\Diamond_e\psi$ (resp., $\Diamond_n\psi$) with $\Diamond_e\psi \vee \Diamond_e\Diamond_e\psi$ (resp., $\Diamond_n\psi \vee \Diamond_n\Diamond_n\psi$). By exploiting a *bisimulation* argument we can prove that the converse does not hold. We will show that, for every $k \geq 0$, there exist two models M_1^k and M_2^k that are bisimilar with respect to WDAC-formulas with maximum metric constant k, but can be easily distinguished by a DAC formula. Let $k \geq 0$, $\mathcal{AP} = \{p\}$: the two spatial models $M_1 = \langle \mathbb{F}_1, \mathbb{O}(\mathbb{F}_1), \mathcal{V}_1\rangle$ and $M_2 = \langle \mathbb{F}_2, \mathbb{O}(\mathbb{F}_2), \mathcal{V}_2\rangle$ are defined as follows.

- $\mathbb{F}_1 = \mathbb{F}_2 = \mathbb{N}\times\mathbb{N}$
- $\mathcal{V}_1(\langle(1, v_a), (k+4, v_b)\rangle) = \mathcal{V}_1(\langle(3, v_a), (k+4, v_b)\rangle) = \{p\}$, for all $v_a, v_b \in \mathbb{N}$;
- $\mathcal{V}_2(\langle(3, v_a), (k+4, v_b)\rangle) = \{p\}$, for all $v_a, v_b \in \mathbb{N}$;
- p is false everywhere else.

The relation $Z^k \subseteq \mathbb{O}(\mathbb{F}_1)\times\mathbb{O}(\mathbb{F}_2)$, defined as follows, is a WDAC-bisimulation between M_1^k and M_2^k:

- $(\langle(h_a, v_b), (h_c, v_d)\rangle, \langle(h_a, v_b), (h_c, v_d)\rangle) \in Z^k$ for all $(h_a, h_c) \neq (1, k+4)$;
- $(\langle(1, v_b), (k+4, v_d)\rangle, \langle(3, v_b), (k+4, v_d)\rangle) \in Z^k$;
- $(\langle(2, v_b), (k+4, v_d)\rangle, \langle(1, v_b), (k+4, v_d)\rangle) \in Z^k$.

Since the DAC-formula $\Diamond_e p$ is true over the object $\langle(0,0),(1,1)\rangle$ in M_1^k but false in M_2^k for every value of k, and since bisimilar models must satisfy the same set of WDAC formulas, $\Diamond_e p$ cannot be translated to any WDAC formula.

Theorem 4 *WDAC is strictly less expressive than DAC.*

Despite being strictly less expressive than DAC, Weak DAC is powerful enough to express the augmented interval and rectangle network consistency problem discussed in Section 3, at the price of a more complex encoding.

Decidability of WDAC trivially follows from the decidability of DAC. However, its weaker semantics allows us to lower the complexity bound. The modal operators are *transitive* in WDAC: if a formula $\Box_e\psi$ holds over an object, then it holds over any object to the east of it (and symmetrically for $\Box_n\psi$), while in full DAC this is not necessarily the case. This implies that if a formula $\Box_e\psi \in \mathrm{REQ}_h(h_a)$ (resp., $\Box_n\psi \in \mathrm{REQ}_v(v_a)$) for some $h_a \in D_h$ (resp., $v_a \in D_v$), then $\Box_e\psi \in \mathrm{REQ}_h(h_b)$ for every $h_b > h_a$ (resp., $\Box_n\psi \in \mathrm{REQ}_v(v_b)$ for every $v_b > v_a$). By exploiting this property, we can provide a bound on the size of LSS satisfying a WDAC formula that is exponentially smaller than the one given for DAC in Theorem 2.

Theorem 5 (Weak Periodic Small Model Theorem) *Let φ be a satisfiable WDAC formula. Then, there exists a ultimately periodic fulfilling* LSS *satisfying φ with horizontal and vertical prefix bounded by* $(2\cdot m+1)\cdot(k+1)+1$, *horizontal and vertical period bounded by* $2\cdot m\cdot(k+1)$, *and threshold k.*

As a direct consequence of Theorem 5, a nondeterministic decision procedure that guesses an ultimately periodic model satisfying the formula φ can be easily built. Such a procedure works in NTIME($k\cdot|\varphi|$), and its exact complexity class depends on how the metric constants are encoded.

Theorem 6 *Satisfiability for WDAC is:*

- *NP-complete, if k is a constant;*
- *NP-complete, if k is represented with unary encoding;*
- *in NEXPTIME, if k is represented with binary encoding.*

NP-completeness of the problem when k is constant or in unary encoding follows from the NP-completeness of SAT. We do not know yet if WDAC with binary encoding is NEXPTIME-hard or not.

6 Conclusions

In this paper, we proposed a new modal logic, called DAC, that pairs qualitative and quantitative spatial reasoning about points, lines, and rectangles over natural numbers frames by means of directional relations. DAC can be viewed as an extension of the spatial logic WSpPNL [8] with special atomic propositions that make it possible to express a weak notion of area. We proved that the satisfiability problem for DAC is decidable. Moreover, we showed that, when a binary encoding of length constraints is provided, it is between EXPSPACE and 2NEXPTIME, while the exact complexity class is an open problem. Then, we analyzed the satisfiability problem for a proper expressive fragment of DAC, called WDAC, and we proved that it belongs to NEXPTIME. As in the case of DAC, the exact complexity class, when a binary encoding of length constraints is provided, is an open problem.

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