When are prime formulae characteristic?[☆],[☆☆]

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Abstract

In the setting of the modal logic that characterizes modal refinement over modal transition systems, Boudol and Larsen showed that the formulae for which model checking can be reduced to preorder checking, that is, the characteristic formulae, are exactly the consistent and prime ones. This paper presents general, sufficient conditions guaranteeing that characteristic formulae are exactly the consistent and prime ones. It is shown that the given conditions apply to various behavioural relations in the literature. In particular, characteristic formulae are exactly the prime and consistent ones for all the semantics in van Glabbeek's linear time-branching time spectrum.

Keywords: process semantics, logics, characteristic formulae, (bi)simulation

1. Introduction

Model checking and equivalence/preorder checking are the two main approaches to the computer-aided verification of reactive systems [2, 3, 4]. In model checking, one typically describes the behaviour of a computing system using a state-transition model, such as a labelled transition system [5], and specifications of properties systems should exhibit are expressed using some modal

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or temporal logic. In this approach, system verification amounts to checking whether a system is a model of the formulae describing a given specification. When using equivalence/preorder checking instead, systems and their speci-

fications are both expressed in the same state-machine-based formalism. In this approach, checking whether a system correctly implements its specification amounts to verifying whether the state machines describing them are related by some suitable notion of behavioural equivalence/preorder. (See [6, 7] for taxonomic studies of the plethora of behavioural relations that have been considered in the field of concurrency theory.)

The study of the connections between the above-mentioned approaches to system verification and, more generally, between behavioural and logical approaches to defining the semantics of reactive systems has been one of the classic topics of research in concurrency theory since the work of Hennessy and Milner, who showed in [8] that bisimilarity [9, 10] coincides with the equivalence induced by a multi-modal logic, which is now commonly called Hennessy-Milner Logic, over labelled transition systems satisfying a mild finiteness constraint. Similar modal characterization results have been established for all the semantics in van Glabbeek's linear-time/branching-time spectrum [7]. Such results are truly fundamental in concurrency theory and have found a variety of applications in process theory—see, for instance, the papers [11, 12]. However, they do not provide a "practically useful" connection between model checking and equivalence/preorder checking; indeed, using a modal characterization theorem to prove that two labelled transition systems are equated by some notion of behavioural equivalence would involve showing that the two systems satisfy exactly the same (typically infinite) set of formulae in the modal logic that characterizes the equivalence of interest.

A bridge between model checking and equivalence/preorder checking is provided by the notion of *characteristic formula* [13, 14]. Intuitively, a characteristic formula provides a complete logical characterization of the behaviour of a process modulo some notion of behavioural equivalence or preorder. The complexity of the problem of deciding whether a formula is characteristic for a process has been studied in [15, 16] in the setting of several modal logics and μ-calculus. At least for finite labelled transition systems, characteristic formulae can be used to reduce equivalence/preorder checking to model checking effectively [17]. A natural question to ask is for what kinds of logical specifications model checking can be reduced to establishing a behavioural relation between an implementation and a labelled transition system that suitably encodes the specification. To the best of our knowledge, this question was first addressed by Boudol and Larsen, who showed in [18] that, in the context of the modal logic that characterizes modal refinement over modal transition systems, the formulae that are "graphically representable" (that is, the ones that are characteristic for some process) are exactly the consistent and prime ones. (A formula is *prime* if whenever it implies a disjunction of two formulae, it implies one of the disjuncts.) A similar result is given in [19] in the setting of covariant-contravariant simulation. Moreover, each formula in the logics considered in [19, 18] can be "graphically represented" by a (possibly empty) finite set of processes.

To our mind, those are very pleasing results that show the very close connection between logical and behavioural approaches to verification in two specific settings. But, how general are they? Do similar results hold for the plethora of other process semantics and their modal characterizations studied in the literature? And, if so, are there general sufficient conditions guaranteeing that characteristic formulae are exactly the consistent and prime ones? The purpose of this article is to provide answers to those questions. In particular, we aim to understand when the notions of characteristic and prime formulae coincide (we refer to such a correspondence as *characterization by primality*), thus providing a characterization of logically defined processes by means of prime formulae.

We work in an abstract setting (described in Section 2), and, instead of investigating each behavioural semantics separately, we define the process semantics as the preorder induced by some logic, i.e. a process p is smaller than a process q if the set of logical properties of p is strictly included in that of q. By investigating preorders defined in this way, we can identify common properties for all logically characterized preorders. It turns out that characteristic formulae are always consistent and prime (Theorem 1). Therefore our main task is to provide sufficiently general conditions guaranteeing that consistent and prime formulae are characteristic formulae for some process.

In Section 3, we introduce the notion of *decomposable logic* and show that, for such logics, consistent and prime formulae are characteristic for some process (Theorem 2). (Intuitively, a logic is decomposable if, for each consistent formula, the set of processes satisfying it includes the set of processes satisfying a characteristic formula and the logic is sufficiently expressive to witness this inclusion.) We then proceed to identify features that make a logic decomposable, thus paving the way to showing the decomposability of a number of logical formalisms (Section 3.1). In particular, we provide two paths to decomposability, namely, a logic is decomposable if

- the set of formulae satisfied by each process can be finitely characterized in a suitable technical sense (see Definition 5) and some additional mild assumptions are met (Corollary 4); or

- each formula can be expressed as the union of the meaning of characteristic formulae (see Definition 2) and some additional assumptions are met (Proposition 12).

In order to show the applicability of our general framework, we use it in Sections 4–5 to prove characterization by primality (i.e., characteristic formulae are exactly the consistent and prime ones) for a variety of logical characterizations of process semantics. In particular, this applies to all the semantics in van Glabbeek's linear time-branching time spectrum. In all these cases, there is a perfect match between the behavioural and logical view of processes: not only do the logics characterize processes up to the chosen notion of behavioural relation, but processes represent all the consistent and prime formulae in the logics.

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Finally, in Section 6 we provide an assessment of the work done and outline future research directions. In particular, in Section 6.1, we give evidence (Proposition 18) that the first path to decomposability we provide (based on Corollary 4) cannot be used to show the decomposability of the logic characterizing conformance simulation [20]. However, we are confident that the alternative path to decomposability (through Proposition 12) might serve the purpose, and we plan to address the issue in the near future.

This paper is an extended version of [1]. Compared with that preliminary study, the present work contains the following new material.

- We provide a more elegant proof of characterization by primality for the case when there are only finitely many processes in Section 4. Such a proof uses an approach that allows us to deal, in a uniform way, with modal refinement and covariant-contravariant simulation as well; it involves the notion of graphical representation of a formula (which is the inverse of characteristic formula of a process) and the one of finitely representability (Section 4).

- We prove characterization by primality for all of the logics characterizing the linear semantics in van Glabbeek's spectrum (Section 5.2). Such a result does not appear in [1] and is thus original.

- We include, in Section 6.1 (Proposition 18), a counterexample showing that conformance simulation cannot be dealt with using Corollary 4 (the first path to decomposability we provide).

- We give detailed proofs for all the results appearing in the conference version.

2. Process semantics defined logically

We assume that $\mathcal{L}$ is a language interpreted over a non-empty set P, which we refer to as a set of processes. Thus, $\mathcal{L}$ is equipped with a semantic function $[\![\cdot]\!]_{\mathcal{L}} : \mathcal{L} \to \mathcal{P}(P)$ (where $\mathcal{P}(P)$ denotes the powerset of P), and we say that $p \in P$ satisfies $\phi \in \mathcal{L}$ whenever $p \in [\![\phi]\!]_{\mathcal{L}}$. For all $p, q \in P$, we define the following notions:
- $\mathcal{L}(p) = \{\phi \in \mathcal{L} \mid p \in [\![\phi]\!]_{\mathcal{L}}\}$: the set of formulae in $\mathcal{L}$ that p satisfies; we assume $\mathcal{L}(p) \neq \emptyset$, for each $p \in P$;
- $p^{\uparrow_{\mathcal{L}}} = \{p' \in P \mid \mathcal{L}(p) \subseteq \mathcal{L}(p')\}$: the *upwards closure* of p (with respect to $\mathcal{L}$);
- p and q are *logically equivalent* (with respect to $\mathcal{L}$), denoted by $p \equiv_{\mathcal{L}} q$, iff $\mathcal{L}(p) = \mathcal{L}(q)$;
- p and q are *incomparable* (with respect to $\mathcal{L}$) iff neither $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ nor $\mathcal{L}(q) \subseteq \mathcal{L}(p)$ holds.

We say that a formula $\phi \in \mathcal{L}$ is *consistent* iff $[\![\phi]\!]_{\mathcal{L}} \neq \emptyset$. Formulae $\phi, \psi \in \mathcal{L}$ are said to be *logically equivalent*) (or simply *equivalent*) iff $[\![\phi]\!]_{\mathcal{L}} = [\![\psi]\!]_{\mathcal{L}}$. When it is clear from the context, we omit the logic $\mathcal{L}$ in the subscript (and in the

text). For example, we write $[\![\phi]\!]$, $\equiv$, and $p^\uparrow$ instead of $[\![\phi]\!]_\mathcal{L}$, $\equiv_\mathcal{L}$, and $p^{\uparrow\mathcal{L}}$, respectively. We note that $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ defines a preorder between processes, which we refer to as the *logical preorder* characterized by $\mathcal{L}$. We say that a preorder over P is *logically characterized* or simply *logical* if it is characterized by some logic $\mathcal{L}$.

For a subset $S \subseteq P$ we say that:

- S is *upwards closed* iff for all $p \in P$, if $p \in S$ then $p^\uparrow \subseteq S$;
- $p \in S$ is *minimal* in S iff for each $q \in S$, if $\mathcal{L}(q) \subseteq \mathcal{L}(p)$ then $\mathcal{L}(q) = \mathcal{L}(p)$;
- $p \in S$ is a *least element* in S iff $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ for each $q \in S$.

Clearly, if p is a least element in a set S, then p is also minimal in S. Notice that, if a set S contains a least element, then it is the unique minimal element in S, up to equivalence.

2.1. Characteristic and prime formulae, and graphical representation

We introduce here the crucial notion of *characteristic formula* for a process [2, 13, 14] (along with the inverse notion of *graphical representation* of a formula) and the one of *prime formula* [19, 18], in the setting of logical preorders over processes. Our aim in this study is to investigate when these notions coincide, thus providing a characterization of logically defined processes by means of prime formulae, which we will often refer to as *characterization by primality*. To begin with, in this section we study such a connection between the above-mentioned notions in a very general setting. As it turns out, for logically characterized preorders, the property of being characteristic always implies primality (Theorem 1). The main focus of this paper becomes therefore to investigate under what conditions a consistent and prime formula is characteristic for some process with respect to a logical preorder (Section 3).

Definition 1 (Characteristic formula)**.** *A formula $\phi \in \mathcal{L}$ is* characteristic *(within logic $\mathcal{L}$) for $p \in P$ iff, for all $q \in P$, it holds that $q \in [\![\phi]\!]$ if and only if $\mathcal{L}(p) \subseteq \mathcal{L}(q)$.*

It is worth observing that if ϕ is characteristic for some process p, then $p \in [\![\phi]\!]$.

The following simple properties related to characteristic formulae will be useful in what follows.

Proposition 1. *The following properties hold for all $p, q \in P$ and $\phi \in \mathcal{L}$:*

(i) *ϕ is characteristic for p if and only if $[\![\phi]\!] = p^\uparrow$;*

(ii) *a characteristic formula for p, if it exists, is unique up to logical equivalence (and can therefore be referred to as $\chi(p)$);*

(iii) *if the characteristic formulae for p and q, namely $\chi(p)$ and $\chi(q)$, exist then $[\![\chi(p)]\!] \subseteq [\![\chi(q)]\!]$ if and only if $\mathcal{L}(q) \subseteq \mathcal{L}(p)$.*

Proof. Property (i) follows directly from the definition of the characteristic formula and the one of upward closure of p, while (ii) and (iii) follow easily from (i). $\qquad\square$

Next we state two useful properties.

Proposition 2. *The following properties hold for each $\phi \in \mathcal{L}$:*

180   *(i)* $[\![\phi]\!]$ *is upwards closed, and*

    (ii) *for every $p \in P$, if $\chi(p)$ exists and $p \in [\![\phi]\!] \subseteq [\![\chi(p)]\!]$, then $[\![\phi]\!] = [\![\chi(p)]\!]$.*

Proof. We prove the two claims separately.

    (i) Assume that $p \in [\![\phi]\!]$ and that $q \in p^{\uparrow}$, or equivalently that $\mathcal{L}(p) \subseteq \mathcal{L}(q)$. As, by assumption, $\phi \in \mathcal{L}(p)$, we have that $\phi \in \mathcal{L}(q)$, and therefore that
185         $q \in [\![\phi]\!]$ as we wanted to prove.

    (ii) This statement follows from the fact that $[\![\phi]\!]$ is upwards closed and, since $p \in [\![\phi]\!]$ by assumption, it includes $p^{\uparrow} = [\![\chi(p)]\!]$.   □

The inverse of the notion of characteristic formula is the one of *graphical representation* of a formula. For the sake of simplicity, in what follows we
190 simply write "represents" instead of "graphically represents".

Definition 2 (Graphical representation)**.** *We say that $S \subseteq P$ represents ϕ iff the elements in S are pairwise incomparable and $[\![\phi]\!] = \bigcup_{p \in S} p^{\uparrow}$. If S is finite we say that ϕ is* finitely represented *by S or simply finitely represented. If $S = \{p\}$ we say that p represents ϕ (or that ϕ is represented by p).*

195   Observe that there is possibly more than one graphical representation of a formula. However, any two graphical representations of a formula are equivalent in the following sense, so the representation is unique up to process equivalence.

Proposition 3. *If $S, T \subseteq P$ represent ϕ then for all $p \in S$ there is some $q \in T$ such that $\mathcal{L}(p) = \mathcal{L}(q)$ and vice versa.*

200 *Proof.* Assume that both S and T represent ϕ and therefore that $[\![\phi]\!] = \bigcup_{p \in S} p^{\uparrow} = \bigcup_{q \in T} q^{\uparrow}$. Assume that $p \in S$. Then $p \in p^{\uparrow} \subseteq \bigcup_{q \in T} q^{\uparrow}$. This implies that $p \in q^{\uparrow}$ for some $q \in T$, and therefore that $\mathcal{L}(q) \subseteq \mathcal{L}(p)$. Symmetrically, we have that $\mathcal{L}(p') \subseteq \mathcal{L}(q)$, for some $p' \in S$. Since elements of S are pairwise incomparable, $\mathcal{L}(p') \subseteq \mathcal{L}(q) \subseteq \mathcal{L}(p)$ implies $p = p'$, and therefore we conclude that $\mathcal{L}(p) = \mathcal{L}(q)$.
205 Following the same argument it is possible to show that for each $q \in T$ there exists some $p \in S$ such that $\mathcal{L}(p) = \mathcal{L}(q)$.   □

We now define what it means for a formula to be prime.

Definition 3 (Prime formula)**.** *We say that $\phi \in \mathcal{L}$ is* prime *iff for each non-empty, finite subset of formulae $\Psi \subseteq \mathcal{L}$ it holds that $[\![\phi]\!] \subseteq \bigcup_{\psi \in \Psi} [\![\psi]\!]$ implies*
210 *$[\![\phi]\!] \subseteq [\![\psi]\!]$ for some $\psi \in \Psi$.*

Observe that our definition is an equivalent, semantic version of the one given in [18]. This serves our purpose to keep the discussion as abstract as possible. In this perspective, we want to abstract (at least at this point of the discussion) from the syntactic details of the logical formalism, while the classic
215 definition of prime formula tacitly applies only to languages that feature at least the Boolean connective $\vee$.

We provide here the first piece of our characterization by primality, by showing that the property of being characteristic implies primality without any extra assumption on the language $\mathcal{L}$ or its interpretation.

Theorem 1. *Let $\phi \in \mathcal{L}$. If ϕ is a characteristic formula for some $p \in P$, then ϕ is prime and consistent.*

Proof. The formula ϕ is obviously consistent because $p \in [\![\chi(p)]\!] = [\![\phi]\!]$. Towards proving that $\chi(p)$ is prime, we assume that $[\![\chi(p)]\!] \subseteq \bigcup_{i \in I}[\![\psi_i]\!]$, where I is finite and non-empty. By our assumption, since $p \in [\![\chi(p)]\!]$, then for some $i \in I$, $p \in [\![\psi_i]\!]$ holds. As, by Proposition 2(i), $[\![\psi_i]\!]$ is upwards closed, using Proposition 1(i) we can conclude that $[\![\chi(p)]\!] = p^{\uparrow} \subseteq [\![\psi_i]\!]$ as we wanted to prove. $\qquad\square$

Notice that the converse is not true in general, that is, there exist formulae that are consistent and prime but not characteristic. To see this, let $P = \mathbb{Q}$, $\mathcal{L} = \mathbb{R}$ and $[\![\phi]\!] = \{p \in \mathbb{Q} \mid \phi \leq p\}$. Clearly, all formulae are consistent. Then, $\mathcal{L}(p) = \{\phi \in \mathbb{R} \mid \phi \leq p\}$ which implies that $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ iff $p \leq q$ iff $q \in [\![p]\!]$. This means that, for each $p \in \mathbb{Q}$, $\phi = p$ is characteristic for p and therefore the characteristic formula is well-defined for all $p \in P$. Furthermore $[\![\phi]\!] \cup [\![\psi]\!] = \{p \in \mathbb{Q} \mid \min\{\phi, \psi\} \leq p\}$ for all $\phi, \psi \in \mathcal{L}$, which implies that all formulae are prime. On the other hand $\phi = \sqrt{2} \notin \mathbb{Q}$ cannot be characteristic for any process as $[\![\sqrt{2}]\!]$ does not have a least element.

3. Characterization by primality for logical preorders

In this section we introduce sufficient conditions under which the converse of Theorem 1 is also true for logical preorders, that is, conditions guaranteeing that any consistent, prime formula is characteristic.

As a first step, we introduce the notion of *decomposable* logic. We show that if a logic is decomposable, then we have a logical characterization of processes by primality. Some of the results involve the Boolean connectives $\wedge$ and $\vee$, whose intended semantics is the standard one.

Definition 4 (Decomposability). *We say that a formula $\phi \in \mathcal{L}$ is* decomposable *iff $[\![\phi]\!] = [\![\chi(p)]\!] \cup [\![\psi_p]\!]$ for some $p \in P$ and $\psi_p \in \mathcal{L}$, with $p \notin [\![\psi_p]\!]$. We say that $\mathcal{L}$ is* decomposable *iff each consistent formula $\phi \in \mathcal{L}$ is decomposable or characteristic for some $p \in P$.*

The attentive reader will have noticed that a characteristic formula is also decomposable provided that "false" is expressible in the logic $\mathcal{L}$. At this point of the paper we are not making any assumption on the logic (not even that it includes false, negation and conjunction), so defining decomposability as we do yields a more abstract and general framework.

Proposition 4. *For all $\phi, \psi \in \mathcal{L}$ and $p \in P$, where $p \notin [\![\psi]\!]$ and ϕ is prime, if $[\![\phi]\!] = [\![\chi(p)]\!] \cup [\![\psi]\!]$, then $[\![\phi]\!] = [\![\chi(p)]\!]$.*

Proof. Assume that ϕ is prime, that $[\![\phi]\!] = [\![\chi(p)]\!] \cup [\![\psi]\!]$ and that $p \notin [\![\psi]\!]$. As $p \in [\![\phi]\!]$, it is clear that $[\![\phi]\!] \not\subseteq [\![\psi]\!]$ and, as ϕ is prime, we have that $[\![\phi]\!] \subseteq [\![\chi(p)]\!]$ which, thanks to Proposition 2(ii), implies that $[\![\phi]\!] = [\![\chi(p)]\!]$. $\qquad\square$

The following theorem allows us to reduce the problem of relating the no-
²⁶⁰ tions of prime and characteristic formulae in a given logic to the problem of
establishing the decomposability property for that logic. This provides us with
a very general setting towards characterization by primality.

Theorem 2. *If $\mathcal{L}$ is decomposable then every formula in $\mathcal{L}$ that is consistent
and prime is also characteristic for some $p \in P$.*

²⁶⁵ *Proof.* The claim follows directly from Proposition 4. $\qquad\square$

3.1. Paths to decomposability

The aim of this section is to identify features that make a logic decomposable,
thus paving the way towards showing the decomposability of a number of logical
formalisms in the next sections. First, we observe that if a characteristic formula
²⁷⁰ $\chi(p)$ exists for every $p \in P$, then what we are left to do is to define, for each $\phi \in$
$\mathcal{L}$, a formula ψ_p, for some $p \in P$, with the properties mentioned in Definition 4,
as captured by the following proposition.

Proposition 5. *Let $\mathcal{L}$ be a logic such that (i) $\chi(p)$ exists for each $p \in P$, and
(ii) for each consistent formula ϕ there exist $p \in [\![\phi]\!]$ and $\psi_p \in \mathcal{L}$ such that*
²⁷⁵ *$p \notin [\![\psi_p]\!]$ and $[\![\phi]\!] \setminus [\![\chi(p)]\!] \subseteq [\![\psi_p]\!] \subseteq [\![\phi]\!]$. Then $\mathcal{L}$ is decomposable.*

Proof. Let $\phi \in \mathcal{L}$ be consistent and let us consider the formula ψ_p (with $p \in [\![\phi]\!]$)
whose existence is guaranteed by proviso (ii) of the proposition. By proviso (i),
the formula $\chi(p)$ exists in $\mathcal{L}$ as well and, since ϕ is upwards closed and $p \in [\![\phi]\!]$,
we have $[\![\chi(p)]\!] = p^\uparrow \subseteq [\![\phi]\!]$. It immediately follows that $[\![\chi(p)]\!] \cup [\![\psi_p]\!] \subseteq [\![\phi]\!]$ as,
²⁸⁰ by hypothesis, $[\![\psi_p]\!] \subseteq [\![\phi]\!]$ holds as well. Moreover, from $[\![\phi]\!] \setminus [\![\chi(p)]\!] \subseteq [\![\psi_p]\!]$
and $[\![\chi(p)]\!] \subseteq [\![\phi]\!]$, we have that $[\![\phi]\!] \subseteq [\![\psi_p]\!] \cup [\![\chi(p)]\!]$. Hence, $[\![\phi]\!] = [\![\psi_p]\!] \cup [\![\chi(p)]\!]$,
and we are done. $\qquad\square$

Clearly, when dealing with formalisms featuring at least the Boolean oper-
ators $\neg$ and $\wedge$, as it is the case with the logic for the bisimulation semantics
²⁸⁵ (Section 5.1), such a formula ψ_p is easily defined as $\neg\chi(p) \wedge \phi$. This is stated in
the following corollary.

Corollary 1. *Let $\mathcal{L}$ be a logic featuring at least the Boolean connective $\wedge$ such
that $\chi(p)$ exists for each $p \in P$, and there exists some formula $\bar{\chi}(p) \in \mathcal{L}$ where
$[\![\bar{\chi}(p)]\!] = P \setminus [\![\chi(p)]\!]$. Then $\mathcal{L}$ is decomposable.*

²⁹⁰ The situation is more complicated for the other logics for the semantics in
the branching time-linear time spectrum (which we consider in Section 5), as
negation is in general not expressible in these logics, not even for characteristic
formulae. Therefore, instead we will prove a slightly stronger statement than
the one in Corollary 1 by identifying a weaker condition than the existence of a
²⁹⁵ negation of the characteristic formulae (that we assume to exist) that also leads
to decomposability of the logic. This is described in the following proposition.

Proposition 6. *Let $\phi \in \mathcal{L}$, p be a minimal element in $[\![\phi]\!]$ such that $\chi(p)$ exists
in $\mathcal{L}$, and let $\bar{\chi}(p)$ be a formula in $\mathcal{L}$ such that $\{q \in P \mid \mathcal{L}(q) \not\subseteq \mathcal{L}(p)\} \subseteq [\![\bar{\chi}(p)]\!]$.
Then, $[\![\phi]\!] \setminus [\![\chi(p)]\!] \subseteq [\![\bar{\chi}(p)]\!]$ holds.*

Proof. Let us consider an element q' such that $q' \in [\![\phi]\!]$ and $q' \notin [\![\chi(p)]\!]$. By the latter assumption, we have that $\mathcal{L}(p) \not\subseteq \mathcal{L}(q')$. Moreover, since p is minimal in $[\![\phi]\!]$ and $q' \in [\![\phi]\!]$, it also holds that $\mathcal{L}(q') \not\subset \mathcal{L}(p)$. Thus, $\mathcal{L}(q') \not\subseteq \mathcal{L}(p)$, which implies $q' \in \{q \in P \mid \mathcal{L}(q) \not\subseteq \mathcal{L}(p)\} \subseteq [\![\bar{\chi}(p)]\!]$, as we wanted to prove. $\qquad\square$

In the next proposition, we build on the above result, and establish some conditions, which are met by the logics characterizing the semantics in van Glabbeek's linear time-branching time spectrum (see Section 5), and which immediately lead to decomposability.

