Undecidability of Interval Temporal Logics with the Overlap Modality

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Brixen, 23th July - TIME 2009

Outline

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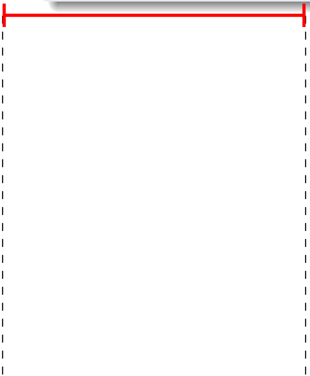
Interval Temporal Logics

- The time period, instead of the time intstant, is the primitive temporal entity
- Propositional letters are evaluated over pairs of points (instead of individual points)
- Relations between worlds are more complicate than the point-based case

Allen's relations

J. F. Allen
Maintaining knowledge about temporal intervals.
Communications of the ACM, 1983.



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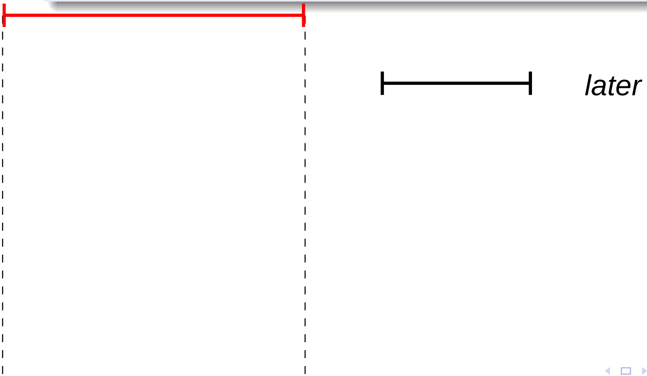

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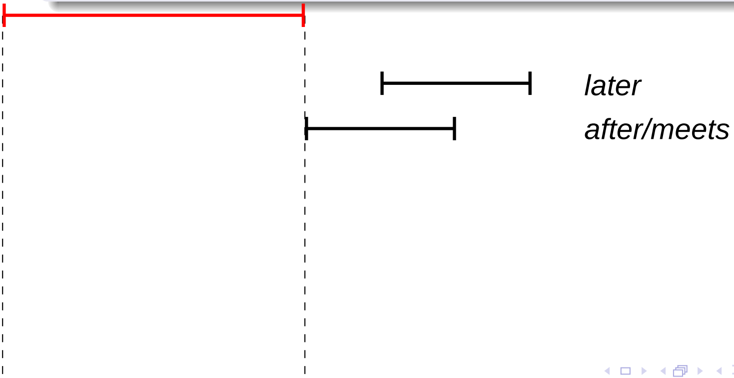

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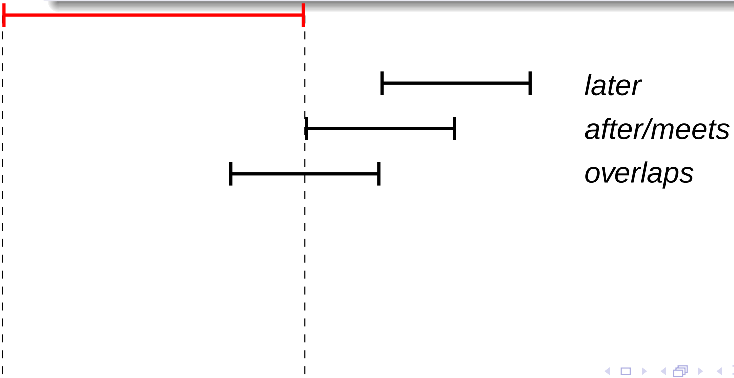

