# Interval Temporal Logics over Strongly Discrete Linear Orders: Expressiveness and Complexity

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**Abstract**

Interval temporal logics provide a natural framework for temporal reasoning about interval structures over linearly ordered domains, where intervals are taken as the primitive ontological entities. Their computational behavior mainly depends on two parameters: the set of modalities they feature and the linear orders over which they are interpreted. In this paper, we identify all fragments of Halpern and Shoham's interval temporal logic HS with a decidable satisfiability problem over the class of strongly discrete linear orders as well as over its relevant subclasses (the class of finite linear orders, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{Z}^-$). We classify them in terms of both their relative expressive power and their complexity, which ranges from NP-completeness to non-primitive recursiveness.

*Keywords:* Interval Temporal Logics, Discrete Linear Orders, Expressiveness, Decidability, Complexity

## 1. Introduction

Most temporal logics proposed in the literature assume a point-based model of time. They have been successfully applied in a variety of fields, ranging from the specification and verification of communication protocols to temporal data mining. However, a number of relevant application domains, such as, for instance, those of planning and synthesis of controllers, are often characterized by advanced features like durative actions, and their temporal relationships, accomplishments, and temporal aggregations, which are neglected or dealt with in an unsatisfactory way by point-based formalisms [1]. Interval temporal logics provide a natural framework for temporal reasoning about interval structures over linearly (or partially) ordered domains. They take time intervals as the

primitive ontological entities and define truth of formulas relative to time intervals, rather than time points. Interval logic modalities correspond to various relations between pairs of intervals. In particular, the well-known logic HS [2] features a set of modalities that make it possible to express all Allen's interval relations [3]. Interval-based formalisms have been extensively used in various areas of computer science and AI, such as, for instance, specification and verification of reactive systems, temporal databases, theories of action and change, natural language processing, and constraint satisfaction. However, most of them make severe syntactic and/or semantic restrictions that considerably weaken their expressive power. Interval temporal logics relax these restrictions, thus allowing one to cope with much more complex application domains and scenarios. Unfortunately, many of them, including HS and the majority of its fragments, turn out to be undecidable. Among the few decidable cases, we mention Propositional Neighborhood Logic (PNL) [4, 5] and the logic D of temporal sub-intervals (over dense linear orders) [6].

The computational properties of any HS fragment mainly depend on two parameters: (i) the set of its interval modalities, and (ii) the class of linear orders over which formulas are interpreted. While the first parameter is fairly natural, the second is definitely less obvious. In most cases, the computational behavior of an interval logic does not change when we move from one class of linear orders to another. However, some meaningful exceptions exist. A real character is the logic D: its satisfiability problem is PSPACE-complete over the class of dense linear orders and undecidable over the classes of finite and (weakly) discrete ones (and its status over the class of all linear orders is still unknown). In the last years, the decidability of interval temporal logics has been extensively studied with respect to various meaningful classes of linear, including the class of finite linear orders, the class of strongly discrete linear orders (there is a finite number of points between any pair of points), the class of weakly discrete linear orders (every point with a successor/predecessor has an immediate successor/predecessor), which includes non-standard temporal structures like, for instance, $\mathbb{N} + \{\omega\}$, the class of dense linear orders, and the class of all linear orders, plus some temporal structures of special interest like $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

In this paper, we focus our attention on the class of strongly discrete linear orders and its relevant subclasses, namely, the class of finite linear orders, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{Z}^-$ (the set of all negative integers). Strongly discrete linear orders come into play in a variety of application domains. Consider, for instance, planning problems. They consist of finding a finite partially-ordered sequence of actions that leads the system from the initial to the final state (goal) within a bounded amount of time, satisfying suitable conditions about which sequence of states the world must go through. In this scenario, finite linear orders are usually the most natural option for time modeling. In other fields, such as, for instance, specification and verification of reactive systems, the system is supposed to run forever, starting from some initial state, satisfying a number of safety and response properties. In this case, $\mathbb{N}$ may be the most appropriate choice.

The aim of this paper is twofold: (i) to give a complete picture of HS fragments with respect to decidability/undecidability of their satisfiability problem

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| Relation | Op | Formal definition | Example |
|----------|-----|-------------------|---------|
| meets | $\langle A\rangle$ | $[x,y]R_A[x',y'] \Leftrightarrow y = x'$ | |
| before | $\langle L\rangle$ | $[x,y]R_L[x',y'] \Leftrightarrow y < x'$ | |
| started-by | $\langle B\rangle$ | $[x,y]R_B[x',y'] \Leftrightarrow x = x', y' < y$ | |
| finished-by | $\langle E\rangle$ | $[x,y]R_E[x',y'] \Leftrightarrow y = y', x < x'$ | |
| contains | $\langle D\rangle$ | $[x,y]R_D[x',y'] \Leftrightarrow x < x', y' < y$ | |
| overlaps | $\langle O\rangle$ | $[x,y]R_O[x',y'] \Leftrightarrow x < x' < y < y'$ | |

Table 1: Allen's interval relations and corresponding HS modalities.

over the considered cases, filling in the remaining gaps, and (ii) to identify the set of all expressively-different decidable fragments and to determine their exact complexity. In the subsequent sections, we first give a short account of notation and basic notions. Then we review known results, pointing out those HS fragments for which we have incomplete information. Next, we study the expressive power (with respect to modal definability) of all decidable fragments with respect to all classes of linear orders considered in the paper. Sections 5–7 are devoted to decidability/undecidability and complexity results, given in increasing order of complexity (from NP to undecidable). Conclusions provide an assessment of the achieved results.

## 2. Preliminaries

Let $\mathbb{D} = \langle D, < \rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[x,y]$, where $x, y \in D$ and $x < y$ (*strict semantics*)[1]. Excluding equality, there are 12 different non-trivial relative position relations between pairs of intervals in a linear order, often called *Allen's relations* [3]: the six relations depicted in Tab. 1 and the inverse ones. In modal interval temporal logics, interval structures are interpreted as Kripke structures and Allen's relations as accessibility relations, thus associating a modality with each Allen's relation $R_X$. Formally, for each relation $R_X$ in Tab. 1, we introduce a modality $\langle X \rangle$ for $R_X$ and a *transposed modality* $\langle \overline{X} \rangle$ for the inverse relation $R_{\overline{X}}$ (that is, $R_{\overline{X}} = (R_X)^{-1}$).

Halpern and Shoham's logic HS is a multi-modal logic with formulas built on a set $\mathcal{AP}$ of proposition letters, the Boolean connectives $\vee$ and $\neg$, and a modality for each Allen's relation. We denote by $X_1 \ldots X_k$ the fragment of HS featuring

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[1]Strict semantics excludes intervals with coincident endpoints (point intervals). A *non-strict semantics*, including point intervals, can be possibly adopted. Even though most results can be easily rephrased in this alternative setting, strict semantics is definitely cleaner. Moreover, it is coherent with recent developments in temporal logic that consider points and intervals as different semantic entities (see, e.g., [7]).

| Ref. | Finite linear orders | Strongly discrete linear orders / Integers | Natural numbers | Negative integers |
|---|---|---|---|---|
| [8] | NP-completeness of B$\overline{B}$, E$\overline{E}$ | NP-completeness of B$\overline{B}$, E$\overline{E}$ | NP-completeness of B$\overline{B}$, E$\overline{E}$ | NP-completeness of B$\overline{B}$, E$\overline{E}$ |
| [9] | NEXPTIME-membership of A$\overline{A}$ | NEXPTIME-membership of A$\overline{A}$ | NEXPTIME-membership of A$\overline{A}$ | NEXPTIME-membership of A$\overline{A}$ |
| [10] | NEXPTIME-hardness of A, $\overline{A}$ | NEXPTIME-hardness of A, $\overline{A}$ | NEXPTIME-hardness of A, $\overline{A}$ | NEXPTIME-hardness of A, $\overline{A}$ |
| [11] | | EXPSPACE-hardness of AB, $\overline{A}$E | EXPSPACE-hardness of AB | EXPSPACE-hardness of $\overline{A}$E |
| [12] | EXPSPACE-membership of AB$\overline{B}$L, $\overline{A}$E$\overline{E}$L | EXPSPACE-membership of AB$\overline{B}$L, $\overline{A}$E$\overline{E}$L | EXPSPACE-membership of AB$\overline{B}$L, $\overline{A}$E$\overline{E}$L | EXPSPACE-membership of AB$\overline{B}$L, $\overline{A}$E$\overline{E}$L |
| [13] | Nonelementary decidability of A$\overline{A}$B$\overline{B}$, A$\overline{A}$E$\overline{E}$ | Undecidability of A$\overline{A}$B, A$\overline{A}\overline{B}$, A$\overline{A}$E, A$\overline{A}\overline{E}$ | Nonelementary decidability of $\overline{A}$B | Nonelementary decidability of AE |
| [13] | Non-primitive recursiveness of A$\overline{A}$B, A$\overline{A}\overline{B}$, A$\overline{A}$E, A$\overline{A}\overline{E}$ | Undecidability of A$\overline{A}$B, A$\overline{A}\overline{B}$, A$\overline{A}$E, A$\overline{A}\overline{E}$ | Undecidability of A$\overline{A}$B, A$\overline{A}\overline{B}$ | Undecidability of A$\overline{A}$E, A$\overline{A}\overline{E}$ |

Table 2: Known results with bibliographic references.

a modality for each Allen's relation in the set $\{R_{X_1}, \ldots, R_{X_k}\}$. Formulas of $\mathsf{X_1 \ldots X_k}$ are defined by the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X_1 \rangle\varphi \mid \ldots \mid \langle X_k \rangle\varphi.$$

The other Boolean connectives can be viewed as abbreviations, while for each modality $\langle X \rangle$, the dual modality $[X]$ is defined as usual: $[X]\varphi \equiv \neg\langle X \rangle\neg\varphi$. Given a formula $\varphi$, the *length* of $\varphi$, denoted by $|\varphi|$, is the number of its symbols. The semantics of HS is given in terms of *interval models* $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$, where $\mathbb{I}(\mathbb{D})$ is the set of all intervals over $\mathbb{D}$ and $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$ is a *valuation function* that assigns to every $p \in \mathcal{AP}$ the set of intervals $V(p)$ over which $p$ holds. The *truth* of a formula over a given interval $[x, y]$ in an interval model $M$ is defined by structural induction on formulas: *(i)* a proposition letter $p$ is true over an interval $[x, y]$ iff $[x, y] \in V(p)$; *(ii)* Boolean connectives are dealt with in the standard way; *(iii)* for each modality $\langle X \rangle$, $M, [x, y] \Vdash \langle X \rangle\psi$ if and only if there exists an interval $[x', y']$ such that $[x, y]R_X[x', y']$ and $M, [x', y'] \Vdash \psi$. Given an interval model $M$ and a formula $\varphi$, we say that $M$ *satisfies* $\varphi$ if there is an interval $[x, y]$ in $\mathbb{I}(\mathbb{D})$ such that $M, [x, y] \Vdash \varphi$. We say that $\varphi$ is *satisfiable* if there exists an interval model that satisfies it, and we say that it is *valid* if it is satisfied by every interval of every interval model.

Hereafter, we will denote HS fragments by the set of their modalities in alphabetical order and omitting those which are definable in terms of the others.
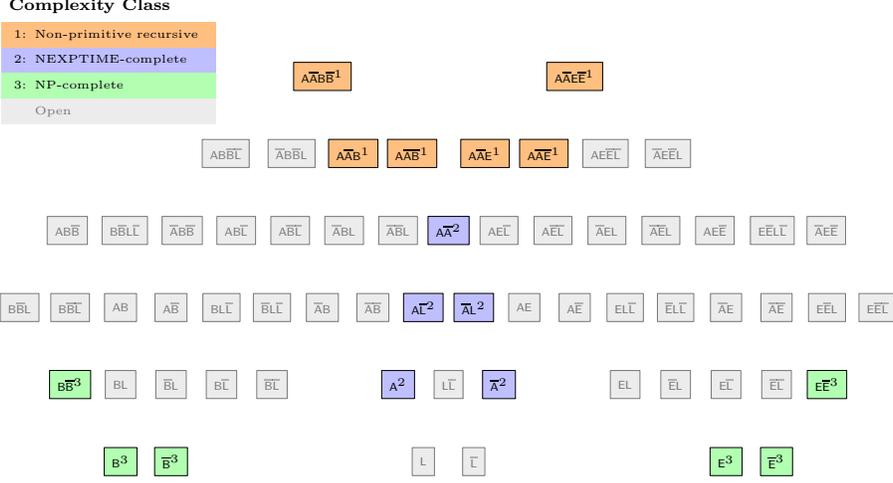
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**Complexity Class**

1: Non-primitive recursive
2: NEXPTIME-complete
3: NP-complete
Open

$A\overline{A}B\overline{B}^1$     $A\overline{A}E\overline{E}^1$

$AB\overline{B}L$   $\overline{A}B\overline{B}L$   $A\overline{A}B^1$   $A\overline{AB}^1$   $A\overline{A}E^1$   $A\overline{AE}^1$   $AE\overline{E}L$   $\overline{A}E\overline{E}L$

$AB\overline{B}$   $B\overline{B}L\overline{L}$   $\overline{A}B\overline{B}$   $AB\overline{L}$   $A\overline{B}L$   $\overline{A}BL$   $\overline{AB}L$   $A\overline{A}^2$   $AE\overline{L}$   $A\overline{E}L$   $\overline{A}EL$   $\overline{AE}L$   $AE\overline{E}$   $E\overline{E}L\overline{L}$   $\overline{A}E\overline{E}$

$B\overline{B}L$   $B\overline{BL}$   $AB$   $A\overline{B}$   $BL\overline{L}$   $\overline{B}L\overline{L}$   $\overline{A}B$   $\overline{AB}$   $A\overline{L}^2$   $\overline{A}L^2$   $AE$   $A\overline{E}$   $EL\overline{L}$   $\overline{E}L\overline{L}$   $\overline{A}E$   $\overline{AE}$   $E\overline{E}L$   $E\overline{EL}$

$B\overline{B}^3$   $BL$   $\overline{B}L$   $B\overline{L}$   $\overline{BL}$     $A^2$   $L\overline{L}$   $\overline{A}^2$     $EL$   $\overline{E}L$   $E\overline{L}$   $\overline{EL}$   $E\overline{E}^3$

$B^3$   $\overline{B}^3$     $L$   $\overline{L}$     $E^3$   $\overline{E}^3$

Figure 1: Hasse diagram of known decidable HS fragments over finite linear orders.

## 3. Overview of known expressiveness and (un)decidability results

In this section, we give a detailed account of known expressiveness and (un)decidability results for HS fragments over the considered (classes of) linear orders. We restrict our attention to modalities $\langle A \rangle, \langle L \rangle, \langle B \rangle$, and $\langle E \rangle$, and the transposed modalities $\langle \overline{A} \rangle, \langle \overline{L} \rangle, \langle \overline{B} \rangle$, and $\langle \overline{E} \rangle$. We do not consider modalities $\langle D \rangle$ and $\langle O \rangle$, and the transposed modalities $\langle \overline{D} \rangle$ and $\langle \overline{O} \rangle$, as HS fragments D, O, $\overline{\text{D}}$, and $\overline{\text{O}}$ (and all their extensions), over the considered classes of linear orders, are undecidable [14, 15]. We also make use of inter-definability equations that hold among HS modalities. In [16], it has been shown that, over the class of all linear orders (and thus also over the class of strongly discrete linear orders and its subclasses), the following equations hold: $\langle D \rangle p \equiv \langle B \rangle \langle E \rangle p$, $\langle \overline{O} \rangle p \equiv \langle B \rangle \langle \overline{E} \rangle p$, $\langle O \rangle p \equiv \langle E \rangle \langle \overline{B} \rangle p$, and $\langle \overline{D} \rangle p \equiv \langle \overline{B} \rangle \langle \overline{E} \rangle p$. Undecidability of BE, B$\overline{\text{E}}$, $\overline{\text{B}}$E, and $\overline{\text{BE}}$ immediately follows. Moreover, since $\langle L \rangle$ and $\langle \overline{L} \rangle$ can be defined in terms of $\langle A \rangle$ and $\langle \overline{A} \rangle$, respectively (it holds that $\langle L \rangle p \equiv \langle A \rangle \langle A \rangle p$, and $\langle \overline{L} \rangle \equiv \langle \overline{A} \rangle \langle \overline{A} \rangle p$ [16]), some (decidable) HS fragments turn out to be expressively equivalent to other ones, e.g., AL is equivalent to A, and thus they can be safely omitted.

In Tab. 2, we summarize already known (un)decidability results for HS fragments over the considered (classes of) linear orders, together with the appropriate bibliographic references. It is worth noticing that Tab. 2 has 4 columns, instead of 5. The reason is that known results for the class of all strongly discrete linear orders and $\mathbb{Z}$ coincide.

In the following, we will partition the considered classes of linear orders in two categories: *left/right symmetric* structures, including the class of finite linear orders, the class of strongly discrete linear orders, and $\mathbb{Z}$, and *asymmetric* structures, including $\mathbb{N}$ and $\mathbb{Z}^-$. For any HS fragment $\mathcal{F}$, we define its mirror
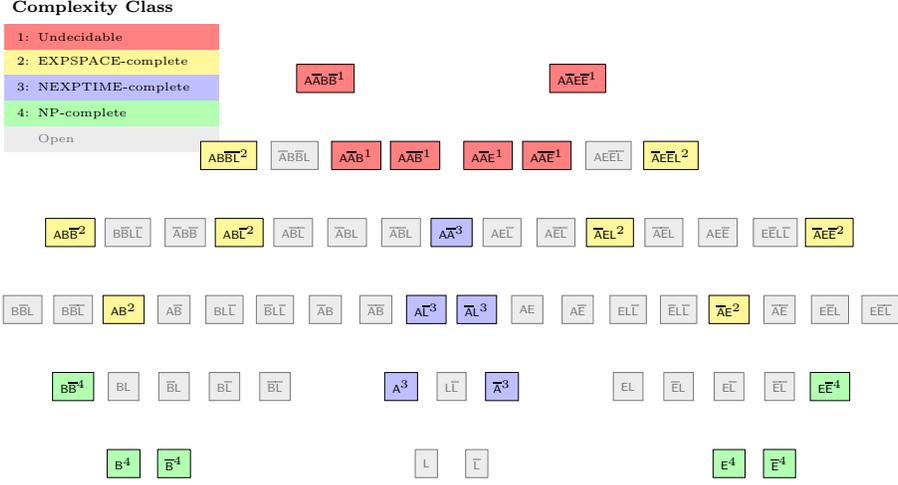
Figure 2: Hasse diagram of known decidable HS fragments over strongly discrete linear orders.

image (or *mirror* fragment) as the fragment obtained by replacing $\langle A \rangle$ by $\langle \overline{A} \rangle$, $\langle \overline{A} \rangle$ by $\langle A \rangle$, $\langle L \rangle$ by $\langle \overline{L} \rangle$, $\langle \overline{L} \rangle$ by $\langle L \rangle$, $\langle B \rangle$ by $\langle E \rangle$, and $\langle \overline{B} \rangle$ by $\langle \overline{E} \rangle$.

It can be easily checked that decidability and complexity results immediately transfer from a given fragment to its mirror image, when interpreted over symmetric structures. As for asymmetric structures, the results for a given fragment, interpreted over $\mathbb{N}$ (resp., $\mathbb{Z}^-$), transfer to its mirror image, interpreted over $\mathbb{Z}^-$ (resp., $\mathbb{N}$). In the following, we will focus our attention on three (classes of) linear orders: the class of finite linear orders (finite subclasses of it are not considered), the class of strongly discrete linear orders (all results hold for $\mathbb{Z}$ as well), and $\mathbb{N}$ (results transfer to mirror images over $\mathbb{Z}^-$).