Proposition 7. *Let $\mathcal{L}$ be a logic that features at least the Boolean connective $\wedge$ and such that:*

(i) $\chi(p)$ *exists for each $p \in P$,*

(ii) *for each consistent ϕ, the set $[\![\phi]\!]$ has a minimal element, and*

(iii) *for each $p \in P$, there exists in $\mathcal{L}$ a formula $\bar{\chi}(p)$ such that either*

- $[\![\bar{\chi}(p)]\!] = P \setminus [\![\chi(p)]\!]$ *or*

- $p \notin [\![\bar{\chi}(p)]\!]$ *and* $\{q \in P \mid \mathcal{L}(q) \not\subseteq \mathcal{L}(p)\} \subseteq [\![\bar{\chi}(p)]\!]$.

Then, $\mathcal{L}$ is decomposable.

Proof. Let $\phi \in \mathcal{L}$ be consistent. We choose a minimal element p in $[\![\phi]\!]$, which exists by proviso (ii) of the proposition, and we define $\psi = \bar{\chi}(p) \wedge \phi$. Clearly, $p \notin [\![\bar{\chi}(p) \wedge \phi]\!]$ since $p \notin [\![\bar{\chi}(p)]\!]$. We show that $[\![\phi]\!] = [\![\chi(p)]\!] \cup [\![\psi]\!]$. The inclusion from right to left immediately follows from the definition of ψ and from the fact that $[\![\phi]\!]$ is upward closed. The converse inclusion is straightforward: if $[\![\bar{\chi}(p)]\!] = P \setminus [\![\chi(p)]\!]$, then it follows from the obvious observation that $[\![\phi]\!] \setminus [\![\chi(p)]\!] = [\![\phi]\!] \cap (P \setminus [\![\chi(p)]\!]) = [\![\psi]\!]$; otherwise, it immediately follows from Proposition 6. $\qquad\square$

In order to apply the above result to prove decomposability for a logic $\mathcal{L}$, we now develop a general framework ensuring conditions (i) and (ii) in Proposition 7. To this end, we exhibit a finite characterization of the (possibly) infinite set $\mathcal{L}(p)$ of true facts associated with every $p \in P$. (In order to ensure condition (iii) of the proposition, we will explicitly construct the formula $\bar{\chi}(p)$ in each of the languages considered in Section 5.)

Definition 5 (Characterization). *We say that the logic $\mathcal{L}$ is characterized by a function $\mathcal{B} : P \to \mathcal{P}(\mathcal{L})$ iff for each $p \in P$ we have $\emptyset \subset \mathcal{B}(p) \subseteq \mathcal{L}(p)$ and for each $\phi \in \mathcal{L}(p)$ it holds that $\bigcap_{\psi \in \mathcal{B}(p)}[\![\psi]\!] \subseteq [\![\phi]\!]$. We say that $\mathcal{L}$ is finitely characterized by $\mathcal{B}$ iff $\mathcal{L}$ is characterized by a function $\mathcal{B}$ such that $\mathcal{B}(p)$ is finite for each $p \in P$. Finally, we say that $\mathcal{B}$ is monotonic iff $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ implies $\mathcal{B}(p) \subseteq \mathcal{B}(q)$ for all $p, q \in P$.*

In what follows, we show that if a logic $\mathcal{L}$ features at least the Boolean connective $\wedge$ and it is finitely characterized by $\mathcal{B}$, for some monotonic $\mathcal{B}$, then it fulfils conditions (i) and (ii) in Proposition 7 (see Proposition 10 and Corollary 2 to follow).

Proposition 8. *If $\mathcal{L}$ is characterized by $\mathcal{B}$, then for each $p, q \in P$, $\mathcal{B}(p) \subseteq \mathcal{L}(q)$ implies $\mathcal{L}(p) \subseteq \mathcal{L}(q)$.*

Proof. Assume that $\mathcal{B}(p) \subseteq \mathcal{L}(q)$ and that $\phi \in \mathcal{L}(p)$. As $\mathcal{L}$ is characterized by $\mathcal{B}$, $\bigcap_{\psi \in \mathcal{B}(p)} \llbracket \psi \rrbracket \subseteq \llbracket \phi \rrbracket$ holds. Since $\mathcal{B}(p) \subseteq \mathcal{L}(q)$, we have $q \in \bigcap_{\psi \in \mathcal{B}(p)} \llbracket \psi \rrbracket \subseteq \llbracket \phi \rrbracket$, which means that $\phi \in \mathcal{L}(q)$, as we wanted to prove. $\qquad\square$

Proposition 9. *If $\mathcal{L}$ is characterized by $\mathcal{B}$, then for each $p, q \in P$, $\mathcal{B}(p) \subseteq \mathcal{B}(q)$ implies $\mathcal{L}(p) \subseteq \mathcal{L}(q)$.*

Proof. Assume that $\mathcal{B}(p) \subseteq \mathcal{B}(q)$. As $\mathcal{B}(q) \subseteq \mathcal{L}(q)$ the result follows by Proposition 8. $\qquad\square$

Every finitely characterized logic featuring at least the Boolean connective $\wedge$ enjoys the pleasing property that every $p \in P$ admits a characteristic formula in $\mathcal{L}$.

Proposition 10. *Let $\mathcal{L}$ be a logic that features at least the Boolean connective $\wedge$ and that is finitely characterized by $\mathcal{B}$. Then, each $p \in P$ has a characteristic formula in $\mathcal{L}$ given by $\chi(p) = \bigwedge_{\phi \in \mathcal{B}(p)} \phi$.*

Proof. Assume that $p \in P$. First, we observe that $\bigwedge_{\phi \in \mathcal{B}(p)} \phi$ is a well-formed formula, as $\mathcal{B}(p)$ is finite. Next, we prove that $\llbracket \bigwedge_{\phi \in \mathcal{B}(p)} \phi \rrbracket = p^{\uparrow}$ holds, from which the thesis immediately follows by Proposition 1(i).

Since $\mathcal{B}(p) \subseteq \mathcal{L}(p)$, it follows that $p \in \llbracket \bigwedge_{\phi \in \mathcal{B}(p)} \phi \rrbracket$. As $\llbracket \bigwedge_{\phi \in \mathcal{B}(p)} \phi \rrbracket$ is upwards closed, we have that $p^{\uparrow} \subseteq \llbracket \bigwedge_{\phi \in \mathcal{B}(p)} \phi \rrbracket$.

Towards proving the converse inclusion, let us assume that $q \in \llbracket \bigwedge_{\phi \in \mathcal{B}(p)} \phi \rrbracket$. Since $\llbracket \bigwedge_{\phi \in \mathcal{B}(p)} \phi \rrbracket = \bigcap_{\phi \in \mathcal{B}(p)} \llbracket \phi \rrbracket$, we have that $q \in \llbracket \phi \rrbracket$, for each $\phi \in \mathcal{B}(p)$. Thus, $\mathcal{B}(p) \subseteq \mathcal{L}(q)$ holds. From Proposition 8, it follows that $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ and, by the definition of upwards closure, we conclude that $q \in p^{\uparrow}$. $\qquad\square$

Proposition 11. *Let $\mathcal{L}$ be a logic that is finitely characterized by $\mathcal{B}$, for some monotonic $\mathcal{B}$. Then, for each $\phi \in \mathcal{L}$ and $q \in \llbracket \phi \rrbracket$, there exists some $p \in P$ such that $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ and p is minimal in $\llbracket \phi \rrbracket$.*

Proof. Towards a contradiction, let us assume that there exist $\phi \in \mathcal{L}$ and $q \in \llbracket \phi \rrbracket$ such that, for each $p \in \llbracket \phi \rrbracket$, with $\mathcal{L}(p) \subseteq \mathcal{L}(q)$, p is not minimal in $\llbracket \phi \rrbracket$. Notice that q is not minimal in $\llbracket \phi \rrbracket$ itself.

Then, there exists an infinite sequence $q_0, q_1, \ldots \in \llbracket \phi \rrbracket$, with $q = q_0$, such that $\mathcal{L}(q_{i+1}) \subsetneq \mathcal{L}(q_i)$ for each $i \geq 0$. As $\mathcal{B}$ is monotonic, we have that $\mathcal{B}(q_{i+1}) \subseteq \mathcal{B}(q_i)$ for each $i \geq 0$. Since $\mathcal{B}(q)$ is finite, there exists some $k \geq 0$ such that $\mathcal{B}(q_k) = \mathcal{B}(q_{k+\ell})$ for each $\ell > 0$. By Proposition 9, $\mathcal{L}(q_k) = \mathcal{L}(q_{k+\ell})$ holds for each $\ell > 0$. This contradicts the fact that $\mathcal{L}(q_{i+1}) \subsetneq \mathcal{L}(q_i)$ for each $i \geq 0$, which means that for each $q \in \llbracket \phi \rrbracket$ there exists some $p \in \llbracket \phi \rrbracket$, with $\mathcal{L}(p) \subseteq \mathcal{L}(q)$, such that p is minimal in $\llbracket \phi \rrbracket$. $\qquad\square$

Corollary 2. *Let $\mathcal{L}$ be a logic that is finitely characterized by $\mathcal{B}$, for some monotonic $\mathcal{B}$. Then, for each consistent formula $\phi \in \mathcal{L}$, $\llbracket \phi \rrbracket$ has a minimal element.*

Proof. The thesis immediately follows from Proposition 11. $\qquad\square$

₃₈₀ The next result will be useful in the next section.

Corollary 3. *Let $\mathcal{L}$ be a logic that features at least the Boolean connective $\wedge$ and that is finitely characterized by $\mathcal{B}$, for some monotonic $\mathcal{B}$. Then, each formula $\phi \in \mathcal{L}$ is represented (uniquely up to equivalence) by a set $rep(\phi)$ (in the sense of Definition 2).*

₃₈₅ *Proof.* Let $minimals(\phi) = \{p \in P \mid p \text{ is minimal in } [\![\phi]\!]\}$. Notice that the elements of $minimals(\phi)$ are not necessarily pairwise incomparable (there can exist $p, q \in minimals(\phi)$ such that $p \neq q$ and $p \equiv q$).

Now, we define $rep(\phi)$ as any subset of $minimals(\phi)$ such that *(i)* elements in $rep(\phi)$ are pairwise incomparable and *(ii)* for each $p \in minimals(\phi)$ there ₃₉₀ exists some $q \in rep(\phi)$ with $p \equiv q$. Intuitively, $rep(\phi)$ contains a representative element for each equivalence class of minimal elements in $[\![\phi]\!]$ modulo $\equiv$. First of all, we observe that, by Proposition 10, $\chi(p)$ exists for each $p \in P$. We show that $[\![\phi]\!] = \bigcup_{p \in rep(\phi)} p^{\uparrow}$, from which the thesis immediately follows.

From Proposition 11, we have that each $q \in [\![\phi]\!]$ belongs to $p^{\uparrow}$, for some ₃₉₅ $p \in rep(\phi)$, and thus $[\![\phi]\!] \subseteq \bigcup_{p \in rep(\phi)} p^{\uparrow}$. Moreover, as $[\![\phi]\!]$ is upwards closed, we get that $[\![\phi]\!] \supseteq \bigcup_{p \in rep(\phi)} p^{\uparrow}$. Therefore, we can conclude that $[\![\phi]\!] = \bigcup_{p \in rep(\phi)} p^{\uparrow}$, which completes the proof. □

We summarize the results we provided so far in the following corollary.

Corollary 4. *Let $\mathcal{L}$ be a logic that features at least the Boolean connective $\wedge$* ₄₀₀ *and such that:*

(i) $\mathcal{L}$ is finitely characterized by $\mathcal{B}$, for some monotonic $\mathcal{B}$, and

(ii) for each $\chi(p)$, there exists in $\mathcal{L}$ a formula $\bar{\chi}(p)$ such that either

- *$[\![\bar{\chi}(p)]\!] = P \setminus [\![\chi(p)]\!]$, or*
- *$p \notin [\![\bar{\chi}(p)]\!]$ and $\{q \in P \mid \mathcal{L}(q) \nsubseteq \mathcal{L}(p)\} \subseteq [\![\bar{\chi}(p)]\!]$.*

₄₀₅ *Then, $\mathcal{L}$ is decomposable.*

Proof. By proviso (i) of the corollary and by Proposition 10, $\chi(p)$ exists for each $p \in P$ (condition (i) in Proposition 7). By Corollary 2, there exists an element $p \in P$ that is minimal in $[\![\phi]\!]$, for each consistent formula $\phi \in \mathcal{L}$ (condition (ii) in Proposition 7). The thesis immediately follows from Proposition 7, given that ₄₁₀ proviso (ii) of the corollary is equivalent to condition (iii) in Proposition 7. □

Remark 1. *It is worth pointing out that the Boolean connective $\wedge$ plays a minor role in (the proof of) Proposition 10 (and thus Corollaries 3 and 4). Indeed, it is applied to formulae in $\mathcal{B}(p)$ only. Thus, such a result can be used also to deal with logics that allow for a limited use of such a connective, such as the logic* ₄₁₅ *for trace semantics and other linear-time semantics (see Section 5.2).*

As another path towards decomposability, we show the following result, which we will use in Section 4 to deal with logical settings requiring a special treatment.

Proposition 12. *If $\mathcal{L}$ is a logic that (i) features the Boolean connective $\vee$ and such that (ii) every $\phi \in \mathcal{L}$ is finitely represented and (iii) a characteristic formula exists in $\mathcal{L}$ for every process $p \in P$, then $\mathcal{L}$ is decomposable.*

Proof. Let $\phi \in \mathcal{L}$ be a consistent formula which is not characteristic for any $p \in P$. We show that ϕ is decomposable, that is, $[\![\phi]\!] = [\![\chi(p)]\!] \cup [\![\psi_p]\!]$ for some $p \in P$ and $\psi_p \in \mathcal{L}$, with $p \notin [\![\psi_p]\!]$.

By assumption (ii), ϕ is finitely represented by some set $rep(\phi) \subseteq P$; by assumption (iii), $\chi(p)$ exists in $\mathcal{L}$ for all $p \in P$; thus, by Proposition 1(i), we have:

$$[\![\phi]\!] = \bigcup_{p \in rep(\phi)} p^{\uparrow} = \bigcup_{p \in rep(\phi)} [\![\chi(p)]\!].$$

Since ϕ is consistent, $[\![\phi]\!] \neq \emptyset$ and therefore $rep(\phi) \neq \emptyset$. Pick some $p \in rep(\phi)$. Notice that $|rep(\phi)| > 1$ otherwise ϕ would be characteristic for p. Thus, using assumption (i), we have:

$$
\begin{aligned}
[\![\phi]\!] &= [\![\chi(p)]\!] \cup \bigcup_{q \in rep(\phi) \backslash \{p\}} [\![\chi(q)]\!] \\
&= [\![\chi(p)]\!] \cup [\![\textstyle\bigvee_{q \in rep(\phi) \backslash \{p\}} \chi(q)]\!].
\end{aligned}
$$

Since $p \notin [\![\bigvee_{q \in rep(\phi) \backslash \{p\}} \chi(q)]\!]$, we can conclude that $\mathcal{L}$ is decomposable. $\quad\square$

In the remainder of the paper we will examine some applications for our general results; in particular, we will use them to prove characterization by primality for several well-known process semantics. First, we consider some cases that can be dealt with using Proposition 12, and then we analyze semantics in van Glabbeek's spectrum.

4. Applications to finitely-represented logics

As a first application of our results, we investigate three kinds of logics:

1. the set of processes P is finite and the logic $\mathcal{L}$ features at least the Boolean connectives $\wedge$ and $\vee$;

2. the logic characterizing modal refinement semantics [18];

3. the logic characterizing covariant-contravariant simulation semantics [19].

Note that in case 1 although P itself is finite, it can contain processes with infinite behaviours, e.g., when $p \in P$ represents a labelled transition system with loops. To deal with these logics we use Proposition 12 and thus we show that the logics satisfy its hypothesis.

Proposition 13. *Let $\mathcal{L}$ be a logic that features at least the Boolean connective $\wedge$ and is interpreted over a finite set P. Then:*
(a) $\mathcal{L}$ is finitely characterized by $\mathcal{B}$, for some monotonic $\mathcal{B}$, and
(b) every formula $\phi \in \mathcal{L}$ is finitely represented.

Proof. (*a*) If P is finite, so is $\mathcal{L}$, up to logical equivalence. Let $\mathcal{L}^{fin}$ be a set of representatives of the equivalence classes of $\mathcal{L}$ modulo logical equivalence, and define $\mathcal{B}^{fin}(p) = \mathcal{L}^{fin}(p) = \mathcal{L}(p) \cap \mathcal{L}^{fin}$, for each $p \in P$. It is easy to see that $\mathcal{L}$ is finitely characterized by $\mathcal{B}^{fin}$, according to Definition 5. Moreover, $\mathcal{B}^{fin}$ is clearly monotonic.

(*b*) The claim immediately follows from Corollary 3, making use of Proposition 13(a) and observing that $rep(\phi) \subseteq P$ is finite for each $\phi \in \mathcal{L}$. $\square$

Proposition 14. *Every logic $\mathcal{L}$ that is interpreted over a finite set P and that features at least the Boolean connectives $\wedge$ and $\vee$ is decomposable.*

Proof. By hypothesis, $\mathcal{L}$ features the Boolean connective $\vee$; by Proposition 13(b), every $\phi \in \mathcal{L}$ is finitely represented; by Proposition 10 and Proposition 13(a), $\chi(p)$ exists in $\mathcal{L}$ for all $p \in P$. Finally, by Proposition 12, $\mathcal{L}$ is decomposable. $\square$

Proposition 15. *The logics characterizing modal refinement semantics or covariant-contravariant simulation semantics given in [19, 18] are decomposable.*

Proof. The claim follows immediately from the definition of the logics and related results shown in [19, 18], and by applying Proposition 12.

More precisely, for covariant-contravariant simulation, Lemma 2 in [19] yields the existence of characteristic formulae and Theorem 3 in [19] yields finite representability of each formula.

On the other hand, for modal refinement, Proposition 3.2 in [18] yields the existence of characteristic formulae and Proposition 4.2 in [18] gives finite representability of each formula. $\square$

Thus, we have the following theorem, resulting from Theorems 1 and 2, along with Propositions 14 and 15.

Theorem 3 (Characterization by primality). *Let $\mathcal{L}$ be:*

- *the logic from [18] that characterizes modal refinement semantics,*

- *the one given in [19] characterizing covariant-contravariant simulation semantics, or*

- *any logic that features at least the Boolean connectives $\wedge$ and $\vee$, and is interpreted over a finite set P .*

Then, each formula $\phi \in \mathcal{L}$ is consistent and prime if and only if ϕ is characteristic for some $p \in P$.

5. Application to semantics in van Glabbeek's spectrum

Our next task is to apply the result described in Corollary 4 to the semantics in the linear time-branching time spectrum [6, 7], over finite trees and with a finite set of actions. All those semantics have been shown to be characterized by specific logics and therefore inherit all the properties of logically defined

preorders. We reason about characterization by primality (Theorems 5 and 7) by showing that each logic is finitely characterized by some monotonic $\mathcal{B}$, and by building, for each characteristic formula $\chi(p)$, a formula $\bar{\chi}(p)$ with the properties specified in Corollary 4(ii).

Processes. To begin with, we give a formal definition of the notion of process. The set of processes P over a finite set of actions Act is given by the following grammar:

$$p ::= 0 \mid ap \mid p + p,$$

where $a \in Act$. Given a process p, we say that p can perform the action a and evolve into p', denoted $p \xrightarrow{a} p'$, iff (i) $p = ap'$ or (ii) $p = p_1 + p_2$ and $p_1 \xrightarrow{a} p'$ or $p_2 \xrightarrow{a} p'$ holds. Note that every process p denotes a finite loop-free labelled transition system whose states are those that are reachable from p via transitions $\xrightarrow{a}$, $a \in Act$, and whose initial state is p [5].

We define the set of *initials* of p, denoted $I(p)$, as the set $\{a \in Act \mid p \xrightarrow{a} p'$ for some $p' \in P\}$. We write $p \xrightarrow{a}$ if $a \in I(p)$, $p \not\xrightarrow{a}$ if $a \notin I(p)$, and $p \not\rightarrow$ if $I(p) = \emptyset$. We define $traces(p)$ as follows (we use ε to denote the empty string):

$$traces(p) = \{\varepsilon\} \cup \{a\tau \mid \exists p' \in P \,.\, p \xrightarrow{a} p' \text{ and } \tau \in traces(p')\}. \qquad (1)$$

For each trace $\tau = a_1 \ldots a_n$, we write $p \xrightarrow{\tau} p'$ for $p \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \ldots p_{n-1} \xrightarrow{a_n} p'$. Finally, for each $p \in P$, $depth(p)$ is the length of a longest trace in $traces(p)$.

Behavioural preorders for process semantics. The semantics of processes is expressed by means of preorders, which, intuitively, classify processes according to their possible behaviours. Roughly speaking, a process *follows* another in the preorder (or it is *above* it) if it exhibits at least the same behaviours as the latter. The semantic relations in van Glabbeek's linear time-branching time spectrum present different levels of granularity: a finer relation is able to distinguish processes that are indistinguishable by a coarser one. Those semantics are as follows (see Figure 1):

- branching time semantics (Figure 1, left-hand side): *simulation* (S), *complete simulation* (CS), *ready simulation* (RS), *trace simulation* (TS), *2-nested simulation* (2S), and *bisimulation* (BS);

- linear time semantics (Figure 1, right-hand side): *trace* (T), *complete trace* (CT), *failure* (F), *failure trace* (FT), *ready* (R), *ready trace* (RT), *impossible future* (IF), *possible future* (PF), *always impossible future* (IFT), *always possible future* (PFT), *impossible 2-simulation* (I2), *possible 2-simulation* (P2), *always impossible 2-simulation* (I2T), *always possible 2-simulation* (P2T).

In the rest of this section Btime-spectrum denotes the set $\{S, CS, RS, TS, 2S, BS\}$, Ltime-spectrum denotes the set $\{T, CT, F, FT, R, RT, IF, PF, IFT, PFT, I2, P2, I2T, P2T\}$, and we let vanG-spectrum = Btime-spectrum $\cup$ Ltime-spectrum.

The remainder of the section is organized as follows. In Section 5.1, we establish characterization by primality for the semantics in van Glabbeek's branching time spectrum. Then, in Section 5.2, we deal with the ones in van Glabbeek's
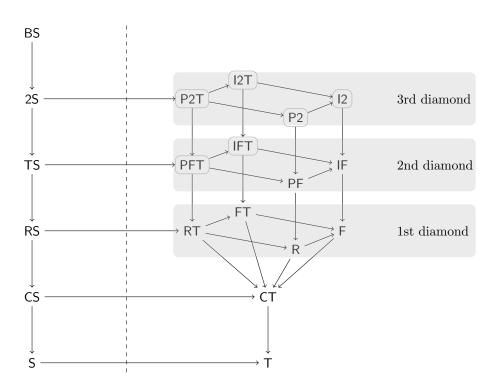
Figure 1: Semantic relations in van Glabbeek's linear time-branching time spectrum (branching semantics are on the left, linear ones are on the right—the six framed names are introduced here: these semantics were studied in [6] but no name was assigned to them).

linear time spectrum; at the beginning of this last section, we also give a short account on the conceptual differences between the two semantics; for a comprehensive account, we refer to [6, 7].

5.1. The branching time spectrum

₅₃₅   In this sub-section, we focus on the semantics in *van Glabbeek's branching time spectrum*, originally introduced in [7] and successively generalized in [6], and their corresponding logical formalisms.