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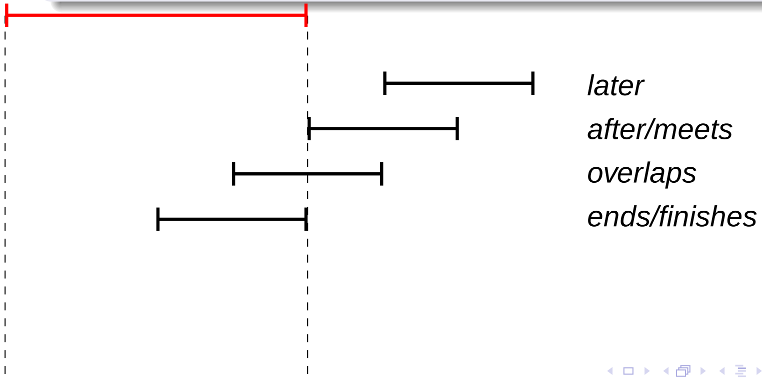

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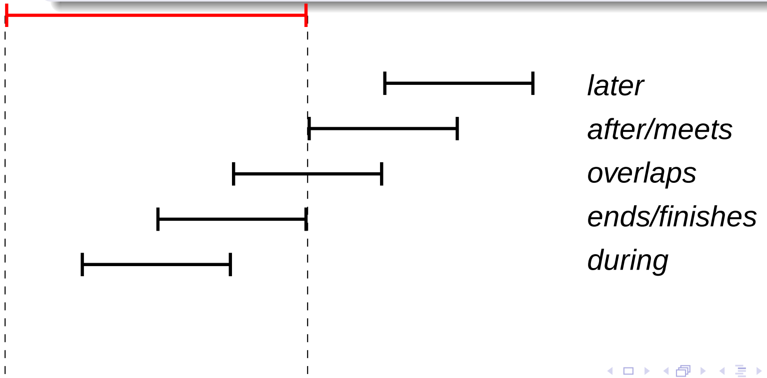

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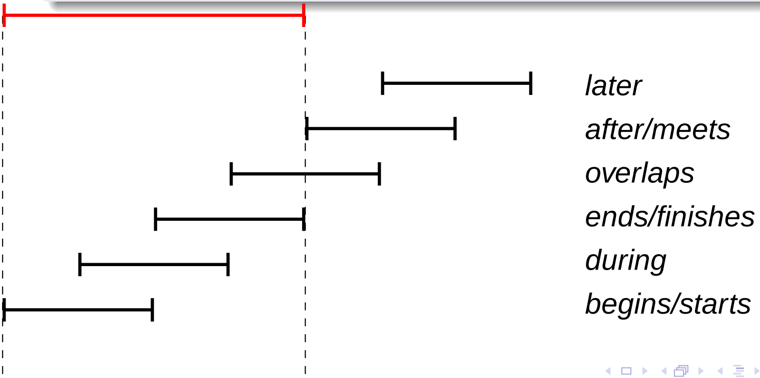

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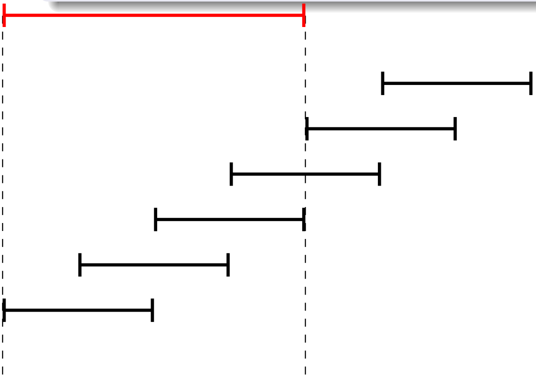

later	$\langle L \rangle$
after/meets	$\langle A \rangle$
overlaps	$\langle O \rangle$
ends/finishes	$\langle E \rangle$
during	$\langle D \rangle$
begins/starts	$\langle B \rangle$