As for expressiveness, most results are undefinability proofs, which highly depend on the considered (class of) linear orders.

We conclude the section with a pictorial representation of known results for the three classes of linear orders identified above. Fragments which are not displayed are all undecidable. Even though the three figures look quite similar, they present some meaningful differences. Grey nodes denote those fragments for which only incomplete information is available. As an example, AB$\overline{\text{BL}}$ is known to be in EXPSPACE over finite linear orders, but no hardness result has been proved, and thus the corresponding node is grey. Notice that, while Fig. 1 (finite linear orders) only displays decidable fragments, some undecidable fragments are included in Fig. 2 (strongly discrete linear orders) and in Fig. 3 (the natural numbers).

In the next section, we provide a classification of decidable fragments with respect their expressive power. Then, in the subsequent sections, building on such a classification, we give a complete picture of decidability and tight complexity results for HS fragments over the considered (classes of) linear orders.
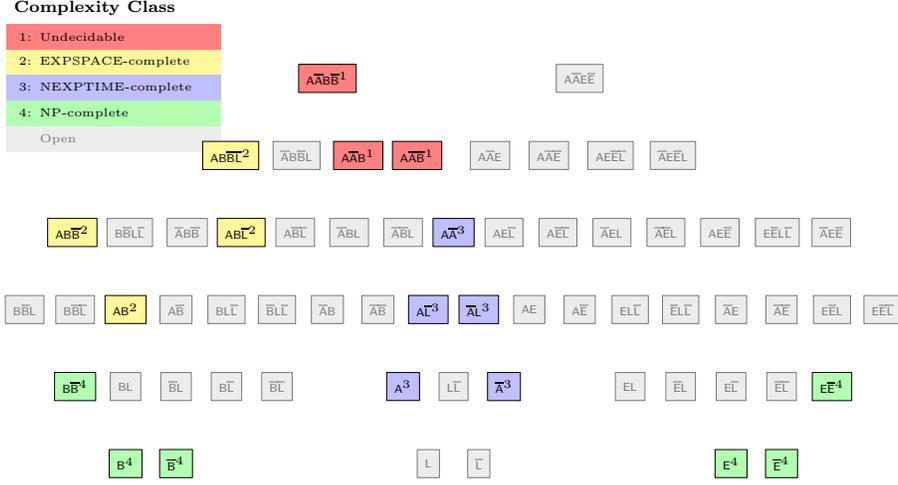
**Complexity Class**

1: Undecidable
2: EXPSPACE-complete
3: NEXPTIME-complete
4: NP-complete
Open

Figure 3: Hasse diagram of known decidable HS fragments over $\mathbb{N}$.

## 4. Expressive Power

In this section, we study the expressive power of (decidable) HS-fragments. Since only modalities $\langle A\rangle$, $\langle\overline{A}\rangle$, $\langle L\rangle$, $\langle\overline{L}\rangle$, $\langle B\rangle$, $\langle\overline{B}\rangle$ (or, symmetrically, $\langle A\rangle$, $\langle\overline{A}\rangle$, $\langle L\rangle$, $\langle\overline{L}\rangle$, $\langle E\rangle$, $\langle\overline{E}\rangle$) are considered, the only known definability results are the definability of $\langle L\rangle$ in terms of $\langle A\rangle$ and that of $\langle\overline{L}\rangle$ in terms of $\langle\overline{A}\rangle$ [16]. In the following, we show that no other inter-definability equations hold over all the considered (classes of) linear orders, thus proving the correctness of Fig. 4.

Given a fragment $\mathcal{F} = \mathsf{X_1 X_2 \ldots X_k}$ and a modality $\langle X\rangle$, we write $\langle X\rangle \in \mathcal{F}$ if $X \in \{X_1, \ldots, X_k\}$. Given two fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if $\langle X\rangle \in \mathcal{F}_1$ implies $\langle X\rangle \in \mathcal{F}_2$, for every modality $\langle X\rangle$.

**Definition 1.** Given an HS fragment $\mathcal{F}$ and an HS modality $\langle X\rangle$, we say that $\langle X\rangle$ is *definable* in $\mathcal{F}$, denoted $\langle X\rangle \lhd \mathcal{F}$, if $\langle X\rangle p \equiv \psi(p)$ for some formula $\psi(p)$ of $\mathcal{F}$, for any fixed proposition letter $p \in \mathcal{AP}$. The equivalence $\langle X\rangle p \equiv \psi(p)$ is called an *inter-definability equation for $\langle X\rangle$ in $\mathcal{F}$.*

In [2], Halpern and Shoham show that, according to the strict semantics, all HS modalities are definable in the fragment featuring modalities $\langle A\rangle$, $\langle B\rangle$, and $\langle E\rangle$, and the transposed modalities $\langle\overline{A}\rangle$, $\langle\overline{B}\rangle$, and $\langle\overline{E}\rangle$ (in case non-strict semantics is assumed, the four modalities $\langle B\rangle$, $\langle E\rangle$, $\langle\overline{B}\rangle$, and $\langle\overline{E}\rangle$ suffice, as shown in [17]).

Given two HS fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we say that $\mathcal{F}_2$ is *at least as expressive as* $\mathcal{F}_1$, denoted $\mathcal{F}_1 \preceq \mathcal{F}_2$, if each operator $\langle X\rangle \in \mathcal{F}_1$ is definable in $\mathcal{F}_2$, and that $\mathcal{F}_1$ is *strictly less expressive* than $\mathcal{F}_2$, denoted $\mathcal{F}_1 \prec \mathcal{F}_2$, if $\mathcal{F}_1 \preceq \mathcal{F}_2$, but not $\mathcal{F}_2 \preceq \mathcal{F}_1$. Moreover, we say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are *expressively incomparable*, denoted $\mathcal{F}_1 \not\equiv \mathcal{F}_2$, if neither $\mathcal{F}_1 \preceq \mathcal{F}_2$ nor $\mathcal{F}_2 \preceq \mathcal{F}_1$, and that they are *expressively equivalent* (or *expressively complete*), denoted $\mathcal{F}_1 \equiv \mathcal{F}_2$, if $\mathcal{F}_1 \preceq \mathcal{F}_2$ and $\mathcal{F}_2 \preceq \mathcal{F}_1$.

In order to show non-definability of a given modality in a certain fragment, we use the standard notion of *N-bisimulation*, suitably adapted to our setting (see Definition 2), and the invariance of modal formulas of modal depth at most $N$ with respect to $N$-bisimulations [18].

**Definition 2.** Let $\mathcal{F}$ be an HS-fragment. An $\mathcal{F}_N$-bisimulation between two models $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}'), V' \rangle$ over a set of proposition letters $\mathcal{AP}$ is a sequence of $N$ relations $Z_N, \ldots, Z_1 \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ such that

(i) for every $([x,y],[x',y']) \in Z_h$, with $N \geq h \geq 1$, $M, [x,y] \Vdash p$ if and only if $M', [x',y'] \Vdash p$, for all $p \in \mathcal{AP}$ *(local condition)*;

(ii) for every $([x,y],[x',y']) \in Z_h$, with $N \geq h > 1$, if $[x,y]R_X[v,w]$ for some $[v,w] \in \mathbb{I}(\mathbb{D})$ and some $\langle X \rangle \in \mathcal{F}$, then there exists $([v,w],[v',w']) \in Z_{h-1}$ such that $[x',y']R_X[v',w']$ *(forward condition)*;

(iii) for every $([x,y],[x',y']) \in Z_h$, with $N \geq h > 1$, if $[x',y']R_X[v',w']$ for some $[v',w'] \in \mathbb{I}(\mathbb{D}')$ and some $\langle X \rangle \in \mathcal{F}$, then there exists $([v,w],[v',w']) \in Z_{h-1}$ such that $[x,y]R_X[v,w]$ *(backward condition)*.

Given an $\mathcal{F}_N$-bisimulation, the truth of $\mathcal{F}$-formulas of modal depth at most $h-1$ is invariant for pairs of intervals belonging to $Z_h$, with $N \geq h \geq 1$.

The standard notion of $\mathcal{F}$-bisimulation can be recovered as a special case of $\mathcal{F}_N$-bisimulation.

**Definition 3.** Let $\mathcal{F}$ be an HS-fragment. An $\mathcal{F}$-bisimulation between two models $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}'), V' \rangle$ over a set of proposition letters $\mathcal{AP}$ is a relation $Z \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ such that, for every $([x,y],[x',y']) \in Z$,

(i) $M, [x,y] \Vdash p$ if and only if $M', [x',y'] \Vdash p$, for all $p \in \mathcal{AP}$ *(local condition)*;

(ii) if $[x,y]R_X[v,w]$ for some $[v,w] \in \mathbb{I}(\mathbb{D})$ and some $\langle X \rangle \in \mathcal{F}$, then there exists $([v,w],[v',w']) \in Z$ such that $[x',y']R_X[v',w']$ *(forward condition)*;

(iii) if $[x',y']R_X[v',w']$ for some $[v',w'] \in \mathbb{I}(\mathbb{D}')$ and some $\langle X \rangle \in \mathcal{F}$, then there exists $([v,w],[v',w']) \in Z$ such that $[x,y]R_X[v,w]$ *(backward condition)*.

To prove that a modality $\langle X \rangle$ is not definable in $\mathcal{F}$, it suffices to provide, for every natural number $N$, a pair of models $M$ and $M'$, and an $\mathcal{F}_N$-bisimulation between them for which there exists a pair $([x,y],[x',y']) \in Z_N$ such that $M, [x,y] \Vdash \langle X \rangle p$ and $M', [x',y'] \not\Vdash \langle X \rangle p$, for some $p \in \mathcal{AP}$. Such a result applies to all classes of linear orders that contain (as their elements) both structures on which $M$ and $M'$ are based. As an example, an undefinability result given for two structures based on $\mathbb{N}$ applies to $\mathbb{N}$ as well as to the class of strongly discrete linear orders, but not to the class of finite linear orders, $\mathbb{Z}^-$, and $\mathbb{Z}$. As there is no linear order contained in all the considered classes, no N-bisimulation can be found that works for all cases. In the following, we will first prove that Fig. 4 is sound and complete with respect to $\mathbb{Z}$ (Lemma 1, Lemma 2, and Lemma 3), and thus with respect to the class of strongly discrete linear orders as well; then, we will show how to tailor the proofs to the remaining classes.

To prove soundness and completeness of Fig. 4, we show that:

*(i)* all pairs of fragments which are not related to each other in Fig. 4 are expressively incomparable;

*(ii)* an edge from a fragment $\mathcal{F}_1$ to a fragment $\mathcal{F}_2$ means that $\mathcal{F}_2 \prec \mathcal{F}_1$;

*(iii)* if an HS-fragment is not displayed in Fig. 4, then it is undecidable.

Let us focus our attention on $\mathbb{Z}$. As for properties *(i)* and *(ii)*, it suffices to show that $\langle L \rangle p \equiv \langle A \rangle \langle A \rangle p$ and $\langle \overline{L} \rangle p \equiv \langle \overline{A} \rangle \langle \overline{A} \rangle p$ are all and only the inter-definability equations among modalities $\langle A \rangle, \langle \overline{A} \rangle, \langle L \rangle, \langle \overline{L} \rangle, \langle B \rangle, \langle \overline{B} \rangle$ over $\mathbb{Z}$ (the same for modalities $\langle A \rangle, \langle \overline{A} \rangle, \langle L \rangle, \langle \overline{L} \rangle, \langle E \rangle, \langle \overline{E} \rangle$). Proving that $\langle L \rangle p \equiv \langle A \rangle \langle A \rangle p$ and $\langle \overline{L} \rangle p \equiv \langle \overline{A} \rangle \langle \overline{A} \rangle p$ are valid inter-definability equations (soundness) is straightforward. To prove that these equations are the only possible ones (completeness), for each operator $\langle X \rangle \in \{\langle A \rangle, \langle \overline{A} \rangle, \langle L \rangle, \langle \overline{L} \rangle, \langle B \rangle, \langle \overline{B} \rangle\}$, we show that $\langle X \rangle$ is not definable in the maximal fragment of A$\overline{\text{A}}$B$\overline{\text{B}}$ not containing, as definable, $\langle X \rangle$ itself. This amounts to prove that:

*(1)* $\langle A \rangle \not\lessdot \overline{\text{A}}$B$\overline{\text{B}}$L and $\langle \overline{A} \rangle \not\lessdot$ AB$\overline{\text{BL}}$;

*(2)* $\langle \overline{B} \rangle \not\lessdot$ A$\overline{\text{A}}$B and $\langle B \rangle \not\lessdot$ A$\overline{\text{AB}}$;

*(3)* $\langle \overline{L} \rangle \not\lessdot$ AB$\overline{\text{B}}$ and $\langle L \rangle \not\lessdot \overline{\text{A}}$B$\overline{\text{B}}$.

We prove the above results one by one.

**Lemma 1.** $\langle A \rangle \not\lessdot \overline{\text{A}}$B$\overline{\text{BL}}$ *and* $\langle \overline{A} \rangle \not\lessdot$ AB$\overline{\text{BL}}$ *over* $\mathbb{Z}$.

**Lemma 2.** $\langle \overline{B} \rangle \not\lessdot$ A$\overline{\text{A}}$B *and* $\langle B \rangle \not\lessdot$ A$\overline{\text{AB}}$ *over* $\mathbb{Z}$.

**Lemma 3.** $\langle \overline{L} \rangle \not\lessdot$ AB$\overline{\text{B}}$ *and* $\langle L \rangle \not\lessdot \overline{\text{A}}$B$\overline{\text{B}}$ *over* $\mathbb{Z}$.

In both Lemma 1 and Lemma 2, we provide a suitable $\mathcal{F}_N$-bisimulation, for an arbitrary natural number $N$, while an $\mathcal{F}$-bisimulation suffices for Lemma 3. Since the proofs are technically rather involved, we decided to move them to an appendix.

The proofs of the above three lemmas can be adapted to $\mathbb{N}$, to the class of finite linear orders, and to $\mathbb{Z}^-$ by suitably restricting the bisimulation relations (details are given in the appendix). This allows us to conclude the following.

**Theorem 1.** $\langle L \rangle p \equiv \langle A \rangle \langle A \rangle p$ *and* $\langle \overline{L} \rangle p \equiv \langle \overline{A} \rangle \langle \overline{A} \rangle p$ *are the only inter-definability equations among the set of operators* $\{\langle A \rangle, \langle \overline{A} \rangle, \langle L \rangle, \langle \overline{L} \rangle, \langle B \rangle, \langle \overline{B} \rangle\}$ *over all the considered classes of linear orders.*

As for property *(iii)* (fragments which are not displayed in Fig. 4 are undecidable), it directly follows from the above lemmas and known undecidability results. By the above three lemmas, it holds that Fig. 4 contains all expressively-different fragments of HS featuring modalities from the set $\{\langle A \rangle, \langle \overline{A} \rangle, \langle B \rangle, \langle \overline{B} \rangle, \langle L \rangle, \langle \overline{L} \rangle\}$ or from the set $\{\langle A \rangle, \langle \overline{A} \rangle, \langle E \rangle, \langle \overline{E} \rangle, \langle L \rangle, \langle \overline{L} \rangle\}$. It can be easily checked that any other fragment contains (possibly as definable) at least one modality from the set $\{\langle D \rangle, \langle \overline{D} \rangle, \langle O \rangle, \langle \overline{O} \rangle\}$, and thus it is undecidable (see Section 2).
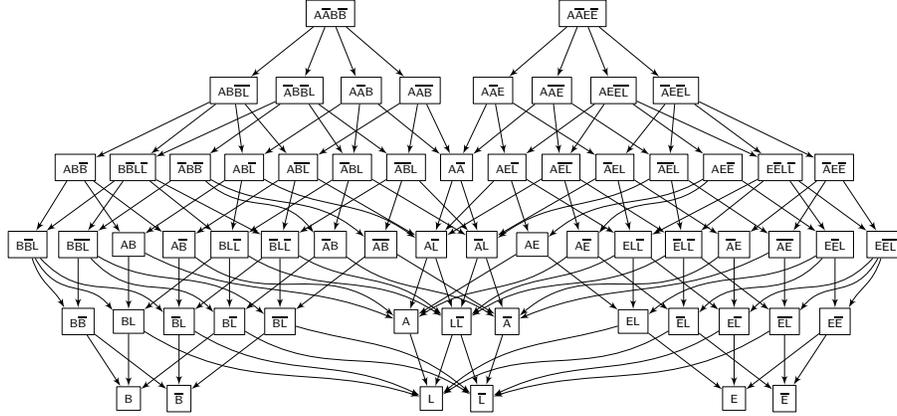
9

Figure 4: Hasse diagram of the considered HS fragments and their relative expressive power.

**Corollary 2.** *The Hasse diagram in Fig. 4 correctly characterizes the relative expressive power of the* HS*-fragments* $A\bar{A}B\bar{B}$, $A\bar{A}E\bar{E}$, *and all their sub-fragments, over any of the considered classes of linear orders.*

The rest of the paper is devoted to the analysis of the computational complexity of HS fragments, moving from lower- to higher-degree complexity classes, and transversally with respect to finite linear orders, strongly discrete linear orders, and $\mathbb{N}$. It is worth mentioning that the class NEXPTIME is not explicitly studied here, as HS fragments with a NEXPTIME-complete satisfiability problem have been already systematically investigated elsewhere (see Tab. 2).

## 5. NP-Completeness

In this section, we prove NP-completeness of $B\bar{B}L\bar{L}$ and of its mirror image $E\bar{E}L\bar{L}$. Since the satisfiability problem for propositional logic is NP-complete, $B\bar{B}L\bar{L}$ and its sub-fragments are at least NP-hard. The core of this section is thus a membership (NP-membership) proof. By a model-theoretic argument, we will show that satisfiability of $B\bar{B}L\bar{L}$-formulas can be reduced to the search for a periodic model, whose prefix and period lengths have a bound which is polynomial in the length of the formula. For the sake of simplicity, we consider the case of $B\bar{B}L\bar{L}$ interpreted over $\mathbb{N}$. The proof can be generalized to the class of strongly discrete linear orders and to $\mathbb{Z}$. The finite case comes for free: it is sufficient to impose that the length of the period is zero. Finally, the case of $\mathbb{Z}^-$ can be sorted out by simply reversing the proof for $\mathbb{N}$.

We start by observing that satisfiability of a $B\bar{B}L\bar{L}$-formula $\varphi$ over $\mathbb{N}$ can be reduced to satisfiability of the formula $\tau(\varphi) = \langle L \rangle \langle \bar{L} \rangle \varphi$ over the interval $[0, 1]$, that is, $M, [x, y] \Vdash \varphi$ for some $[x, y]$ if and only $M, [0, 1] \Vdash \tau(\varphi)$. Hence, we can safely restrict our attention to the satisfiability problem for a $B\bar{B}L\bar{L}$-formula

over $[0,1]$ (*initial* satisfiability). As a preliminary step, we introduce some useful notation and notions, including that of periodic model.

**Definition 4.** An interval model $M = \langle \mathbb{I}(\mathbb{N}), V \rangle$ is *ultimately periodic, with prefix Pre and period Per*, if for every interval $[x, y] \in \mathbb{I}(\mathbb{N})$ and proposition letter $p \in \mathcal{AP}$: *(i)* if $x \geq Pre$, then $[x, y] \in V(p)$ iff $[x + Per, y + Per] \in V(p)$ and *(ii)* if $y \geq Pre$, then $[x, y] \in V(p)$ iff $[x, y + Per] \in V(p)$.