Definition 6 (Branching time semantic relations [6, 7])**.** *For each $X \in$ Btime-spectrum, $\precsim_X$ is the largest relation over P satisfying the following conditions for each* ₅₄₀ *$p, q \in P$.*

simulation (S): *$p \precsim_S q \Leftrightarrow$ for all $p \xrightarrow{a} p'$ there exists some $q \xrightarrow{a} q'$ such that $p' \precsim_S q'$;*

complete simulation (CS): *$p \precsim_{CS} q \Leftrightarrow (i)$ for all $p \xrightarrow{a} p'$ there exists some $q \xrightarrow{a} q'$ such that $p' \precsim_{CS} q'$, and (ii) $I(p) = \emptyset$ iff $I(q) = \emptyset$;*

ready simulation (RS): $p \lesssim_{RS} q \Leftrightarrow (i)$ *for all* $p \xrightarrow{a} p'$ *there exists some* $q \xrightarrow{a} q'$ *such that* $p' \lesssim_{RS} q'$, *and* (ii) $I(p) = I(q)$;

trace simulation (TS): $p \lesssim_{TS} q \Leftrightarrow (i)$ *for all* $p \xrightarrow{a} p'$ *there exists some* $q \xrightarrow{a} q'$ *such that* $p' \lesssim_{TS} q'$, *and* (ii) $traces(p) = traces(q)$;

2-nested simulation (2S): $p \lesssim_{2S} q \Leftrightarrow (i)$ *for all* $p \xrightarrow{a} p'$ *there exists some* $q \xrightarrow{a} q'$ *such that* $p' \lesssim_{2S} q'$, *and* (ii) $q \lesssim_{S} p$;

bisimulation (BS): $p \lesssim_{BS} q \Leftrightarrow (i)$ *for all* $p \xrightarrow{a} p'$ *there exists some* $q \xrightarrow{a} q'$ *such that* $p' \lesssim_{BS} q'$, *and* (ii) *for all* $q \xrightarrow{a} q'$ *there exists some* $p \xrightarrow{a} p'$ *such that* $p' \lesssim_{BS} q'$.

For each $X \in$ Btime-spectrum, *the equivalence relation* $\equiv_X$ *is defined as expected, i.e.,*

$$p \equiv_X q \Leftrightarrow p \lesssim_X q \text{ and } q \lesssim_X p.$$

Branching time logics. The languages of the different logics yield the following chain of strict inclusions (Table 1, left-hand side): $\mathcal{L}_S \subset \mathcal{L}_{CS} \subset \mathcal{L}_{RS} \subset \mathcal{L}_{TS} \subset \mathcal{L}_{2S} \subset \mathcal{L}_{BS}$, corresponding to a chain of formalisms with strictly increasing expressive power. Notice that, as it will appear clear after the definition of the satisfaction relation below, some of the languages present some redundancy, in the sense that they could be replaced with smaller ones, without any loss in expressiveness. For instance, a disjunction is expressible in $\mathcal{L}_{BS}$ using conjunction and negation, and suitably replacing **tt** with **ff** and vice versa. We followed this approach because we find it helpful to have syntactically larger languages corresponding to more expressive semantics.

The syntax of the logics characterizing the semantics in van Glabbeek's branching time spectrum is given in the following definition. There, we treat formulae of the form $[a]\psi$ and **0** as syntactic shorthands for, respectively, $\neg\langle a\rangle\neg\psi$ and $\bigwedge_{a \in Act}[a]\mathbf{ff}$.

Definition 7 (Syntax [6, 7]). *For each* $X \in$ Btime-spectrum, $\mathcal{L}_X$ *is the language defined by the corresponding grammar given below:*

$\mathcal{L}_S$:
$$\phi_S ::= \mathbf{tt} \mid \mathbf{ff} \mid \phi_S \wedge \phi_S \mid \phi_S \vee \phi_S \mid \langle a\rangle\phi_S.$$

$\mathcal{L}_{CS}$:
$$\phi_{CS} ::= \mathbf{tt} \mid \mathbf{ff} \mid \phi_{CS} \wedge \phi_{CS} \mid \phi_{CS} \vee \phi_{CS} \mid \langle a\rangle\phi_{CS} \mid \mathbf{0}.$$

$\mathcal{L}_{RS}$:
$$\phi_{RS} ::= \mathbf{tt} \mid \mathbf{ff} \mid \phi_{RS} \wedge \phi_{RS} \mid \phi_{RS} \vee \phi_{RS} \mid \langle a\rangle\phi_{RS} \mid [a]\mathbf{ff}.$$

$\mathcal{L}_{TS}$:
$$\phi_{TS} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_{TS} \wedge \phi_{TS} \mid \phi_{TS} \vee \phi_{TS} \mid \langle a\rangle\phi_{TS} \mid \psi_{TS}.$$
$$\psi_{TS} \quad ::= \quad \mathbf{ff} \mid [a]\psi_{TS}.$$

$\mathcal{L}_{2S}$:
$$\phi_{2S} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_{2S} \wedge \phi_{2S} \mid \phi_{2S} \vee \phi_{2S} \mid \langle a\rangle\phi_{2S} \mid \neg\phi_S.$$

$\mathcal{L}_{BS}$:
$$\phi_{BS} ::= \mathbf{tt} \mid \mathbf{ff} \mid \phi_{BS} \wedge \phi_{BS} \mid \phi_{BS} \vee \phi_{BS} \mid \langle a\rangle\phi_{BS} \mid \neg\phi_{BS}.$$

	○ Syntax	☆ Monotonic function $\mathcal{B}$ for finite characterization ⋈ Formula $\bar{\chi}(p)$
○	$\phi_X^\exists ::= \text{tt} \mid \text{ff} \mid \phi_X \wedge \phi_X \mid \phi_X \vee \phi_X \mid \langle a \rangle \phi_X$	$X \in$ Btime-spectrum
☆ ☆	$\mathcal{B}_X(p) = \mathcal{B}_X^+(p) \cup \mathcal{B}_X^-(p)$ $\mathcal{B}_X^+(p) = \{\text{tt}\} \cup \{\langle a \rangle \varphi \mid a \in Act, \varphi = \bigwedge_{\psi \in \Psi} \psi, \Psi \subseteq \mathcal{B}_X(p'), p \xrightarrow{a} p'\}$	
S	○ $\phi_S ::= \phi_S^\exists$	☆ $\mathcal{B}_S^-(p) = \emptyset$ ⋈ $\bar{\chi}_S(p) = \bigvee_{a \in Act} \langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}_S(p')$
CS	○ $\phi_{CS} ::= \phi_{CS}^\exists \mid \phi_{CS}^\forall$ ○ $\phi_{CS}^\forall ::= \mathbf{0}$	☆ $\mathcal{B}_{CS}^-(p) = \{\mathbf{0} \mid p \xrightarrow{a} \forall a \in Act\}$ ⋈ $\bar{\chi}_{CS}(p) = (\bigvee_{a \in Act} \langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}_{CS}(p')) \vee \mathbf{0}$ if $I(p) \neq \emptyset$ ⋈ $\bar{\chi}_{CS}(p) = \bigvee_{a \in Act} \langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}_{CS}(p')$ if $I(p) = \emptyset$
RS	○ $\phi_{RS} ::= \phi_{RS}^\exists \mid \phi_{RS}^\forall$ ○ $\phi_{RS}^\forall ::= [a]\text{ff}$	☆ $\mathcal{B}_{RS}^-(p) = \{[a]\text{ff} \mid a \in Act, p \xrightarrow{a}\}$ ⋈ $\bar{\chi}_{RS}(p) = (\bigvee_{a \in Act} \langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}_{RS}(p')) \vee \bigvee_{a \in I(p)} [a]\text{ff}$
TS	○ $\phi_{TS} ::= \phi_{TS}^\exists \mid \phi_{TS}^\forall$ ○ $\phi_{TS}^\forall ::= \text{ff} \mid [a]\phi_{TS}^\forall$	☆ $\mathcal{B}_{TS}^-(p) = \{[\tau a]\text{ff} \mid \tau \in traces(p), a \in Act, \tau a \notin traces(p)\}$ ⋈ $\bar{\chi}_{TS}(p) = (\bigvee_{a \in Act} \langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}_{TS}(p')) \vee \bigvee_{\tau \in traces(p), \tau a \notin traces(p)} \langle \tau a \rangle \text{tt} \vee \bigvee_{p \xrightarrow{\tau a}} [\tau a]\text{ff}$
2S	○ $\phi_{2S} ::= \phi_{2S}^\exists \mid \phi_{2S}^\forall$ ○ $\phi_{2S}^\forall ::= \neg \phi_S$	☆ $\mathcal{B}_{2S}^-(p) = \{[a]\varphi \in \mathcal{L}_{2S}(p) \mid a \in Act, \varphi = \bigvee_{p' \in max\text{-}succ(p,a)} \bigwedge_{\psi \in \mathcal{B}_{2S}^-(p')} \psi\}$ where $max\text{-}succ(p,a) = \{p' \in P \mid p \xrightarrow{a} p'$ and $\nexists p''. p \xrightarrow{a} p''$ and $p' <_S p''\}$ ⋈ $\bar{\chi}_{2S}(p) = (\bigvee_{a \in Act} \langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}_{2S}(p')) \vee \bar{\Phi}(p)$ where $\bar{\Phi}(p) = \bigvee_{a \in I(p)} [a]\text{ff} \vee \bigvee_{a \in I(p)} \bigvee_{p \xrightarrow{a} p'} [a]\bar{\Phi}(p')$
BS	○ $\phi_{BS} ::= \phi_{BS}^\exists \mid \phi_{BS}^\forall$ ○ $\phi_{BS}^\forall ::= \neg \phi_{BS}$	☆ $\mathcal{B}_{BS}^-(p) = \{[a]\varphi \in \mathcal{L}_{BS}(p) \mid a \in Act, \varphi = \bigvee_{p \xrightarrow{a} p'} \bigwedge_{\psi \in \mathcal{B}_{BS}(p')} \psi\}$ ⋈ $\bar{\chi}_{BS}(p) = \neg \chi_{BS}(p)$ ($\chi_{BS}(p)$ is characteristic for p within $\mathcal{L}_{BS}$)

Table 1: Syntax, monotonic function $\mathcal{B}$ for finite characterization, and formula $\bar{\chi}(p)$, relative to the logics for the semantics in van Glabbeek's branching time spectrum.

Remark 2. *It is important to notice that disjunction does not appear in the original formulation found in [7] of the logics characterizing the branching semantics in the spectrum. However, adding it does not affect the expressive power of these logics with respect to the corresponding semantics, as shown in [6, Proposition 6.2 at page 42].*

We give here the semantics of the logics, by describing the satisfaction relation for the most expressive one, namely $\mathcal{L}_{BS}$, that characterizes bisimulation semantics. The semantics for the other logics can be obtained by considering the corresponding subset of clauses.

Definition 8 (Satisfaction relation)**.** *The satisfaction relation for the logic $\mathcal{L}_{BS}$ is defined as follows:*

- $p \in [\![\text{tt}]\!]$, *for every $p \in P$,*

- $p \notin [\![\mathbf{ff}]\!]$, *for every* $p \in P$,

- $p \in [\![\phi_1 \wedge \phi_2]\!]$ *iff* $p \in [\![\phi_1]\!]$ *and* $p \in [\![\phi_2]\!]$,

- $p \in [\![\phi_1 \vee \phi_2]\!]$ *iff* $p \in [\![\phi_1]\!]$ *or* $p \in [\![\phi_2]\!]$,

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- $p \in [\![\langle a \rangle \phi]\!]$ *iff* $p' \in [\![\phi]\!]$ *for some* $p' \in P$ *such that* $p \xrightarrow{a} p'$,

- $p \in [\![\neg \phi]\!]$ *iff* $p \notin [\![\phi]\!]$.

We say that a process p *satisfies* a formula $\phi \in \mathcal{L}_{\mathsf{BS}}$ if, and only if, $p \in [\![\phi]\!]$.

The following well-known theorem states the relationship between logics and process semantics that allows us to use our general results about logically char-
605 acterized semantics.

Theorem 4 (Logical characterization of branching time semantics [6, 7]). *For each $X \in$ Btime-spectrum and for all $p, q \in P$, $p \lesssim_X q$ iff $\mathcal{L}_X(p) \subseteq \mathcal{L}_X(q)$.*

We observe that all the logics defined above feature the Boolean connective $\wedge$, as required by one of the assumptions of Corollary 4. In what follows, we
610 show that every logic meets also the other conditions of the corollary, that is, it is finitely characterized by some monotonic $\mathcal{B}$, and for each $\chi(p)$ there exists a formula $\bar{\chi}(p)$ such that either $[\![\bar{\chi}(p)]\!] = P \setminus [\![\chi(p)]\!]$ (as it is the case for $\mathcal{L}_{\mathsf{BS}}$) or $p \notin [\![\bar{\chi}(p)]\!]$ and $\{q \in P \mid \mathcal{L}(q) \not\subseteq \mathcal{L}(p)\} \subseteq [\![\bar{\chi}(p)]\!]$ (which holds in all the other cases). This yields the characterization by primality for the logics for the
615 semantics in Btime-spectrum (Theorem 5).

To this end, we first summarize, in Table 1, both the functions $\mathcal{B}$ and the formulae $\bar{\chi}(p)$ for all the branching time semantics (rightmost column), and then we prove their correctness. In particular, in Lemma 1, we prove the finite characterization result, while in Lemma 2 we show the correctness of the formula
620 $\bar{\chi}(p)$. We have already pointed out the connection between our function $\mathcal{B}$ and the notion of characteristic formulae (see Proposition 10). As a matter of fact, roughly speaking, the definition of $\mathcal{B}_X(p)$ (with $X \in$ Btime-spectrum) provided in Table 1 somehow correspond to breaking the characteristic formula of p (within logic $\mathcal{L}_X$) into its conjuncts. The same applies for the semantics in
625 the linear time spectrum we will consider in Section 5.2.

For the sake of readability, besides including the functions $\mathcal{B}$ and the formu-lae $\bar{\chi}(p)$, Table 1 also recalls, in a compact but equivalent way, the syntax of the different logical formalisms. Roughly speaking, each language consists of an "existential" and a "universal" sub-language, as highlighted by the definitions
630 in the second column of Table 1 ($\phi_X ::= \phi_X^{\exists} \mid \phi_X^{\forall}$ for each $X \in$ Btime-spectrum apart from simulation). The "existential" sub-language (formulae derivable from the non-terminal $\phi_X^{\exists}$) is common to all the logics and so is its definition (top of Table 1). The "universal" sub-language (formulae derivable from the non-terminal $\phi_X^{\forall}$) is what actually distinguishes the various languages: its definition
635 is provided for each logic in the corresponding row. The operators $\langle \tau \rangle$ and $[\tau]$ (with $\tau = a_1 a_2 \dots a_k \in Act^*$), occurring in the definition of $\mathcal{B}$ for trace simula-tion (TS), are abbreviations for $\langle a_1 \rangle \langle a_2 \rangle \dots \langle a_k \rangle$ and $[a_1][a_2] \dots [a_k]$, respectively (notice that, in particular, both $\langle \varepsilon \rangle \varphi$ and $[\varepsilon] \varphi$ will be treated as φ).

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Lemma 1. *Let $X \in$ Btime-spectrum. $\mathcal{L}_X$ is finitely characterized by $\mathcal{B}_X$, for some monotonic $\mathcal{B}_X$ (see Table 1).*

Proof. We detail the case of ready simulation only (the proof for the other cases can be found in Appendix A.1). For the sake of clarity we recall from Table 1 that $\mathcal{B}_{\mathsf{RS}}$ is defined as $\mathcal{B}_{\mathsf{RS}}^+(p) \cup \mathcal{B}_{\mathsf{RS}}^-(p)$, where

- $\mathcal{B}_{\mathsf{RS}}^+(p) = \{\mathbf{tt}\} \cup \{\langle a \rangle \varphi \mid a \in Act, \varphi = \bigwedge_{\psi \in \Psi} \psi, \Psi \subseteq \mathcal{B}_X(p'), p \xrightarrow{a} p'\}$, and
- $\mathcal{B}_{\mathsf{RS}}^-(p) = \{[a]\mathbf{ff} \mid a \in Act, p \xrightarrow{a}\!\!\!\!\!/\ \}$.

For the sake of a lighter notation, we omit the subscript $_{\mathsf{RS}}$, i.e., we write $\mathcal{B}$ (resp., $\mathcal{B}^+$, $\mathcal{B}^-$, $\lesssim$) for $\mathcal{B}_{\mathsf{RS}}$ (resp., $\mathcal{B}_{\mathsf{RS}}^+$, $\mathcal{B}_{\mathsf{RS}}^-$, $\lesssim_{\mathsf{RS}}$) since there is no risk of ambiguity.

We show that, for every $p \in P$,

(i) $\emptyset \subset \mathcal{B}(p) \subseteq \mathcal{L}(p)$,

(ii) for each $\phi \in \mathcal{L}(p)$, it holds $\bigcap_{\psi \in \mathcal{B}(p)} \llbracket \psi \rrbracket \subseteq \llbracket \phi \rrbracket$,

(iii) $\mathcal{B}(p)$ is finite, and

(iv) for each $q \in P$, if $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ then $\mathcal{B}(p) \subseteq \mathcal{B}(q)$.

To begin with, we prove property (iii), which also tells us that $\mathcal{B}$ is well-defined. It is immediate to see that, since Act is finite, so is $\mathcal{B}^-(p)$ for all $p \in P$. We show that also $\mathcal{B}(p)$ is finite for every $p \in P$ by induction on the depth of p. When $I(p) = \emptyset$ (base case), $\mathcal{B}(p) = \{\mathbf{tt}\} \cup \mathcal{B}^-(p)$, which is clearly finite. Let us deal now with the inductive step ($I(p) \neq \emptyset$). By the construction of $\mathcal{B}^+(p)$, a formula belongs to $\mathcal{B}^+(p)$ if, and only if, it is either $\mathbf{tt}$ or $\langle a \rangle \varphi$, where $a \in Act$ and $\varphi = \bigwedge_{\psi \in \Psi} \psi$, for some $\Psi \subseteq \mathcal{B}(p')$ and some p' such that $p \xrightarrow{a} p'$. By the inductive hypothesis, $\mathcal{B}(p')$ is finite, and thus φ is well defined. Since Act is also finite and processes are finitely branching, there are only finitely many such formulae $\langle a \rangle \varphi$, meaning that $\mathcal{B}^+(p)$ is finite. Therefore $\mathcal{B}(p) = \mathcal{B}^+(p) \cup \mathcal{B}^-$ is finite as well.

In order to prove property (i), we preliminarily observe that $\mathbf{tt} \in \mathcal{B}(p)$ for every $p \in P$, and thus $\emptyset \subset \mathcal{B}(p)$. Now, to prove that $\mathcal{B}(p) \subseteq \mathcal{L}(p)$ holds for every $p \in P$, we first observe that $\mathcal{B}^-(p) \subseteq \mathcal{L}(p)$ trivially holds, by definition of $\mathcal{B}^-(p)$, and then we show that $\mathcal{B}(p) \subseteq \mathcal{L}(p)$ also holds for every $p \in P$, by induction on the depth of p. When $I(p) = \emptyset$ (base case), we have $\mathcal{B}^+(p) = \{\mathbf{tt}\} \subseteq \mathcal{L}(p)$, and therefore $\mathcal{B}(p) \subseteq \mathcal{L}(p)$ holds as well. To deal with the inductive step ($I(p) \neq \emptyset$), let $\phi \in \mathcal{B}^+(p)$. If $\phi = \mathbf{tt}$, then $\phi \in \mathcal{L}(p)$, and we are done. Assume $\phi = \langle a \rangle \bigwedge_{\psi \in \Psi} \psi$, where $a \in Act$ and $\Psi \subseteq \mathcal{B}(p')$ for some p' such that $p \xrightarrow{a} p'$. By the inductive hypothesis, we have that $\mathcal{B}(p') \subseteq \mathcal{L}(p')$, meaning that $p' \in \llbracket \psi \rrbracket$ for all $\psi \in \Psi$. Since $p \xrightarrow{a} p'$, we have that $p \in \llbracket \langle a \rangle \bigwedge_{\psi \in \Psi} \psi \rrbracket$, which amounts to $\phi \in \mathcal{L}(p)$.

In order to prove property (ii), we let $\phi \in \mathcal{L}(p)$, for a generic $p \in P$, and we proceed by induction on the structure of ϕ (notice that we can ignore the case $\phi = \mathbf{ff}$, as $\phi \in \mathcal{L}(p)$ implies $\phi \neq \mathbf{ff}$).

- $\phi = \mathbf{tt}$ or $\phi = [a]\mathbf{ff}$: it is enough to observe that $\phi \in \mathcal{B}(p)$, which implies $\bigcap_{\psi \in \mathcal{B}(p)}[\![\psi]\!] \subseteq [\![\phi]\!]$.

- $\phi = \varphi_1 \vee \varphi_2$: it holds that $\varphi_i \in \mathcal{L}(p)$ for some $i \in \{1,2\}$. By the inductive hypothesis, we have that $\bigcap_{\psi \in \mathcal{B}(p)}[\![\psi]\!] \subseteq [\![\varphi_i]\!]$ and, since $[\![\varphi_i]\!] \subseteq [\![\phi]\!]$, we obtain the claim.

- $\phi = \varphi_1 \wedge \varphi_2$: it holds that $\varphi_i \in \mathcal{L}(p)$ for all $i \in \{1,2\}$. By the inductive hypothesis, we have that $\bigcap_{\psi \in \mathcal{B}(p)}[\![\psi]\!] \subseteq [\![\varphi_i]\!]$ for all $i \in \{1,2\}$. This implies that $\bigcap_{\psi \in \mathcal{B}(p)}[\![\psi]\!] \subseteq [\![\varphi_1]\!] \cap [\![\varphi_2]\!] = [\![\phi]\!]$.

- $\phi = \langle a \rangle \varphi$: by definition we have that $\varphi \in \mathcal{L}(p')$ for some $p \xrightarrow{a} p'$. By the inductive hypothesis, we have that $\bigcap_{\psi \in \mathcal{B}(p')}[\![\psi]\!] \subseteq [\![\varphi]\!]$. We define $\zeta = \langle a \rangle \bigwedge_{\psi \in \mathcal{B}(p')} \psi$. Clearly, ζ belongs to $\mathcal{B}^+(p)$ (by construction—notice that ζ is well defined due to the finiteness of $\mathcal{B}(p')$) and $[\![\zeta]\!] \subseteq [\![\phi]\!]$ (because $\bigcap_{\psi \in \mathcal{B}(p')}[\![\psi]\!] \subseteq [\![\varphi]\!]$). Hence, $\bigcap_{\psi \in \mathcal{B}(p)}[\![\psi]\!] \subseteq [\![\zeta]\!] \subseteq [\![\phi]\!]$ holds.

Finally, we show that $\mathcal{B}$ is monotonic (property (iv)). Consider $p, q \in P$, with $\mathcal{L}(p) \subseteq \mathcal{L}(q)$. We want to show that $\phi \in \mathcal{B}(p)$ implies $\phi \in \mathcal{B}(q)$, for each ϕ. Firstly, we observe that, by $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ and Theorem 4, $p \lesssim q$ holds. Thus, we have that $I(p) = I(q)$ and, for each $a \in Act$ and $p' \in P$ with $p \xrightarrow{a} p'$, there exists some $q' \in P$ such that $q \xrightarrow{a} q'$ and $p' \lesssim q'$. We also observe that $\mathcal{B}^-(p) = \mathcal{B}^-(q)$, since $I(p) = I(q)$. In order to show that $\mathcal{B}(p) \subseteq \mathcal{B}(q)$, we proceed by induction on the depth of p. If $I(p) = \emptyset$, then $I(q) = \emptyset$ as well. Thus, we have that $\mathcal{B}^+(p) = \mathcal{B}^+(q) = \{\mathbf{tt}\}$, and the thesis follows. Otherwise ($I(p) \neq \emptyset$), let us consider a formula $\phi \in \mathcal{B}(p)$. If $\phi \in \mathcal{B}^-(p)$, then the claim follows from $\mathcal{B}^-(p) = \mathcal{B}^-(q) \subseteq \mathcal{B}(q)$. If $\phi = \mathbf{tt}$, then, by definition of $\mathcal{B}^+$, we have $\phi \in \mathcal{B}^+(q) \subseteq \mathcal{B}(q)$. Finally, if $\phi = \langle a \rangle \varphi \in \mathcal{B}^+(p)$, then, by definition of $\mathcal{B}^+$, there exist $p' \in P$, with $p \xrightarrow{a} p'$, such that $\varphi = \bigwedge_{\psi \in \Psi} \psi$ for some $\Psi \subseteq \mathcal{B}(p')$. This implies the existence of some $q' \in P$ such that $q \xrightarrow{a} q'$ and $p' \lesssim q'$ (and therefore $\mathcal{L}(p') \subseteq \mathcal{L}(q')$ by Theorem 4). By the inductive hypothesis, $\mathcal{B}(p') \subseteq \mathcal{B}(q')$ holds as well, which means that $\Psi \subseteq \mathcal{B}(q')$. Hence, we have that $\langle a \rangle \varphi \in \mathcal{B}^+(q) \subseteq \mathcal{B}(q)$. $\qquad \square$

Lemma 2. *Let $X \in$ Btime-spectrum. For each $p \in P$ and $\chi_X(p)$ characteristic within $\mathcal{L}_X$ for p, there exists a formula in $\mathcal{L}_X$, denoted by $\bar{\chi}_X(p)$, such that either*

- $[\![\bar{\chi}(p)]\!] = P \setminus [\![\chi(p)]\!]$, *or*

- $p \notin [\![\bar{\chi}_X(p)]\!]$ *and* $\{p' \in P \mid p' \not\lesssim_X p\} \subseteq [\![\bar{\chi}_X(p)]\!]$.