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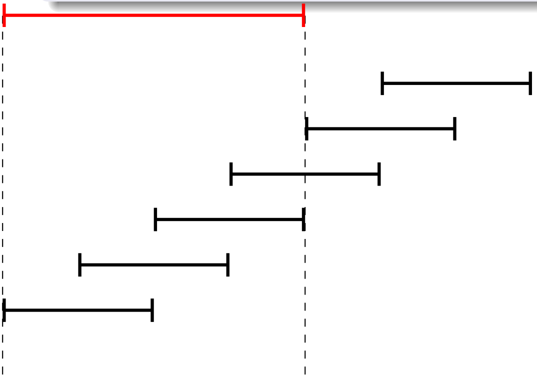

later	$\langle L \rangle$ $\langle \overline{L} \rangle$
after/meets	$\langle A \rangle$ $\langle \overline{A} \rangle$
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begins/starts	$\langle B \rangle$ $\langle \overline{B} \rangle$

Some ontological choices

- Time structure:
  - linear or branching !
  - discrete or dense !
  - with or without beginning/end !
- Nature of intervals:
  - can or cannot intervals be unbounded !
  - Are intervals with coinciding endpoints admissible or not admissible !

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Outline

First discouraging undecidability results

HS is undecidable

J. Halpern and Y. Shoham
A propositional modal interval logic.
Journal of the ACM, 1991.

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Undecidability of a small fragment of HS: BE

K. Lodaya
Sharpening the Undecidability of Interval Temporal Logic.
ASIAN 2000, volume 1961 of LNCS, pages 290-298. Springer, 2000.

First decidable fragments

- Restrictions of the interval-based semantics
  - **locality**: truth of atomic propositions over an interval is defined as truth at its initial point
  - **homogeneity**: truth of a formula over an interval implies truth of that formula over every sub-interval

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- Restrictions of the underlying structures
  - **split logic**: each interval can be divided in subintervals in only one way

A. Montanari, G. Sciavicco, and N. Vitacolonna
Decidability of interval temporal logics over split-frames via granularity.
JELIA 2002, volume 2424 of LNCS, pages 259-270. Springer, 2002.

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- Simple fragments of HS
  - $\mathbf{B\overline{B}}$, $\mathbf{E\overline{E}}$

More meaningful decidable fragments

- **RPNL** (A)

D. Bresolin, A. Montanari, and G. Sciavicco
An optimal decision procedure for Right Propositional Neighborhood Logic.
Journal of Automated Reasoning, 2007.

More meaningful decidable fragments

- **RPNL** (A)
- **PNL** ($A\overline{A}$)

D. Bresolin, A. Montanari, and P. Sala
An optimal tableau-based decision algorithm for Propositional Neighborhood Logic.
STACS 2007, volume 4393 of LNCS, pages 549-560. Springer, 2007.

More meaningful decidable fragments

- **RPNL** (A)
- **PNL** (A$\overline{\text{A}}$)
- **Subinterval logic** (D)

D. Bresolin, V. Goranko, A. Montanari, P. Sala
Tableau-based decision procedures for the logics of subinterval structures over dense orderings.
Journal of Logic and Computation, December 2008.

State of the art

- $D\overline{D}$ is decidable over dense linear orders
- Most extensions of A (resp., $\overline{A}$) are undecidable (except for the ones with $B\overline{B}$ and $E\overline{E}$)
- The class of fragments $B^*E^*(= BE, B\overline{E}, \overline{B}E, \overline{BE})$ is undecidable

A. Montanari, G. Puppis and P. Sala
A Decidable Spatial Logic with Cone-shaped Cardinal Directions.
CSL 2009 (in press).

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D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco

Decidable and Undecidable Fragments of Halpern and Shohams Interval Temporal Logic: Towards a Complete Classification.

LPAR 2008, volume 5330 of LNCS, pages 590-604. Springer, 2008.