Let $\varphi$ be the $\mathsf{B\overline{B}L\overline{L}}$-formula to be checked for satisfiability. We define the closure of $\varphi$, denoted $Cl(\varphi)$, as the set of all subformulas of $\varphi$ and of their negations. Let $M$ be a model such that $M, [0, 1] \Vdash \varphi$. For every point $x$ of the model, let $\mathcal{R}_L(x)$ (resp., $\mathcal{R}_{\overline{L}}(x)$) be the maximal subset of $Cl(\varphi)$ consisting of all and only those $\langle L \rangle$-formulas (resp., $\langle \overline{L} \rangle$-formulas) and their negations that are satisfied over intervals ending (resp., beginning) at $x$. Notice that all intervals ending (resp., beginning) at the same point must satisfy the same $\langle L \rangle$-formulas (resp., $\langle \overline{L} \rangle$-formulas). Let $\mathcal{R}(x) = \mathcal{R}_L(x) \cup \mathcal{R}_{\overline{L}}(x)$. $\mathcal{R}(x)$ must be consistent, that is, it cannot contain a formula and its negation. Let $\mathcal{R}$ be the subset of $Cl(\varphi)$ that contains all $\langle L \rangle$- and $\langle \overline{L} \rangle$-formulas and all their negations. It is immediate to see that $|\mathcal{R}| \leq 2|\varphi|$. In the following, we will also compare intervals with respect to satisfiability of $\langle B \rangle$- and $\langle \overline{B} \rangle$-formulas. Given a model $M$, we say that two intervals $[x, y]$ and $[x', y']$ are $B$-equivalent (denoted $[x, y] \equiv_B [x', y']$) if for every $\langle B \rangle \psi \in Cl(\varphi)$, $M, [x, y] \Vdash \langle B \rangle \psi$ iff $M, [x', y'] \Vdash \langle B \rangle \psi$ and for every $\langle \overline{B} \rangle \psi \in Cl(\varphi)$, $M, [x, y] \Vdash \langle \overline{B} \rangle \psi$ iff $M, [x', y'] \Vdash \langle \overline{B} \rangle \psi$. Hereafter, we will denote by $m_B$ the number of $\langle B \rangle$- and $\langle \overline{B} \rangle$-formulas in $Cl(\varphi)$.

To prove that the satisfiability problem for $\mathsf{B\overline{B}L\overline{L}}$ is in NP, we first prove that every satisfiable formula $\varphi$ has an ultimately periodic model; then, we show how to contract it to obtain a model whose prefix and period are polynomial in $|\varphi|$.

**Lemma 4.** *Let $\varphi$ be a $\mathsf{B\overline{B}L\overline{L}}$-formula and $M = \langle \mathbb{I}(\mathbb{N}), V \rangle$ be a model for $\varphi$. Then, there exists also an ultimately periodic model $M^* = \langle \mathbb{I}(\mathbb{N}), V^* \rangle$ for $\varphi$.*

*Proof.* Let $M = \langle \mathbb{I}(\mathbb{N}), V \rangle$ be such that $M, [0, 1] \Vdash \varphi$. If $M$ is not ultimately periodic, we turn it into an ultimately periodic model as follows.

First, we show that, by transitivity of $\langle L \rangle$ and $\langle \overline{L} \rangle$, there must exist a point $\bar{x} > 1$ such that $\mathcal{R}(z) = \mathcal{R}(\bar{x})$ for every $z \geq \bar{x}$. Let us consider modality $\langle L \rangle$. By definition, if $M, [x, y] \models \langle L \rangle \psi$, then $M, [x', y'] \models \langle L \rangle \psi$ for every $y' \leq y$. It immediately follows that $\mathcal{R}_L(y) \subseteq \mathcal{R}_L(y')$, that is, either $\mathcal{R}_L(y) = \mathcal{R}_L(y')$ or $\mathcal{R}_L(y) \subset \mathcal{R}_L(y')$. Since the set of $\langle L \rangle$-requests is finite, there exists a point $x_1 > 1$ such that $\mathcal{R}_L(z) = \mathcal{R}_L(x_1)$ for every $z \geq x_1$. A similar argument holds for modality $\langle \overline{L} \rangle$. By definition, if $M, [x, y] \models \langle \overline{L} \rangle \psi$, then $M, [x', y'] \models \langle \overline{L} \rangle \psi$ for every $x' \geq x$. It immediately follows that $\mathcal{R}_{\overline{L}}(x) \subseteq \mathcal{R}_{\overline{L}}(x')$, that is, either $\mathcal{R}_{\overline{L}}(x) = \mathcal{R}_{\overline{L}}(x')$ or $\mathcal{R}_{\overline{L}}(x) \subset \mathcal{R}_{\overline{L}}(x')$. Since the set of $\langle \overline{L} \rangle$-requests is finite, there exists a point $x_2 > 1$ such that $\mathcal{R}_{\overline{L}}(z) = \mathcal{R}_{\overline{L}}(x_2)$ for every $z \geq x_2$. We take $\bar{x} = max(\{x_1, x_2\})$ as the prefix $Pre$.

Next, we take as the period of the model a value $Per > m_B$ that satisfies the following conditions: *(i)* for every point $x \leq Pre$ and formula $\langle L \rangle \psi \in \mathcal{R}(x)$, there exists an interval $[x_\psi, y_\psi]$ such that $M, [x_\psi, y_\psi] \Vdash \psi$ and $x < x_\psi < y_\psi <$

$Pre + Per$; *(ii)* for every interval $[x, y]$ such that $x < Pre$ and $y \geq Pre + Per$ and every formula $\langle \overline{B} \rangle \psi$ such that $M, [x, y] \Vdash \langle \overline{B} \rangle \psi$, there exists an interval $[x, y_\psi]$ such that $[x, y] \equiv_B [x, y_\psi]$, $M, [x, y_\psi] \Vdash \psi$, and $Pre \leq y_\psi < Pre + Per$. The existence of such a $Per$ can be guaranteed as follows. Since $M$ is a model, all requests $\langle L \rangle \psi \in \mathcal{R}_L(x)$ are fulfilled. Let $x_1$ be the least natural number such that the right endpoints of the fulfilling intervals $[x_\psi, y_\psi]$ are less than or equal to it. As for condition *(ii)*, by definition of modality $\langle B \rangle$, if $M, [x, y] \models \langle B \rangle \psi$, then $M, [x, y'] \models \langle B \rangle \psi$ for every $y' \geq y$. Since the set of $\langle B \rangle$-requests is finite, there exists a point $x_2 > 1$ such that $M, [x, z] \models \langle B \rangle \psi$ if and only if $M, [x, x_2] \models \langle B \rangle \psi$, for all $\langle B \rangle \psi \in Cl(\varphi)$ and every $z \geq x_2$ (without loss of generality, we can assume $x_2$ to be greater than $Pre + m_B$). A symmetric argument can be used to prove that there exists a point $x_3 > 1$ such that $M, [x, z] \models \langle \overline{B} \rangle \psi$ if and only if $M, [x, x_3] \models \langle \overline{B} \rangle \psi$, for all $\langle \overline{B} \rangle \psi \in Cl(\varphi)$ and every $z \geq x_3$ (again, without loss of generality, we can assume $x_3$ to be greater than $Pre + m_B$). Any natural number greater than $max(\{x_1, x_2, x_3\}) - Pre$ can be taken as $Per$.

The above two conditions are not sufficient to guarantee periodicity; we need to add the following one: *(iii)* for every interval $[x, y]$ such that $Pre \leq x < Pre + Per$ and $y \geq Pre + 2Per$ and every formula $\langle \overline{B} \rangle \psi$ such that $M, [x, y] \Vdash \langle \overline{B} \rangle \psi$, there exists an interval $[x, y_\psi]$ such that $[x, y] \equiv_B [x, y_\psi]$, $M, [x, y_\psi] \Vdash \psi$, and $y_\psi < Pre + 2Per$. We show how to possibly change the valuation $V$ of the model $M$ to force condition *(iii)* to be satisfied. Let $[x, y]$ be an interval that violates condition *(iii)*. We choose a (finite) minimal set of "witness points" $WP = \{y_1 < \ldots < y_k\}$ such that *(a)* for every interval $[x, y']$ and every formula $\langle B \rangle \psi$, if $M, [x, y'] \Vdash \langle B \rangle \psi$, then there exists $y_i \in WP$ such that $M, [x, y_i] \Vdash \psi$ and $x < y_i < y'$, and *(b)* for every interval $[x, y'']$ and every formula $\langle \overline{B} \rangle \theta$, if $M, [x, y''] \Vdash \langle \overline{B} \rangle \theta$, then there exists $y_i \in WP$ such that $M, [x, y_i] \Vdash \theta$ and either $y_i > y''$ or $[x, y_i] \equiv_B [x, y'']$. As for condition *(a)*, for each formula $\langle B \rangle \psi \in Cl(\varphi)$, if there exists $y'$ such that $M, [x, y'] \Vdash \langle B \rangle \psi$, we add to $WP$ the least point $y_i$ such that $M, [x, y_i] \Vdash \psi$ (notice that this implies that, for all $y''$, if $M, [x, y''] \Vdash \langle B \rangle \psi$, then $y_i < y''$). As for condition *(b)*, we must distinguish two cases. If there exists a finite number of $y'$ such that $M, [x, y'] \Vdash \theta$ (that is, $\langle \overline{B} \rangle \theta$ is satisfied by a finite number of intervals), we add to $WP$ the greatest natural number $y_i$ such that $M, [x, y_i] \Vdash \theta$. Otherwise (there exists an infinite number of $y'$ such that $M, [x, y'] \Vdash \theta$), we add to $WP$ a sufficiently large natural number $y_i$ such that $M, [x, y_i] \Vdash \theta$ and for all $z \geq y_i, [x, y_i] \equiv_B [x, z]$. We already showed that, for every $x < Pre$, there exists $y'$ such that for each $\langle B \rangle \psi \in Cl(\varphi)$ (resp., $\langle \overline{B} \rangle \psi \in Cl(\varphi)$), $M, [x, y'] \Vdash \langle B \rangle \psi$ (resp., $M, [x, y'] \Vdash \langle \overline{B} \rangle \psi$) if and only if $M, [x, z] \Vdash \langle B \rangle \psi$ (resp., $M, [x, z] \Vdash \langle \overline{B} \rangle \psi$), for all $z \geq y'$. As a matter of fact, the assumption about $x$ does not play any role in the proof, and thus the claim can be safely extended to all $Pre \leq x < Pre + Per$. We take as $y_i$ a natural number $y''$, greater than or equal to $y'$, such that $M, [x, y''] \Vdash \theta$. As for the cardinality of $WP$, it immediately follows that $|WP| \leq m_B$ (the number of $\langle B \rangle$- and $\langle \overline{B} \rangle$-formulas in $Cl(\varphi)$).

We now focus our attention on those witness points $\{y_j < \ldots < y_k\}$ that are greater than $Pre + Per$, and we turn $V$ into a new valuation $V'$ defined as follows:

- for every $p \in \mathcal{AP}$ and every $j \le i \le k$, we put $[x, Pre + Per + (i - j + 1)] \in V'(p)$ iff $[x, y_i] \in V(p)$;

- for every $p \in \mathcal{AP}$ and every $Pre + Per + (k - j + 1) < y' \le y_k$, we put $[x, y'] \in V'(p)$ iff $[x, y_k] \in V(p)$;

- the valuation of all other intervals remains unchanged.

Notice that the second item is used to unproblematically fill intervals $[x, Pre + Per + (k - j + 1) + 1], \ldots, [x, y_k]$ by forcing their valuation (in $M'$) to be equal to the valuation of $[x, y_k]$ (in $M$)[2]. As for the last item, it is worth pointing out that the valuation of intervals $[x, y']$, with $x < y' \le Pre + Per$, and the valuation of intervals $[x', y']$, with $Pre + Per < x', y' \le y_k$, do not change.

It can be easily checked that the validity of conditions *(i)* and *(ii)* is not affected by such a rewriting of the valuation function. Moreover, once it has been completed, all intervals starting at $x$ fulfill condition *(iii)*.

We show that $M' = \langle \mathbb{I}(\mathbb{N}), V' \rangle$ is a model for $\varphi$, that is, $M', [0, 1] \Vdash \varphi$. As for formulas of the form $\langle L \rangle \psi \in \mathcal{R}(x)$, with $x \le Pre$, by condition *(i)*, we know that they are fulfilled (in $M$) by an interval $[x_\psi, y_\psi]$, with $x < x_\psi < y_\psi < Pre + Per$, whose valuation has not been changed (in $M'$). The case of formulas of the form $\langle \overline{L} \rangle \psi \in \mathcal{R}(x)$, with $x \le Pre$, is trivial. Let us consider now points $x > Pre$. By definition of $Pre$, all formulas $\langle L \rangle \psi \in \mathcal{R}(x)$, with $x > Pre$, are satisfied by infinitely many intervals, and thus any change in the valuation function that affects a finite number of intervals has not impact on their satisfiability. As for formulas $\langle \overline{L} \rangle \psi \in \mathcal{R}(x)$, with $x > Pre$, by definition of $Pre$, it immediately follows that they are satisfied (in $M$) by intervals contained in $[0, Pre]$, whose valuation has not been changed (in $M'$).

We now prove that intervals $[x, y_i]$ in $M$ and intervals $[x, Pre + Per + (i - j + 1)]$ in $M'$ behave the same with respect to the operators $\langle B \rangle$ and $\langle \overline{B} \rangle$ as well. For the sake of simplicity, we restrict our attention to formulas of the forms $\langle B \rangle p$ and $\langle \overline{B} \rangle p$, with $p \in \mathcal{AP}$. An easy inductive argument can be exploited to cope with the general case. Suppose now that there exist a $\langle B \rangle p \in Cl(\varphi)$ and $y_l \in WP$ such that $M, [x, y_l] \Vdash \langle B \rangle p$ and $M', [x, Pre + Per + (l - j + 1)] \nVdash \langle B \rangle p$. From $M, [x, y_l] \Vdash \langle B \rangle p$, it follows that there exists $\overline{y}$ such that $\overline{y} < y_l$ and $M, [x, \overline{y}] \Vdash p$. By condition *(a)*, the witness $y_h$ of $\langle B \rangle p$ in $WP$ is less than or equal to $\overline{y}$, which, in its turn, is less than $y_l$. It immediately follows that $M', [x, Pre + Per + (h - j + 1)] \Vdash p$ and thus $M', [x, Pre + Per + (l - j + 1)] \Vdash \langle B \rangle p$ (contradiction). Suppose now that there exist a $\langle \overline{B} \rangle p \in Cl(\varphi)$ and $y_l \in WP$ such that $M, [x, y_l] \Vdash \langle \overline{B} \rangle p$ and $M', [x, Pre + Per + (l - j + 1)] \nVdash \langle \overline{B} \rangle p$. First of all, we observe that the latter cannot be the case if there exists an infinite number of intervals $[x, y]$ such that $M, [x, y] \Vdash p$ ($V'$ differs from $V$ in the valuation of a finite number of intervals). Now, from $M, [x, y_l] \Vdash \langle \overline{B} \rangle p$, it follows that there

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[2]A contraction of the domain of $M$ is not a viable alternative, as removing the points belonging to the interval $[Pre + Per + (k - j + 1) + 1, y_k]$ may invalidate (in $M'$) some $\langle B \rangle$- or $\langle \overline{B} \rangle$-formulas over intervals starting at some $x' \ne x$.

exists $\overline{y}$ such that $\overline{y} > y_l$ and $M, [x, \overline{y}] \Vdash p$. Since there exists a finite number of intervals $[x, y]$ such that $M, [x, y] \Vdash p$, by the finite case of condition *(b)*, the witness $y_h$ of $\langle \overline{B} \rangle p$ in $WP$ is greater than or equal to $\overline{y}$, which, in its turn, is greater than $y_l$. It immediately follows that $M', [x, Pre + Per + (h - j + 1)] \Vdash p$ and thus $M', [x, Pre + Per + (l - j + 1)] \Vdash \langle \overline{B} \rangle p$ (contradiction). As for the converse, from minimality of $WP$ and from the fact that $V'$ "preserves" the order, that is, points $x_l, x_h \in WP$, with $l < h$, are "mapped" into points $Pre + Per + (l - j + 1) < Pre + Per + (h - j + 1)$, it easily follows that if $M', [x, Pre + Per + (h - j + 1)] \Vdash \langle B \rangle p$ (resp., $M', [x, Pre + Per + (l - j + 1)] \Vdash \langle \overline{B} \rangle p$) and $M', [x, Pre + Per + (l - j + 1)] \Vdash p$ (resp., $M', [x, Pre + Per + (h - j + 1)] \Vdash p$), then $M, [x, y_h] \Vdash \langle B \rangle p$ (resp., $M, [x, y_l] \Vdash \langle \overline{B} \rangle p$) and $M, [x, y_l] \Vdash p$ (resp., $M, [x, y_h] \Vdash p$), for some $j <= l < h <= k$. Moreover, if $M', [x, Pre + Per + (l - j + 1)] \Vdash \langle B \rangle p$ (resp., $M', [x, Pre + Per + (l - j + 1)] \Vdash \langle \overline{B} \rangle p$), for some $j \leq l \leq k$, and $M', [x, y'] \Vdash p$, for some $y' < y_j$ (resp., $y' > y_k$), then $M, [x, y_l] \Vdash \langle B \rangle p$ (resp., $M, [x, y_l] \Vdash \langle \overline{B} \rangle p$) and $M, [x, y'] \Vdash p$ (notice that this is always the case with $[x, Pre + Per + 1]$ and $[x, Pre + Per + (k - j + 1)]$, respectively). Again, an easy inductive argument can be exploited to cope with the general case.

By repeating such a procedure a sufficient number of times (at most, as many times as the points in between $Pre$ and $Pre + Per$ are), we obtain a model $M = \langle \mathbb{I}(\mathbb{N}), \overline{V} \rangle$ for $\varphi$ that satisfies conditions *(i)*, *(ii)*, and *(iii)*.

We are now ready to build the ultimately periodic model $M^* = \langle \mathbb{I}(\mathbb{N}), V^* \rangle$. First, we define the valuation function $V^*$ for some of the intervals whose left endpoint belongs to the prefix or to the first occurrence of the period: (a) for each $p \in \mathcal{AP}$ and each $[x, y]$, with $y < Pre + Per$, $[x, y] \in V^*(p)$ if and only if $[x, y] \in \overline{V}(p)$; (b) for each $p \in \mathcal{AP}$ and each $[x, y]$, with $Pre \leq x < Pre + Per$ and $y \leq x + Per$, $[x, y] \in V^*(p)$ if and only if $[x, y] \in \overline{V}(p)$. Then, we extend $V^*$ to cover the entire model: *(1)* for each $p \in \mathcal{AP}$ and each $[x, y]$, with $x < Pre$ and $y \geq Pre + Per$, $[x, y] \in V^*(p)$ if and only if $[x, y - Per] \in V^*(p)$; *(2)* for each $p \in \mathcal{AP}$ and each $[x, y]$, with $Pre \leq x < Pre + Per$ and $y > x + Per$, $[x, y] \in V^*(p)$ if and only if $[x, y - Per] \in V^*(p)$; *(3)* for each $p \in \mathcal{AP}$ and each $[x, y]$, with $x \geq Pre + Per$, $[x, y] \in V^*(p)$ if and only if $[x - Per, y - Per] \in V^*(p)$. It is easy to check that $M^*, [0, 1] \Vdash \varphi$ and thus $M^*$ is the ultimately periodic model we were looking for. □

The next lemma shows that, by applying a point-elimination technique similar to the one used in [19], we can reduce the length of the prefix and the period of an ultimately periodic model to a size polynomial in $|\varphi|$.