Proof. We detail the case of ready simulation only (the proof for the other cases can be found in Appendix A.2). For the sake of clarity we recall here the definition of $\bar{\chi}_{\mathsf{RS}}$ from Table 1:
$$\bar{\chi}_{\mathsf{RS}}(p) = \big(\bigvee_{a \in Act} \langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}_{\mathsf{RS}}(p') \big) \vee \bigvee_{a \in I(p)} [a]\mathbf{ff}.$$
As we did for the proof of the previous lemma, we omit the subscript RS with no risk of ambiguity, e.g., we write $\bar{\chi}$ (resp., $\equiv$) for $\bar{\chi}_{\mathsf{RS}}$ (resp., $\equiv_{\mathsf{RS}}$).

Let us first show that for every $p \in P$ we have $p \notin [\![\bar{\chi}(p)]\!]$. We proceed by induction on the depth of p.

- $I(p) = \emptyset$: we have $\bar{\chi}(p) = \bigvee_{a \in Act} \langle a \rangle \mathbf{tt}$ (up to logical equivalence), and thus $p \notin [\![\bar{\chi}(p)]\!]$.
- $I(p) \neq \emptyset$: obviously, $p \notin [\![[a]\mathbf{ff}]\!]$ holds for every $a \in I(p)$. Moreover, for every $a \in Act$ and every $p \xrightarrow{a} p''$, by the inductive hypothesis, $p'' \notin [\![\bar{\chi}(p'')]\!]$. Thus, $p'' \notin [\![\bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$ and therefore $p \notin [\![\langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$ for every $a \in Act$. Hence, we obtain that $p \notin [\![\bar{\chi}(p)]\!]$.

Now, let us show that $\{p' \in P \mid p' \not\lesssim p\} \subseteq [\![\bar{\chi}(p)]\!]$, that is, $[\![\bar{\chi}(p)]\!]$ contains at least the elements that are either strictly above p or incomparable with it. The proof is by induction on the depth of p.

- $I(p) = \emptyset$: we have that $\{p' \in P \mid p' \not\lesssim p\} = P \setminus \{p \in P \mid I(p) = \emptyset\}$ because in this case we have that $p \equiv 0$ and thus $q \lesssim 0$ does not hold for any process q with $I(q) \neq \emptyset$. It is easy to see that $P \setminus \{p \in P \mid I(p) = \emptyset\} \subseteq [\![\bigvee_{a \in Act} \langle a \rangle \mathbf{tt}]\!] = [\![\bar{\chi}(p)]\!]$.
- $I(p) \neq \emptyset$: let $q \not\lesssim p$. Thus, either $I(q) \neq I(p)$ or there exists some q', with $q \xrightarrow{a} q'$, such that, for every p', $p \xrightarrow{a} p'$ implies $q' \not\lesssim p'$. If it is the case that $I(q) \neq I(p)$, then either $q \in [\![\langle a \rangle \mathbf{tt}]\!]$ holds for some $a \notin I(p)$, or $q \in [\![[a]\mathbf{ff}]\!]$ for some $a \in I(p)$. In either case, $q \in [\![\bar{\chi}(p)]\!]$ holds.

  Otherwise, if there exist $a \in Act$ and $q' \in P$, with $q \xrightarrow{a} q'$, such that $q' \not\lesssim p'$ for every $p \xrightarrow{a} p'$, then, by the inductive hypothesis, $q' \in [\![\bar{\chi}(p')]\!]$ for every p' such that $p \xrightarrow{a} p'$. Thus, $q' \in [\![\bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$ and therefore $q \in [\![\langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$. Hence, we conclude $q \in [\![\bar{\chi}(p)]\!]$. $\qquad\square$

Finally, the following theorem states the main result of this section.

Theorem 5 (Characterization by primality for the branching time spectrum)**.** *Let $X \in$ Btime-spectrum and $\phi \in \mathcal{L}_X$. Then, ϕ is consistent and prime if and only if ϕ is characteristic for some $p \in P$.*

Proof. The claim immediately follows from Theorem 1, Theorem 2, Corollary 4, Lemma 1, and Lemma 2. $\qquad\square$

5.2. The linear time spectrum

In this sub-section, we focus on the semantics in *van Glabbeek's linear time spectrum*, [6, 7], and their corresponding logical formalisms.

To begin with, we define sets $X(p)$, for $X \in$ Ltime-spectrum (defined on page 14) and $p \in P$ (they were originally defined in [6, 7]). Intuitively, $X(p)$ (with $X \in$ Ltime-spectrum) establishes the granularity of *process observations*, and thus the level of detail at which processes are compared. For example, $\mathsf{T}(p)$ contains all traces that p is able to perform; similarly, $\mathsf{CT}(p)$ contains all traces that p is able to perform and that lead to a process where no action can be performed. When we move up to semantics in the three diamonds (see Figure 1), things get a bit more involved: $\mathsf{R}(p)$ contains pairs $\langle \tau, Y \rangle$ whose first element is a trace that takes p to a process p' whose set of *initials* (actions it can perform) is exactly Y; on the contrary, $\langle \tau, Y \rangle \in \mathsf{F}(p)$ denotes the fact that p reaches p' through τ and p' cannot perform any action in Y; $\mathsf{RT}(p)$ contains words over the alphabet $(Act \cup \mathcal{P}(Act))^*$, that is, $\sigma \in \mathsf{RT}(p)$ is a finite sequence

$\sigma_1\sigma_2\ldots\sigma_k$ where each σ_i is either an action (*action elements*) or a set of actions (*set elements*): action elements identify a trace τ that p can perform, while set elements represent initials of (some of) the processes reached while performing 765 τ; finally, elements of $\mathsf{FT}(p)$ differ from the ones in $\mathsf{RT}(p)$ in that set elements represent actions that cannot be performed (rather than initials). Set $X(p)$, where X is a semantics in the 2nd or the 3rd diamond, is defined analogously to its counterpart in the 1st diamond (see Figure 1); the only difference is that set elements carry information on traces that can/cannot be performed (2nd 770 diamond) or on processes that are/are not simulated (3rd diamond) rather than on actions that can/cannot be performed.

The following observation will be useful later on, when we will finitely characterize the logics characterizing the semantics in the linear time spectrum through some function $\mathcal{B}$.

775 **Remark 3.** *We can safely reduce to words in* $\mathsf{RT}(p)$ *and* $\mathsf{FT}(p)$ *whose length is bounded by* $2 \cdot depth(p) + 1$. *Indeed, the number of action elements occurring in any such word* σ *cannot be greater than the length of a longest trace p can perform; moreover, two consecutive set elements can be suitably merged into one: for instance, word* $a\{a\}\{b\} \in \mathsf{FT}(p)$ *says that p can perform a and reach* 780 *process p', which in turn can perform neither a nor b; the same information is captured by* $a\{a, b\}$, *which is also an element of* $\mathsf{FT}(p)$.

The same observation applies to corresponding sets for the semantics in the 2nd and 3rd diamond, i.e., $\mathsf{PFT}(p)$, $\mathsf{IFT}(p)$, $\mathsf{P2T}(p)$, *and* $\mathsf{I2T}(p)$.

Sets $X(p)$ (for $X \in$ Ltime-spectrum and $p \in P$) are formally defined below. 785 For every $p \in P$, $[p]_{\mathsf{BS}}$ is the equivalence class of p with respect to bisimulation equivalence, that is, $[p]_{\mathsf{BS}} = \{q \in P \mid q \equiv_{\mathsf{BS}} p\}$; for every $Q \subseteq P$, we use $[Q]_{\mathsf{BS}}$ to denote the set of equivalence classes of processes in Q with respect to bisimulation equivalence, that is, $[Q]_{\mathsf{BS}} = \{[p]_{\mathsf{BS}} \mid p \in Q\}$; moreover, we use $p_{\downarrow\mathsf{S}}$ to denote the set of equivalence classes of processes that are simulated by p, 790 that is, $p_{\downarrow\mathsf{S}} = \{[q]_{\mathsf{BS}} \in [P]_{\mathsf{BS}} \mid q \lesssim_{\mathsf{S}} p\}$. Notice that $p_{\downarrow\mathsf{S}}$ is finite for all p.

trace (T): $\mathsf{T}(p) = \{\varepsilon\} \cup \{a\tau \mid \exists p' \in P \:.\: p \xrightarrow{a} p' \text{ and } \tau \in \mathsf{T}(p')\}$;[1]

complete trace (CT): $\mathsf{CT}(p) = \{\tau \in \mathsf{T}(p) \mid \exists p'.p \xrightarrow{\tau} p' \text{ and } p' \not\rightarrow\}$;

ready (R): $\mathsf{R}(p) = \{\langle \tau, Y \rangle \in Act^* \times \mathcal{P}(Act) \mid \exists p' \in P \:.\: p \xrightarrow{\tau} p' \text{ and } I(p') = Y\}$;

795 **ready trace** (RT):

- $p \xrightarrow{\varepsilon}\bullet p$ for all $p \in P$,
- $p \xrightarrow{Y}\bullet p$ for all $p \in P$ and $Y \subseteq Act$ such that $I(p) = Y$,

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[1] $\mathsf{T}(p)$ is defined in the same way as $traces(p)$ (cf. equation (1) at page 14); we re-define it using a different notation to ease the reading and to be uniform with the notation used for the other linear time semantics.

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- if $p \xrightarrow{a} q$, then $p \xrightarrow{a}_\bullet q$, for all $p, q \in P$ and $a \in Act$,
- if $p \xrightarrow{\sigma}_\bullet q$ and $q \xrightarrow{\rho}_\bullet r$, then $p \xrightarrow{\sigma \cdot \rho}_\bullet r$, for all $p, q, r \in P$ and $\sigma, \rho \in (Act \cup \mathcal{P}(Act))^*$,
- $\mathsf{RT}(p) = \{\sigma \in (Act \cup \mathcal{P}(Act))^* \mid \exists q \in P . p \xrightarrow{\sigma}_\bullet q\}$;

failure (F): $\mathsf{F}(p) = \{\langle \tau, Y \rangle \in Act^* \times \mathcal{P}(Act) \mid \exists p' \in P . p \xrightarrow{\tau} p'$ and $I(p') \cap Y = \emptyset\}$;

failure trace (FT):

- $p \xrightarrow{\varepsilon}_\# p$ for all $p \in P$,
- $p \xrightarrow{Y}_\# p$ for all $p \in P$ and $Y \subseteq Act$ such that $I(p) \cap Y = \emptyset$,
- if $p \xrightarrow{a} q$, then $p \xrightarrow{a}_\# q$ for all $p, q \in P$ and $a \in Act$,
- if $p \xrightarrow{\sigma}_\# q$ and $q \xrightarrow{\rho}_\# r$, then $p \xrightarrow{\sigma \cdot \rho}_\# r$ for all $p, q, r \in P$ and $\sigma, \rho \in (Act \cup \mathcal{P}(Act))^*$,
- $\mathsf{FT}(p) = \{\sigma \in (Act \cup \mathcal{P}(Act))^* \mid \exists q \in P . p \xrightarrow{\sigma}_\# q\}$;

possible future (PF): $\mathsf{PF}(p) = \{\langle \tau, \Gamma \rangle \in Act^* \times \mathcal{P}(Act^*) \mid \exists p' \in P . p \xrightarrow{\tau} p'$ and $\mathsf{T}(p') = \Gamma\}$;

possible-future trace (PFT):

- $p \xrightarrow{\varepsilon}_{\bullet\bullet} p$ for all $p \in P$,
- $p \xrightarrow{\Gamma}_{\bullet\bullet} p$ for all $p \in P$ and $\Gamma \subseteq Act^*$ such that $\mathsf{T}(p) = \Gamma$,
- if $p \xrightarrow{a} q$, then $p \xrightarrow{a}_{\bullet\bullet} q$, for all $p, q \in P$ and $a \in Act$,
- if $p \xrightarrow{\sigma}_{\bullet\bullet} q$ and $q \xrightarrow{\rho}_{\bullet\bullet} r$, then $p \xrightarrow{\sigma \cdot \rho}_{\bullet\bullet} r$, for all $p, q, r \in P$ and $\sigma, \rho \in (Act \cup \mathcal{P}(Act^*))^*$,
- $\mathsf{PFT}(p) = \{\sigma \in (Act \cup \mathcal{P}(Act^*))^* \mid \exists q \in P . p \xrightarrow{\sigma}_{\bullet\bullet} q\}$;

impossible future (IF): $\mathsf{IF}(p) = \{\langle \tau, \Gamma \rangle \in Act^* \times \mathcal{P}(Act^*) \mid \exists p' \in P . p \xrightarrow{\tau} p'$ and $\mathsf{T}(p') \cap \Gamma = \emptyset\}$;

impossible-future trace (IFT):

- $p \xrightarrow{\varepsilon}_{\#\#} p$ for all $p \in P$,
- $p \xrightarrow{\Gamma}_{\#\#} p$ for all $p \in P$ and $\Gamma \subseteq Act^*$ such that $\mathsf{T}(p) \cap \Gamma = \emptyset$,
- if $p \xrightarrow{a} q$, then $p \xrightarrow{a}_{\#\#} q$, for all $p, q \in P$ and $a \in Act$,
- if $p \xrightarrow{\sigma}_{\#\#} q$ and $q \xrightarrow{\rho}_{\#\#} r$, then $p \xrightarrow{\sigma \cdot \rho}_{\#\#} r$, for all $p, q, r \in P$ and $\sigma, \rho \in (Act \cup \mathcal{P}(Act^*))^*$,
- $\mathsf{IFT}(p) = \{\sigma \in (Act \cup \mathcal{P}(Act^*))^* \mid \exists q \in P . p \xrightarrow{\sigma}_{\#\#} q\}$.

possible 2-simulation (P2): $\mathsf{P2}(p) = \{\langle \tau, \mathbb{P} \rangle \in Act^* \times \mathcal{P}([P]_{\mathsf{BS}}) \mid \exists p' \in P . p \xrightarrow{\tau} p'$ and $p'_{\downarrow_\mathsf{S}} = \mathbb{P}\}$;

possible-2-simulation trace (P2T):

- $p \xrightarrow{\varepsilon}\!\!\bullet\!\bullet\!\bullet\, p$ for all $p \in P$,
- $p \xrightarrow{\mathbb{P}}\!\!\bullet\!\bullet\!\bullet\, p$ for all $p \in P$ and $\mathbb{P} \subseteq [P]_{\mathsf{BS}}$ such that $p_{\downarrow\mathsf{S}} = \mathbb{P}$,
- if $p \xrightarrow{a} q$, then $p \xrightarrow{a}\!\!\bullet\!\bullet\!\bullet\, q$, for all $p, q \in P$ and $a \in Act$,
- if $p \xrightarrow{\sigma}\!\!\bullet\!\bullet\!\bullet\, q$ and $q \xrightarrow{\rho}\!\!\bullet\!\bullet\!\bullet\, r$, then $p \xrightarrow{\sigma\cdot\rho}\!\!\bullet\!\bullet\!\bullet\, r$, for all $p, q, r \in P$ and $\sigma, \rho \in (Act \cup \mathcal{P}([P]_{\mathsf{BS}}))^*$,
- $\mathsf{P2T}(p) = \{\sigma \in (Act \cup \mathcal{P}([P]_{\mathsf{BS}}))^* \mid \exists q \in P\,.\, p \xrightarrow{\sigma}\!\!\bullet\!\bullet\!\bullet\, q\}$;

impossible 2-simulation (I2): $\mathsf{I2}(p) = \{\langle \tau, \mathbb{P} \rangle \in Act^* \times \mathcal{P}([P]_{\mathsf{BS}}) \mid \exists p' \in P\,.\, p \xrightarrow{\tau} p'$ and $p'_{\downarrow\mathsf{S}} \cap \mathbb{P} = \emptyset\}$;

impossible-2-simulation trace (I2T):

- $p \xrightarrow{\varepsilon}\!\!\#\#\#\, p$ for all $p \in P$,
- $p \xrightarrow{\mathbb{P}}\!\!\#\#\#\, p$ for all $p \in P$ and $\mathbb{P} \subseteq [P]_{\mathsf{BS}}$ such that $p_{\downarrow\mathsf{S}} \cap \mathbb{P} = \emptyset$,
- if $p \xrightarrow{a} q$, then $p \xrightarrow{a}\!\!\#\#\#\, q$, for all $p, q \in P$ and $a \in Act$,
- if $p \xrightarrow{\sigma}\!\!\#\#\#\, q$ and $q \xrightarrow{\rho}\!\!\#\#\#\, r$, then $p \xrightarrow{\sigma\cdot\rho}\!\!\#\#\#\, r$, for all $p, q, r \in P$ and $\sigma, \rho \in (Act \cup \mathcal{P}([P]_{\mathsf{BS}}))^*$,
- $\mathsf{I2T}(p) = \{\sigma \in (Act \cup \mathcal{P}([P]_{\mathsf{BS}}))^* \mid \exists q \in P\,.\, p \xrightarrow{\sigma}\!\!\#\#\#\, q\}$.

Definition 9 (Linear time semantic relations [6, 7]). *For each $X \in$ Ltime-spectrum, $\lesssim_X$ is defined as follows:*
$$p \lesssim_X q \Leftrightarrow X(p) \subseteq X(q),$$
while $\equiv_X$ is defined as
$$p \equiv_X q \Leftrightarrow p \lesssim_X q \text{ and } q \lesssim_X p \Leftrightarrow X(p) = X(q).$$

Linear time logics. Unlike the logics for branching time semantics, the linear time logics do not yield a chain of strict inclusions; their expressive power is captured by a partial ordering relation, as shown in Figure 1 (right-hand side).

The syntax defining the logics characterizing the semantics in van Glabbeek's linear time spectrum is as follows. Let us recall here the meaning of the following abbreviations:

- $[a]\psi$ stands for $\neg\langle a\rangle\neg\psi$,
- $\mathbf{0}$ stands for $\bigwedge_{a \in Act}[a]\mathbf{ff}$,
- $\langle\tau\rangle$ (with $\tau = a_1 a_2 \ldots a_k \in Act^*$) stands for $\langle a_1\rangle\langle a_2\rangle \ldots \langle a_k\rangle$, and
- $[\tau]$ (with $\tau = a_1 a_2 \ldots a_k \in Act^*$) stands for $[a_1][a_2] \ldots [a_k]$.

Additionally, for $Y \subseteq Act$ (resp., finite set $\Gamma \subseteq Act^*$), we use $\langle\!| Y |\!\rangle$ (resp., $\langle\!| \Gamma |\!\rangle$) as an abbreviation for $\bigwedge_{a \in Y}\langle a\rangle\mathbf{tt} \wedge \bigwedge_{a \in Act\setminus Y}[a]\mathbf{ff}$ (resp., $\bigwedge_{\tau \in \Gamma}\langle\tau\rangle\mathbf{tt} \wedge \bigwedge_{\tau \in Act^*_{|\Gamma}\setminus\Gamma}[\tau]\mathbf{ff}$), where $Act^*_{|\Gamma} = \{\tau \in Act^* \mid |\tau| \leq (\max_{\tau' \in \Gamma}|\tau'|) + 1\}$

Definition 10 (Syntax [6, 7]). *Let $X \in$ Ltime-spectrum, $\mathcal{L}_X$ is the language defined by the corresponding grammar among the following ones:*
$\mathcal{L}_{\mathsf{T}}$:

$$\phi_\mathsf{T} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_\mathsf{T} \wedge \phi_\mathsf{T} \mid \phi_\mathsf{T} \vee \phi_\mathsf{T} \mid \psi_\mathsf{T}$$
$$\psi_\mathsf{T} \quad ::= \quad \mathbf{tt} \mid \langle a \rangle \psi_\mathsf{T},$$

$\mathcal{L}_\mathsf{CT}$:

$$\phi_\mathsf{CT} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_\mathsf{CT} \wedge \phi_\mathsf{CT} \mid \phi_\mathsf{CT} \vee \phi_\mathsf{CT} \mid \psi_\mathsf{CT}$$
$$\psi_\mathsf{CT} \quad ::= \quad \mathbf{tt} \mid \mathbf{0} \mid \langle a \rangle \psi_\mathsf{CT},$$

$\mathcal{L}_\mathsf{F}$:

$$\phi_\mathsf{F} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_\mathsf{F} \wedge \phi_\mathsf{F} \mid \phi_\mathsf{F} \vee \phi_\mathsf{F} \mid \psi_\mathsf{F}$$
$$\psi_\mathsf{F} \quad ::= \quad \mathbf{tt} \mid \langle a \rangle \psi_\mathsf{F} \mid \gamma_\mathsf{F}$$
$$\gamma_\mathsf{F} \quad ::= \quad [a]\mathbf{ff} \mid \gamma_\mathsf{F} \wedge \gamma_\mathsf{F},$$

$\mathcal{L}_\mathsf{R}$:

$$\phi_\mathsf{R} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_\mathsf{R} \wedge \phi_\mathsf{R} \mid \phi_\mathsf{R} \vee \phi_\mathsf{R} \mid \psi_\mathsf{R}$$
$$\psi_\mathsf{R} \quad ::= \quad \mathbf{tt} \mid \langle\!\langle Y \rangle\!\rangle \mid \langle a \rangle \psi_\mathsf{R},$$

$\mathcal{L}_\mathsf{FT}$:

$$\phi_\mathsf{FT} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_\mathsf{FT} \wedge \phi_\mathsf{FT} \mid \phi_\mathsf{FT} \vee \phi_\mathsf{FT} \mid \psi_\mathsf{FT}$$
$$\psi_\mathsf{FT} \quad ::= \quad \mathbf{tt} \mid [a]\mathbf{ff} \mid \langle a \rangle \psi_\mathsf{FT} \mid [a]\mathbf{ff} \wedge \psi_\mathsf{FT},$$

$\mathcal{L}_\mathsf{RT}$:

$$\phi_\mathsf{RT} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_\mathsf{RT} \wedge \phi_\mathsf{RT} \mid \phi_\mathsf{RT} \vee \phi_\mathsf{RT} \mid \psi_\mathsf{RT}$$
$$\psi_\mathsf{RT} \quad ::= \quad \mathbf{tt} \mid \langle\!\langle Y \rangle\!\rangle \wedge \psi_\mathsf{RT} \mid \langle a \rangle \psi_\mathsf{RT},$$

$\mathcal{L}_\mathsf{IF}$:

$$\phi_\mathsf{IF} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_\mathsf{IF} \wedge \phi_\mathsf{IF} \mid \phi_\mathsf{IF} \vee \phi_\mathsf{IF} \mid \psi_\mathsf{IF}$$
$$\psi_\mathsf{IF} \quad ::= \quad \mathbf{tt} \mid \langle a \rangle \psi_\mathsf{IF} \mid \gamma_\mathsf{IF}$$
$$\gamma_\mathsf{IF} \quad ::= \quad [\tau]\mathbf{ff} \mid \gamma_\mathsf{IF} \wedge \gamma_\mathsf{IF},$$

$\mathcal{L}_\mathsf{PF}$:

$$\phi_\mathsf{PF} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_\mathsf{PF} \wedge \phi_\mathsf{PF} \mid \phi_\mathsf{PF} \vee \phi_\mathsf{PF} \mid \psi_\mathsf{PF}$$
$$\psi_\mathsf{PF} \quad ::= \quad \mathbf{tt} \mid \langle\!\langle \Gamma \rangle\!\rangle \mid \langle a \rangle \psi_\mathsf{PF},$$

$\mathcal{L}_\mathsf{IFT}$:

$$\phi_\mathsf{IFT} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_\mathsf{IFT} \wedge \phi_\mathsf{IFT} \mid \phi_\mathsf{IFT} \vee \phi_\mathsf{IFT} \mid \psi_\mathsf{IFT}$$
$$\psi_\mathsf{IFT} \quad ::= \quad \mathbf{tt} \mid [\tau]\mathbf{ff} \mid \langle a \rangle \psi_\mathsf{IFT} \mid [\tau]\mathbf{ff} \wedge \psi_\mathsf{IFT},$$