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Outline

In this paper

We study the satisfiability problem for logics containing the overlap modality

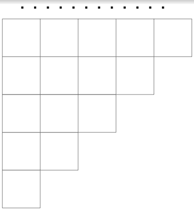
We provide a number of undecidability results

- All the extensions of the fragments O and $\overline{\mathrm{O}}$ (except for the extensions with L and $\overline{\mathrm{L}}$) are undecidable
- The logic $\mathrm{O}\overline{\mathrm{O}}$ is undecidable over discrete linear orders

Proof overview

Reduction from the Octant Tiling Problem

This is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile $\mathcal{O} = \{(i,j) : i,j \in \mathbb{N} \wedge 0 \leq i \leq j\}$ respecting the color constraints.



by König's Lemma

$\mathbb{N} \times \mathbb{N} \to \mathcal{O}$

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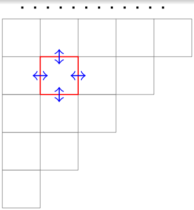

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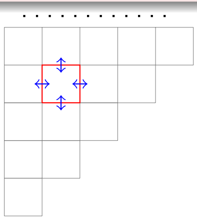

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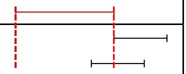
Proof overview (cont'd)

We focus on the proof for the fragment AO

We build a formula $\phi_{\mathcal{T}} \in$ AO s.t.

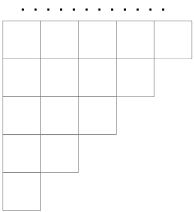
$$\phi_{\mathcal{T}} \text{ is satisfiable} \Leftrightarrow \mathcal{T} \text{ can tile the octant.}$$

Op.	Semantics
$\langle A \rangle$	$\mathbf{M}, [a,b] \Vdash \langle A \rangle \phi \Leftrightarrow \exists c (b < c . \mathbf{M}, [b,c] \Vdash \phi)$
$\langle O \rangle$	$\mathbf{M}, [a,b] \Vdash \langle O \rangle \phi \Leftrightarrow \exists c, d (a < c < b < d . \mathbf{M}, [c,d] \Vdash \phi)$



Proof overview (cont'd)

1. Encoding the octant
2. Encoding the neighbourhood relations



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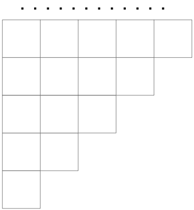
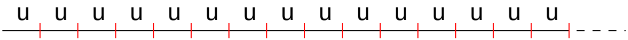


$u \rightarrow \langle A \rangle u$



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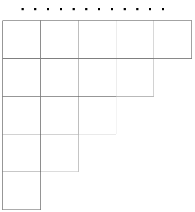
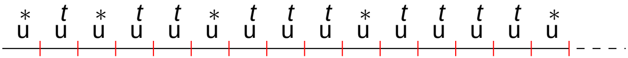


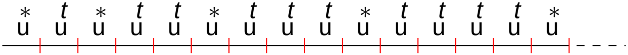


Proof overview (cont'd)

1. Encoding the octant
2. Encoding the neighbourhood relations

. . . .

t_1^5	t_2^5	t_3^5	t_4^5	t_5^5
t_1^4	t_2^4	t_3^4	t_4^4	
t_1^3	t_2^3	t_3^3		
t_1^2	t_2^2			
t_1^1				



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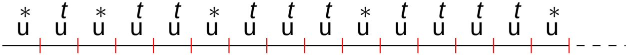

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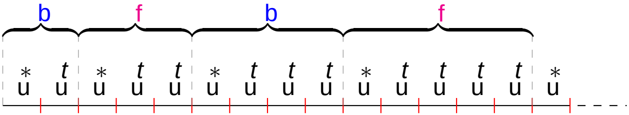
b	t_1^5	t_2^5	t_3^5	t_4^5	t_5^5
f	t_1^4	t_2^4	t_3^4	t_4^4	
b	t_1^3	t_2^3	t_3^3		
f	t_1^2	t_2^2			
b	t_1^1				