**Lemma 5.** *Let $\varphi$ be a $\mathsf{B\overline{B}L\overline{L}}$-formula. Then, $\varphi$ is initially satisfiable over $\mathbb{N}$ if and only if it is initially satisfiable over an ultimately periodic model $M = \langle \mathbb{I}(\mathbb{N}), V \rangle$, with $Pre + Per \leq (m_L + 2) \cdot m_B + m_L + 3$, where $m_L = 2|\mathcal{R}|$.*

*Proof.* By Lemma 4, we can assume that $\varphi$ is initially satisfied over an ultimately periodic model $M = \langle \mathbb{I}(\mathbb{N}), V \rangle$. If $Pre + Per$ is not less than or equal to $(m_L + 2) \cdot m_B + m_L + 3$, we proceed as follows.

To start with, we show that, for each $\langle L \rangle \psi \in \mathcal{R}(x)$, with $1 < x < Pre + 2Per$, there exist $1 < x' \leq Pre + Per$ and $y' < Pre + 2Per$ such that the interval $[x', y']$ satisfies $\psi$. Let $[x_\psi, y_\psi]$ be an interval such that $M, [x_\psi, y_\psi] \Vdash \psi$, with $x < x_\psi$ (since $M$ is a model, there exists at least one such interval). If $x_\psi > Pre + Per$, we take the smallest $k$ such that $x_\psi - (k \cdot Per) \leq Pre + Per$. By periodicity, $V([x_\psi - (k \cdot Per), y_\psi - (k \cdot Per)]) = V([x_\psi, y_\psi])$, and thus $M, [x_\psi - (k \cdot Per), y_\psi - (k \cdot Per)] \Vdash \psi$ as well. Consider now the right endpoint of the resulting interval. If $y_\psi - (k \cdot Per) \geq Pre + 2Per$, we take the smallest $k'$ such that $y_\psi - ((k + k') \cdot Per) < Pre + 2Per$. By periodicity, $V([x_\psi - (k \cdot Per), y_\psi - ((k + k') \cdot Per)]) = V([x_\psi - (k \cdot Per), y_\psi - (k \cdot Per)])$, and thus $M, [x_\psi - (k \cdot Per), y_\psi - ((k + k') \cdot Per)] \Vdash \psi$ as well. We choose $1 < x^\psi_{max} \leq Pre + Per$ and $y^\psi_{max} < Pre + 2Per$ such that $M, [x^\psi_{max}, y^\psi_{max}] \Vdash \psi$ and, for each $x^\psi_{max} < x \leq Pre + Per$, no interval starting at $x$ satisfies $\psi$. We collect all such points $x^\psi_{max}, y^\psi_{max}$ into a set (of $L$-blocked points) $Bl_L \subset \{0, \ldots, Pre + 2Per\}$. Similarly, for each $\langle \overline{L} \rangle \psi \in \mathcal{R}(x)$, with $1 < x < Pre + 2Per$, we choose an interval $[x^\psi_{min}, y^\psi_{min}]$ such that $M, [x^\psi_{min}, y^\psi_{min}] \Vdash \psi$ and, for each $y < y^\psi_{min}$, no interval ending at $y$ satisfies $\psi$. We collect all points $x^\psi_{min}, y^\psi_{min}$ into a set (of $\overline{L}$-blocked points) $Bl_{\overline{L}} \subset \{0, \ldots, Pre\}$.

Let $Bl = Bl_L \cup Bl_{\overline{L}} \cup \{Pre, Pre + Per\}$. It trivially holds that $|Bl| \leq m_L + 2$. Let $Bl = \{x_1 < x_2 < \ldots < x_n\}$. For each $0 < i < n$, let $Bl_i = \{x \mid x_i < x < x_{i+1}\}$; moreover, let $Bl_0 = \{x \mid 1 < x < x_1\}$ and $Bl_n = \{x \mid x_n < x < Pre + 2Per\}$. We prove that if $y, y' \in Bl_i$, for some $i$, then $\mathcal{R}(y) = \mathcal{R}(y')$. The proof is by contradiction. Let us assume $\mathcal{R}(y) \neq \mathcal{R}(y')$. By definition of ultimately periodic model, it follows that at least one between $y$ and $y'$ must belong to the prefix of $M$. Let us assume that $\langle L \rangle \psi \in \mathcal{R}(y)$ and $\langle L \rangle \psi \notin \mathcal{R}(y')$. By definition, $[L]\neg\psi \in \mathcal{R}(y')$. This implies that $y < y'$, as $\langle L \rangle$ is transitive, and hence that $y < Pre$. Now, consider the interval $[x^\psi_{max}, y^\psi_{max}]$ defined above. Since both $y$ and $y'$ belong to the same set $Bl_i$, two cases may arise: either $x^\psi_{max} < y$ or $y' < x^\psi_{max}$. In the former case, since $\langle L \rangle \psi \in \mathcal{R}(y)$, there must exist an interval $[x'', y'']$ satisfying $\psi$ and such that $x^\psi_{max} < x'' \leq y'$, against the definition of $x^\psi_{max}$. In the latter case, we immediately get a contradiction with $[L]\neg\psi \notin \mathcal{R}(y')$. In a similar way, we can prove that it cannot be the case that $\langle \overline{L} \rangle \psi \in \mathcal{R}(y)$ and $\langle \overline{L} \rangle \psi \notin \mathcal{R}(y')$. Since, by assumption, $Pre + Per > (m_L + 2) \cdot m_B + m_L + 3$, a simple combinatorial argument can be used to prove that there exists a set $Bl_i$, for some $x_{i+1} \leq Pre + Per$, such that $|Bl_i| > m_B$. Let $|Bl| = m_L + 2$ and $x_n = Pre + Per$ (worst case). The prefix $[0, Pre + Per]$ includes $Pre + Per + 1$ points. The $m_L + 4$ points $0, 1, x_1, \ldots, x_n$ do not belong to any set $Bl_i$. The remaining points are more than $(m_L + 2) \cdot m_B + m_L + 3 + 1 - (m_L + 4)$, that is, $(m_L + 2) \cdot m_B$, and they are distributed over $m_L + 2$ sets. Hence, at least one of these sets, say, $Bl_i$, contains more than $m_B$ points.

Let $\bar{x}$ be the first point in such a $Bl_i$. We show how to build a model $M' = \langle \mathbb{I}(\mathbb{N} \setminus \{\bar{x}\}), V' \rangle$, where $V'$ is a suitable adaptation of $V$, such that $M', [0, 1] \Vdash \varphi$. Consider $M'' = \langle \mathbb{I}(\mathbb{N} \setminus \{\bar{x}\}), V'' \rangle$, where $V''$ is the projection of $V$ over the intervals that neither start nor end at $\bar{x}$. By definition, the removal of $\bar{x}$ does not affect the satisfaction of box-formulas from $Cl(\varphi)$. The only potential problem

is with some diamond-formulas which were satisfied in $M$ and are not satisfied in $M''$ anymore. Let $[x, y]$, with $y < \bar{x}$, be such that $M, [x, y] \Vdash \langle L \rangle \psi$. By definition of $Bl$, there exists an interval $[x_{max}^{\psi}, y_{max}^{\psi}]$ satisfying $\psi$ and such that $x_{max}^{\psi}, y_{max}^{\psi} \in Bl$, $x_{max}^{\psi} \leq Pre + Per$, and that there exists no interval $[x', y']$ satisfying $\psi$, with $x_{max}^{\psi} < x' \leq Pre + Per$. Then, either $x_{max}^{\psi} > y$ or there exists an interval $[x', y']$ such that $M, [x', y'] \Vdash \psi$ and $x' > Pre + Per$. Therefore, $M'', [x, y] \Vdash \langle L \rangle \psi$. A symmetric argument applies to the case of $\langle \overline{L} \rangle \psi$. This allows us to conclude that the removal of $\bar{x}$ does not cause any problem with diamond-formulas of the forms $\langle L \rangle \psi$ or $\langle \overline{L} \rangle \psi$. Let us assume now that, for some $y < x < \bar{x}$ (resp., $y < \bar{x} < x$) and some formula $\langle \overline{B} \rangle \psi \in Cl(\varphi)$ (resp., $\langle B \rangle \psi \in Cl(\varphi)$), $M, [y, x] \Vdash \langle \overline{B} \rangle \psi$ (resp., $M, [y, x] \Vdash \langle B \rangle \psi$) and $M, [y, x'] \Vdash \neg \psi$ for every $x' > x$ (resp., $x' < x$) and $x' \neq \bar{x}$. Since $\bar{x}$ is the first point in $Bl_i$, by transitivity of $\langle \overline{B} \rangle$ (resp., $\langle B \rangle$), it holds that $M, [y, x_i] \Vdash \langle \overline{B} \rangle \psi$ (resp., $M, [y, x_{i+1}] \Vdash \langle B \rangle \psi$). Consider now the first $m_B$ successors $\bar{x} + 1, \ldots \bar{x} + m_B$ of $\bar{x}$ (in $M$). Since $|Bl_i| > m_B$, all these points belong to $Bl_i$. It is possible to prove that there exists at least one point $\bar{x} + k$, with $1 \leq k \leq m_B$, that satisfies the following properties: (i) for every $\langle B \rangle \xi \in Cl(\varphi)$, if $M, [y, \bar{x} + k + 1] \Vdash \langle B \rangle \xi$, then $M, [y, \bar{x} + k] \Vdash \langle B \rangle \xi$, and (ii) for every $\langle \overline{B} \rangle \zeta \in Cl(\varphi)$, if $M, [y, \bar{x} + k - 1] \Vdash \langle \overline{B} \rangle \zeta$, then $M, [y, \bar{x} + k] \Vdash \langle \overline{B} \rangle \zeta$. (Notice that properties (i) and (ii) are trivially satisfied by the formula that causes the defect we want to remove.) By transitivity of $\langle B \rangle$, if $M, [y, \bar{x} + k + 1] \Vdash \langle B \rangle \xi$, then $M, [y, x'] \Vdash \langle B \rangle \xi$ for every $x' \geq \bar{x} + k + 1$. Hence, if $\bar{x} + k$ does not satisfy property (i) for $\langle B \rangle \xi$, all its successors are forced to respect it for $\langle B \rangle \xi$. Symmetrically, by transitivity of $\langle \overline{B} \rangle$, if $M, [y, \bar{x} + k - 1] \Vdash \langle \overline{B} \rangle \zeta$ but $M, [y, \bar{x} + k] \not\Vdash \langle \overline{B} \rangle \zeta$, then $M, [y, x'] \not\Vdash \langle \overline{B} \rangle \zeta$ for every $x' \geq \bar{x} + k$. Hence, all successors of $\bar{x} + k$ trivially satisfy property (ii) for $\langle \overline{B} \rangle \zeta$. Since the number of $\langle B \rangle$- and $\langle \overline{B} \rangle$-formulas is bounded by $m_B$, a point that satisfies properties (i) and (ii) can always be found. We fix the defect by defining the labeling $V'$ as follows: for each $p \in \mathcal{AP}$ and $1 \leq t \leq k$, $[y, \bar{x} + t] \in V'(p)$ if and only if $[y, \bar{x} + t - 1] \in V(p)$; the labeling of the other intervals remains unchanged. It can be easily checked that this change in the labeling does not introduce new defects of any kind.

By iterating the above procedure, we obtain a model $\overline{M} = \langle \mathbb{I}(\mathbb{N}), \overline{V} \rangle$, with $Pre + Per \leq (m_L + 2) \cdot m_B + m_L + 3$. To complete the proof, we must propagate the changes we made to the finite model prefix of length $Pre + 2Per$ to the remaining infinite suffix. To build an ultimately periodic model $M^* = \langle \mathbb{I}(\mathbb{N}), V^* \rangle$, we proceed as in the proof of Lemma 4: (i) for each $p \in \mathcal{AP}$ and every $[x, y]$, with $y \leq Pre + Per$, $[x, y] \in V^*(p)$ iff $[x, y] \in \overline{V}(p)$; (ii) for each $p \in \mathcal{AP}$ and every $[x, y]$, with $Pre < x \leq Pre + Per$ and $y \leq x + Per$, $[x, y] \in V^*(p)$ iff $[x, y] \in \overline{V}(p)$; (iii) for each $p \in \mathcal{AP}$ and every $[x, y]$, with $x \leq Pre$ and $y > Pre + Per$, $[x, y] \in V^*(p)$ iff $[x, y - Per] \in V^*(p)$; (iv) for each $p \in \mathcal{AP}$ and every $[x, y]$, with $Pre < x \leq Pre + Per$ and $y > x + Per$, $[x, y] \in V^*(p)$ iff $[x, y - Per] \in V^*(p)$; (v) for each $p \in \mathcal{AP}$ and every $[x, y]$, with $x \geq Pre + Per$, $[x, y] \in V^*(p)$ iff $[x - Per, y - Per] \in V^*(p)$. $\qquad \square$

NP-membership of $\mathsf{B\overline{B}L\overline{L}}$ is a consequence of the above lemmas and the fact that $m_L$ and $m_B$ are both polynomial in $|\varphi|$.

**Theorem 3.** *The satisfiability problem for* $\mathsf{B\overline{B}L\overline{L}}$, *its mirror image, and all its sub-fragments, over all the considered classes of linear orders, is NP-complete.*

## 6. EXPSPACE-Completeness

In this section, we study the computational complexity of the fragment $\mathsf{AB\overline{BL}}$, of its sub-fragments, except for those included in $\mathsf{B\overline{B}L\overline{L}}$, and of their mirror images. We know from [12] that $\mathsf{AB\overline{BL}}$ and its mirror image $\mathsf{\overline{A}E\overline{E}L}$ are in EXPSPACE for all the considered classes of linear orders. In the following, we sharpen EXPSPACE-hardness results given in [11] by showing that all fragments of $\mathsf{AB\overline{BL}}$ (resp., $\mathsf{\overline{A}E\overline{E}L}$), which are not included in $\mathsf{B\overline{B}L\overline{L}}$ (resp., $\mathsf{E\overline{E}L\overline{L}}$), are EXPSPACE-hard for all these classes.

To prove EXPSPACE-hardness, we provide a reduction from the $2^n$-*corridor tiling problem* (also known as *exponential-corridor tiling problem*), which is known to be EXPSPACE-complete [20, Section 5.5]. Formally, an instance of the exponential-corridor tiling problem is a tuple $\mathcal{T} = (T, t_0, t_1, T_L, T_R, C_H, C_V, n)$ consisting of a finite set $T$ of tiles, two tiles $t_0, t_1 \in T$, a set of left tiles $T_L \subseteq T$, a set of right tiles $T_R \subseteq T$, two binary relations $C_H$ and $C_V$ over $T$, that specify a set of horizontal and vertical constraints, and a positive natural number $n$. The problem amounts to deciding whether there exist a positive natural number $l$ and a tiling $f : \{0, \ldots, 2^n - 1\} \times \{0, \ldots, l - 1\} \to T$ of the corridor of width $2^n$ and height $l$, that associates the tile $t_0$ with $(0,0)$, the tile $t_1$ with $(0, l-1)$, and a tile in $T_L$ (resp., $T_R$) with the first (resp., last) position of every row of the corridor (apart from $(0,0)$ and $(0, l-1)$), and that satisfies the following horizontal and vertical constraints $C_H$ and $C_V$:

(i) for each $x < 2^n - 1$ and each $y < l$, $f(x,y)\ C_H\ f(x+1,y)$;

(ii) for each $x < 2^n$ and each $y < l - 1$, $f(x,y)\ C_V\ f(x,y+1)$.

In [11], a reduction for $\mathsf{AB}$ over $\mathbb{N}$ (and, thus, for $\mathsf{\overline{A}E}$ over $\mathbb{Z}^-$) is given. In the following, we define a variant of such a reduction where we use $\langle\overline{B}\rangle$ instead of $\langle B \rangle$ and finite, instead of infinite, structures. The proof will make an extensive use of a derived *"always in the future"* modality $[G]$, defined as follows:

$$[G]\varphi \equiv \varphi \wedge [A]\varphi \wedge [A][A]\varphi$$

When evaluated over an interval $[x, y]$, $[G]\varphi$, forces $\varphi$ to be true on $[x, y]$ and on every interval $[z, t]$, with $z \geq y$.

**Lemma 6.** *There exists a polynomial-time reduction from the $2^n$-corridor tiling problem to the satisfiability problem for* $\mathsf{A\overline{B}}$ *over all the considered classes of linear orders.*

*Proof.* Let $\mathcal{T} = (T, t_0, t_1, T_L, T_R, C_H, C_V, n)$ be an instance of the $2^n$-corridor tiling problem, where $T = \{t_0, t_1, \ldots, t_k\}$. We provide an $\mathsf{A\overline{B}}$-formula, whose size is polynomial in $|\mathcal{T}|$, which is satisfiable if and only if there exist a natural number $l$ and a correct tiling $f : \{0, \ldots, 2^n - 1\} \times \{0, \ldots, l - 1\} \to T$. We use

$k+1$ proposition letters $t_0, t_1, \ldots, t_k$ to represent the tiles from $T$, $n$ proposition letters $b_0, \ldots, b_{n-1}$ to represent the binary expansion of the $x$-coordinate of a point in the corridor, and one proposition letter $c$ to identify those intervals that correspond to points $(x, y)$ of the corridor. Such a correspondence is obtained by ensuring that we interpret these proposition letters only over intervals of length 1. The valuation function $V$ of the model of the formula is then related to the tiling function $f$ as follows: for each point $(x, y) \in \{0, \ldots, 2^n - 1\} \times \{0, \ldots, l-1\}$ and each tile $t_i \in T$, if $f(x, y) = t_i$, then $[x + 2^n y, x + 2^n y + 1] \in V(c) \cap V(t_i) \cap \bigcap_{b \in \{b_{j_1}, \ldots, b_{j_h}\}} V(b)$, where $\{j_1, \ldots, j_h\} \subseteq \{0, \ldots, n-1\}$ and $x = \sum_{j \in \{j_1, \ldots, j_h\}} 2^j$.