$\mathcal{L}_\mathsf{PFT}$:

$$\phi_\mathsf{PFT} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_\mathsf{PFT} \wedge \phi_\mathsf{PFT} \mid \phi_\mathsf{PFT} \vee \phi_\mathsf{PFT} \mid \psi_\mathsf{PFT}$$
$$\psi_\mathsf{PFT} \quad ::= \quad \mathbf{tt} \mid \langle\!\langle \Gamma \rangle\!\rangle \wedge \psi_\mathsf{PFT} \mid \langle a \rangle \psi_\mathsf{PFT},$$

$\mathcal{L}_\mathsf{I2}$:

$$\phi_\mathsf{I2} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_\mathsf{I2} \wedge \phi_\mathsf{I2} \mid \phi_\mathsf{I2} \vee \phi_\mathsf{I2} \mid \psi_\mathsf{I2}$$
$$\psi_\mathsf{I2} \quad ::= \quad \mathbf{tt} \mid \langle a \rangle \psi_\mathsf{I2} \mid \gamma_\mathsf{I2}$$
$$\gamma_\mathsf{I2} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid [a]\gamma_\mathsf{I2} \mid \gamma_\mathsf{I2} \wedge \gamma_\mathsf{I2} \mid \gamma_\mathsf{I2} \vee \gamma_\mathsf{I2},$$

$\mathcal{L}_\mathsf{P2}$:

$$\phi_\mathsf{P2} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_\mathsf{P2} \wedge \phi_\mathsf{P2} \mid \phi_\mathsf{P2} \vee \phi_\mathsf{P2} \mid \psi_\mathsf{P2}$$
$$\psi_\mathsf{P2} \quad ::= \quad \mathbf{tt} \mid \langle a \rangle \psi_\mathsf{P2} \mid \gamma_\mathsf{P2}^{\exists} \wedge \gamma_\mathsf{P2}^{\forall}$$
$$\gamma_\mathsf{P2}^{\exists} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \langle a \rangle \gamma_\mathsf{P2}^{\exists} \mid \gamma_\mathsf{P2}^{\exists} \wedge \gamma_\mathsf{P2}^{\exists} \mid \gamma_\mathsf{P2}^{\exists} \vee \gamma_\mathsf{P2}^{\exists}$$
$$\gamma_\mathsf{P2}^{\forall} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid [a]\gamma_\mathsf{P2}^{\forall} \mid \gamma_\mathsf{P2}^{\forall} \wedge \gamma_\mathsf{P2}^{\forall} \mid \gamma_\mathsf{P2}^{\forall} \vee \gamma_\mathsf{P2}^{\forall},$$

$\mathcal{L}_\mathsf{I2T}$:

$$\phi_\mathsf{I2} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid \phi_\mathsf{I2} \wedge \phi_\mathsf{I2} \mid \phi_\mathsf{I2} \vee \phi_\mathsf{I2} \mid \psi_\mathsf{I2}$$
$$\psi_\mathsf{I2} \quad ::= \quad \mathbf{tt} \mid \langle a \rangle \psi_\mathsf{I2} \mid \gamma_\mathsf{I2} \wedge \psi_\mathsf{I2}$$
$$\gamma_\mathsf{I2} \quad ::= \quad \mathbf{tt} \mid \mathbf{ff} \mid [a]\gamma_\mathsf{I2} \mid \gamma_\mathsf{I2} \wedge \gamma_\mathsf{I2} \mid \gamma_\mathsf{I2} \vee \gamma_\mathsf{I2},$$

$\mathcal{L}_\mathsf{P2T}$:

$$\phi_{\mathsf{P2}} \quad ::= \quad \mathtt{tt} \mid \mathtt{ff} \mid \phi_{\mathsf{P2}} \wedge \phi_{\mathsf{P2}} \mid \phi_{\mathsf{P2}} \vee \phi_{\mathsf{P2}} \mid \psi_{\mathsf{P2}}$$
$$\psi_{\mathsf{P2}} \quad ::= \quad \mathtt{tt} \mid \langle a \rangle \psi_{\mathsf{P2}} \mid \gamma_{\mathsf{P2}}^{\exists} \wedge \gamma_{\mathsf{P2}}^{\forall} \wedge \psi_{\mathsf{P2}}$$
$$\gamma_{\mathsf{P2}}^{\exists} \quad ::= \quad \mathtt{tt} \mid \mathtt{ff} \mid \langle a \rangle \gamma_{\mathsf{P2}}^{\exists} \mid \gamma_{\mathsf{P2}}^{\exists} \wedge \gamma_{\mathsf{P2}}^{\exists} \mid \gamma_{\mathsf{P2}}^{\exists} \vee \gamma_{\mathsf{P2}}^{\exists}$$
$$\gamma_{\mathsf{P2}}^{\forall} \quad ::= \quad \mathtt{tt} \mid \mathtt{ff} \mid [a] \gamma_{\mathsf{P2}}^{\forall} \mid \gamma_{\mathsf{P2}}^{\forall} \wedge \gamma_{\mathsf{P2}}^{\forall} \mid \gamma_{\mathsf{P2}}^{\forall} \vee \gamma_{\mathsf{P2}}^{\forall}.$$

Remark 4. *$\mathcal{L}_{\mathsf{R}}$ (resp., $\mathcal{L}_{\mathsf{RT}}$, $\mathcal{L}_{\mathsf{PF}}$, $\mathcal{L}_{\mathsf{PFT}}$) is strictly more expressive than $\mathcal{L}_{\mathsf{F}}$ (resp., $\mathcal{L}_{\mathsf{FT}}$, $\mathcal{L}_{\mathsf{IF}}$, $\mathcal{L}_{\mathsf{IFT}}$). However, we notice that, unlike the other cases, the embedding of the latter into the former is not succinct in general: translating $\mathcal{L}_{\mathsf{F}}$ formulae (resp., $\mathcal{L}_{\mathsf{FT}}$) into $\mathcal{L}_{\mathsf{R}}$ (resp., $\mathcal{L}_{\mathsf{RT}}$) ones might cause an exponential growth (in the size of the set of actions) of the formula size; the situation is even worse when translating $\mathcal{L}_{\mathsf{IF}}$ formulae (resp., $\mathcal{L}_{\mathsf{IFT}}$) into $\mathcal{L}_{\mathsf{PF}}$ (resp., $\mathcal{L}_{\mathsf{PFT}}$) ones, which might cause a doubly exponential growth (in the size of the set of actions and in the size of the formula). For instance, formula $\bigwedge_{a \in Y}[a]\mathtt{ff} \in \mathcal{L}_{\mathsf{F}}$ (for $Y \subseteq Act$) is translated into formula $\bigvee_{Y' \subseteq Act \setminus Y} \langle\!\langle Y' \rangle\!\rangle = \bigvee_{Y' \subseteq Act \setminus Y} (\bigwedge_{a \in Y'} \langle a \rangle \mathtt{tt} \wedge \bigwedge_{a \in Act \setminus Y'} [a]\mathtt{ff}) \in \mathcal{L}_{\mathsf{R}}$, whose size is exponential in the size of Act.*

Remark 5. *We notice that we allow for larger languages than the ones proposed in [6, 7], e.g., we allow for the use of $\wedge$ and $\vee$ as outermost operator. However, it is not difficult to see that our logics do not distinguish more processes than the original ones defined in [6, 7]. We need such extended languages because we use $\vee$ as outermost operator in order to define the formula $\bar{\chi}$ (in Lemma 4) and we need $\wedge$ to apply Corollary 4 (in the proof of Theorem 7).*

The satisfaction relation and semantics associated to logics for the linear time semantics are the same as the ones for logics for branching time semantics (see Definition 8).

The following well-known theorem is the linear time counterpart of Theorem 4 for branching time semantics. It states the relationship between logics and process semantics that allows us to use our general results about logically characterized semantics.

Theorem 6 (Logical characterization of linear time semantics [6, 7])**.** *For each $X \in \mathsf{Ltime\text{-}spectrum}$ and for all $p, q \in P$, $p \lesssim_X q$ iff $\mathcal{L}_X(p) \subseteq \mathcal{L}_X(q)$.*

We proceed now to prove our characterization by primality result for the above defined logics. Following the same approach adopted in the previous section for branching time semantics, we show that also the logics considered in this section meet the conditions of Corollary 4, that is, they are finitely characterized by some monotonic $\mathcal{B}$, and for each $\chi(p)$ there exists a formula $\bar{\chi}(p)$ such that $p \notin [\![\bar{\chi}(p)]\!]$ and $\{q \in P \mid \mathcal{L}(q) \not\subseteq \mathcal{L}(p)\} \subseteq [\![\bar{\chi}(p)]\!]$.

An elucidation is in order: even though, according to Corollary 4, logics are required to feature (arbitrary) disjunction, we have already observed (Remark 1 at page 11) that the Boolean connective $\wedge$ plays a minor role in (the proof of) Corollary 4, and thus we can use it to deal with the logics studied in this section, that allow for a limited use of such a connective.

In order to finitely characterize the logics for the semantics in the linear time spectrum, we base our definition of $\mathcal{B}_X(p)$ on suitably defined finite versions $X^{fin}(p)$ of sets $X(p)$ (with $X \in \mathsf{Ltime\text{-}spectrum}$ and $p \in P$) that carry all

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$X{=}\mathsf{T}$	$\mathsf{T}^{fin}(p) = \mathsf{T}(p)$ $\qquad$ (as $\mathsf{T}(p)$ is already finite) $formula_{\mathsf{T}}(\tau) = \langle\tau\rangle\mathbf{tt}$
$X{=}\mathsf{CT}$	$\mathsf{CT}^{fin}(p) = \mathsf{CT}(p)$ $\qquad$ (as $\mathsf{CT}(p)$ is already finite) $formula_{\mathsf{CT}}(\tau) = \langle\tau\rangle\mathbf{0}$
$X{=}\mathsf{F}$	$\mathsf{F}^{fin}(p) = \mathsf{F}(p)$ $\qquad$ (as $\mathsf{F}(p)$ is already finite) $formula_{\mathsf{F}}(\langle\tau,Y\rangle) = \langle\tau\rangle\bigwedge_{a\in Y}[a]\mathbf{ff}$
$X{=}\mathsf{R}$	$\mathsf{R}^{fin}(p) = \mathsf{R}(p)$ $\qquad$ (as $\mathsf{R}(p)$ is already finite) $formula_{\mathsf{R}}(\langle\tau,Y\rangle) = \langle\tau\rangle\langle\!\langle Y\rangle\!\rangle$
$X{=}\mathsf{FT}$	$\mathsf{FT}^{fin}(p) = \mathsf{FT}(p)_{\leq 2\cdot depth(p)+1}$ $formula_{\mathsf{FT}}(\sigma) = \begin{cases} \mathbf{tt} & \sigma = \varepsilon \\ \langle a\rangle formula_{\mathsf{FT}}(\sigma') & \sigma = a\sigma', a\in Act, \sigma' \in (Act\cup\mathcal{P}(Act))^* \\ \bigwedge_{a\in Y}[a]\mathbf{ff}\wedge formula_{\mathsf{FT}}(\sigma') & \sigma = Y\sigma', Y\in\mathcal{P}(Act), \sigma'\in(Act\cup\mathcal{P}(Act))^* \end{cases}$
$X{=}\mathsf{RT}$	$\mathsf{RT}^{fin}(p) = \mathsf{RT}(p)_{\leq 2\cdot depth(p)+1}$ $formula_{\mathsf{RT}}(\sigma) = \begin{cases} \mathbf{tt} & \sigma = \varepsilon \\ \langle a\rangle formula_{\mathsf{RT}}(\sigma') & \sigma = a\sigma', a\in Act, \sigma' \in (Act\cup\mathcal{P}(Act))^* \\ \langle\!\langle Y\rangle\!\rangle\wedge formula_{\mathsf{RT}}(\sigma') & \sigma = Y\sigma', Y\in\mathcal{P}(Act), \sigma'\in(Act\cup\mathcal{P}(Act))^* \end{cases}$
$X{=}\mathsf{IF}$	$\mathsf{IF}^{fin}(p) = \mathsf{IF}(p)\cap(Act^*_{\leq depth(p)+1}\times\mathcal{P}(Act^*_{\leq depth(p)+1}))$ $formula_{\mathsf{IF}}(\langle\tau,\Gamma\rangle) = \langle\tau\rangle\bigwedge_{\tau'\in\Gamma}[\tau']\mathbf{ff}$
$X{=}\mathsf{PF}$	$\mathsf{PF}^{fin}(p) = \mathsf{PF}(p)$ $\qquad$ (as $\mathsf{PF}(p)$ is already finite) $formula_{\mathsf{PF}}(\langle\tau,\Gamma\rangle) = \langle\tau\rangle\langle\!\langle\Gamma\rangle\!\rangle$
$X{=}\mathsf{IFT}$	$\mathsf{IFT}^{fin}(p) = \mathsf{IFT}(p)\cap[(Act\cup\mathcal{P}(Act^*_{\leq depth(p)+1}))^*]_{\leq 2\cdot depth(p)+1}$ $formula_{\mathsf{IFT}}(\sigma) = \begin{cases} \mathbf{tt} & \sigma = \varepsilon \\ \langle a\rangle formula_{\mathsf{IFT}}(\sigma') & \sigma = a\sigma', a\in Act, \sigma' \in (Act\cup\mathcal{P}(Act^*))^* \\ \bigwedge_{\tau\in\Gamma}[\tau]\mathbf{ff}\wedge formula_{\mathsf{IFT}}(\sigma') & \sigma = \Gamma\sigma', \Gamma\in\mathcal{P}(Act^*), \sigma'\in(Act\cup\mathcal{P}(Act^*))^* \end{cases}$
$X{=}\mathsf{PFT}$	$\mathsf{PFT}^{fin}(p) = \mathsf{PFT}(p)_{\leq 2\cdot depth(p)+1}$ $formula_{\mathsf{PFT}}(\sigma) = \begin{cases} \mathbf{tt} & \sigma = \varepsilon \\ \langle a\rangle formula_{\mathsf{PFT}}(\sigma') & \sigma = a\sigma', a\in Act, \sigma' \in (Act\cup\mathcal{P}(Act^*))^* \\ \langle\!\langle\Gamma\rangle\!\rangle\wedge formula_{\mathsf{PFT}}(\sigma') & \sigma = \Gamma\sigma', \Gamma\in\mathcal{P}(Act^*), \sigma'\in(Act\cup\mathcal{P}(Act^*))^* \end{cases}$
$X{=}\mathsf{I2}$	$\mathsf{I2}^{fin}(p) = \mathsf{I2}(p)\cap(Act^*_{\leq depth(p)+1}\times\mathcal{P}([P_{\leq depth(p)+1}]_{\mathrm{BS}}))$ $formula_{\mathsf{I2}}(\langle\tau,\mathbb{P}\rangle) = \langle\tau\rangle\bigwedge_{[p']_{\mathrm{BS}}\in\mathbb{P}}\neg\chi_{\mathsf{S}}(p')$
$X{=}\mathsf{P2}$	$\mathsf{P2}^{fin}(p) = \mathsf{P2}(p)$ $\qquad$ (as $\mathsf{P2}(p)$ is already finite) $formula_{\mathsf{P2}}(\langle\tau,\mathbb{P}\rangle) = \langle\tau\rangle\alpha(\mathbb{P})$ $\quad$ with $\alpha(\mathbb{P})\quad=\bigwedge_{[p']_{\mathrm{BS}}\in\mathbb{P}}\chi_{\mathsf{S}}(p')\wedge\bigvee_{[p']_{\mathrm{BS}}\in\mathbb{P}} simulated\text{-}by(p')$ $\qquad\qquad\qquad\qquad\qquad\qquad$ and $simulated\text{-}by(p')=\bigwedge_{a\in Act}[a]\bigvee_{p'\xrightarrow{a}p''} simulated\text{-}by(p'')$
$X{=}\mathsf{I2T}$	$\mathsf{I2T}^{fin}(p) = \mathsf{I2T}(p)\cap[(Act\cup\mathcal{P}([P_{\leq depth(p)+1}]_{\mathrm{BS}}))^*]_{\leq 2\cdot depth(p)+1}$ $formula_{\mathsf{I2T}}(\sigma) = \begin{cases} \mathbf{tt} & \sigma = \varepsilon \\ \langle a\rangle formula_{\mathsf{I2T}}(\sigma') & \sigma = a\sigma', a\in Act, \sigma' \in (Act\cup\mathcal{P}([P]_{\mathrm{BS}}))^* \\ \bigwedge_{[p]_{\mathrm{BS}}\in\mathbb{P}}\neg\chi_{\mathsf{S}}(p)\wedge formula_{\mathsf{I2T}}(\sigma') & \sigma = \mathbb{P}\sigma', \mathbb{P}\in\mathcal{P}([P]_{\mathrm{BS}}), \sigma'\in(Act\cup\mathcal{P}([P]_{\mathrm{BS}}))^* \end{cases}$
$X{=}\mathsf{P2T}$	$\mathsf{P2T}^{fin}(p) = \mathsf{P2T}(p)_{\leq 2\cdot depth(p)+1}$ $formula_{\mathsf{P2T}}(\sigma) = \begin{cases} \mathbf{tt} & \sigma = \varepsilon \\ \langle a\rangle formula_{\mathsf{P2T}}(\sigma') & \sigma = a\sigma', a\in Act, \sigma' \in (Act\cup\mathcal{P}([P]_{\mathrm{BS}}))^* \\ \alpha(\mathbb{P})\wedge formula_{\mathsf{P2T}}(\sigma') & \sigma = \mathbb{P}\sigma', \mathbb{P}\in\mathcal{P}([P]_{\mathrm{BS}}), \sigma'\in(Act\cup\mathcal{P}([P]_{\mathrm{BS}}))^* \end{cases}$

Table 2: Instantiations of $X^{fin}(\cdot)$ and $formula_X(\cdot)$ for all $X\in$ Ltime-spectrum. For a set S and $n\in\mathbb{N}$, we let $S_{\leq n} = \{x\in S\mid |x|\leq n\}$ (we adopt the convention that $|x| = depth(x)$, for $x\in P$).

significant pieces of information, in the sense of Proposition 16. Thus, we define
$\mathcal{B}_X$ as follows: for all $X\in$ Ltime-spectrum and $p\in P$

$$\mathcal{B}_X(p) = \{\mathbf{tt}\}\cup\{formula_X(x)\mid x\in X^{fin}(p)\}$$

where $formula_X(x)$ and $X^{fin}(p)$ are instantiated as in Table 2. Intuitively, $formula_X(x)$ is a logical characterization of a process observation x according to semantics X, as stated by Proposition 17 below. Thus, $\mathcal{B}_X(p)$ contains, besides $\mathbf{tt}$, a formula for each element of $X^{fin}(p)$, meaning that its finiteness immediately follows from the one of $X^{fin}(p)$. The construction of the finite sets $X^{fin}(p)$ uses the following notation: $S_{\sim n} = \{x\in S\mid |x|\sim n\}$ for every set

S, $n \in \mathbb{N}$, and $\sim \in \{\leq, =\}$ (we adopt the convention that $|x| = depth(x)$, for $x \in P$), and is based on the following considerations:

(a) Act is finite;

(b) the set of traces that a process can perform is finite;

(c) even though the set of traces that a process p cannot perform is infinite, it is characterized by the finite set of minimal traces that p cannot perform, and the length of the longest trace in such a characterizing set is $depth(p)+$ 1 (e.g., if a process p cannot perform a, then it clearly cannot perform any trace that starts with a; even if there are infinitely many such traces, it is enough to keep track of the fact that a is a minimal trace that cannot be performed by p); thus, using the notation introduced above, we can focus on $Act^*_{\leq depth(p)+1} = \{\tau \in Act^* \mid |\tau| \leq depth(p) + 1\}$ (rather than full Act^*) when we need to characterize traces that p can or cannot do;

(d) even though the set of processes that are simulated by a process p is infinite, it is actually finite up to bisimilarity (i.e., $p_{\downarrow s}$ is finite);

(e) the set of processes that are not simulated by a process p is infinite; however, using an argument similar to the one from item (c), it is possible to identify a finite (up to bisimilarity) set of minimal (wrt. depth) processes that characterizes it; for a process p, such a set is defined as

$$P_{\leq depth(p)+1} = \{p' \in P \mid depth(p') \leq depth(p) + 1\};$$

(f) as already observed in Remark 3, as far as sets $\mathsf{FT}(p)$, $\mathsf{RT}(p)$, $\mathsf{IFT}(p)$, $\mathsf{PFT}(p)$, $\mathsf{I2T}(p)$, and $\mathsf{P2T}(p)$ are concerned, we can restrict ourselves to considering words whose length is bounded by $2 \cdot depth(p) + 1$.

The following two results follow from the definition of the semantics X and of $formula_X(x)$ in Table 2, and their proofs are omitted.

Proposition 16. *For all $X \in$ Ltime-spectrum and $p, p' \in P$, the following statements are equivalent:*

(a) $X(p) \subseteq X(p')$,

(b) $X^{fin}(p) \subseteq X^{fin}(p')$, and

(c) $X^{fin}(p) \subseteq X(p')$.

Proposition 17. *For all $X \in$ Ltime-spectrum, $p \in P$, and $x \in \bigcup_{q \in P} X^{fin}(q)$, we have:*

$$x \in X(p) \text{ if and only if } p \in \llbracket formula_X(x) \rrbracket.$$

Lemma 3. *Let $X \in$ Ltime-spectrum. $\mathcal{L}_X$ is finitely characterized by $\mathcal{B}_X$, for some monotonic $\mathcal{B}_X$.*

Proof. For every $X \in$ Ltime-spectrum and $p \in P$, we show that,

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(i) $\emptyset \subset \mathcal{B}_X(p) \subseteq \mathcal{L}_X(p)$,

(ii) for each $\phi \in \mathcal{L}_X(p)$, it holds that $\bigcap_{\psi \in \mathcal{B}_X(p)} [\![\psi]\!] \subseteq [\![\phi]\!]$,

(iii) $\mathcal{B}_X(p)$ is finite, and

(iv) for each $q \in P$, if $\mathcal{L}_X(p) \subseteq \mathcal{L}_X(q)$ then $\mathcal{B}_X(p) \subseteq \mathcal{B}_X(q)$.

Let $p \in P$ and $X \in$ Ltime-spectrum.

Property (iii) immediately follows from the definition of $\mathcal{B}_X(p)$, since the set $X^{fin}(p)$ is finite.

As for property (i), we observe that $\mathbf{tt} \in \mathcal{B}_X(p)$, and thus it suffices to prove $\mathcal{B}_X(p) \subseteq \mathcal{L}_X(p)$. To this end, we need to show that $formula_X(x) \in \mathcal{L}_X(p)$ for every $x \in X^{fin}(p)$. Intuitively, this is very easy to see: an element x of $X^{fin}(p)$ carries information on how p can/cannot evolve; the formula $formula_X(x)$ express exactly the same information about p, and thus it is satisfied by p. We refer the reader to Appendix B, Lemma 6, for a formal proof.

Property (ii) follows from the observation that $\bigwedge_{\psi \in \mathcal{B}(p)} \psi$ is characteristic for p within $\mathcal{L}_X$ (a detailed proof of this claim is given in Appendix B, Lemma 7). Indeed, let $\phi \in \mathcal{L}_X(p)$. By the definition of characteristic formula (Definition 1), we have $[\![\bigwedge_{\psi \in \mathcal{B}_X(p)} \psi]\!] = \{q \in P \mid \mathcal{L}_X(p) \subseteq \mathcal{L}_X(q)\}$, and thus we have $\bigcap_{\psi \in \mathcal{B}_X(p)} [\![\psi]\!] = [\![\bigwedge_{\psi \in \mathcal{B}_X(p)} \psi]\!] = \{q \in P \mid \mathcal{L}_X(p) \subseteq \mathcal{L}_X(q)\} \subseteq [\![\phi]\!]$ since $[\![\phi]\!]$ is upwards closed by Proposition 2(i).