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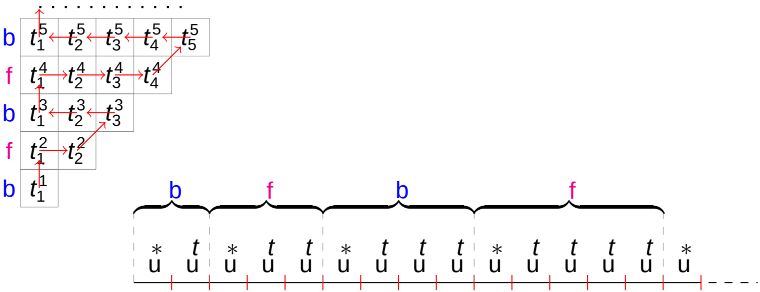

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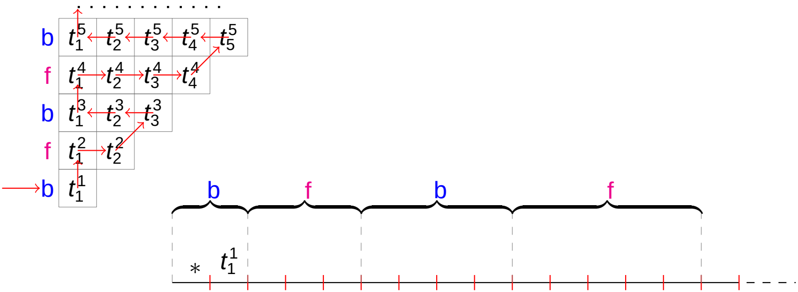

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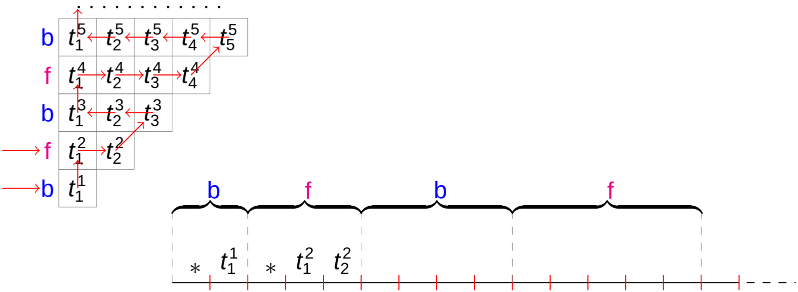

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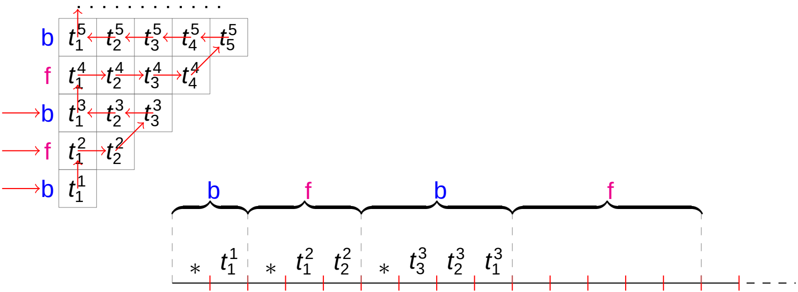

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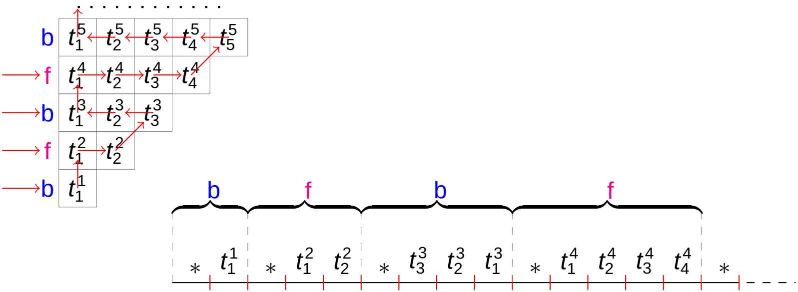