First, we force the existence of a finite chain of intervals of length 1 where $c$ is true, we guarantee that this $c$-chain is unique, and we associate a unique proposition letter $t_i$ with each $c$-labeled interval:

$$\varphi_0 \begin{cases} \varphi_c &= c \wedge [G]((c \wedge \langle A \rangle \top) \rightarrow \langle A \rangle (c \vee c_{stop})) \wedge [G]\neg \langle \overline{B} \rangle c, \\[4pt] \varphi_s &= [G]\neg \langle \overline{B} \rangle c_{stop} \wedge \langle A \rangle \langle A \rangle c_{stop} \wedge [G](c_{stop} \rightarrow ([G]\neg c \wedge [A][G]\neg c_{stop})), \\[4pt] \varphi_f &= [G]\Big(c \leftrightarrow \bigvee_{0 \leq i \leq k} t_i\Big) \wedge [G]\Big(\bigwedge_{0 \leq i < j \leq k} \neg(t_i \wedge t_j)\Big). \end{cases}$$

Then, we guarantee that, for every point $(x, y)$ in the corridor, truth values of proposition letters $b_0, \ldots, b_{n-1}$ over intervals $[x + 2^n y, x + 2^n y + m]$, for every $m \geq 1$, represent the binary expansion of $x$. Such a constraint can be enforced by the conjunction $\varphi_x$ of the following formulas:

$$\varphi_x \begin{cases} \varphi_x^1 &= \bigwedge_{0 \leq i < n} \neg b_i \wedge \langle A \rangle \langle A \rangle (c \wedge \langle A \rangle c_{stop} \wedge \bigwedge_{0 \leq i < n} b_i), \\[4pt] \varphi_x^2 &= [G]\Big(c \rightarrow \varphi_{inc}^0\Big), \\[4pt] \varphi_x^3 &= [G]\bigwedge_{0 \leq i < n}\Big((b_i \rightarrow [\overline{B}]b_i) \wedge (\neg b_i \rightarrow [\overline{B}]\neg b_i)\Big), \end{cases}$$

where $\varphi_{inc}^i$ is defined as follows:

$$\varphi_{inc}^i = \begin{cases} \top & \text{if } i = n; \\[4pt] \Big(b_i \wedge [A](c \rightarrow \neg b_i) \wedge \varphi_{inc}^{i+1}\Big) \vee \Big(\neg b_i \wedge [A](c \rightarrow b_i) \wedge \varphi_{eq}^{i+1}\Big) & \text{otherwise,} \end{cases}$$

and $\varphi_{eq}^i$ is defined as follows:

$$\varphi_{eq}^i = \begin{cases} \top & \text{if } i = n; \\[4pt] \Big((b_i \wedge [A](c \rightarrow b_i)) \vee (\neg b_i \wedge [A](c \rightarrow \neg b_i))\Big) \wedge \varphi_{eq}^{i+1} & \text{otherwise.} \end{cases}$$

Finally, we establish a correspondence between intervals that represent vertically adjacent tiles by setting a fresh proposition letter $co$:

$$\varphi_{co} = [G](co \rightarrow \varphi_{eq}^0) \wedge [G]((c \wedge \langle \overline{B} \rangle \varphi_{eq}^0) \rightarrow \langle \overline{B} \rangle co) \wedge [G]\neg(\varphi_{eq}^0 \wedge \langle \overline{B} \rangle co).$$

To conclude the proof, we must enforce the horizontal and vertical constraints $C_H$ and $C_V$ and the constraints on the border of the corridor. This can be

done by means of the following formulas (remember that, by definition of tiling, $t_0, t_1 \in T$ and $T_L, T_R \subseteq T$):

$$
\varphi_{con} \begin{cases}
\varphi_{01} & = \quad t_0 \wedge \langle A \rangle \langle A \rangle \Big( c \wedge \bigwedge_{0 \le i < n} \neg b_i \wedge t_1 \wedge \neg \langle \overline{B} \rangle co \Big) \\[4pt]
\varphi_L & = \quad [G]\Big( c \wedge \bigwedge_{0 \le i < n} \neg b_i \rightarrow \bigvee_{t_L \in T_L} t_L \Big), \\[4pt]
\varphi_R & = \quad [G]\Big( c \wedge \bigwedge_{0 \le i < n} b_i \rightarrow \bigvee_{t_R \in T_R} t_R \Big), \\[4pt]
\varphi_H & = \quad [G] \bigwedge_{0 \le i \le k} \Big( t_i \rightarrow [A]\big( c \rightarrow \bigvee_{(t_i, t_j) \in C_H} t_j \big) \Big), \\[4pt]
\varphi_V & = \quad [G] \bigwedge_{0 \le i \le k} \Big( t_i \rightarrow [\overline{B}](co \rightarrow \bigvee_{(t_i, t_j) \in C_V} \langle A \rangle t_j ) \Big).
\end{cases}
$$

The formula $\varphi_{\mathcal{T}} = \varphi_0 \wedge \varphi_x \wedge \varphi_{co} \wedge \varphi_{con}$ is of polynomial size with respect to $|\mathcal{T}|$, and it is satisfiable if and only if $\mathcal{T}$ can tile the $2^n$-corridor. $\qquad \square$

**Theorem 4.** *The satisfiability problem for* $AB\overline{BL}$, $AB\overline{B}$, $AB$, $A\overline{B}$, $AB\overline{L}$, *and* $A\overline{BL}$, *as well as for their mirror images, is EXPSPACE-complete for all the considered classes of linear orders.*

## 7. Non-Primitive Recursiveness and Undecidability

In this section, we study the complexity of the fragments of $A\overline{A}B\overline{B}$ and $A\overline{A}E\overline{E}$ which have not been taken into consideration in the previous sections, namely, all fragments which are not sub-fragments of $AB\overline{BL}$, $\overline{A}E\overline{EL}$, or $A\overline{A}$. We give both hardness and undecidability proofs. In all cases, we proceed by reducing an appropriate problem for *faulty counter machines* to the satisfiability problem for the considered fragment.

### 7.1. Faulty Counter Machines

*Faulty counter machines* [21] are a variant of Minsky counter automata where transitions may non-deterministically increase (*incrementing faulty machines*) or decrease (*decrementing faulty machines*) the values of counters. A comprehensive survey on faulty machines and on the relevant complexity, decidability, and undecidability results can be found in [22]. Formally, a *counter automaton* is a tuple $\mathcal{A} = (\Sigma, Q, q_0, C, \Delta, F)$, where $\Sigma$ is a finite alphabet, $Q$ is a finite set of control *states*, $q_0 \in Q$ is the initial state, $C = \{c_1, \dots, c_k\}$ is the set of *counters*, whose values range over $\mathbb{N}$, $\Delta$ is a *transition relation*, and $F \subseteq Q$ is the set of final states. Let us denote by $\epsilon$ the *empty word* (we assume $\epsilon \notin \Sigma$). The relation $\Delta$ is a subset of $Q \times (\Sigma \cup \{\epsilon\}) \times L \times Q$, where $L$ is the *instruction set* $L = \{inc, dec, ifz\} \times \{1, \dots, k\}$. A *configuration* of $\mathcal{A}$ is a pair $(q, \bar{v})$, where $q \in Q$ and $\bar{v}$ is the vector of counter values. A *run* of a Minsky (i.e., with no error) counter automaton is a finite or infinite sequence of configurations such that, for every pair of consecutive configurations $(q, \bar{v}), (q', \bar{v}')$, a *transition* $(q, \bar{v}) \xrightarrow{l,a} (q', \bar{v}')$ has been taken (for some $(q, a, l, q') \in \Delta$). The value of $\bar{v}'$ is obtained from the value of $v$ by performing instruction $l$, where $l = (dec, i)$ requires $v_i > 0$ and $l = (ifz, i)$ requires $v_i = 0$. When the machine is (faulty)

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incrementing, counters may be erroneously incremented at any time: we use in this case $(q, \bar{v}) \xrightarrow{l,a}_{\dagger} (q', \bar{v}')$ to indicate that there exist $\bar{v}_\dagger, \bar{v}'_\dagger$ such that $\bar{v} \leq \bar{v}_\dagger$, $(q, \bar{v}_\dagger) \xrightarrow{l,a} (q', \bar{v}'_\dagger)$, and $\bar{v}'_\dagger \leq \bar{v}'$. The ordering $\leq$ is defined component-wise in the obvious way. In incrementing machines, once a faulty transition has been taken, counter values may have been increased nondeterministically before or after the execution of the exact transition by an arbitrary natural number. Symmetrically, when the machine is (faulty) decrementing, counters may be nondeterministically decreased, and we use the same notation $(q, \bar{v}) \xrightarrow{l,a}_{\dagger} (q', \bar{v}')$ to denote that there exist $\bar{v}_\dagger, \bar{v}'_\dagger$ such that $\bar{v} \geq \bar{v}_\dagger$, $(q, \bar{v}_\dagger) \xrightarrow{l,a} (q', \bar{v}'_\dagger)$, and $\bar{v}'_\dagger \geq \bar{v}'$.

We say that a *finite* run of $\mathcal{A}$ over a word $w \in \Sigma^*$ is *accepting* if and only if it ends with a finale state in $F$. In the case of an $\omega$-word $w \in \Sigma^\omega$, we say that an *infinite* run of $\mathcal{A}$ over $w$ is *accepting* if and only if it traverses a state in $F$ infinitely often. We are interested here in the *non-emptiness problem* for faulty machines, defined as the problem of deciding whether there exists at least one ($\omega$-)word accepted by a faulty counter machines. For finite words, it is non-primitive recursive, while for infinite words it is undecidable [21].

### 7.2. Symmetric Structures

To start with, let us consider the results given in [13] (reported in Tab. 2) for the fragments $A\overline{A}B$ and $A\overline{AB}$, and their mirrors images, over symmetric structures. When interpreted over finite linear structures, these fragments turn out to be decidable (it is an immediate consequence of the decidability result for $A\overline{A}B\overline{B}$ and its mirror image), but not primitive recursive. When interpreted over Dedekind-complete infinite structures, they become undecidable.

In [13], a reduction from the (undecidable) reachability problem for lossy counter machines to the satisfiability problem for $A\overline{A}B$ and $A\overline{AB}$ (that works for their mirror images as well) is given. In the following, we provide a reduction from a slightly different (undecidable) problem, namely, the non-emptiness problem for incrementing counter automata over $\omega$-words, to the satisfiability problem for $AE$ over the class of strongly discrete linear orders. By symmetry, such a result immediately transfers to $\overline{A}B$. Moreover, the proposed encoding can be easily adapted to the cases of $A\overline{E}$ and $\overline{AE}$.

**Lemma 7.** *There exists a reduction from the non-emptiness problem for incrementing counter automata over $\omega$-words to the satisfiability problem for $AE$ over the class of strongly discrete linear orders.*

*Proof.* Let $\mathcal{A} = (\Sigma, Q, q_0, C, \Delta, F)$ be an incrementing counter automaton. We write an $AE$-formula $\varphi_{\mathcal{A}}$ which is satisfiable over the class of strongly discrete linear orders if and only if there is at least one $\omega$-word over $\Sigma$ accepted by $\mathcal{A}$. The formula will make use of the universal operator $[G]$ defined in Section 6.

Let us assume that $|Q| = \mu + 1$, $|\Sigma| = \nu$, $|F| = \eta$, and $|C| = k$, and there are *(i)* $\mu + 1$ proposition letters $q_0, q_1, \dots, q_\mu$, one for each state in $Q$ ($q_0$ being the
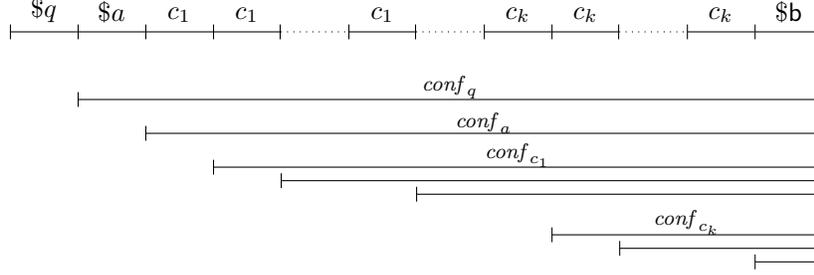
Figure 5: Encoding of a configuration of an incrementing counter automaton in AE.

initial state); *(ii)* $\nu + 1$ proposition letters $a_0, a_1, \ldots, a_\nu$, one for each symbol in $\Sigma \cup \{\epsilon\}$ ($a_1, \ldots, a_\nu$ encode symbols in $\Sigma$, while $a_0$ encodes $\epsilon$); and *(iii)* $k$ proposition letters $c_1, \ldots, c_k$, one for each counter in $C$. Moreover, to simplify the formula, we introduce a proposition letter \$q (resp., \$a, \$c) which holds at some interval if and only if at least one $q_i$ (resp., $a_i$, $c_i$) holds at that interval. Finally, a proposition letter *conf* is used to denote a configuration. Additional auxiliary proposition letters will be introduced later on.

To encode the components of a configuration, we use intervals of the form $[x, x + 1]$ (unit intervals), which are univocally identified by the AE-formula $[E]\perp$. A configuration is modeled by a (non-unit) interval $[x, x + s]$, labeled with *conf*, consisting of a sequence of unit intervals labeled as follows: $[x, x+1]$ is labeled with (a proposition letter for) a state in $Q$, $[x+1, x+2]$ by a letter in $\Sigma$, and all the remaining unit intervals, but the last one (for technical reasons, $[x+s-1, x+s]$ is labeled with a special proposition letter \$b), are labeled with counters in $C$. Figure 5 depicts (part of) the encoding of a configuration. We constrain any configuration interval $[x, x+s]$ to contain one unit interval labeled with a state, one labeled with an alphabet letter, and, for $1 \le i \le k$, as many unit intervals labeled with $c_i$ as the value of counter $c_i$ is in that configuration. Without loss of generality, we assume all counter values to be initialized to $0$ ($\tilde{v} = \bar{0}$), and thus the initial configuration contains no counter proposition letters. We first constrain proposition letters that denote states (in $Q$), input symbols (in $\Sigma \cup \{\epsilon\}$), and counter values to be correctly placed, by means of the following formulas:

$$\varphi_p \begin{cases} \varphi_{p1} &= [G](\$q \leftrightarrow \bigvee_{i=0}^{\mu} q_i \wedge \$a \leftrightarrow \bigvee_{i=0}^{\nu} a_i \wedge \$c \leftrightarrow \bigvee_{i=1}^{k} c_i) \\ \varphi_{p2} &= [G]([E]\perp \leftrightarrow \$q \vee \$a \vee \$c \vee \$b) \\ \varphi_{p3} &= [G]\bigwedge_{p \in \{q,a,c,b\}}(\$p \to \neg \bigvee_{p' \in \{q,a,c,b\}, p' \ne p} \$p') \\ \varphi_{p4} &= [G](\bigwedge_{i \ne j}(q_i \to \neg q_j) \wedge \bigwedge_{i \ne j}(a_i \to \neg a_j) \wedge \bigwedge_{i \ne j}(c_i \to \neg c_j)), \end{cases}$$

which make sure that placeholders are correctly set ($\varphi_{p1}$), that they are all unit intervals ($\varphi_{p2}$) and no more than one placeholder labels a unit interval ($\varphi_{p3}$), and that counters, states, and alphabet letters are unique ($\varphi_{p4}$). Next, we encode the sequence of configurations as a (unique) infinite chain that starts at the ending point of the interval where $\varphi_{\mathcal{A}}$ is evaluated, and we constrain the

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counter values of the initial configuration to be equal to 0. To force such a chain to be unique and to prevent configurations from containing or overlapping other configurations, we introduce an additional proposition letter $conf'$, which holds over all and only those intervals which are suffixes of a $conf$-interval:

$$\varphi_i \begin{cases} \varphi_{i1} & = & \langle A \rangle (conf \wedge \langle E \rangle \langle E \rangle \top \wedge [E][E][E]\bot) \\ \varphi_{i2} & = & [G](conf \to \langle A \rangle conf \wedge \langle E \rangle \langle E \rangle \top) \\ \varphi_{i3} & = & [G]((conf \to [E]conf') \wedge (conf' \to \neg conf)) \\ \varphi_{i4} & = & [G]\big((\langle A \rangle conf' \to \neg conf) \wedge (conf' \to \langle A \rangle conf \wedge \neg \langle E \rangle conf)\big), \end{cases}$$

where we force the initial configuration to have two internal points ($\varphi_{i1}$) only, the existence of a chain of configurations, each one of which has room for a state and a letter ($\varphi_{i2}$), that $conf$s are ended by $conf'$s ($\varphi_{i3}$), that $conf$s neither overlap nor contain other $conf$s, and that $conf'$s meet $conf$s and are not ended by $conf$s ($\varphi_{i4}$). Next, we force configurations to be properly structured: they must start with a unit interval labeled with a state (the initial configuration with $q_0$), followed by a unit interval labeled with an input letter, possibly followed by a number of unit intervals labeled with counters, followed by a last unit interval labeled with \$b. As modalities $\langle A \rangle$ and $\langle E \rangle$ do not allow one, in general, to refer to the subintervals of a given interval, a little technical detour is necessary. We introduce the auxiliary proposition letters $conf_q$, $conf_a$, and $conf_{c_i}$ (one for each type of counter), and we label the suffix of a configuration interval met by a unit interval labeled with \$q (resp., \$a, $c_i$) with $conf_q$ (resp., $conf_a$, $conf_{c_i}$). In such a way, modality $\langle E \rangle$ can be exploited to get an indirect access to the components of a configuration. As an example, we use it to force every configuration to include at most one state and one input letter. Notice that proposition letter \$b plays an essential role here: it allows us to associate the last $c_i$ of each configuration with the corresponding $conf_{c_i}$:

$$\varphi_f \begin{cases} \varphi_{f1} = \langle A \rangle q_0 \wedge [G](\langle A \rangle conf \leftrightarrow \langle A \rangle \$q) \\ \varphi_{f2} = [G]((\$q \to \langle A \rangle \$a) \wedge (\$a \vee \$c \to \langle A \rangle (\$c \vee \$b)) \wedge (\$b \to \langle A \rangle \$q)) \\ \varphi_{f3} = [G](\neg(conf_q \wedge \langle E \rangle conf_q) \wedge \neg(conf_a \wedge \langle E \rangle conf_a)) \\ \varphi_{f4} = [G](( \$q \to [A](conf' \to conf_q)) \wedge (\$a \to [A](conf' \to conf_a))) \\ \varphi_{f5} = [G](\bigwedge_{i=1}^{k}(c_i \to [A](conf' \to conf_{c_i}))), \end{cases}$$

by means of which we ensure that every $conf$ starts with a state and, in particular, the initial $conf$ starts with $q_0$ ($\varphi_{f1}$), that $conf$s are properly structured ($\varphi_{f2}$), that $conf$s contain at most one state and one letter ($\varphi_{f3}$), that \$q meets $conf_q$ and \$a meets $conf_a$ ($\varphi_{f4}$), and, finally that the $c_i$s meet their respective $conf_{c_i}$ ($\varphi_{f5}$). Now, to model decrements and increments, auxiliary proposition letters $c_{dec}, c_{new}, conf_{dec}$, and $conf_{new}$ are introduced. The letter $c_{dec}$, which labels at most one unit interval $c_i$ of a given configuration, constrains the value of the $i$-th counter to be decremented by 1 by the next transition, provided that $\Delta$ contains such a transition. Similarly, we constrain $c_{new}$ to label a (unique) unit interval $c_i$ added by the last transition to represent an increment by 1 of the value of the $i$-th counter, provided that $\Delta$ contains such a transition. Such

conditions are imposed by the following formulas:

$$\varphi_{nd}\begin{cases} \varphi_{nd1} &= [G]\big(\bigwedge_{l\in\{new,dec\}}(c_l \to (\$c \wedge [A](conf' \to conf_l))))\\ \varphi_{nd2} &= [G]\big(\bigwedge_{l\in\{new,dec\}}(([E]\bot \wedge \langle A\rangle conf_l) \to c_l))\\ \varphi_{nd3} &= [G](\neg(conf_{dec} \wedge \langle E\rangle conf_{dec}) \wedge \neg(conf_{new} \wedge \langle E\rangle conf_{new})), \end{cases}$$

that is, $c_l$ ($l \in new, dec$) in a configuration implies that the following configuration is precisely a $conf_l$ ($\varphi_{nd1}$), that a configuration is $conf_l$ only if it follows a configuration that features precisely $c_l$ ($\varphi_{nd2}$), and that a given $conf_l$ labels exactly one configuration. To constrain the values that counters may assume in consecutive configurations, we introduce three auxiliary proposition letters $corr$, $corr'$, and $corr_{conf}$. To model the faulty behavior of $\mathcal{A}$, that can increment, but not decrement, the values of counters non-deterministically, we allow two $corr$-intervals to start, but not to end, at the same point, as follows:

$$\varphi_c\begin{cases} \varphi_{c1} &= [A](\langle A\rangle c_{new} \to \neg\langle E\rangle corr)\\ \varphi_{c2} &= [G]((\$q \vee \$a \vee c_{dec}) \to [A]\neg corr)\\ \varphi_{c3} &= [G]((\$c \wedge \neg c_{dec}) \to \langle A\rangle corr)\\ \varphi_{c4} &= [G](([E]\bot \wedge \langle A\rangle corr) \to \$c)\\ \varphi_{c5} &= [G]((corr \to ([E]corr' \wedge \langle A\rangle\$c))\wedge\\ &\quad\ (\langle A\rangle conf \to [A](corr' \to corr_{conf})))\\ \varphi_{c6} &= [G](\neg(corr_{conf} \wedge \langle E\rangle corr_{conf}) \wedge (corr \to \langle E\rangle corr_{conf}))\\ \varphi_{c7} &= [G](\langle A\rangle corr_{conf} \to \langle A\rangle conf)\\ \varphi_{c8} &= [G](\bigwedge_{i=1}^{k}(c_i \to [A](corr \to \langle A\rangle c_i)))\\ \varphi_{c9} &= [G]\neg(corr \wedge \langle E\rangle corr), \end{cases}$$

that is, $new$ counters have not a counterpart in previous $conf$s ($\varphi_{c1}$), $q$s, $a$s, and $dec$ counters have not a counterpart in the following $conf$s ($\varphi_{c2}$), counters that are not $dec$ do have a counterpart in next $conf$s ($\varphi_{c3}$), $corr$s are always met by a counter ($\varphi_{c4}$), $corr$s, $corr'$s and $corr_{conf}$s are properly related to each other ($\varphi_{c5}$, $\varphi_{c7}$, and $\varphi_{c9}$), $corr$s connect counters of consecutive $conf$s ($\varphi_{c6}$), and each $corr$ does correspond to some counter ($\varphi_{c8}$). Finally, we constrain consecutive configurations to be related by some transition $(q, a, l, q')$ in $\Delta$ by means of the following formulas:

$$\begin{aligned} \varphi_{inc} &= \bigvee_{(q,a,(inc,i),q')\in\Delta}\big(\langle A\rangle(q \wedge \langle A\rangle a) \wedge \langle A\rangle(conf \wedge \neg\langle E\rangle conf_{dec} \wedge \langle A\rangle q' \wedge\\ &\quad\ \langle A\rangle(conf \wedge \langle E\rangle(conf_{c_i} \wedge conf_{new}))))\\ \varphi_{dec} &= \bigvee_{(q,a,(dec,i),q')\in\Delta}\big(\langle A\rangle(q \wedge \langle A\rangle a) \wedge \langle A\rangle(conf \wedge \langle A\rangle q' \wedge\\ &\quad\ \langle E\rangle(conf_{c_i} \wedge conf_{dec})))\\ \varphi_{ifz} &= \bigvee_{(q,a,(ifz,i),q')\in\Delta}\big(\langle A\rangle(q \wedge \langle A\rangle a) \wedge \langle A\rangle(conf \wedge \neg\langle E\rangle conf_{dec} \wedge \langle A\rangle q' \wedge\\ &\quad\ [E]\neg conf_{c_i}))\\ \varphi_d &= [G]\big(\langle A\rangle conf \to \varphi_{inc} \vee \varphi_{dec} \vee \varphi_{ifz}\big). \end{aligned}$$

The first three formulas encode increment ($\varphi_{inc}$), decrement ($\varphi_{dec}$), and conditional instructions ($\varphi_{ifz}$), respectively; the fourth one specifies the behavior of a generic instruction ($\varphi_d$). It is straightforward to prove that the formula $\varphi_{\mathcal{A}} = \varphi_p \wedge \varphi_i \wedge \varphi_f \wedge \varphi_{nd} \wedge \varphi_c \wedge \varphi_d \wedge [A]\langle A\rangle\langle A\rangle \bigvee_{q_f\in F} q_f \wedge [A]\langle A\rangle\langle A\rangle \bigvee_{i=1}^{\nu} a_i$

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is satisfiable if and only if $\mathcal{A}$ accepts at least one $\omega$-word. Notice that the last conjunct forces the word to be infinite by imposing that a letter $a_i$, with $i \neq 0$ (recall that $a_0$ encodes the symbol $\epsilon$), occurs infinitely often. □

Non-primitive recursiveness of $\mathsf{A\overline{E}}$, $\mathsf{\overline{A}B}$, $\mathsf{A\overline{E}}$, and $\mathsf{\overline{A}\overline{E}}$ over finite linear orders can be proved by a reduction from the (non-primitive recursive) non-emptiness problem for incrementing counter automata over finite words to the satisfiability problem for these logics. The encoding is quite similar to the one for $\omega$-words (and thus the proof is omitted): it suffices to remove the constraint that forces the computation to be infinite and to revise the acceptance condition as reachability of a final state.

**Theorem 5.** *The satisfiability problem for $\mathsf{A\overline{A}B\overline{B}}$ and each fragment of it containing, at least, $\langle \overline{A} \rangle$ and one among $\langle B \rangle$ and $\langle \overline{B} \rangle$ (and for all their mirror images), is decidable, but non-primitive recursive, over finite linear orders, and undecidable over strongly discrete linear orders and $\mathbb{Z}$.*

### 7.3. Asymmetric Structures

The asymmetric nature of $\mathbb{N}$ and $\mathbb{Z}^-$ is reflected by the computational behavior of (some of) the fragments of $\mathsf{A\overline{A}B\overline{B}}$ and of their mirror images. In the following, we focus our attention on $\mathbb{N}$. Every result for a fragment $\mathcal{F}$ over $\mathbb{N}$ can be easily transferred to its mirror image $\mathcal{F}'$ over $\mathbb{Z}^-$. We prove that, when interpreted over $\mathbb{N}$, (i) $\mathsf{\overline{A}B\overline{B}}$, but not $\mathsf{A\overline{E}\overline{E}}$, becomes decidable, but non-primitive recursive (hardness already holds for $\mathsf{\overline{A}B}$ and $\mathsf{\overline{A}\overline{B}}$), and (ii) $\mathsf{\overline{A}BL}$ and $\mathsf{\overline{A}\overline{B}L}$ remain undecidable. While decidability of $\mathsf{\overline{A}B}$ is a direct consequence of [13], extending it to include $\langle \overline{B} \rangle$ requires a suitable adaptation of results in [13].

**Lemma 8.** $\mathsf{\overline{A}B\overline{B}}$ *is decidable over $\mathbb{N}$.*

*Proof.* Let $\varphi$ be a satisfiable $\mathsf{\overline{A}B\overline{B}}$-formula and $M = \langle \mathbb{I}(\mathbb{N}), V \rangle$ be a model such that $M, [x_\varphi, y_\varphi] \Vdash \varphi$ for some interval $[x_\varphi, y_\varphi]$. It can be easily checked that, starting from $[x_\varphi, y_\varphi]$, modalities $\langle \overline{A} \rangle$, $\langle B \rangle$, and $\langle \overline{B} \rangle$ do not allow one to access any interval $[x, y]$, with $x > x_\varphi$ and thus valuation over such intervals can be safely ignored. By exploiting this limitation, one can restrict the search for a model of $\varphi$ to a set of ultimately periodic models only, as it can be shown that, for every satisfiable $\mathsf{\overline{A}B\overline{B}}$-formula, there are an ultimately periodic model $M^* = \langle \mathbb{I}(\mathbb{N}), V^* \rangle$ and an interval $[x_\varphi, y_\varphi]$ such that $M, [x_\varphi, y_\varphi] \Vdash \varphi$, $y_\varphi < Pre$, and $Per \leq m_B$, where $m_B$ is the number of $\langle B \rangle$- and $\langle \overline{B} \rangle$-formulas in $Cl(\varphi)$. To guess the non-periodic part of the model, the algorithm for satisfiability checking of $\mathsf{A\overline{A}B\overline{B}}$-formulas over finite linear orders can be used [13]. Then, the algorithm for satisfiability checking of $\mathsf{AB\overline{B}}$-formulas over $\mathbb{N}$ [11] can be applied to check whether the guessed prefix can be extended to a complete model over $\mathbb{I}(\mathbb{N})$ by guessing the valuation of intervals $[x, y]$ with $x < Pre$ and $Pre \leq y \leq Pre + Per$. To prove termination, it suffices to observe that if the guessed prefix is not *minimal* (in the sense of [13]), it can be shrunk into a smaller one that satisfies the minimality condition (see Proposition 2 and Figure 3 in [13]). Since the number of minimal prefix models is bounded, and so is the

length of the period, decidability of the satisfiability problem for $\overline{A}B\overline{B}$ over $\mathbb{N}$ immediately follows. □

We now show that both $\overline{A}B$ and $\overline{AB}$ over $\mathbb{N}$ are already non-primitive recursive, and that adding $\langle L \rangle$ to either of them makes them undecidable.

Non-primitive recursiveness of $\overline{A}B$ (resp., $\overline{AB}$) over $\mathbb{N}$ can be proved by a reduction from the non-emptiness problem for decrementing counter automata over finite words to the satisfiability problem for $\overline{A}B$ (resp., $\overline{AB}$). Undecidability of $\overline{A}BL$ (resp., $\overline{AB}L$) over $\mathbb{N}$ can be proved by substituting $\omega$-words for finite words. Since the encoding of the latter problem is quite similar to the one of the former (modality $\langle L \rangle$ can be exploited to force computations to be infinite and to encode infinitary accepting conditions), we omit its description.

**Lemma 9.** *There exists a reduction from the non-emptiness problem for decrementing counter automata over finite words to the satisfiability problem for $\overline{A}B$ over $\mathbb{N}$.*

*Proof.* Let $\mathcal{A} = (\Sigma, Q, q_0, C, \Delta, F)$ be a decrementing counter automaton. We write an $\overline{A}B$-formula $\varphi_{\mathcal{A}}$ which is satisfiable over $\mathbb{N}$ if and only if there is at least one finite word over $\Sigma$ accepted by $\mathcal{A}$. We use a construction similar to the one given in the previous section, where the time order is reversed, and we modify the formulas accordingly. In this case, we will make use of a 'transposed' universal modality (*"always in the past"* modality) $[\overline{G}]$, defined as follows:

$$[\overline{G}]\varphi \equiv \varphi \wedge [\overline{A}]\varphi \wedge [\overline{A}][\overline{A}]\varphi$$

Such a universal modality, when evaluated over the interval $[x, y]$, forces a formula to be true on $[x, y]$ and on every interval $[z, t]$, with $t \leq x$.

$$\varphi_p \begin{cases} \varphi_{p1} & = & [\overline{G}](\$q \leftrightarrow \bigvee_{i=0}^{\mu} q_i \wedge \$a \leftrightarrow \bigvee_{i=1}^{\nu} a_i \wedge \$c \leftrightarrow \bigvee_{i=1}^{k} c_i) \\ \varphi_{p2} & = & [\overline{G}]([B]\bot \leftrightarrow \$q \vee \$a \vee \$c) \\ \varphi_{p3} & = & [\overline{G}] \bigwedge_{p \in \{q,a,c\}} (\$p \rightarrow \neg \bigvee_{p' \in \{q,a,c\}, p' \neq p} \$p') \\ \varphi_{p4} & = & [\overline{G}](\bigwedge_{i \neq j}(q_i \rightarrow \neg q_j) \wedge \bigwedge_{i \neq j}(a_i \rightarrow \neg a_j) \wedge \bigwedge_{i \neq j}(c_i \rightarrow \neg c_j)) \end{cases}$$

$$\varphi_f \begin{cases} \varphi_{f_1} & = & [\overline{A}]\bot \wedge conf \wedge [B][B]\bot \wedge \langle B \rangle \top \wedge \langle B \rangle q_0 \\ \varphi_{f_2} & = & [\overline{G}](\bigwedge_{s \in Q \cup \Sigma \cup C}(s \rightarrow [\overline{A}]conf_s \wedge \bigwedge_{s' \in Q \cup \Sigma \cup C, s' \neq s} [\overline{A}]\neg conf_{s'})) \\ \varphi_{f_3} & = & [\overline{G}](conf \rightarrow \langle B \rangle \$q \wedge \bigvee_{q \in Q} conf_q \wedge [B] \bigwedge_{q \in Q} \neg conf_q) \\ \varphi_{f_4} & = & [\overline{G}](conf \rightarrow \langle B \rangle(\bigvee_{a \in \Sigma} conf_a) \wedge \\ & & [B](\bigvee_{a \in \Sigma} conf_a \rightarrow [B] \bigwedge_{a \in \Sigma} \neg conf_a)) \end{cases}$$

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$$\varphi_c \begin{cases} \varphi_{c_1} & = & [\overline{G}](c_{new} \lor c_{dec} \to \$c) \\ \varphi_{c_2} & = & [\overline{G}]((c_{new} \to [\overline{A}]conf_{new}) \land (\neg c_{new} \to [\overline{A}]\neg conf_{new})) \\ \varphi_{c_3} & = & [\overline{G}]((c_{dec} \to [\overline{A}]conf_{dec}) \land (\neg c_{dec} \to [\overline{A}]\neg conf_{dec})) \\ \varphi_{c_4} & = & [\overline{G}](conf \to [B](conf_{new} \to [B]\neg conf_{new})) \\ \varphi_{c_5} & = & [\overline{G}](\$c \to (\neg c_{new} \leftrightarrow \langle\overline{A}\rangle corr)) \\ \varphi_{c_6} & = & [\overline{G}](corr \to (\langle B\rangle \bigvee_{q\in Q} conf_q \land \\ & & \quad [B](\bigvee_{q\in Q} conf_q \to [B] \bigwedge_{q\in Q} \neg conf_q) \land [B]\neg corr)) \\ \varphi_{c_7} & = & [\overline{G}](corr \to [B]\neg c_{dec} \land \langle B\rangle\$c \land \bigwedge_{c\in C}(\langle B\rangle c \to conf_c)) \end{cases}$$

$$\begin{array}{rcl} \varphi_{inc} & = & \bigvee_{(q,a,(inc,i),q')\in\Delta}\langle\overline{A}\rangle(conf \land conf_{q'} \land \langle B\rangle q \land \langle B\rangle conf_a) \\ \varphi_{dec} & = & \bigvee_{(q,a,(dec,i),q')\in\Delta}([B]\neg conf_{new}\land \\ & & \quad \langle\overline{A}\rangle(conf \land conf_{q'} \land \langle B\rangle q \land \langle B\rangle conf_a \land \langle B\rangle(conf_{c_i} \land conf_{dec}))) \\ \varphi_{ifz} & = & \bigvee_{(q,a,(ifz,i),q')\in\Delta}([B]\neg conf_{new}\land \\ & & \quad \langle\overline{A}\rangle(conf \land conf_{q'} \land \langle B\rangle q \land \langle B\rangle conf_a \land [B]\neg conf_{c_i}) \\ \varphi_d & = & [\overline{G}]((conf \land \langle\overline{A}\rangle\top) \to (\varphi_{inc} \lor \varphi_{dec} \lor \varphi_{ifz})). \end{array}$$

It is worth pointing out that, due to the possibility of faulty decrements, the effects of an increment operation can be nullified, and thus $\varphi_{inc}$ does not force the presence of a unit interval labeled with $c_{new}$ in the target configuration.

It holds that $\varphi_{\mathcal{A}} = \varphi_p \land \varphi_f \land \varphi_c \land \varphi_d \land \langle\overline{A}\rangle\langle\overline{A}\rangle(\bigvee_{q\in Q_f} q)$ is satisfiable if and only if $\mathcal{A}$ accepts at least one finite word. $\qquad\square$

An easy adaptation of the above construction leads to the following theorem.

**Theorem 6.** $\overline{A}B$ *(resp., $\overline{AB}$) is non-primitive recursive over $\mathbb{N}$. The addition of $\langle\overline{B}\rangle$ (resp., $\langle B\rangle$) preserves decidability, while the addition of $\langle L\rangle$ to either of them makes them undecidable.*

## 8. Conclusions

In this paper, we drew the definitive line between decidable and undecidable HS fragments over the class of strongly discrete linear orders and over its relevant subclasses (the class of finite linear orders, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{Z}^-$). Moreover, we gave a complete picture of expressiveness and complexity of decidable fragments. A graphical account of the status of the various fragments and of their relationships over finite linear orders (resp., strongly discrete linear orders, $\mathbb{N}$) is given in Fig. 6 (resp., Fig. 7, Fig. 8), where already known results, reported in Tab. 2, are paired with the results given in the present paper.

Our original contributions can be summarized as follows. We provided a complete classification of the expressive power of HS fragments over the three categories of linear orders taken into consideration. From a technical point of view, showing that $\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$ and $\langle\overline{L}\rangle p \equiv \langle\overline{A}\rangle\langle\overline{A}\rangle p$ are the only valid inter-definability equations turned out to be quite involved (no results came for
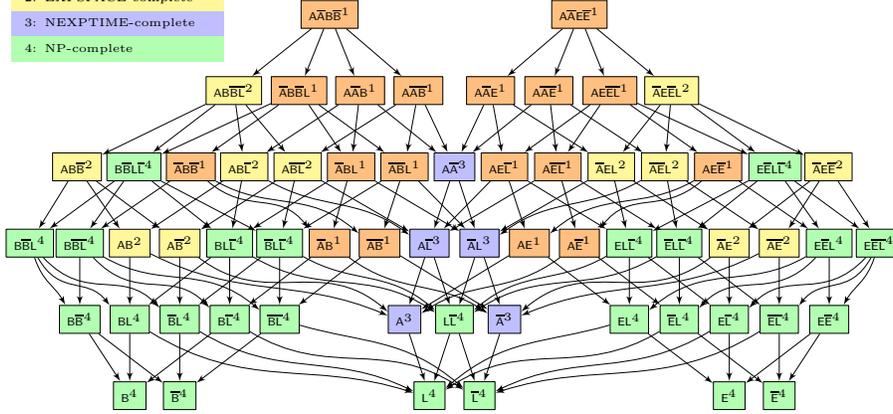
Figure 6: Hasse diagram of all decidable HS fragments of over finite linear orders.
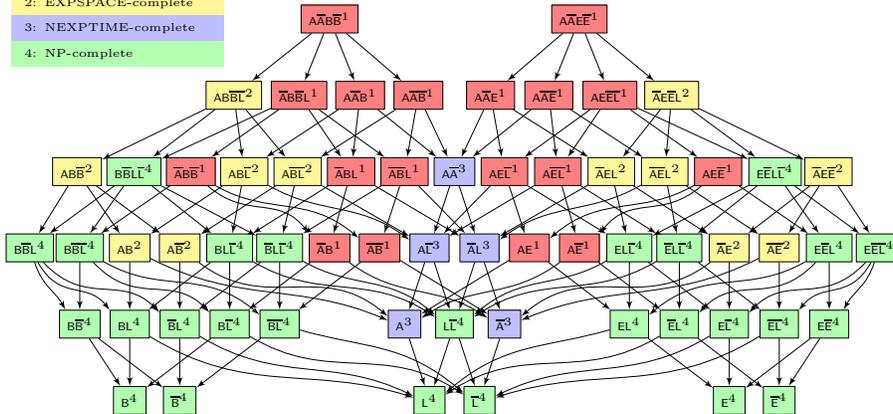


Figure 7: Hasse diagram of all fragments of $A\overline{A}B\overline{B}$ and $A\overline{A}E\overline{E}$ over strongly discrete linear orders.