Finally, in order to prove property (iv), let $q \in P$ be such that $\mathcal{L}_X(p) \subseteq \mathcal{L}_X(q)$. By Theorem 6, we have $p \lesssim_X q$. By Corollary 5 in Appendix B, we know that $depth(p) \leq depth(q)$. Using this property, it is easy to see that $X(p) \subseteq X(q)$ implies $X^{fin}(p) \subseteq X^{fin}(q)$. Then, we have $p \lesssim_X q \Leftrightarrow X(p) \subseteq X(q) \Rightarrow X^{fin}(p) \subseteq X^{fin}(q) \Rightarrow \mathcal{B}_X(p) \subseteq \mathcal{B}_X(q)$. $\qquad\square$

Lemma 4. *Let $X \in$ Ltime-spectrum. For each $p \in P$ and $\chi_X(p)$ characteristic within $\mathcal{L}_X$ for p, there exists a formula in $\mathcal{L}_X$, denoted by $\bar{\chi}_X(p)$, such that (i) $p \notin [\![\bar{\chi}_X(p)]\!]$ and (ii) $\{p' \in P \mid p' \not\lesssim_X p\} \subseteq [\![\bar{\chi}_X(p)]\!]$.*

Proof. First, we define, for $X \in$ Ltime-spectrum,
$$\bar{X}^{fin}(p) = \left(\bigcup_{[p']_{\mathsf{BS}} \in [P_{\leq depth(p)}]_{\mathsf{BS}}} X^{fin}(p') \right) \setminus X(p),$$
and then we define $\bar{\chi}_X$ as follows: for each $p \in P$
$$\bar{\chi}_X(p) = \bigvee_{x \in \bar{X}^{fin}(p)} formula_X(x) \vee \bigvee_{\tau \in Act^*_{=depth(p)+1}} \langle\tau\rangle\mathbf{tt}$$
Let $p \in P$ and $X \in$ Ltime-spectrum. We have to show that (i) $p \notin [\![\bar{\chi}_X(p)]\!]$ and (ii) $\{p' \in P \mid p' \not\lesssim_X p\} \subseteq [\![\bar{\chi}_X(p)]\!]$.

In order to prove (i), assume, towards a contradiction, that $p \in [\![\bar{\chi}_X(p)]\!]$. Then, we distinguish two possibilities:

1. if $p \in [\![\bigvee_{\tau \in Act^*_{=depth(p)+1}} \langle\tau\rangle\mathbf{tt}]\!]$, then there is $\tau \in \mathsf{T}(p)$ whose length is greater than the depth of p, which is a contradiction;

2. if $p \in [\![formula_X(x)]\!]$ for some $x \in \bar{X}^{fin}(p)$, then, by Proposition 17, we have that $x \in X(p)$, which is in contradiction with x being an element of $\bar{X}^{fin}(p)$.

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In order to prove (ii), let $p' \in P$ be such that $p' \not\lesssim_X p$. If $depth(p') >$ $depth(p)$, then there is a trace $\tau \in \mathsf{T}(p')$ with length $depth(p) + 1$, and thus $p' \in \llbracket \bigvee_{\tau \in Act^*_{=depth(p)+1}} \langle \tau \rangle \mathbf{tt} \rrbracket \subseteq \llbracket \bar{\chi}_X(p) \rrbracket$. Otherwise $(depth(p') \leq depth(p))$, we proceed as follows. By Definition 9, $X(p') \not\subseteq X(p)$, and, by Proposition 16 (not $(a) \Rightarrow$ not (c)), we have $X^{fin}(p') \not\subseteq X(p)$. Thus, there is some $x \in X^{fin}(p') \setminus X(p)$, which means that $x \in \bar{X}^{fin}(p)$. By Proposition 17, $x \in X^{fin}(p') \subseteq X(p')$ implies $p' \in \llbracket formula_X(x) \rrbracket$ and, since $x \in \bar{X}^{fin}(p)$, we have $p' \in \llbracket \bar{\chi}_X(p) \rrbracket$. $\qquad \square$

Finally, the following theorem states the main result of this section.

Theorem 7 (Characterization by primality for the linear time spectrum). *Let* $X \in$ Ltime-spectrum *and* $\phi \in \mathcal{L}_X$. *Then,* ϕ *is consistent and prime if and only if* ϕ *is characteristic for some* $p \in P$.

Proof. The claim immediately follows from Theorem 1, Theorem 2, Corollary 4, Lemma 3, and Lemma 4. $\qquad \square$

6. Conclusions and future directions

In this paper, we have provided general sufficient conditions guaranteeing that formulae for which model checking can be reduced to equivalence/preorder checking, that is, the characteristic formulae, are exactly the consistent and prime ones. We have applied our framework to show that characteristic formulae are exactly the consistent and prime ones when the set of processes is finite, as well as for modal refinement semantics [18], covariant-contravariant semantics [19] (the result was known for these last two semantics), and all the semantics in van Glabbeek's spectrum [7] and those considered in [6]. Our results indicate that the "characterization by primality" result, first proved by Boudol and Larsen [18] in the context of the modal logic that characterizes modal refinement over modal transition systems, holds in a wide variety of settings in concurrency theory. We feel, therefore, that this study reinforces the view that there is a very close connection between the behavioural and logical view of processes: not only do the logics characterize processes up to the chosen notion of behavioural relation, but processes characterize all the prime and consistent formulae.

6.1. Applying the theory to conformance simulation

As a future work, we would like to provide a characterization by primality for the logics characterizing *conformance simulation* (C) [20]. Here, we give evidence (Proposition 18 below) that Corollary 4 cannot be used to show the decomposability of the corresponding logic $\mathcal{L}_C$ (defined below). However, we are confident that the alternative path to decomposability we provide (through Proposition 12) might serve the purpose.

Conformance simulation [20] is defined as the largest relation $\lesssim_C$ satisfying:
$$p \lesssim_C q \quad \Leftrightarrow \quad I(p) \subseteq I(q) \text{ and } \quad \text{for all } q', a \text{ such that } q \xrightarrow{a} q' \text{ and } p \xrightarrow{a}$$
$$\text{there exists } p' \text{ such that } p \xrightarrow{a} p' \text{ and } p' \lesssim_C q'.$$

The logic $\mathcal{L}_\mathsf{C}$, characterizing conformance simulation, is interpreted over set of processes P, defined as in Section 5 on page 14. Its language includes the Boolean constants **tt** and **ff**, the Boolean connectives $\wedge$ and $\vee$, as well as the modality (a) ($a \in Act$), whose semantic interpretation is as follows:

$$p \in [\![(a)\phi]\!] \text{ iff } p \xrightarrow{a} \text{ and } p' \in [\![\phi]\!] \text{ for all } p' \text{ such that } p \xrightarrow{a} p'.$$

The semantic clauses for the other operators are given in Definition 8.

Analogously to the case of the semantics in the linear time-branching time spectrum, the logic $\mathcal{L}_\mathsf{C}$ captures exactly conformance simulation, as stated by the following result.
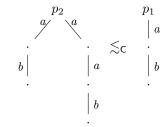
Theorem 8 (Logical characterization of conformance simulation [20]). *For all* $p, q \in P$, $p \lesssim_\mathsf{C} q$ *iff* $\mathcal{L}_\mathsf{C}(p) \subseteq \mathcal{L}_\mathsf{C}(q)$.

The next result shows that Corollary 4 cannot be used to show the decomposability of logic $\mathcal{L}_\mathsf{C}$ and thus to prove its characterization by primality.

Proposition 18. $\mathcal{L}_\mathsf{C}$ *is not finitely characterized by* $\mathcal{B}$*, for any monotonic* $\mathcal{B}$ *(see Definition 5).*

Proof. Suppose, towards a contradiction, that $\mathcal{L}_\mathsf{C}$ is finitely characterized by $\mathcal{B}$, for some monotonic $\mathcal{B}$, and let us define the following processes, for $k > 0$: $p_k = \sum_{i=1}^{k} a^i b0$.

Since $I(p_k) = I(p_{k+1})$ and p_k is a subtree of p_{k+1}, it is clear that $p_{k+1} \lesssim_\mathsf{C} p_k$ holds, for all k (see the picture below to verify the claim for $k = 1$, i.e., $p_2 \lesssim_\mathsf{C} p_1$).

On the other hand, $p_k \not\lesssim_\mathsf{C} p_{k+1}$, because the branch $a^{k+1}b0$ cannot be matched by any branch in p_k, where the longest sequence of consecutive a's is a^k. Thus, we have a non-well-founded sequence of strictly decreasing processes $\ldots \lesssim_\mathsf{C} p_{k+1} \lesssim_\mathsf{C} p_k \ldots \lesssim_\mathsf{C} p_1$.

For all k, $p_{k+1} \lesssim_\mathsf{C} p_k$ implies $\mathcal{L}_\mathsf{C}(p_{k+1}) \subset \mathcal{L}_\mathsf{C}(p_k)$, which, by monotonicity of $\mathcal{B}$ (Definition 5), in turn implies $\mathcal{B}(p_{k+1}) \subset \mathcal{B}(p_k)$. Thus, there exist infinitely many formulae $\psi_1, \psi_2, \ldots, \psi_k, \ldots$ such that $\psi_k \in \mathcal{B}(p_k) \setminus \mathcal{B}(p_{k+1})$ for all k. We show that the formulae $\{\psi_k\}_{k \geq 1}$ are pairwise different, that is, $\psi_k \neq \psi_j$ for each $k \neq j$. To this end, let us suppose, towards a contradiction, that there exist j, k such that $j < k$ and $\psi_k = \psi_j$. By construction, we have that $\psi_j \in \mathcal{B}(p_j) \setminus \mathcal{B}(p_{j+1})$ and $\psi_k = \psi_j \in \mathcal{B}(p_k) \setminus \mathcal{B}(p_{k+1})$. Since $j + 1 \leq k$, we have that $\mathcal{B}(p_k) \subseteq \mathcal{B}(p_{j+1})$, which means that $\psi_k = \psi_j \in \mathcal{B}(p_{j+1})$, leading to a contradiction.

Therefore the sequence $\{\psi_k\}_{k \geq 1}$ includes infinitely many different formulae. Since $\mathcal{B}(p_1)$ contains them all, the hypothesis of finiteness of $\mathcal{B}(p_1)$ is contradicted, and the thesis follows. $\qquad\square$

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Appendix A. Proofs for semantics in van Glabbeek's branching time spectrum

Appendix A.1. Finite characterization

We prove here that logics in van Glabbeek's branching-time spectrum are finitely characterized by some monotonic $\mathcal{B}$ (condition (i) in Corollary 4).

Lemma 1. *Let $X \in$ Btime-spectrum. $\mathcal{L}_X$ is finitely characterized by $\mathcal{B}_X$, for some monotonic $\mathcal{B}_X$ (see Table 1).*

Proof. We consider each semantics $X \in$ Btime-spectrum and we show that $\mathcal{B}_X$ (as defined in Table 1) is such that for each $p \in P$:

(i) $\emptyset \subset \mathcal{B}_X(p) \subseteq \mathcal{L}_X(p)$,

(ii) for each $\phi \in \mathcal{L}_X(p)$, it holds $\bigcap_{\psi \in \mathcal{B}_X(p)} \llbracket \psi \rrbracket \subseteq \llbracket \phi \rrbracket$,

(iii) $\mathcal{B}_X(p)$ is finite, and

(iv) for each $q \in P$, if $\mathcal{L}_X(p) \subseteq \mathcal{L}_X(q)$ then $\mathcal{B}_X(p) \subseteq \mathcal{B}_X(q)$.

Before considering each semantics separately, we notice that, for every $X \in$ Btime-spectrum, we have that $\mathbf{tt} \in \mathcal{B}_X(p)$, and thus $\emptyset \subset \mathcal{B}_X(p)$. Consequently, we can focus on showing that $\mathcal{B}_X(p) \subseteq \mathcal{L}_X(p)$ holds when proving property (i).

In addition, we find it convenient to partially factorize the proofs of properties (i) and (iii). To this end, we show that, for every $X \in$ Btime-spectrum,

$$\text{if } \mathcal{B}_X^-(p) \subseteq \mathcal{L}_X(p) \text{ for all } p \in P, \text{ then } \mathcal{B}_X(p) \subseteq \mathcal{L}_X(p) \text{ for all } p \in P, \text{ and} \quad (i')$$

$$\text{if } \mathcal{B}_X^-(p) \text{ is finite for all } p \in P, \text{ then } \mathcal{B}_X(p) \text{ is finite for all } p \in P. \quad (iii')$$

Let $X \in$ Btime-spectrum.

First, let us assume that $\mathcal{B}_X^-(p) \subseteq \mathcal{L}_X(p)$ holds for all $p \in P$; we show that $\mathcal{B}_X(p) \subseteq \mathcal{L}_X(p)$ holds for all $p \in P$ as well, by induction on the depth of p. When $I(p) = \emptyset$ (base case), we have $\mathcal{B}_X^+(p) = \{\mathbf{tt}\} \subseteq \mathcal{L}_X(p)$, and therefore $\mathcal{B}_X(p) \subseteq \mathcal{L}_X(p)$ holds as well. To deal with the inductive step ($I(p) \neq \emptyset$), let $\phi \in \mathcal{B}_X^+(p)$. If $\phi = \mathbf{tt}$, then $\phi \in \mathcal{L}_X(p)$, and we are done. Assume $\phi = \langle a \rangle \bigwedge_{\psi \in \Psi} \psi$, where $a \in Act$ and $\Psi \subseteq \mathcal{B}_X(p')$ for some p' such that $p \xrightarrow{a} p'$. By the inductive hypothesis, we have that $\mathcal{B}_X(p') \subseteq \mathcal{L}_X(p')$, meaning that $p' \in \llbracket \psi \rrbracket$ for all $\psi \in \Psi$. Since $p \xrightarrow{a} p'$, we have that $p \in \llbracket \langle a \rangle \bigwedge_{\psi \in \Psi} \psi \rrbracket$, which amounts to $\phi \in \mathcal{L}_X(p)$.

Now, let us assume $\mathcal{B}_X^-(p)$ to be finite for every $p \in P$; we show that also $\mathcal{B}_X(p)$ is finite for every $p \in P$, by induction on the depth of p. When $I(p) = \emptyset$ (base case), $\mathcal{B}_X(p) = \{\mathbf{tt}\} \cup \mathcal{B}_X^-(p)$, which is clearly finite. Let us deal now with the inductive step ($I(p) \neq \emptyset$). By the construction of $\mathcal{B}_X^+(p)$, a formula belongs to $\mathcal{B}_X^+(p)$ if, and only if, it is either $\mathbf{tt}$ or $\langle a \rangle \varphi$, where $a \in Act$ and $\varphi = \bigwedge_{\psi \in \Psi} \psi$, for some $\Psi \subseteq \mathcal{B}_X(p')$ and some p' such that $p \xrightarrow{a} p'$. By the inductive hypothesis, $\mathcal{B}_X(p')$ is finite. Since Act is also finite and processes are finitely branching, there are only finitely many such formulae $\langle a \rangle \varphi$, meaning that $\mathcal{B}_X^+(p)$ is finite. Therefore $\mathcal{B}_X(p) = \mathcal{B}_X^+(p) \cup \mathcal{B}_X^-$ is finite as well.

As a consequence of (i'), showing that $\mathcal{B}_X^-(p) \subseteq \mathcal{L}_X(p)$ holds for all $p \in P$ is enough to prove property (i); similarly, thanks to (iii'), showing finiteness of $\mathcal{B}^-(p)$ for all $p \in P$ is enough to prove property (iii). We will use (i') and (iii') to prove, respectively, properties (i) and (iii) for all semantics.

To keep the notation light, we often omit the subscripts identifying the semantics as it is clear from the context, e.g., when working out the case of simulation semantics, we write $\mathcal{L}$, $\mathcal{B}$, and $p \lesssim q$ instead of $\mathcal{L}_\mathsf{S}$, $\mathcal{B}_\mathsf{S}$, and $p \lesssim_\mathsf{S} q$, respectively, and similarly for the other semantics.

Case simulation (S). For the sake of clarity we recall from Table 1 that $\mathcal{B}$ is defined as
$$\mathcal{B}(p) = \{\mathbf{tt}\} \cup \{\langle a \rangle \varphi \mid a \in Act, \varphi = \bigwedge_{\psi \in \Psi} \psi, \Psi \subseteq \mathcal{B}(p'), p \xrightarrow{a} p'\}.$$

Since $\mathcal{B}^-(p) = \emptyset$ for all $p \in P$, properties (i) and (iii) immediately follow from (i') and (iii'), respectively. Notice also that property (iii) implies that $\mathcal{B}$ is well defined.

In order to prove property (ii), we let $\phi \in \mathcal{L}(p)$, for a generic $p \in P$, and we proceed by induction on the structure of ϕ (notice that we can ignore the case $\phi = \mathbf{ff}$, as $\phi \in \mathcal{L}(p)$ implies $\phi \neq \mathbf{ff}$).

- $\phi = \mathbf{tt}$: the claim follows trivially.

- $\phi = \varphi_1 \vee \varphi_2$: it holds that $\varphi_i \in \mathcal{L}(p)$ for some $i \in \{1, 2\}$. By the inductive hypothesis, we have that $\bigcap_{\psi \in \mathcal{B}(p)} [\![\psi]\!] \subseteq [\![\varphi_i]\!]$ and, since $[\![\varphi_i]\!] \subseteq [\![\phi]\!]$, we obtain the claim.

- $\phi = \varphi_1 \wedge \varphi_2$: it holds that $\varphi_i \in \mathcal{L}(p)$ for all $i \in \{1, 2\}$. By the inductive hypothesis, we have that $\bigcap_{\psi \in \mathcal{B}(p)} [\![\psi]\!] \subseteq [\![\varphi_i]\!]$ for all $i \in \{1, 2\}$. This implies that $\bigcap_{\psi \in \mathcal{B}(p)} [\![\psi]\!] \subseteq [\![\varphi_1]\!] \cap [\![\varphi_2]\!] = [\![\phi]\!]$.

- $\phi = \langle a \rangle \varphi$: by definition we have that $\varphi \in \mathcal{L}(p')$ for some $p \xrightarrow{a} p'$. By the inductive hypothesis, we have that $\bigcap_{\psi \in \mathcal{B}(p')} [\![\psi]\!] \subseteq [\![\varphi]\!]$. We define $\zeta = \langle a \rangle \bigwedge_{\psi \in \mathcal{B}(p')} \psi$. Clearly, ζ belongs to $\mathcal{B}^+(p)$ (by construction—notice that ζ is well defined due to the finiteness of $\mathcal{B}(p')$) and $[\![\zeta]\!] \subseteq [\![\phi]\!]$ (because $\bigcap_{\psi \in \mathcal{B}(p')} [\![\psi]\!] \subseteq [\![\varphi]\!]$). Hence, $\bigcap_{\psi \in \mathcal{B}(p)} [\![\psi]\!] \subseteq [\![\zeta]\!] \subseteq [\![\phi]\!]$ holds.

Finally, we show that $\mathcal{B}$ is monotonic (property (iv)). Consider $p, q \in P$, with $\mathcal{L}(p) \subseteq \mathcal{L}(q)$. We want to show that $\phi \in \mathcal{B}(p)$ implies $\phi \in \mathcal{B}(q)$, for each ϕ. Firstly, we observe that, by $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ and Theorem 4, $p \lesssim q$ holds. Thus, for each $a \in Act$ and $p' \in P$ with $p \xrightarrow{a} p'$, there exists some $q' \in P$ such that $q \xrightarrow{a} q'$ and $p' \lesssim q'$. We also observe that $\mathcal{B}^-(p) = \mathcal{B}^-(q) = \emptyset$. In order to show that $\mathcal{B}(p) \subseteq \mathcal{B}(q)$, we proceed by induction on the depth of p. If $I(p) = \emptyset$, then $\mathcal{B}^+(p) = \{\mathbf{tt}\} \subseteq \mathcal{B}^+(q)$, and the thesis follows. Otherwise $(I(p) \neq \emptyset)$, let us consider a formula $\phi \in \mathcal{B}(p)$. If $\phi \in \mathcal{B}^-(p)$, then the claim follows from $\mathcal{B}^-(p) = \mathcal{B}^-(q) \subseteq \mathcal{B}(q)$. If $\phi = \mathbf{tt}$, then, by definition of $\mathcal{B}^+$, we have $\phi \in \mathcal{B}^+(q) \subseteq \mathcal{B}(q)$. Finally, if $\phi = \langle a \rangle \varphi \in \mathcal{B}^+(p)$, then, by definition of $\mathcal{B}^+$, there exist $p' \in P$, with $p \xrightarrow{a} p'$, such that $\varphi = \bigwedge_{\psi \in \Psi} \psi$ for some $\Psi \subseteq \mathcal{B}(p')$. This implies the existence of some $q' \in P$ such that $q \xrightarrow{a} q'$ and $p' \lesssim q'$ (and therefore

$\mathcal{L}(p') \subseteq \mathcal{L}(q')$ by Theorem 4). By the inductive hypothesis, $\mathcal{B}(p') \subseteq \mathcal{B}(q')$ holds as well, which means that $\Psi \subseteq \mathcal{B}(q')$. Hence, we have that $\langle a \rangle \varphi \in \mathcal{B}^+(q) \subseteq \mathcal{B}(q)$.

Case complete simulation (CS). For the sake of clarity we recall from Table 1 that $\mathcal{B}$ is defined as $\mathcal{B}(p) = \mathcal{B}^+(p) \cup \mathcal{B}^-(p)$, where

- $\mathcal{B}^+(p) = \{\mathbf{tt}\} \cup \{\langle a \rangle \varphi \mid a \in Act, \varphi = \bigwedge_{\psi \in \Psi} \psi, \Psi \subseteq \mathcal{B}(p'), p \xrightarrow{a} p'\}$, and

- $\mathcal{B}^-(p) = \{\mathbf{0} \mid p \xrightarrow{a}\!\!\!\!\!/\,, \forall a \in Act\}$.

For all $p \in P$, if $I(p) = \emptyset$, then $\mathcal{B}^-(p) = \{\mathbf{0}\}$ and $\mathbf{0} \in \mathcal{L}(p)$, otherwise $\mathcal{B}^-(p) = \emptyset$; thus, $\mathcal{B}^-(p)$ is finite and $\mathcal{B}^-(p) \subseteq \mathcal{L}(p)$. Hence, properties ($i$) and ($iii$) immediately follow from (i') and (iii'), respectively. Notice also that property (iii) implies that $\mathcal{B}$ is well defined.

In order to prove property (ii), we let $\phi \in \mathcal{L}(p)$, for a generic $p \in P$, and we proceed by induction on the structure of ϕ. Apart from a new base case ($\phi = \mathbf{0}$, which we deal with below), the proof is the same as in the case of simulation semantics above.

- $\phi = \mathbf{0}$: it is enough to observe that $\phi \in \mathcal{B}(p)$, which implies $\bigcap_{\psi \in \mathcal{B}(p)} \llbracket \psi \rrbracket \subseteq \llbracket \phi \rrbracket$.

Finally, we show that $\mathcal{B}$ is monotonic (property (iv)). Consider $p, q \in P$, with $\mathcal{L}(p) \subseteq \mathcal{L}(q)$. We want to show that $\phi \in \mathcal{B}(p)$ implies $\phi \in \mathcal{B}(q)$, for each ϕ. Firstly, we observe that, by $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ and Theorem 4, $p \lesssim q$ holds. Thus, $I(p) = \emptyset$ if and only if $I(q) = \emptyset$, which implies $\mathcal{B}^-(p) = \mathcal{B}^-(q)$; moreover, we have that for each $a \in Act$ and $p' \in P$ with $p \xrightarrow{a} p'$, there exists some $q' \in P$ such that $q \xrightarrow{a} q'$ and $p' \lesssim q'$. In order to show that $\mathcal{B}(p) \subseteq \mathcal{B}(q)$, we proceed by induction on the depth of p. If $I(p) = \emptyset$, then $I(q) = \emptyset$ as well. Thus, we have that $\mathcal{B}^+(p) = \mathcal{B}^+(q) = \{\mathbf{tt}\}$, and the thesis follows. Otherwise ($I(p) \neq \emptyset$), let us consider a formula $\phi \in \mathcal{B}(p)$. If $\phi \in \mathcal{B}^-(p)$, then the claim follows from $\mathcal{B}^-(p) = \mathcal{B}^-(q) \subseteq \mathcal{B}(q)$. If $\phi = \mathbf{tt}$, then, by definition of $\mathcal{B}^+$, we have $\phi \in \mathcal{B}^+(q) \subseteq \mathcal{B}(q)$. Finally, if $\phi = \langle a \rangle \varphi \in \mathcal{B}^+(p)$, then, by definition of $\mathcal{B}^+$, there exist $p' \in P$, with $p \xrightarrow{a} p'$, such that $\varphi = \bigwedge_{\psi \in \Psi} \psi$ for some $\Psi \subseteq \mathcal{B}(p')$. This implies the existence of some $q' \in P$ such that $q \xrightarrow{a} q'$ and $p' \lesssim q'$ (and therefore $\mathcal{L}(p') \subseteq \mathcal{L}(q')$ by Theorem 4). By the inductive hypothesis, $\mathcal{B}(p') \subseteq \mathcal{B}(q')$ holds as well, which means that $\Psi \subseteq \mathcal{B}(q')$. Hence, we have that $\langle a \rangle \varphi \in \mathcal{B}^+(q) \subseteq \mathcal{B}(q)$.

Case ready simulation (RS). See the proof of Lemma 1 at page 19.