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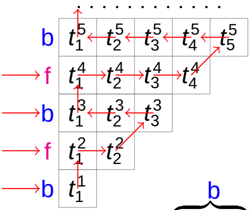
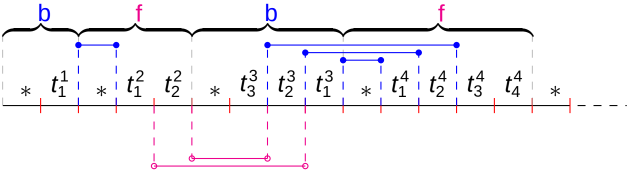


$\mathrm{up\_rel}^{b} \rightarrow \neg\langle O\rangle \mathrm{up\_rel}^{b}$

$\mathrm{up\_rel}^{f} \rightarrow \neg\langle O\rangle \mathrm{up\_rel}^{f}$



Theorems

Theorem [AO undecidability]

The satisfiability problem for the logic AO is undecidable over any class of linear orders that contains at least one linear order with an infinite ascending sequence.

Theorem [$A^*O^*, B^*O^*, E^*O^*, D^*O^*$ undecidability]

The satisfiability problem for the logics $\overline{\mathsf{A}}\mathsf{O}$, BO, $\overline{\mathsf{B}}\mathsf{O}$, EO, $\overline{\mathsf{E}}\mathsf{O}$, DO, and $\overline{\mathsf{D}}\mathsf{O}$ (resp., $\mathsf{A}\overline{\mathsf{O}}$, $\overline{\mathsf{A}}\,\overline{\mathsf{O}}$, $\mathsf{B}\overline{\mathsf{O}}$, $\overline{\mathsf{B}}\,\overline{\mathsf{O}}$, $\mathsf{E}\overline{\mathsf{O}}$, $\overline{\mathsf{E}}\,\overline{\mathsf{O}}$, $\mathsf{D}\overline{\mathsf{O}}$, and $\overline{\mathsf{D}}\,\overline{\mathsf{O}}$) is undecidable over any class of linear orders that contains at least one linear order with an infinite ascending (resp., descending) sequence.

Outline

Conclusions

- Undecidable extensions of the fragments O and $\overline{\text{O}}$: $\text{A}^*\text{O}^*, \text{B}^*\text{O}^*, \text{E}^*\text{O}^*, \text{D}^*\text{O}^*$
- Fragment $\text{O}\overline{\text{O}}$ undecidable over discrete linear orders
- Extended to the fragment O (resp., $\overline{\text{O}}$) over discrete linear orders (unpublished)

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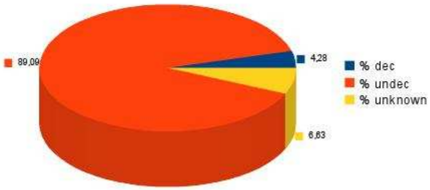
Current classification for the main classes of linear orders



Pure fragments	1448
Decidable	62
Unknown	96
Decidable+unknown	158
Undecidable	1290

% dec	4,28
% undec	89,09
% unknown	6,63
Tot	100

Future works

To complete the classification of HS fragments:

- $\mathsf{L}^*\mathsf{O}^*$ and $\mathsf{L}^*\mathsf{D}^*$: conjecture is undecidability
- O, $\overline{\mathsf{O}}$, and $\mathsf{O}\overline{\mathsf{O}}$ over dense linear orders: conjecture is ?
- D, $\overline{\mathsf{D}}$, and $\mathsf{D}\overline{\mathsf{D}}$ over discrete linear orders: conjecture is ?
- $\mathsf{B}^*\mathsf{D}^*$: conjecture is decidability at least over dense structures

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- $\mathsf{O}, \overline{\mathsf{O}}$, and $\mathsf{O}\overline{\mathsf{O}}$ over dense linear orders: conjecture is ?
- D, $\overline{\mathsf{D}}$, and $\mathsf{D}\overline{\mathsf{D}}$ over discrete linear orders: conjecture is ?
- $\mathsf{B}^*\mathsf{D}^*$: conjecture is decidability at least over dense structures