Figure 8: Hasse diagram of all fragments of $A\overline{A}B\overline{B}$ and $A\overline{A}E\overline{E}$ over $\mathbb{N}$.

free from [16]). As for (un)decidability and complexity, we first proved that NP-membership of $B\overline{B}$ (resp., $E\overline{E}$) [8] can be extended to $B\overline{B}L\overline{L}$ (resp., $E\overline{E}L\overline{L}$) in all the considered linear orders. Then, we showed EXPSPACE-hardness of $AB$, $\overline{A}E$, $A\overline{B}$, and $\overline{A}\overline{E}$ over the class of finite linear orders by suitably revising the proof given in [11] for $AB$ (resp., $\overline{A}E$) over strongly discrete linear orders and $\mathbb{N}$ (resp., strongly discrete linear orders and $\mathbb{Z}^-$). EXPSPACE-hardness of all fragments over all infinite structures easily follows. Next, we proved that a non-trivial adaptation of the results given in [13] allows us to show that $\overline{A}B$, $\overline{A}\overline{B}$, $AE$, and $A\overline{E}$ are non elementarily decidable (non-primitive recursive) over finite linear orders and undecidable over strongly discrete ones. Finally, we studied the effects of switching from symmetric to asymmetric structures on decidability, undecidability, and non-primitive recursiveness results.

## Acknowledgements

## References

[1] D. E. Smith, The case for durative actions: a commentary on PDDL2.1, J. Artif. Int. Res. 20 (2003) 149–154.

[2] J. Halpern, Y. Shoham, A propositional modal logic of time intervals, Journal of the ACM 38 (1991) 935–962.

[3] J. F. Allen, Maintaining knowledge about temporal intervals, Communications of the ACM 26 (1983) 832–843.

[4] D. Bresolin, V. Goranko, A. Montanari, G. Sciavicco, Propositional interval neighborhood logics: expressiveness, decidability, and undecidable extensions, Annals of Pure and Applied Logic 161 (2009) 289–304.

[5] V. Goranko, A. Montanari, G. Sciavicco, Propositional interval neighborhood temporal logics, Journal of Universal Computer Science 9 (2003) 1137–1167.

[6] D. Bresolin, V. Goranko, A. Montanari, P. Sala, Tableaux for logics of subinterval structures over dense orderings, Journal of Logic and Computation 20 (2010) 133–166.

[7] P. Balbiani, V. Goranko, G. Sciavicco, Two sorted point-interval temporal logic, in: Proc. of the 7th Workshop on Methods for Modalities (M4M), volume 278 of *ENTCS*, Springer, 2011, pp. 31–45.

[8] V. Goranko, A. Montanari, G. Sciavicco, A road map of interval temporal logics and duration calculi, Journal of Applied Non-Classical Logics 14 (2004) 9–54.

[9] D. Bresolin, A. Montanari, P. Sala, An optimal tableau-based decision algorithm for Propositional Neighborhood Logic, in: Proc. of the 24th International Symposium on Theoretical Aspects of Computer Science (STACS), volume 4393 of *LNCS*, Springer, 2007, pp. 549–560.

[10] D. Bresolin, A. Montanari, G. Sciavicco, An optimal decision procedure for Right Propositional Neighborhood Logic, Journal of Automated Reasoning 38 (2007) 173–199.

[11] A. Montanari, G. Puppis, P. Sala, G. Sciavicco, Decidability of the interval temporal logic $AB\bar{B}$ on natural numbers, in: Proc. of the 27th Symposium on Theoretical Aspects of Computer Science (STACS), Inria Nancy Grand Est & Loria, 2010, pp. 597–608.

[12] D. Bresolin, P. Sala, G. Sciavicco, On begins, meets, and before, International Journal of Foundations of Computer Science 23 (2012) 559–583.

[13] A. Montanari, G. Puppis, P. Sala, Maximal decidable fragments of halpern and shoham's modal logic of intervals, in: Proc. of the 37th International Colloquium on Automata, Languages and Programming - Part II (ICALP), volume 6199 of *LNCS*, Springer, 2010, pp. 345–356.

[14] J. Marcinkowski, J. Michaliszyn, The Ultimate Undecidability Result for the Halpern-Shoham Logic, in: Proc. of the 26th Symposium on Logic in Computer Science (LICS), IEEE Comp. Society Press, 2011, pp. 377–386.

[15] D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, G. Sciavicco, The dark side of interval temporal logic: Sharpening the undecidability border, in: Proc. of the 18th International Symposium on Temporal Representation and Reasoning (TIME), IEEE Comp. Society Press, 2011, pp. 131–138.

[16] D. D. Monica, V. Goranko, A. Montanari, G. Sciavicco, Expressiveness of the Interval Logics of Allen's Relations on the Class of all Linear Orders: Complete Classification, in: Proc. of the 22nd International Joint Conference on Artificial Intelligence (IJCAI), AAAI Press, 2011, pp. 845–850.

[17] Y. Venema, Expressiveness and completeness of an interval tense logic, Notre Dame Journal of Formal Logic 31 (1990) 529–547.

[18] V. Goranko, M. Otto, Model theory of modal logic, in: P. Blackburn, J. van Benthem, F. Volter (Eds.), Handbook of Modal Logic, Elsevier, 2007, pp. 249–329.

[19] D. Bresolin, A. Montanari, G. Sciavicco, An optimal decision procedure for Right Propositional Neighborhood Logic, Journal of Automated Reasoning 38 (2007) 173–199.

[20] D. Gabbay, A. Kurusz, F. Wolter, M. Zakharyaschev (Eds.), Many-Dimensional Modal Logics: Theory and Applications, Elsevier, 2003.

[21] S. Demri, R. Lazic, LTL with the freeze quantifier and register automata, in: Proc. of the 21st Annual IEEE Symposium on Logic in Computer Science (LICS), IEEE Computer Society, 2006, pp. 17–26.

[22] P. Bouyer, N. Markey, J. Ouaknine, P. Schnoebelen, J. Worrell, On termination and invariance for faulty channel machines, Formal Aspects of Computing 24 (2012) 595–607.

## Appendix

**Lemma 1.** $\langle A \rangle \not\lessgtr \overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}\mathsf{L}$ *and* $\langle \overline{A} \rangle \not\lessgtr \mathsf{A}\mathsf{B}\overline{\mathsf{B}}\mathsf{L}$ *over* $\mathbb{Z}$.

*Proof.* We only prove that $\langle A \rangle \not\lessgtr \overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}\mathsf{L}$ (the other one can be proved similarly). Let $N \in \mathbb{N}$ and $M = M' = \langle \mathbb{I}(\mathbb{Z}), V \rangle$, where $V(p) = \{[x, x+1] \mid x = a(k+1), a \in \mathbb{N}\}$, $k = N^2 + 3$, and $p$ is the only proposition letter in $\mathcal{AP}$. We show now how to define a $\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}\mathsf{L}_N$-bisimulation between $M$ and $M'$. To this end, we define the relation $Z_h$ for $1 \le h \le N$. As a preliminary step, we introduce the function:

$$\chi(h) = (N^2 + 1)(k+1) + 1 + (k+1) \sum_{4+h}^{N+3} i + (h+2)(k+1),$$

which will be useful to establish suitable limits on the mutual position of the endpoints of intervals involved in $Z_h$. A crucial property of $\chi$ is that $\chi(h-1) - \chi(h) = (h+2)(k+1) > 0$.

| Relation | Condition |
|---|---|
| $[x,y]Z_h^1[x,y]$ | None |
| $[x,y]Z_h^2[x+k+1,y+k+1]$ | $x > h^2(k+1) \wedge$ |
| | $y < \chi(h)$ |
| $[x,y]Z_h^3[x,y+k+1]$ | $y - x > (2h-1)(k+1) \wedge$ |
| | $y > h^2(k+1) \wedge$ |
| | $y < \chi(h)$ |
| $[x,y]Z_h^4[x+1,y+1]$ | $x > h^2 + N^2(k+1) + 1 \wedge$ |
| | $x < (N^2+1)(k+1) - 1 \wedge$ |
| | $y < (N^2+1)(k+1) + (N-h)(k+1)$ |
| $[x,y]Z_h^5[x,y+1]$ | $y - x > 2h - 1 \wedge$ |
| | $y > h^2 + N^2(k+1) + 1 \wedge$ |
| | $y < (N^2+1)(k+1) + (N-h)(k+1)$ |
| $[x,y]Z_h^6[x+k+1,y]$ | $y - x > (h+2)(k+1) \wedge$ |
| | $x > h^2(k+1)$ |
| | $x < \chi(h)$ |
| $[x,y]Z_h^7[x+1,y]$ | $y - x > h \wedge$ |
| | $x > h^2 + N^2(k+1) + 1 \wedge$ |
| | $x < (N^2+1)(k+1) - 1$ |

Table 3: Case-by-case definition of $Z_h$ (Lemma 1). $Z_h = \bigcup_1^7 Z_h^i$. For each $i \in \{1, \ldots, 7\}$, $Z_h^i$ is defined as $[x,y]Z_h^i[w,z]$ iff the corresponding condition in the right column holds.

We define $Z_h$ as $\bigcup_{i=1}^7 Z_h^i$, where relations $Z_h^i$ are given in Tab. 3. It is not difficult to check that $Z_h$ meets the *local condition*: $Z_h^1$ is the identity relation and, since $M = M'$, it trivially satisfies the local condition; $Z_h^2$ satisfies it as $V$ is, in a way, $(k+1)$-periodic, and thus $Z_h^2$-related intervals agree on the truth value of $p$; $Z_h^3$, $Z_h^5$, and $Z_h^6$ only relate intervals longer than 1, all satisfying $\neg p$; $Z_h^4$- and $Z_h^7$-related intervals are such that their left endpoint is not a multiple of $k+1$, and thus they all satisfy $\neg p$ as well.

Let us consider now the *forward condition* (the *backward* one can be dealt with in a similar way). We prove that, given three intervals $[x,y], [v,w]$, and $[x',y']$ such that $([x,y],[x',y']) \in Z_h$ and $[x,y]R_X[v,w]$, for some $\langle X \rangle \in \overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}\mathsf{L}$, there exists $[v',w']$ such that $[x',y']R_X[v',w']$ and $([v,w],[v',w']) \in Z_{h-1}$. To improve readability, we give a graphical account of proof steps in Fig. 9, Fig. 10, and Fig. 11. Each figure consists of a graph showing how bisimilarity is preserved by (some of the) modalities of $\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}\mathsf{L}$. More precisely, Fig. 9 deals with modality $\langle \overline{A} \rangle$, Fig. 10 with $\langle B \rangle$ and $\langle \overline{B} \rangle$, and Fig. 11 with $\langle L \rangle$. The graphs should be read as follows. Given $Z^i, Z^j$, an edge from $Z^i$ to $Z^j$ is labelled by $\langle X \rangle$, a set of conditions $Co(v,w)$ on the endpoints of $[v,w]$, and an assignment $As(v',w')$, meaning that if $([x,y],[x',y']) \in Z_h^i$ and we choose $[v,w]$ under conditions $Co(v,w)$, then the assignment $As(v',w')$ guarantees that $([v,w],[v',w']) \in Z_{h-1}^j$ and $[x',y']R_X[v',w']$. For each graph, we now prove that: *(i)* for each $Z^i$, the logical disjunction of $Co(v,w)$, taken from outgoing edges, is a tautology; *(ii)* for each edge from $Z^i$ to $Z^j$, the conjunction of conditions for $Z_h^i$ with $Co(v,w)$
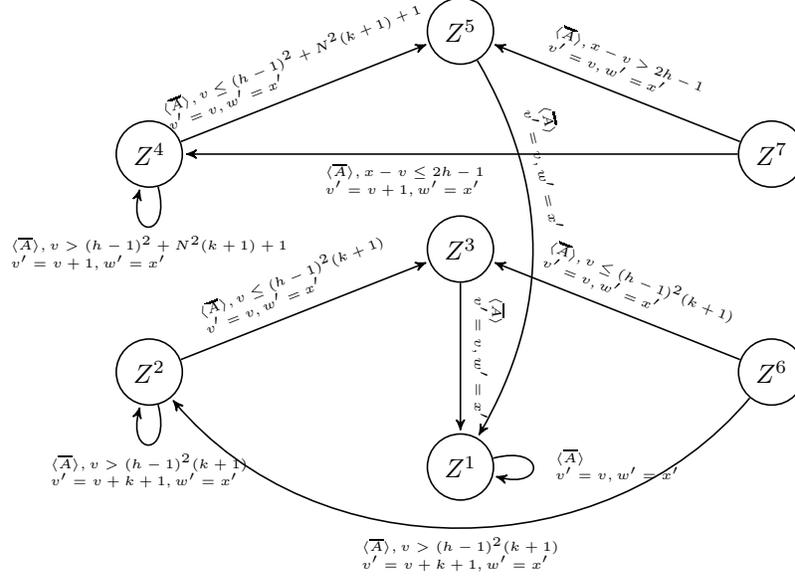
Figure 9: Subgraph of $Z_h$ for $\langle\overline{A}\rangle$-moves (Lemma 1).

and $As(v',w')$ implies the conditions for $Z_{h-1}^j$, as specified in Tab. 3. While the former is straightforward, the latter requires some work to be checked.

Let us start with Fig. 9 for $\langle X \rangle = \langle\overline{A}\rangle$. According to $As(v',w')$, it always holds $w' = x'$. As for $v'$, we must distinguish different cases. Those edges that end up in $Z^1$ can be easily checked. If $([x,y],[x',y']) \in Z_h^2$, then two cases may arise. If $v > (h-1)^2(k+1)$, then $v' = v + k + 1 < w'$ and thus conditions $Z_{h-1}^2$ are met. Otherwise $(v \leq (h-1)^2(k+1))$, we take $v' = v < w'$. From $w = x > h^2(k+1)$ and $v \leq (h-1)^2(k+1)$, it follows that $w - v > (2h-1)(k+1) > (2(h-1)-1)(k+1)$. Since $w = x > h^2(k+1) > (h-1)^2(k+1)$ and $w = x < y < \chi(h) < \chi(h-1)$, we conclude that $([v,w],[v',w']) \in Z_{h-1}^3$. Let $([x,y],[x',y']) \in Z_h^4$. If $v > (h-1)^2 + N^2(k+1) + 1$, then $v' = v + 1 < w'$ guarantees that $([v,w],[v',w']) \in Z_{h-1}^4$. Otherwise $(v \leq (h-1)^2 + N^2(k+1)+1)$, it holds that $x - v > h^2 - (h-1)^2 = 2h-1 > 2(h-1)-1$, and thus letting $v' = v$ guarantees that $([v,w],[v',w']) \in Z_{h-1}^5$. Let $([x,y],[x',y']) \in Z_h^6$. If $v > (h-1)^2(k+1)$, then $v' = v + k + 1 < w'$ guarantees that $([v,w],[v',w']) \in Z_{h-1}^2$. Otherwise $(v \leq (h-1)^2(k+1))$, from $w = x > h^2(k+1)$, it follows that $w - v > h^2(k+1) - (h-1)^2(k+1) = (2h-1)(k+1) > (2(h-1)-1)(k+1)$, and, therefore, letting $v' = v < w'$, it holds that $([v,w],[v',w']) \in Z_{h-1}^3$. Finally, let $([x,y],[x',y']) \in Z_h^7$. If $x - v > 2h-1$, then $v' = v < w'$ guarantees that $([v,w],[v',w']) \in Z_{h-1}^5$. Otherwise $(x - v \leq 2h-1)$, it holds that $v \geq x - (2h-1) > h^2 + N^2(k+1) + 1 - 2h + 1 = (h-1)^2 + N^2(k+1) + 1$, and thus letting $v' = v + 1 < w'$ guarantees that $([v,w],[v',w']) \in Z_{h-1}^4$.

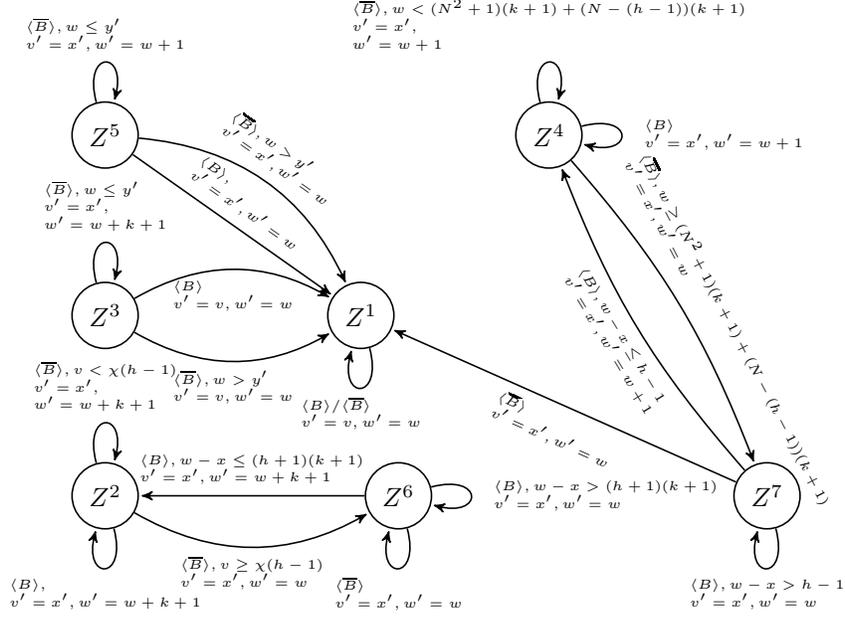Let us turn our attention to Fig. 10 for modalities $\langle B \rangle$ and $\langle \overline{B} \rangle$. In this

Figure 10: Subgraph of $Z_h$ for $\langle B\rangle/\langle\overline{B}\rangle$-moves (Lemma 1).

case, the verification of graph properties must be done via the projection on the specific modality (either $\langle B\rangle$ or $\langle\overline{B}\rangle$). As it happened with the previous graph, edges that end up in $Z^1$ are easy to check. By the semantics of $\langle B\rangle$ and $\langle\overline{B}\rangle$, it always holds that $v' = x'$. Let $([x,y],[x',y']) \in Z_h^2$. If $\langle X\rangle = \langle B\rangle$, it suffices to choose $w' = w + k + 1 < y'$ to guarantee that $([v,w],[v',w']) \in Z_{h-1}^2$. If $\langle X\rangle = \langle\overline{B}\rangle$, two cases are possible. If $w < \chi(h-1)$, then $w' = w + k + 1 > y'$ guarantees that $([v,w],[v',w']) \in Z_{h-1}^2$. Otherwise $(w \geq \chi(h-1))$, we choose $w' = w > y'$ to get $([v,w],[v',w']) \in Z_{h-1}^6$, as $v = x < y < \chi(h) < \chi(h-1)$, $w - v > \chi(h-1) - \chi(h) = (h+2)(k+1) > ((h-1)+2)(k+1)$, and $v = x > h^2(k+1) > (h-1)^2(k+1)$. When $([x,y],[x',y']) \in Z_h^3$, $\langle X\rangle = \langle\overline{B}\rangle$, and $w \leq y'$, then we take $w' = w + k + 1 > y'$, so that $([v,w],[v',w']) \in Z_{h-1}^3$, as $w \leq y' = y + k + 1 < \chi(h) + (k+1) < \chi(h-1)$ implies $w < \chi(h-1)$. We do not need to consider the other outgoing edges as they end up in $Z^1$, and we already dealt with such a case. Let $([x,y],[x',y']) \in Z_h^4$. We distinguish two cases. If $\langle X\rangle = \langle B\rangle$, then $w' = w+1 < y'$, and thus $([v,w],[v',w']) \in Z_{h-1}^4$. If $\langle X\rangle = \langle\overline{B}\rangle$ and $w < (N^2+1)(k+1) + (N-(h-1))(k+1)$, then $w' = w+1 > y'$ and thus $([v,w],[v',w']) \in Z_{h-1}^4$; otherwise $(w \geq (N^2+1)(k+1) + (N-(h-1))(k+1))$, it holds that $w - x > w - y > (N^2+1)(k+1) + (N-h+1)(k+1) - (N^2+1)(k+1) - (N-h)(k+1) = k+1 > h-1$, and letting $w' = w > y'$ guarantees that $([v,w],[v',w']) \in Z_{h-1}^7$. When $([x,y],[x',y']) \in Z_h^5$, $\langle X\rangle = \langle\overline{B}\rangle$, and $w \leq y'$, then we put $w' = w+1 > y'$ to get $([v,w],[v',w']) \in Z_{h-1}^5$. Let $([x,y],[x',y']) \in Z_h^6$. If $\langle X\rangle = \langle B\rangle$ and $w-x > (h+1)(k+1) = (h-1+2)(k+1)$, then letting $w' = w < y'$

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Figure 11: Subgraph of $Z_h$ for $\langle L \rangle$-moves (Lemma 1).

implies $([v,w],[v',w']) \in Z^6_{h-1}$; otherwise $(w - x \leq (h+1)(k+1) = (h-1+2)(k+1))$, from $x < \chi(h)$, it follows that $w < \chi(h) + (h-1+2)(k+1) < \chi(h-1)$, and thus by letting $w' = w + k + 1 < y'$, we have that $([v,w],[v',w']) \in Z^2_{h-1}$. If $\langle X \rangle = \langle \overline{B} \rangle$, then, by letting $w' = w > y'$, it holds that $([v,w],[v',w']) \in Z^6_{h-1}$. Finally, let $([x,y],[x',y']) \in Z^7_h$. If $\langle X \rangle = \langle B \rangle$ and $w - x > h - 1$, then we put $w' = w < y'$ so that $([v,w],[v',w']) \in Z^7_{h-1}$; otherwise $(w - x \leq h - 1)$, it holds that $w \leq x + h < (N^2+1)(k+1) - 1 + h < (N^2+1)(k+1) + (N-h)(k+1) + h < (N^2+1)(k+1) + (N-h)(k+1) + k + 1 = (N^2+1)(k+1) + (N-(h-1))(k+1)$, which, together with $w' = w + 1 < y'$, implies $([v,w],[v',w']) \in Z^4_{h-1}$. If $\langle X \rangle = \langle \overline{B} \rangle$, then $w' = w > y'$ implies that $([v,w],[v',w']) \in Z^7_{h-1}$.