Case trace simulation (TS). For the sake of clarity we recall from Table 1 that $\mathcal{B}$ is defined as $\mathcal{B}(p) = \mathcal{B}^+(p) \cup \mathcal{B}^-(p)$, where

- $\mathcal{B}^+(p) = \{\mathbf{tt}\} \cup \{\langle a \rangle \varphi \mid a \in Act, \varphi = \bigwedge_{\psi \in \Psi} \psi, \Psi \subseteq \mathcal{B}(p'), p \xrightarrow{a} p'\}$, and

- $\mathcal{B}^-(p) = \{[\tau a]\mathbf{ff} \mid \tau \in traces(p), a \in Act, \tau a \notin traces(p)\}$.

It is easy to see that, for all p, (a) $\mathcal{B}^-(p)$ is finite, because Act is finite and so is $traces(p)$, and (b) $\mathcal{B}^-(p) \subseteq \mathcal{L}(p)$ trivially holds, by definition of $\mathcal{B}^-(p)$. Hence, properties (i) and (iii) immediately follow from (i') and (iii'), respectively. Notice also that property (iii) implies that $\mathcal{B}$ is well defined.

In order to prove property (ii), we let $\phi \in \mathcal{L}(p)$, for a generic $p \in P$, and we proceed by induction on the structure of ϕ. Inductive steps work as in previous cases; the following additional base case, with respect to previous cases, must be considered.

- $\phi = [\tau a]\mathbf{ff}$: if $\phi \in \mathcal{B}^-(p)$, the property holds trivially. Otherwise, $\phi \notin \mathcal{B}^-(p)$ implies $\tau \notin traces(p)$ (notice that $\tau a \notin traces(p)$: indeed, if $\tau a \in traces(p)$ then $\phi \notin \mathcal{L}(p)$, thus raising a contradiction). There must be two prefixes $\tau' = a_1 \ldots a_k$ and $\tau'' = a_1 \ldots a_k a_{k+1}$ of τ (i.e., τ' is the largest proper prefix of τ''—possibly $\tau' = \varepsilon$ and/or $\tau'' = \tau$) such that $p \xrightarrow{\tau'}$ and $p \xrightarrow{\tau''} \!\!\!\!\!/\,$. The formula $\psi = [\tau'']\mathbf{ff}$ is such that $\psi \in \mathcal{B}^-(p)$ and $[\![\psi]\!] \subseteq [\![\phi]\!]$. The thesis follows immediately.

Finally, we show that $\mathcal{B}$ is monotonic (property (iv)). Consider $p, q \in P$, with $\mathcal{L}(p) \subseteq \mathcal{L}(q)$. We want to show that $\phi \in \mathcal{B}(p)$ implies $\phi \in \mathcal{B}(q)$, for each ϕ. Firstly, we observe that, by $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ and Theorem 4, $p \lesssim q$ holds. Thus, we have that $traces(p) = traces(q)$, which implies $\mathcal{B}^-(p) = \mathcal{B}^-(q)$; moreover, we have that for each $a \in Act$ and $p' \in P$ with $p \xrightarrow{a} p'$, there exists some $q' \in P$ such that $q \xrightarrow{a} q'$ and $p' \lesssim q'$. In order to show that $\mathcal{B}(p) \subseteq \mathcal{B}(q)$, we proceed by induction on the depth of p. If $I(p) = \emptyset$, then $I(q) = \emptyset$ as well. Thus, we have that $\mathcal{B}^+(p) = \mathcal{B}^+(q) = \{\mathbf{tt}\}$, and the thesis follows. Otherwise $(I(p) \neq \emptyset)$, let us consider a formula $\phi \in \mathcal{B}(p)$. If $\phi \in \mathcal{B}^-(p)$, then the claim follows from $\mathcal{B}^-(p) = \mathcal{B}^-(q) \subseteq \mathcal{B}(q)$. If $\phi = \mathbf{tt}$, then, by definition of $\mathcal{B}^+$, we have $\phi \in \mathcal{B}^+(q) \subseteq \mathcal{B}(q)$. Finally, if $\phi = \langle a \rangle \varphi \in \mathcal{B}^+(p)$, then, by definition of $\mathcal{B}^+$, there exist $p' \in P$, with $p \xrightarrow{a} p'$, such that $\varphi = \bigwedge_{\psi \in \Psi} \psi$ for some $\Psi \subseteq \mathcal{B}(p')$. This implies the existence of some $q' \in P$ such that $q \xrightarrow{a} q'$ and $p' \lesssim q'$ (and therefore $\mathcal{L}(p') \subseteq \mathcal{L}(q')$ by Theorem 4). By the inductive hypothesis, $\mathcal{B}(p') \subseteq \mathcal{B}(q')$ holds as well, which means that $\Psi \subseteq \mathcal{B}(q')$. Hence, we have that $\langle a \rangle \varphi \in \mathcal{B}^+(q) \subseteq \mathcal{B}(q)$.

Case 2-nested simulation (2S). For the sake of clarity we recall from Table 1 that $\mathcal{B}$ is defined as $\mathcal{B}(p) = \mathcal{B}^+(p) \cup \mathcal{B}^-(p)$, where

- $\mathcal{B}^+(p) = \{\mathbf{tt}\} \cup \{\langle a \rangle \varphi \mid a \in Act, \varphi = \bigwedge_{\psi \in \Psi} \psi, \Psi \subseteq \mathcal{B}(p'), p \xrightarrow{a} p'\}$, and
- $\mathcal{B}^-(p) = \{[a]\varphi \in \mathcal{L}(p) \mid a \in Act, \varphi = \bigvee_{p' \in max\text{-}succ(p,a)} \bigwedge_{\psi \in \mathcal{B}^-(p')} \psi\}$,

where $max\text{-}succ(p, a) = \{p' \in P \mid p \xrightarrow{a} p' \text{ and } \nexists p''.p \xrightarrow{a} p'' \text{ and } p' <_{\mathsf{S}} p''\}$.

By definition, $\mathcal{B}^-(p) \subseteq \mathcal{L}(p)$ for every $p \in P$; moreover, it is easy to verify that $\mathcal{B}^-(p)$ is finite, by induction on the depth on p and by recalling that Act is finite and processes are finitely branching. Hence, properties (i) and (iii) immediately follow from (i') and (iii'), respectively. Notice also that property (iii) implies that $\mathcal{B}$ is well defined.

In order to prove property (ii), we let $\phi \in \mathcal{L}(p)$, for a generic $p \in P$, and we proceed, as usual, by induction on the structure of ϕ. With respect to the previous semantics, we have to consider new formulae of the form $\neg \varphi$. They are treated as an additional base case, which, in turn, is handled by means of (inner) induction on the structure of φ.

- $\phi = \mathbf{tt}$ (base case): the claim follows trivially.

- $\phi = \neg\varphi$ (base case): according to the definition of the syntax of $\mathcal{L}_{2\mathsf{S}}$, φ must belong to $\mathcal{L}_{\mathsf{S}}$. We show that $\bigcap_{\psi \in \mathcal{B}_{2\mathsf{S}}(p)} \llbracket \psi \rrbracket \subseteq \llbracket \neg\varphi \rrbracket$ holds for all $\varphi \in \mathcal{L}_{\mathsf{S}}$. We proceed by (inner) induction on the structure of φ (notice that we can ignore the case $\varphi = \mathbf{tt}$, as $\phi \in \mathcal{L}(p)$ implies $\varphi \neq \mathbf{tt}$).

  - $\varphi = \mathbf{ff}$: the claim follows trivially.

  - $\varphi = \varphi_1 \wedge \varphi_2$: then, $\neg\varphi$ is logically equivalent to $\neg\varphi_1 \vee \neg\varphi_2$. Thus, $\phi \in \mathcal{L}(p)$ implies $\neg\varphi_i \in \mathcal{L}(p)$ for some $i \in \{1,2\}$. By inductive hypothesis, $\bigcap_{\psi \in \mathcal{B}(p)} \llbracket \psi \rrbracket \subseteq \llbracket \neg\varphi_i \rrbracket$ and, since $\llbracket \neg\varphi_i \rrbracket \subseteq \llbracket \neg\varphi \rrbracket$, the thesis follows.

  - $\varphi = \varphi_1 \vee \varphi_2$: then, $\neg\varphi$ is logically equivalent to $\neg\varphi_1 \wedge \neg\varphi_2$. Thus, $\phi \in \mathcal{L}(p)$ implies $\neg\varphi_i \in \mathcal{L}(p)$ for every $i \in \{1,2\}$. By inductive hypothesis, $\bigcap_{\psi \in \mathcal{B}(p)} \llbracket \psi \rrbracket \subseteq \llbracket \neg\varphi_i \rrbracket$ for every $i \in \{1,2\}$, which means that $\bigcap_{\psi \in \mathcal{B}(p)} \llbracket \psi \rrbracket \subseteq \llbracket \neg\varphi_1 \rrbracket \cap \llbracket \neg\varphi_2 \rrbracket = \llbracket \neg\varphi \rrbracket$, hence the thesis.

  - $\varphi = \langle a \rangle \varphi_1$: then, $\neg\varphi$ is logically equivalent to $[a]\neg\varphi_1$. Thus, $\phi \in \mathcal{L}(p)$ implies $\neg\varphi_1 \in \mathcal{L}(p')$ for every $p' \in P$ such that $p \xrightarrow{a} p'$. By the inductive hypothesis, $\bigcap_{\psi \in \mathcal{B}(p')} \llbracket \psi \rrbracket \subseteq \llbracket \neg\varphi_1 \rrbracket$ for every $p' \in P$ such that $p \xrightarrow{a} p'$, from which it follows $\bigcup_{p' \in max\text{-}succ(p,a)} \bigcap_{\psi \in \mathcal{B}(p')} \llbracket \psi \rrbracket \subseteq \llbracket \neg\varphi_1 \rrbracket$. We define $\zeta = [a] \bigvee_{p' \in max\text{-}succ(p,a)} \bigwedge_{\psi \in \mathcal{B}^-(p')} \psi$ (notice that $\zeta = [a]\mathbf{ff}$ if $p \xrightarrow{a}\!\!\!\!\!/\,$). If is not difficult to show, by induction on the depth of p, that, for all $p,q \in P$, if $p \lesssim_{\mathsf{S}} q$ then $p \in \llbracket \psi \rrbracket$ for all $\psi \in \mathcal{B}^-(q)$. Moreover, we observe that for every p' such that $p \xrightarrow{a} p'$, there is $q \in max\text{-}succ(p,a)$ with $p' \lesssim_{\mathsf{S}} q$. Therefore, $\zeta \in \mathcal{L}(p)$, because for all p' with $p \xrightarrow{a} p'$ we have that $p' \in \llbracket \bigwedge_{\psi \in \mathcal{B}^-(q)} \psi \rrbracket$, where q is such that $q \in max\text{-}succ(p,a)$ and $p' \lesssim_{\mathsf{S}} q$. Moreover, we have that $\zeta \in \mathcal{B}^-(p)$ (by definition of $\mathcal{B}^-(p)$) and therefore $\bigcap_{\psi \in \mathcal{B}(p)} \llbracket \psi \rrbracket \subseteq \llbracket \zeta \rrbracket$. By $\bigcup_{p' \in max\text{-}succ(p,a)} \bigcap_{\psi \in \mathcal{B}(p')} \llbracket \psi \rrbracket \subseteq \llbracket \neg\varphi_1 \rrbracket$, we have $\llbracket \zeta \rrbracket \subseteq \llbracket [a]\neg\varphi_1 \rrbracket = \llbracket \neg\varphi \rrbracket$, and the thesis immediately follows.

- $\phi \in \{\varphi_1 \vee \varphi_2, \varphi_1 \wedge \varphi_2, \langle a \rangle \varphi\}$ (inductive step): the proof is exactly the same as in the previous cases.

Before showing that $\mathcal{B}$ is monotonic (property (iv)), we prove, as a preliminary result, that if $p \equiv_{\mathsf{S}} q$ then $\mathcal{B}^-(p) = \mathcal{B}^-(q)$. To this end, we first prove that if $p \equiv_{\mathsf{S}} q$, then for every $a \in Act$ and $p' \in max\text{-}succ(p,a)$ there is $q' \in max\text{-}succ(q,a)$ such that $p' \equiv_{\mathsf{S}} q'$. (Notice that, since $\equiv_{\mathsf{S}}$ is symmetric, this also implies that for every $a \in Act$ and $q' \in max\text{-}succ(q,a)$ there is $p' \in max\text{-}succ(p,a)$ such that $p' \equiv_{\mathsf{S}} q'$.) Let $p,q \in P$ be such that $p \equiv_{\mathsf{S}} q$, and let $a \in Act$ and $p' \in max\text{-}succ(p,a)$. By $p \lesssim_{\mathsf{S}} q$, there is q' such that $q \xrightarrow{a} q'$ and $p' \lesssim_{\mathsf{S}} q'$. We show that $q' \in max\text{-}succ(q,a)$ and $p' \equiv_{\mathsf{S}} q'$. In order to show the former, assume, towards a contradiction, that $q' \notin max\text{-}succ(q,a)$; thus, there is q'' such that $q \xrightarrow{a} q''$ and $q' <_{\mathsf{S}} q''$. By $q \lesssim_{\mathsf{S}} p$, we have that there is p'' such that $p \xrightarrow{a} p''$ and $q'' \lesssim_{\mathsf{S}} p''$. Thus, p'' is such that $p \xrightarrow{a} p''$ and $p' <_{\mathsf{S}} p''$, thus

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getting a contradiction with the hypothesis that $p' \in max\text{-}succ(p, a)$. Therefore, $q' \in max\text{-}succ(q, a)$. Now, in order to show that $p' \equiv_{\sf S} q'$, it is enough to prove that $q' \lesssim_{\sf S} p'$ (as we already know that $p' \lesssim_{\sf S} q'$). Assume, towards a contradiction, that $q' \not\lesssim_{\sf S} p'$ (i.e., $p' <_{\sf S} q'$). By $q \lesssim_{\sf S} p$, there is p'' such that $p \xrightarrow{a} p''$ and $q' \lesssim_{\sf S} p''$. Therefore, $p' <_{\sf S} p''$, thus getting a contradiction with the hypothesis that $p' \in max\text{-}succ(p, a)$. Therefore, $q' \lesssim_{\sf S} p'$, which means $q' \equiv_{\sf S} p'$. We turn to proving that if $p \equiv_{\sf S} q$ then $\mathcal{B}^-(p) = \mathcal{B}^-(q)$. By the symmetry of $\equiv_{\sf S}$, it is enough to prove that $\mathcal{B}^-(p) \subseteq \mathcal{B}^-(q)$. We proceed by induction on the depth of p. If $I(p) = \emptyset$ (base case), then $I(q) = \emptyset$ as well, and $\mathcal{B}^-(p) = \mathcal{B}^-(q) = \{[a]\mathbf{ff} \mid a \in Act\}$. If $I(p) \neq \emptyset$ (inductive step), then let $[a]\varphi \in \mathcal{B}^-(p)$. We distinguish two cases.

- If $a \notin I(p)$, then $\varphi = \mathbf{ff}$ and, by $q \lesssim_{\sf S} p$, $a \notin I(q)$, which implies $[a]\varphi \in \mathcal{B}^-(q)$.

- If $a \in I(p)$, then $\varphi = \bigvee_{p' \in max\text{-}succ(p,a)} \bigwedge_{\psi \in \mathcal{B}^-(p')} \psi$. By $p \equiv_{\sf S} q$, we have that for every $p' \in max\text{-}succ(p, a)$ there is $q' \in max\text{-}succ(q, a)$ with $p' \equiv_{\sf S} q'$, which, by inductive hypothesis, implies $\mathcal{B}^-(p') = \mathcal{B}^-(q')$. The converse direction holds as well, that is, for every $q' \in max\text{-}succ(q, a)$ there is $p' \in max\text{-}succ(p, a)$ with $p' \equiv_{\sf S} q'$, which, by inductive hypothesis, implies $\mathcal{B}^-(p') = \mathcal{B}^-(q')$. Consequently, $[a]\varphi \in \mathcal{B}^-(q)$, and we are done.

We are now ready to show that $\mathcal{B}$ is monotonic (property (iv)). Consider $p, q \in P$, with $\mathcal{L}(p) \subseteq \mathcal{L}(q)$. We want to show that $\phi \in \mathcal{B}(p)$ implies $\phi \in \mathcal{B}(q)$, for each ϕ. Firstly, we observe that, by $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ and Theorem 4, $p \lesssim_{\sf 2S} q$ holds. Thus, we have that $p \equiv_{\sf S} q$, which implies $\mathcal{B}^-(p) = \mathcal{B}^-(q)$ (thanks to the previous result); moreover, we have that for each $a \in Act$ and $p' \in P$ with $p \xrightarrow{a} p'$, there exists some $q' \in P$ such that $q \xrightarrow{a} q'$ and $p' \lesssim_{\sf 2S} q'$. In order to show that $\mathcal{B}(p) \subseteq \mathcal{B}(q)$, we proceed by induction on the depth of p. If $I(p) = \emptyset$, then $I(q) = \emptyset$ as well. Thus, we have that $\mathcal{B}^+(p) = \mathcal{B}^+(q) = \{\mathbf{tt}\}$, and the thesis follows. Otherwise ($I(p) \neq \emptyset$), let us consider a formula $\phi \in \mathcal{B}(p)$. If $\phi \in \mathcal{B}^-(p)$, then the claim follows from $\mathcal{B}^-(p) = \mathcal{B}^-(q) \subseteq \mathcal{B}(q)$. If $\phi = \mathbf{tt}$, then, by definition of $\mathcal{B}^+$, we have $\phi \in \mathcal{B}^+(q) \subseteq \mathcal{B}(q)$. Finally, if $\phi = \langle a \rangle \varphi \in \mathcal{B}^+(p)$, then, by definition of $\mathcal{B}^+$, there exist $p' \in P$, with $p \xrightarrow{a} p'$, such that $\varphi = \bigwedge_{\psi \in \Psi} \psi$ for some $\Psi \subseteq \mathcal{B}(p')$. This implies the existence of some $q' \in P$ such that $q \xrightarrow{a} q'$ and $p' \lesssim_{\sf 2S} q'$ (and therefore $\mathcal{L}(p') \subseteq \mathcal{L}(q')$ by Theorem 4). By the inductive hypothesis, $\mathcal{B}(p') \subseteq \mathcal{B}(q')$ holds as well, which means that $\Psi \subseteq \mathcal{B}(q')$. Hence, we have that $\langle a \rangle \varphi \in \mathcal{B}^+(q) \subseteq \mathcal{B}(q)$.

Case bisimulation (BS). For the sake of clarity we recall from Table 1 that $\mathcal{B}$ is defined as $\mathcal{B}^+(p) \cup \mathcal{B}^-(p)$, where

- $\mathcal{B}^+(p) = \{\mathbf{tt}\} \cup \{\langle a \rangle \varphi \mid a \in Act, \varphi = \bigwedge_{\psi \in \Psi} \psi, \Psi \subseteq \mathcal{B}(p'), p \xrightarrow{a} p'\}$, and
- $\mathcal{B}^-(p) = \{[a]\varphi \in \mathcal{L}(p) \mid a \in Act, \varphi = \bigvee_{p \xrightarrow{a} p'} \bigwedge_{\psi \in \mathcal{B}(p')} \psi\}$.

By definition, $\mathcal{B}^-(p) \subseteq \mathcal{L}(p)$ for every $p \in P$; moreover, it is easy to verify that $\mathcal{B}^-(p)$ is finite, by induction on the depth on p and by recalling that Act is finite and processes are finitely branching. Hence, properties (i) and (iii) immediately follow from (i') and (iii'), respectively. Notice also that property (iii)

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implies that $\mathcal{B}$ is well defined.

In order to prove property (ii), we let $\phi \in \mathcal{L}(p)$, for a generic $p \in P$, and we proceed, as usual, by induction on the structure of ϕ. With respect to the previous semantics, bisimulation one is characterized by the use of free negation (formula of the form $\neg\varphi$, for every $\varphi \in \mathcal{L}$). By using the fact that $\neg\langle a\rangle\varphi$ is equivalent to $[a]\neg\varphi$, it suffices to show how to deal with formulae in the form $[a]\varphi$, for every $\varphi \in \mathcal{L}$; the other cases are dealt with as in the previous cases.

- $\phi = [a]\varphi$: by $\phi \in \mathcal{L}(p)$, we have that $\varphi \in \mathcal{L}(p')$ for all p' such that $p \xrightarrow{a} p'$. By the inductive hypothesis, $\bigcap_{\psi \in \mathcal{B}(p')}[\![\psi]\!] \subseteq [\![\varphi]\!]$, for each p' such that $p \xrightarrow{a} p'$, from which it follows $\bigcup_{p\xrightarrow{a}p'}\bigcap_{\psi\in\mathcal{B}(p')}[\![\psi]\!] \subseteq [\![\varphi]\!]$. We define $\zeta = [a]\bigvee_{p\xrightarrow{a}p'}\bigwedge_{\psi\in\mathcal{B}(p')}\psi$ (notice that $\zeta = [a]\mathbf{ff}$ if $p \xrightarrow{a}\!\!\!\!\!/\,$). Clearly, for all p' such that $p \xrightarrow{a}\!\!\!\!\!/\, p'$, it holds $p' \in [\![\psi]\!]$ for all $\psi \in \mathcal{B}(p')$. Therefore, $p \in [\![\zeta]\!]$, which means $\zeta \in \mathcal{L}(p)$. Moreover, we have that $\zeta \in \mathcal{B}^-(p)$ (by definition of $\mathcal{B}^-(p)$) and therefore $\bigcap_{\psi\in\mathcal{B}(p)}[\![\psi]\!] \subseteq [\![\zeta]\!]$. By $\bigcup_{p\xrightarrow{a}p'}\bigcap_{\psi\in\mathcal{B}(p')}[\![\psi]\!] \subseteq [\![\varphi]\!]$, we have $[\![\zeta]\!] \subseteq [\![[a]\varphi]\!] = [\![\phi]\!]$, and the thesis immediately follows.

Finally, we show that $\mathcal{B}$ is monotonic (property (iv)). Consider $p, q \in P$, with $\mathcal{L}(p) \subseteq \mathcal{L}(q)$. We want to show that $\mathcal{B}(p) \subseteq \mathcal{B}(q)$. Firstly, we observe that, by $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ and Theorem 4, $p \lesssim q$ holds. Moreover, by the definition of bisimulation semantics, $p \lesssim q$ implies $q \lesssim p$, and thus $p \equiv q$. It is straightforward to show, by induction on the depth of p, that if $p \equiv q$, then $\mathcal{B}(p) = \mathcal{B}(q)$, hence the thesis. $\qquad\square$

Appendix A.2. Existence of $\bar{\chi}(\cdot)$

In what follows, we show that it is possible to build, for each $\chi(p)$, a formula $\bar{\chi}(p)$, with the properties described in Corollary 4(ii).

Lemma 2. *Let $X \in$ Btime-spectrum. For each $p \in P$ and $\chi_X(p)$ characteristic within $\mathcal{L}_X$ for p, there exists a formula in $\mathcal{L}_X$, denoted by $\bar{\chi}_X(p)$, such that either*

- $[\![\bar{\chi}(p)]\!] = P \setminus [\![\chi(p)]\!]$*, or*

- $p \notin [\![\bar{\chi}_X(p)]\!]$ *and* $\{p' \in P \mid p' \not\lesssim_X p\} \subseteq [\![\bar{\chi}_X(p)]\!]$*.*

Proof. We consider each semantics $X \in$ Btime-spectrum in turn. As in the previous section, for the sake of a lighter notation, we often omit the subscripts identifying the semantics as it is clear from the context, e.g., we write $\mathcal{L}$, $\bar{\chi}$, and $p \not\lesssim q$ instead of $\mathcal{L}_S$, $\bar{\chi}_S$, and $p \not\lesssim_S q$, respectively.

Case simulation (S). For the sake of clarity we recall here the definition of $\bar{\chi}$ from Table 1:

$$\bar{\chi}(p) = \bigvee_{a\in Act}\langle a\rangle \bigwedge_{p\xrightarrow{a}p'} \bar{\chi}(p').$$

Let us first show that for every $p \in P$ we have $p \notin [\![\bar{\chi}(p)]\!]$. We proceed by induction on the depth of p.

- $I(p) = \emptyset$: we have $\bar{\chi}(p) = \bigvee_{a\in Act}\langle a\rangle\mathbf{tt}$, and thus $p \notin [\![\bar{\chi}(p)]\!]$.

- $I(p) \neq \emptyset$: obviously, $p \notin [\![\langle a \rangle \mathbf{tt}]\!]$ holds for every $a \notin I(p)$. Now, for every $p \xrightarrow{a} p'$, by induction hypothesis, $p' \notin [\![\bar{\chi}(p')]\!]$. Thus, $p' \notin [\![\bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$ and therefore $p \notin [\![\langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$. Hence, we obtain that $p \notin [\![\bar{\chi}(p)]\!]$.

Now, let us show that $\{p' \in P \mid p' \not\lesssim p\} \subseteq [\![\bar{\chi}(p)]\!]$. The proof is by induction on the depth of p.

- $I(p) = \emptyset$: we have that $\{p' \in P \mid p' \not\lesssim p\} = P \setminus \{p' \in P \mid I(p') = \emptyset\}$. It is easy to see that $P \setminus \{p' \in P \mid I(p') = \emptyset\} = [\![\bigvee_{a \in Act} \langle a \rangle \mathbf{tt}]\!] = [\![\bar{\chi}(p)]\!]$.

- $I(p) \neq \emptyset$: let $q \not\lesssim p$. Thus there exists some q', with $q \xrightarrow{a} q'$, such that, for every p', $p \xrightarrow{a} p'$ implies $q' \not\lesssim p'$. Then, by inductive hypothesis, $q' \in [\![\bar{\chi}(p')]\!]$ for every p' such that $p \xrightarrow{a} p'$. Thus, $q' \in [\![\bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$ and therefore $q \in [\![\langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$. Hence, we conclude $q \in [\![\bar{\chi}(p)]\!]$.

Case complete simulation (CS). For the sake of clarity we recall here the definition of $\bar{\chi}$ from Table 1:

$$\bar{\chi}(p) = \left(\bigvee_{a \in Act} \langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p') \right) \vee \mathbf{0} \qquad \text{if } I(p) \neq \emptyset$$
$$\bar{\chi}(p) = \bigvee_{a \in Act} \langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p') \qquad \text{if } I(p) = \emptyset$$

Let us first show that for every $p \in P$ we have $p \notin [\![\bar{\chi}(p)]\!]$. We proceed by induction on the depth of p.

- $I(p) = \emptyset$: we have $\bar{\chi}(p) = \bigvee_{a \in Act} \langle a \rangle \mathbf{tt}$, and obviously $p \notin [\![\bar{\chi}(p)]\!]$.

- $I(p) \neq \emptyset$: obviously, $p \notin [\![\mathbf{0}]\!]$ and $p \notin [\![\langle a \rangle \mathbf{tt}]\!]$ for every $a \notin I(p)$. Now, for every $p \xrightarrow{a} p_1$, by the inductive hypothesis, $p_1 \notin [\![\bar{\chi}(p_1)]\!]$. Thus, $p_1 \notin [\![\bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$ and therefore $p \notin [\![\langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$. Hence, we obtain that $p \notin [\![\bar{\chi}(p)]\!]$.

Now, let us show that $\{p' \in P \mid p' \not\lesssim p\} \subseteq [\![\bar{\chi}(p)]\!]$. The proof is by induction on the depth of p.

- $I(p) = \emptyset$: we have that $\{p' \in P \mid p' \not\lesssim p\} = P \setminus \{p' \in P \mid I(p') = \emptyset\}$ because, when $I(p) = \emptyset$, $q \not\lesssim p$ holds exactly for processes q with $I(q) \neq \emptyset$. It is easy to see that $P \setminus \{p' \in P \mid I(p') = \emptyset\} = [\![\bigvee_{a \in Act} \langle a \rangle \mathbf{tt}]\!] = [\![\bar{\chi}(p)]\!]$.

- $I(p) \neq \emptyset$: let $q \not\lesssim p$. Thus, either $I(q) = \emptyset$ and $I(p) \neq \emptyset$ (or analogously $I(p) = \emptyset$ and $I(q) \neq \emptyset$), or there exists some q', with $q \xrightarrow{a} q'$, such that, for every p', $p \xrightarrow{a} p'$ implies $q' \not\lesssim p'$. If it is the case that $I(q) = \emptyset$ and $I(p) \neq \emptyset$ (respectively, $I(p) = \emptyset$ and $I(q) \neq \emptyset$), then $q \in [\![\mathbf{0}]\!]$ (respectively, $q \in [\![\langle a \rangle \mathbf{tt}]\!]$ holds for some $a \notin I(p)$). In any case, $q \in [\![\bar{\chi}(p)]\!]$ holds.

  Otherwise, if there exist $a \in Act$ and $q' \in P$, with $q \xrightarrow{a} q'$, such that $q' \not\lesssim p'$ for every $p \xrightarrow{a} p'$, then, by the inductive hypothesis, $q' \in [\![\bar{\chi}(p')]\!]$ for every p'. Thus, $q' \in [\![\bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$ and therefore $q \in [\![\langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$. Hence, we conclude $q \in [\![\bar{\chi}(p)]\!]$.

Case ready simulation: (RS). See the proof of Lemma 2 at page 20.

Case trace simulation (TS). For the sake of clarity we recall here the definition of $\bar{\chi}$ from Table 1:

$\bar{\chi}(p) = \left(\bigvee_{a \in act} \langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p') \right) \vee \bigvee_{\tau \in traces(p), \tau a \notin traces(p)} \langle \tau a \rangle \mathbf{tt} \vee \bigvee_{p \xrightarrow{\tau a} p'} [\tau a] \mathbf{ff}$.

Let us first show that for every $p \in P$ we have $p \notin [\![\bar{\chi}(p)]\!]$. We proceed by induction on the depth of p.

- $I(p) = \emptyset$: we have $\bar{\chi}(p) = \bigvee_{a \in Act}\langle a\rangle\mathbf{tt}$, and thus $p \notin [\![\bar{\chi}(p)]\!]$.
- $I(p) \neq \emptyset$: obviously, $p \notin [\![\langle\tau a\rangle\mathbf{tt}]\!]$ holds for every $\tau \in traces(p)$ and $a \in Act$ such that $\tau a \notin traces(p)$. Moreover, $p \notin [\![[\tau a]\mathbf{ff}]\!]$ holds for every $\tau a \in traces(p)$. Now, for every $p \xrightarrow{a} p_1$, by the inductive hypothesis, $p_1 \notin [\![\bar{\chi}(p_1)]\!]$. Thus, $p_1 \notin [\![\bigwedge_{p \xrightarrow{a} p'}\bar{\chi}(p')]\!]$ and therefore $p \notin [\![\langle a\rangle\bigwedge_{p \xrightarrow{a} p'}\bar{\chi}(p')]\!]$. Hence, we obtain that $p \notin [\![\bar{\chi}(p)]\!]$.

Now, let us show that $\{p' \in P \mid p' \not\lesssim p\} \subseteq [\![\bar{\chi}(p)]\!]$, that is, $[\![\bar{\chi}(p)]\!]$ contains at least the elements that are either strictly above p or incomparable with it. The proof is by induction on the depth of p.

- $I(p) = \emptyset$: we have that $\{p' \in P \mid p' \not\lesssim p\} = P \setminus \{p' \in P \mid I(p') = \emptyset\}$ because, when $I(p) = \emptyset$, $q \not\lesssim p$ holds exactly for processes q with $I(q) \neq \emptyset$. It is easy to see that $P \setminus \{p' \in P \mid I(p') = \emptyset\} = [\![\bigvee_{a \in Act}\langle a\rangle\mathbf{tt}]\!] = [\![\bar{\chi}(p)]\!]$.
- $I(p) \neq \emptyset$: let $q \not\lesssim p$. Thus, either $traces(q) \neq traces(p)$ or there exists some q', with $q \xrightarrow{a} q'$, such that, for every p', $p \xrightarrow{a} p'$ implies $q' \not\lesssim p'$. If it is the case that $traces(q) \neq traces(p)$, then either $q \in [\![\langle\tau a\rangle\mathbf{tt}]\!]$ for some τa such that $\tau \in traces(p)$ and $\tau a \notin traces(p)$, or $q \in [\![[\tau a]\mathbf{ff}]\!]$ for some $\tau a \in traces(p)$. In either case, $q \in [\![\bar{\chi}(p)]\!]$ holds.

If, on the other hand, $traces(p) = traces(q)$, then there exist $a \in Act$ and $q' \in P$, with $q \xrightarrow{a} q'$, such that $q' \not\lesssim p'$ for every $p \xrightarrow{a} p'$, then, by the inductive hypothesis, $q' \in [\![\bar{\chi}(p')]\!]$ for every p'. Thus, $q' \in [\![\bigwedge_{p \xrightarrow{a} p'}\bar{\chi}(p')]\!]$ and therefore $q \in [\![\langle a\rangle\bigwedge_{p \xrightarrow{a} p'}\bar{\chi}(p')]\!]$ (notice that $a \in I(p)$, due to $a \in I(q)$ and $traces(p) = traces(q)$). Hence, we conclude $q \in [\![\bar{\chi}(p)]\!]$.

Case 2-Nested simulation (2S). For the sake of clarity we recall here the definition of $\bar{\chi}$ from Table 1:

$$\bar{\chi}(p) = \left(\bigvee_{a \in Act}\langle a\rangle\bigwedge_{p \xrightarrow{a} p'}\bar{\chi}(p')\right) \vee \bar{\Phi}(p)$$
$$\bar{\Phi}(p) = \bigvee_{a \in I(p)}[a]\mathbf{ff} \vee \bigvee_{a \in I(p)}\bigvee_{p \xrightarrow{a} p'}[a]\bar{\Phi}(p')$$

Let us first show that for every $p \in P$ we have $p \notin [\![\bar{\chi}(p)]\!]$. We proceed by induction on the depth of p.

- $I(p) = \emptyset$: we have $\bar{\chi}(p) = \bigvee_{a \in Act}\langle a\rangle\mathbf{tt}$, and thus $p \notin [\![\bar{\chi}(p)]\!]$.
- $I(p) \neq \emptyset$: obviously, $p \notin [\![\langle a\rangle\mathbf{tt}]\!]$ holds for every $a \notin I(p)$. Next, for every $p \xrightarrow{a} p'$, by the inductive hypothesis, $p' \notin [\![\bar{\chi}(p')]\!]$. Thus, $p \notin [\![\langle a\rangle\bar{\chi}(p')]\!]$, for each p' such that $p \xrightarrow{a} p'$. Therefore $p \notin [\![\langle a\rangle\bigwedge_{p \xrightarrow{a} p'}\bar{\chi}(p')]\!]$. We are left with showing that $p \notin [\![\bar{\Phi}(p)]\!]$. To this end, observe that $p \notin [\![[a]\mathbf{ff}]\!]$ for all $a \in I(p)$; moreover, for all $p \xrightarrow{a} p'$, since $p' \notin [\![\bar{\chi}(p')]\!]$ holds by the inductive hypothesis, in particular we have $p' \notin [\![\bar{\Phi}(p')]\!]$. Thus, $p \notin [\![[a]\bar{\Phi}(p')]\!]$ for each p' such that $p \xrightarrow{a} p'$ and $p \notin [\![\bigvee_{p \xrightarrow{a} p'}[a]\bar{\Phi}(p')]\!]$. Hence, we obtain that $p \notin [\![\bar{\chi}(p)]\!]$.

Now, let us show that $\{p' \in P \mid p' \not\lesssim p\} \subseteq [\![\bar{\chi}(p)]\!]$. The proof is by induction on the depth of p.

- $I(p) = \emptyset$: we have that $\{p' \in P \mid p' \not\lesssim p\} = P \setminus \{p' \in P \mid I(p') = \emptyset\}$. It is easy to see that $P \setminus \{p' \in P \mid I(p') = \emptyset\} = [\![\bigvee_{a \in Act}\langle\mathbf{tt}\rangle]\!] = [\![\bar{\chi}(p)]\!]$.
- $I(p) \neq \emptyset$: let $q \not\lesssim_{2S} p$. Thus, by definition, either there exists $a \in Act$ and q', with $q \xrightarrow{a} q'$, such that, for every p', $p \xrightarrow{a} p'$ implies $q' \not\lesssim_{2S} p'$, or $p \not\lesssim_S q$.

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First, if there exist $a \in Act$ and $q' \in P$, with $q \xrightarrow{a} q'$, such that $q' \not\lesssim_{2S} p'$ for every $p \xrightarrow{a} p'$, by the inductive hypothesis, $q' \in [\![\bar{\chi}(p')]\!]$ for every p' such that $p \xrightarrow{a} p'$. Thus, $q' \in [\![\bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$ and therefore $q \in [\![\langle a \rangle \bigwedge_{p \xrightarrow{a} p'} \bar{\chi}(p')]\!]$.

Otherwise, if $p \not\lesssim_S q$, then there exist $a \in Act$ and p' with $p \xrightarrow{a} p'$, such that $p' \not\lesssim_S q'$ for every $q \xrightarrow{a} q'$. We show by induction on the depth of q that if $p \not\lesssim_S q$ then $q \in \bar{\Phi}(p)$. If $I(q) = \emptyset$, then $q \in [\![[a]\mathbf{ff}]\!]$ for some $a \in I(p)$ (notice that $I(p) \neq \emptyset$ by assumption) and thus $q \in \bar{\Phi}(p)$. Now, when $I(q) \neq \emptyset$, by the inductive hypothesis, $q' \in [\![\bar{\Phi}(p')]\!]$ for every $q \xrightarrow{a} q'$. Thus $q \in [\![[a]\bar{\Phi}(p')]\!]$, which implies that $q \in [\![\bar{\bar{\Phi}}(p)]\!]$. Since $p \not\lesssim_S q$ by assumption, we have that $q \in [\![\bar{\Phi}(p)]\!]$. Therefore $q \in [\![\bar{\chi}(p)]\!]$ holds by definition of $\bar{\chi}(p)$, and we are done.

Case bisimulation (BS). We simply define $\bar{\chi}(p) = \neg\chi(p)$ ($\chi(p)$ exists thanks to Proposition 10) and $[\![\bar{\chi}(p)]\!] = P \setminus [\![\chi(p)]\!]$ holds trivially. $\qquad\square$

Appendix B. Proofs for semantics in van Glabbeek's linear time spectrum

Lemma 5. *For all $p, q \in P$ such that $p \lesssim_T q$, we have $depth(p) \leq depth(q)$.*

Proof. The claim immediately follows from the definition of $depth(p)$ (i.e., $depth(p)$ is the length of a longest trace in $\mathsf{T}(p)$) and the fact that $p \lesssim_T q$ if and only if $\mathsf{T}(p) \subseteq \mathsf{T}(q)$. $\qquad\square$

Corollary 5. *Let $X \in$ Ltime-spectrum. For all $p, q \in P$ such that $p \lesssim_X q$, we have $depth(p) \leq depth(q)$.*

Lemma 6. $\mathcal{B}_X(p) \subseteq \mathcal{L}_X(p)$ *holds for every $X \in$ Ltime-spectrum and $p \in P$.*

Proof. Let $X \in$ Ltime-spectrum, $p \in P$, and $\phi \in \mathcal{B}_X(p)$. If $\phi = \mathbf{tt}$, then the thesis follows trivially. If $\phi = formula_X(x)$ for some $x \in X^{fin}(p)$, then we distinguish the following cases.

- If $X = \mathsf{T}$, then $\phi = \langle \tau \rangle \mathbf{tt}$ for $\tau \in \mathsf{T}^{fin}(p) \subseteq \mathsf{T}(p)$. Since $\tau \in \mathsf{T}(p)$, it clearly holds $p \in [\![\phi]\!]$, which means $\phi \in \mathcal{L}_\mathsf{T}(p)$.

- If $X = \mathsf{CT}$, then $\phi = \langle \tau \rangle \mathbf{0}$ for $\tau \in \mathsf{CT}^{fin}(p) \subseteq \mathsf{CT}(p)$. Since $\tau \in \mathsf{CT}(p)$, it clearly holds $p \in [\![\phi]\!]$, which means $\phi \in \mathcal{L}_\mathsf{CT}(p)$.

- If $X = \mathsf{F}$, then $\phi = \langle \tau \rangle \bigwedge_{a \in Y} [a]\mathbf{ff}$ for $\langle \tau, Y \rangle \in \mathsf{F}^{fin}(p) \subseteq \mathsf{F}(p)$. From the definition of $\mathsf{F}(p)$, it immediately follows $p \in [\![\phi]\!]$, and we are done.

- If $X = \mathsf{R}$, then $\phi = \langle \tau \rangle \langle\!\langle Y \rangle\!\rangle$ for $\langle \tau, Y \rangle \in \mathsf{R}^{fin}(p) \subseteq \mathsf{R}(p)$. From the definition of $\mathsf{R}(p)$, it immediately follows $p \in [\![\phi]\!]$, and we are done.

- If $X = \mathsf{FT}$, then let $\phi = formula_\mathsf{FT}(\sigma)$ for $\sigma \in \mathsf{FT}^{fin}(p)$; we proceed by induction on σ.

    - If $\sigma = \varepsilon$, then $\phi = \mathbf{tt}$, and the thesis follows trivially;

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– If $\sigma = a\sigma'$, with $a \in Act$ and $\sigma' \in \mathsf{FT}^{fin}(p')$ for some p' such that $p \xrightarrow{a} p'$, then $\phi = \langle a \rangle formula_{\mathsf{FT}}(\sigma')$ and $formula_{\mathsf{FT}}(\sigma') \in \mathcal{B}_{\mathsf{FT}}(p')$[2]. By the inductive hypothesis, we have that $formula_{\mathsf{FT}}(\sigma') \in \mathcal{L}_{\mathsf{FT}}(p')$, that is, $p' \in [\![formula_{\mathsf{FT}}(\sigma')]\!]$. It follows that $p \in [\![\phi]\!]$, which equals to $\phi \in \mathcal{L}_{\mathsf{FT}}(p)$.

– If $\sigma = Y\sigma'$, with $Y \in \mathcal{P}(Act)$ and $\sigma' \in \mathsf{FT}^{fin}(p)$, then $\phi = \bigwedge_{a \in Y}[a]\mathbf{ff} \wedge formula_{\mathsf{FT}}(\sigma')$. By $Y\sigma' \in \mathsf{FT}(p')$, we have that $p \in [\![\bigwedge_{a \in Y}[a]\mathbf{ff}]\!]$. By $\sigma' \in \mathsf{FT}^{fin}(p)$, it holds $formula_{\mathsf{FT}}(\sigma') \in \mathcal{B}_{\mathsf{FT}}(p)$ and thus, by the inductive hypothesis, $formula_{\mathsf{FT}}(\sigma') \in \mathcal{L}_{\mathsf{FT}}(p)$, that is, $p \in [\![formula_{\mathsf{FT}}(\sigma')]\!]$. It follows that $p \in [\![\phi]\!]$, which equals to $\phi \in \mathcal{L}_{\mathsf{FT}}(p)$.

• If $X \in \{\mathsf{RT}, \mathsf{IFT}, \mathsf{PFT}, \mathsf{I2T}, \mathsf{P2T}\}$, then the proof uses the same inductive argument as in the previous case, and thus we omit the details.

• If $X = \mathsf{IF}$, then $\phi = \langle\tau\rangle \bigwedge_{\tau' \in \Gamma}[\tau']\mathbf{ff}$ for $\langle\tau, \Gamma\rangle \in \mathsf{IF}^{fin}(p) \subseteq \mathsf{IF}(p)$. From the definition of $\mathsf{IF}(p)$, it immediately follows that $p \in [\![\phi]\!]$, and we are done.

• If $X = \mathsf{PF}$, then $\phi = \langle\tau\rangle\langle\!\langle\Gamma\rangle\!\rangle$ for $\langle\tau, \Gamma\rangle \in \mathsf{PF}^{fin}(p) \subseteq \mathsf{PF}(p)$. From the definition of $\mathsf{PF}(p)$, it immediately follows that $p \in [\![\phi]\!]$, and we are done.

• If $X = \mathsf{I2}$, then $\phi = \langle\tau\rangle \bigwedge_{[p']_{\mathsf{BS}} \in \mathbb{P}} \neg\chi_{\mathsf{s}}(p')$ for $\langle\tau, \mathbb{P}\rangle \in \mathsf{I2}^{fin}(p) \subseteq \mathsf{I2}(p)$. By the definition of $\mathsf{I2}(p)$, $\langle\tau, \mathbb{P}\rangle \in \mathsf{I2}(p)$ implies that there is some q such that $p \xrightarrow{\tau} q$ and $q_{\downarrow\mathsf{s}} \cap \mathbb{P} = \emptyset$. From $q_{\downarrow\mathsf{s}} \cap \mathbb{P} = \emptyset$, it follows that $p' \not\lesssim_{\mathsf{S}} q$ for all $[p']_{\mathsf{BS}} \in \mathbb{P}$, which means that $q \notin [\![\chi_{\mathsf{s}}(p')]\!]$ for all $[p']_{\mathsf{BS}} \in \mathbb{P}$. Therefore, $p \in [\![\phi]\!]$, and we are done.

• If $X = \mathsf{P2}$, then $\phi = \langle\tau\rangle\alpha(\mathbb{P})$, with

$$\alpha(\mathbb{P}) = \bigwedge_{[p']_{\mathsf{BS}} \in \mathbb{P}} \chi_{\mathsf{s}}(p') \wedge \bigvee_{[p']_{\mathsf{BS}} \in \mathbb{P}} \textit{simulated-by}(p')$$

and

$$\textit{simulated-by}(p') = \bigwedge_{a \in Act}[a] \bigvee_{p' \xrightarrow{a} p''} \textit{simulated-by}(p'').$$

By the definition of $\mathsf{P2}(p)$, $\langle\tau, \mathbb{P}\rangle \in \mathsf{P2}(p)$ implies that there is some q such that $p \xrightarrow{\tau} q$ and $q_{\downarrow\mathsf{s}} = \mathbb{P}$. It follows that $p' \lesssim_{\mathsf{S}} q$ for all $[p']_{\mathsf{BS}} \in \mathbb{P}$, which means that $q \in [\![\chi_{\mathsf{s}}(p')]\!]$ for all $[p']_{\mathsf{BS}} \in \mathbb{P}$. Moreover, by $q_{\downarrow\mathsf{s}} = \mathbb{P}$, we have that $[q]_{\mathsf{BS}} \in \mathbb{P}$. Since $q \in \textit{simulated-by}(q)$ clearly holds, we conclude $p \in [\![\phi]\!]$, and we are done. $\qquad\square$

Lemma 7. *Formula* $\bigwedge_{\psi \in \mathcal{B}_X(p)} \psi$ *is characteristic for p within $\mathcal{L}_X$, for all $X \in$* Ltime-spectrum *and $p \in P$.*

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[2]Notice that, while it is clear that $\sigma' \in \mathsf{FT}(p')$, we are not guaranteed in general that $|\sigma'| \leq 2 \cdot depth(p')$ holds (which is needed for σ' to belong to $\mathsf{FT}^{fin}(p')$). However, it can be shown that we can always reduce to the case where this holds (see item *(f)* at page 28 for a detailed explanation).

Proof. According to Proposition 1(i), Theorem 6, and Definition 9, it suffices to show, for all $X \in$ Ltime-spectrum and $p \in P$, that $[\![\bigwedge_{\psi \in \mathcal{B}_X(p)} \psi]\!] = \{p' \in P \mid X(p) \subseteq X(p')\}$.

In order to show the inclusion $\{p' \in P \mid X(p) \subseteq X(p')\} \subseteq [\![\bigwedge_{\psi \in \mathcal{B}_X(p)} \psi]\!]$, let $p' \in P$ be such that $X(p) \subseteq X(p')$. By Proposition 16 (*(a)* $\Rightarrow$ *(b)*), we have $X^{fin}(p) \subseteq X^{fin}(p')$. We show that $p' \in [\![\psi]\!]$ for all $\psi \in \mathcal{B}_X(p)$. To this end, consider a generic element ψ of $\mathcal{B}_X(p)$, which means that $\psi = formula_X(x)$ for some $x \in X^{fin}(p) \subseteq X(p) \subseteq X(p')$. By Proposition 17 (left-to-right direction), we have $p' \in [\![formula_X(x)]\!] = [\![\psi]\!]$. Since ψ is a generic element of $\mathcal{B}_X(p)$, the claim follows.

Let us turn now to the converse inclusion. Let $p' \in P$ be such that $p' \in [\![\psi]\!]$ for all $\psi \in \mathcal{B}_X(p)$, which means $p' \in [\![formula_X(x)]\!]$ for all $x \in X^{fin}(p)$. By Proposition 17 (right-to-left direction), it holds that $x \in X(p')$ for all $x \in X^{fin}(p)$, that is, $X^{fin}(p) \subseteq X(p')$. By Proposition 16 (*(c)* $\Rightarrow$ *(a)*), we have $X(p) \subseteq X(p')$, and we are done. $\square$