Finally, let us consider Fig. 11 for modality $\langle L \rangle$ (by the semantics of $\langle L \rangle$, it holds that $v > y$). Once more, the edges that end up in $Z^1$ are easy to check. Let $([x,y],[x',y']) \in Z^2_h$. If $v \leq y'$ and $w < \chi(h-1)$, then we put $v' = v + k + 1 > y'$ and $w' = w + k + 1 > v'$ so that $([v,w],[v',w']) \in Z^2_{h-1}$. Otherwise, if $v \leq y'$ and $w \geq \chi(h-1)$, then $w - y > \chi(h-1) - \chi(h) = (2+h)(k+1)$ and, thus, $w - v \geq w - y' = w - y - k - 1 > (2+h)(k+1) - (k+1) = (2+h-1)(k+1)$. Hence, by taking $v' = v + k + 1 > y'$ and $w' = w > v'$, it holds that $([v,w],[v',w']) \in Z^6_{h-1}$, as $v \leq y' = y + k + 1 < \chi(h) + k + 1 < \chi(h-1)$ implies $v < \chi(h-1)$. Let $([x,y],[x',y']) \in Z^3_h$. If $v \leq y'$ and $w < \chi(h-1)$, then we just set $v' = v + k + 1 > y'$ and $w' = w + k + 1 > v'$ to obtain $([v,w],[v',w']) \in Z^2_{h-1}$;

| Relation | Condition |
|---|---|
| $[x,y]Z_h^1[x,y]$ | None |
| $[k,y]Z_h^2[k,y+1]$ | $k+h < y < k+N+2$ |
| $[x,y]Z_h^3[x+1,y+1]$ | $x > \xi(h) \wedge y < \kappa(h)$ |
| $[x,y]Z_h^4[x,y+1]$ | $y > \xi(h) \wedge y - x > N + 2h + 1 \wedge y < \kappa(h)$ |
| $[x,y]Z_h^5[x+1,y]$ | $y - x > N + 2h + 1 \wedge \xi(h) < x < \kappa(h)$ |

Table 4: Case-by-case definition of $Z_h$ (Lemma 2). $Z_h = \bigcup_1^5 Z_h^i$. For each $i \in \{1, \ldots, 5\}$, $Z_h^i$ is defined as $[x,y]Z_h^i[w,z]$ iff the corresponding condition in the right column holds.

Otherwise, if $v \leq y'$ and $w \geq \chi(h-1)$, then $w - y > (2+h)(k+1)$ and, thus, $w - v \geq w - y' = w - y - k - 1 > (2+h)(k+1) - (k+1) = (2+h-1)(k+1)$. Hence, by taking $v' = v + k + 1 > y'$ and $w' = w > v'$, we have that $([v,w],[v',w']) \in Z_{h-1}^6$, since $v > y > h^2(k+1) > (h-1)^2(k+1)$ implies $v > (h-1)^2(k+1)$ and $v \leq y' = y + k + 1 < \chi(h) + k + 1 < \chi(h-1)$ implies $v < \chi(h-1)$. Let $([x,y],[x',y']) \in Z_h^4$. If $v \leq y+1$, then $v = y+1$ and, from $y < (N^2+1)(k+1) + (N-h)(k+1)$, it follows that $v < \chi(h-1) - (h-1+2)(k+1) < \chi(h-1)$. Now, if $w < \chi(h-1)$, then we take $v' = v + k + 1 > y'$ and $w' = w + k + 1 > v'$ so that $([v,w],[v',w']) \in Z_{h-1}^2$. Otherwise, if $w \geq \chi(h-1)$, then $w - v > (h-1+2)(k+1)$, and thus, by taking $v' = v + k + 1 > y'$ and $w' = w > v'$, it holds that $([v,w],[v',w']) \in Z_{h-1}^6$. Finally, let $([x,y],[x',y']) \in Z_h^5$ and $v \leq y+1$. First all, we observe that it necessarily holds $v = y + 1$. From $y < (N^2+1)(k+1) + (N-h)(k+1)$, it follows that $v < \chi(h-1) - (h-1+2)(k+1) < \chi(h-1)$. Now, if $w < \chi(h-1)$, then we take $v' = v + k + 1 > y'$ and $w' = w + k + 1 > v'$, and thus $([v,w],[v',w']) \in Z_{h-1}^2$. Otherwise, if $w \geq \chi(h-1)$, then $w - v > (h-1+2)(k+1)$, and choosing $v' = v + k + 1 > y'$ and $w' = w > v'$ guarantees that $([v,w],[v',w']) \in Z_{h-1}^6$.

To conclude the proof, it suffices to observe that $M, [(N^2+1)(k+1)-2, (N^2+1)(k+1)-1] \Vdash \neg\langle A\rangle p$ and $M', [(N^2+1)(k+1)-1, (N^2+1)(k+1)] \Vdash \langle A\rangle p$. It can be easily checked that these intervals are related by $Z_N^4$, and thus no formula of modal depth at most $N$ can define $\langle A\rangle$ in the language $\overline{A}B\overline{B}L$. As the entire construction is parametric in $N$, we can conclude that $\langle A\rangle$ is not definable by any finite formula, which amounts to say that it is not definable. One can adapt the whole argument to prove that $\langle\overline{A}\rangle \not\lhd AB\overline{BL}$. $\qquad\square$

## Lemma 2. $\langle\overline{B}\rangle \not\lhd A\overline{A}B$ and $\langle B\rangle \not\lhd A\overline{A}B$ over $\mathbb{Z}$.

*Proof.* We prove that $\langle\overline{B}\rangle \not\lhd A\overline{A}B$. The structure of the proof is similar to the one of the previous lemma. Let $N \in \mathbb{N}$ and $M = M' = \langle \mathbb{I}(\mathbb{Z}), V\rangle$, where $V(p) = \{[x,y] \mid y - x \leq N + 2\}$ and $p$ is the only proposition letter. We show how to define an $A\overline{A}B_N$-bisimulation between $M$ and $M'$. Let $Z_h = \bigcup_{i=1}^5 Z_h^i$, where relations $Z_h^i$ are given in Tab. 4, with $\xi(h) = N + \Sigma_{i=1}^h(2i+1) + Nh$, $k = \xi(N) + 1$, and $\kappa(h) = k + N + 2 + (N-h)(2N+2+h)$. It is worth noticing that, for each $h$, $\xi(h), \kappa(h) > 0$ and $\xi(h) - \xi(h-1) = \kappa(h-1) - \kappa(h) = N + 2h + 1$.

As for the *local condition*, it suffices to observe that $Z_h^1$ trivially satisfies it (identity relation); $Z_h^2$ only relates $p$-intervals; $Z_h^3$ relates intervals of the

same length, that thus agree on the truth value of $p$; $Z_h^4$ and $Z_h^5$ only relate $\neg p$-intervals.

Let us consider now the *forward condition* (the *backward* one can be dealt with in a similar way), with the help of Fig. 12 (the picture should be read as those in the previous lemma, but, unlike what happened there, it jointly deals with all modalities). The edges that end up in $Z_1$ are easy to check. Let $([x, y], [x', y']) \in Z_h^2$ and $\langle X \rangle = \langle A \rangle$. If $w < \kappa(h - 1)$, then $v = y > k + h > \xi(h - 1)$ and, by letting $v' = y'$ and $w' = w + 1$, we get $([v, w], [v', w']) \in Z_{h-1}^3$. Otherwise, if $w \geq \kappa(h - 1)$, then $w - v = w - y > \kappa(h - 1) - (k + N + 2) > \kappa(h-1) - \kappa(h) = N + 2h + 1 > N + 2(h - 1) + 1$, so that $v' = y'$ and $w' = w$ imply $([v, w], [v', w']) \in Z_{h-1}^5$. Let $([x, y], [x', y']) \in Z_h^3$. If $\langle X \rangle = \langle A \rangle$ and $w < \kappa(h-1)$, then we put $v' = y'$ and $w' = w+1$ to obtain $([v, w], [v', w']) \in Z_{h-1}^3$; otherwise, if $w \geq \kappa(h-1)$, then $w - v = w - y > \kappa(h-1) - \kappa(h) = N + 2h + 1 > N + 2(h-1) + 1$, and thus, by taking $v' = y'$ and $w' = w$, it holds that $([v, w], [v', w']) \in Z_{h-1}^5$. If $\langle X \rangle = \langle \overline{A} \rangle$ and $v > \xi(h - 1)$, then it is easy to check that $v' = v + 1$ and $w' = x'$ imply $([v, w], [v', w']) \in Z_{h-1}^3$; otherwise, if $v \leq \xi(h - 1)$, then $w - v = x - v > \xi(h) - \xi(h - 1) = N + 2h + 1 > N + 2(h - 1) + 1$, and thus $v' = v$ and $w' = x'$ guarantee that $([v, w], [v', w']) \in Z_{h-1}^4$. If $\langle X \rangle = \langle B \rangle$, then we simply put $v' = x'$ and $w' = w + 1$ to get $([v, w], [v', w']) \in Z_{h-1}^3$. The outgoing edges from $Z_h^4$ labelled with $\langle A \rangle$ are dealt with in the same way as the ones from $Z_h^2$. Finally, let $([x, y], [x', y']) \in Z_h^5$. If $\langle X \rangle = \langle \overline{A} \rangle$ and $v > \xi(h - 1)$, then we set $v' = v + 1$ and $w' = x'$ to guarantee that $([v, w], [v', w']) \in Z_{h-1}^3$; otherwise, if $v \leq \xi(h - 1)$, then $w - v = x - v > \xi(h) - \xi(h - 1) = N + 2h + 1 > N + 2(h - 1) + 1$, and thus $v' = v$ and $w' = x'$ imply $([v, w], [v', w']) \in Z_{h-1}^4$. If $\langle X \rangle = \langle B \rangle$ and $w - x > N + 2(h - 1) + 1$, then we put $v' = x'$ and $w' = w$ to get $([v, w], [v', w']) \in Z_{h-1}^5$; otherwise, if $w - x \leq N + 2(h - 1) + 1$, then $w \leq N + 2(h - 1) + 1 + x < N + 2h + 1 + \kappa(h) = \kappa(h - 1)$, and thus, by letting $v' = x'$ and $w' = w + 1$, we have that $([v, w], [v', w']) \in Z_{h-1}^3$.

To conclude the proof, it suffices to observe that $M, [k, k + N + 1] \Vdash \langle \overline{B} \rangle p$, $M', [k, k + N + 2] \Vdash \neg \langle \overline{B} \rangle p$, and $[k, k + N + 1] Z_N^2 [k, k + N + 2]$. Thus, no formula of modal depth at most $N$ can define $\langle \overline{B} \rangle$ in the language $\mathsf{A\overline{A}B}$. As it happened with the previous lemma, the entire construction is parametric in $N$, so we can conclude that $\langle \overline{B} \rangle$ is not definable by any finite formula, which is to say that it is not definable. One can easily adapt the whole argument to prove that $\langle B \rangle \not\lessdot \mathsf{A\overline{A}B}$. $\qquad \square$

**Lemma 3.** $\langle \overline{L} \rangle \not\lessdot \mathsf{AB\overline{B}}$ *and* $\langle L \rangle \not\lessdot \mathsf{\overline{A}B\overline{B}}$ *over* $\mathbb{Z}$.

*Proof.* The proof of this lemma makes use of an $\mathsf{AB\overline{B}}$-bisimulation, and it turns out to be much easier than those of the previous two. Let $M = \langle \mathbb{I}(\mathbb{Z}), V \rangle$ and $M' = \langle \mathbb{I}(\mathbb{Z}), V' \rangle$, with $V(p) = \{[0, 1]\}$ and $V'(p) = \emptyset$. We show that $Z = \{([x, y], [x, y]) \mid x \geq 2\}$ is an $\mathsf{AB\overline{B}}$-bisimulation between the two models. To check that all conditions are satisfied, it suffices to observe that, starting from any pair of $Z$-related intervals, the application of modalities $\mathsf{AB\overline{B}}$ does not allow one to reach (in any of the two structures) any interval $[x, y]$, with $x < 2$. Since $[2, 3] Z [2, 3]$, $M, [2, 3] \Vdash \langle \overline{L} \rangle p$, and $M', [2, 3] \Vdash \neg \langle \overline{L} \rangle p$, it immediately follows that
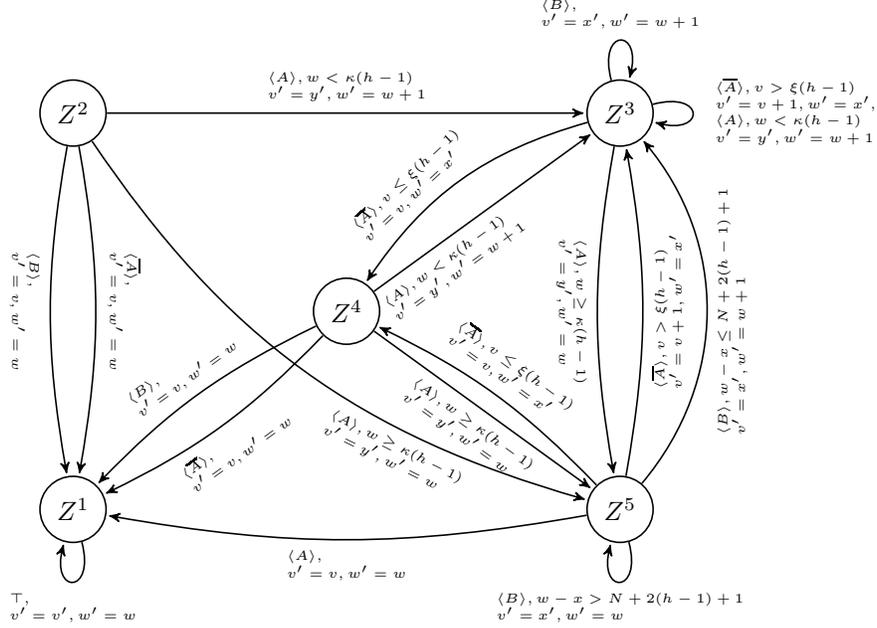
Figure 12: Graph of $Z_h$ for all modalities (Lemma 2).

no $\mathsf{AB\overline{B}}$-formula can define $\langle\overline{L}\rangle$, that is, $\langle\overline{L}\rangle \not\lhd \mathsf{AB\overline{B}}$. A similar argument works for $\langle L\rangle \not\lhd \mathsf{\overline{A}B\overline{B}}$. □

**Theorem 1.** *$\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$ and $\langle\overline{L}\rangle p \equiv \langle\overline{A}\rangle\langle\overline{A}\rangle p$ are the only inter-definability equations among the set of operators $\{\langle A\rangle, \langle\overline{A}\rangle, \langle L\rangle, \langle\overline{L}\rangle, \langle B\rangle, \langle\overline{B}\rangle\}$ over all the considered classes of linear orders.*

*Proof.* By Lemma 1, Lemma 2, and Lemma 3, the statement holds for $\mathbb{Z}$. Its truth for the class of strongly discrete linear orders immediately follows. As for $\mathbb{N}$ (resp., the class of finite linear orders, $\mathbb{Z}^-$), it suffices to observe that, in the proof of each of above lemmas, we can suitably restrict the portion of $\mathbb{Z}$ that plays an essential role in the relations $Z_h$. More precisely, it is possible to define a lower bound $l$ and an upper bound $u$, with $l < u$, such that the replacement of $\mathbb{Z}$ by $\mathbb{Z}_{>l} = \{x \in \mathbb{Z} \mid x > l\}$ (resp., $\mathbb{Z}_{>l}^{<u} = \{x \in \mathbb{Z} \mid l < x < u\}$, $\mathbb{Z}^{<u} = \{x \in \mathbb{Z} \mid x < u\}$) does not affect the proof in any significant way. Bounds $l$ and $u$ indeed enjoy the following property: for every pair of intervals $[x,y], [x',y']$, with $([x,y],[x',y']) \in Z_h$ for some $h$, if $x \le l$ or $x' \le l$ (resp., $y \ge u$ or $y' \ge u$), then $x = x'$ (resp., $y = y'$). In particular, lower and upper bounds for the $N$-bisimulation used in the proof of Lemma 1 are, respectively, $k + 1$ and $\chi(1) + k + 1$, where $k$ is the constant defined at the beginning of the proof, those for the $N$-bisimulation in Lemma 2 are, respectively, $\xi(1)$ and $\kappa(1) + 1$, and those for the bisimulation in Lemma 3 are 0 and 3, respectively.

For instance, to adapt the proof of Lemma 1 to the case of $\mathbb{N}$, it suffices to replace $\mathbb{Z}$ by $\mathbb{Z}_{>k+1}$ (which is isomorphic to $\mathbb{N}$). Similarly, to deal with the class of finite linear orders (resp., $\mathbb{Z}^-$) it suffices to replace $\mathbb{Z}$ by $\mathbb{Z}_{>k+1}^{<\chi(1)+k+1}$ (resp., $\mathbb{Z}^{<\chi(1)+k+1}$). Analogously for the other two lemmas. $\qquad\square$