On First-Order Propositional Neighborhood Logics: a First Attempt

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Abstract. Propositional Neighborhood Logic (PNL) is the decidable interval-based temporal logic that features the modal operators corresponding to the Allen's relations *meets* and *met by*. Right PNL (RPNL) is the fragment of PNL featuring only one of the two modality allowed in PNL. In this paper, we introduce a new extension of RPNL, whose propositional letters are generalized into first-order formulas. In contrast with recent results on the decidability of first-order point-based temporal logics with only one variable, we show that the interval-based case yields undecidability. In particular, in this paper we prove that the first order version of RPNL, allowing first-order formulas with only one (possibly reused) variable, is undecidable with respect to most meaningful choices for temporal and first-order domains.

1 Introduction

Interval temporal logics are based on temporal structures over (usually) linearly ordered domains, where time intervals, rather than time instants, are the primitive ontological entities. The problem of representing and reasoning about time intervals arises naturally in various fields of computer science, artificial intelligence, and temporal databases, such as theories of action and change, natural language processing, and constraint satisfaction problems. In particular, temporal logics with interval-based semantics have been proposed as a useful formalism for the specification and verification of hardware [19] and of real-time systems [11].

A systematic analysis of the variety of relations between two intervals in a linear order was performed by Allen [1], who proposed the use of interval reasoning in systems for time management and planning. Allen identified the thirteen different binary relations between intervals on linear orders, hereafter referred to as Allen's relations. In [14], Halpern and Shoham introduced a multi-modal logic, hereafter called HS, involving modal operators corresponding to all Allen's interval relations and showed that such a logic is undecidable under very weak assumptions on the class of interval structures in which it is interpreted. One of the few known cases of decidable interval logics with truly interval semantics (not reducible to point-based semantics) is the *Propositional Neighborhood Interval Logic* (*PNL*) [5, 13]. PNL is a fragment of HS with only two modal operators, corresponding to the Allen's relations *meets* and its inverse *met by*. Its satisfiability problem has been shown to be decidable (NEXPTIME-complete) when interpreted over various classes of linearly ordered sets and, in particular, over domains based on natural numbers [6]; the results presented in the same paper and in [18] showed that all possible extensions of PNL with Allen's modal operator make the logic undecidable, which means that PNL is maximal in terms of decidability (as a matter of fact, there are extensions of PNL that are non-elementary decidable only if interpreted over finite prefixes of $\mathbb{N}$ and undecidable in most of the other cases), with respect to modal operators corresponding to Allen's relations. In [7, 8], authors proposed a 'metric' extension of PNL, called *Metric* PNL (MPNL, for short), which involves special propositional letters expressing equality or inequality constraints on the length of the current interval with respect to fixed integer constants. The satisfiability problem for MPNL interpreted in the interval structure over natural numbers is proved decidable in [8], with complexity between EXPSPACE and 2NEXPTIME when the integer constraints in formulae are represented in binary (and NEXPTIME-complete when the integer constraints in formulae are constant or represented in unary). In [17], the authors analyzed extensions of PNL and MPNL with binders and variables that allow one to store the length of the current interval with respect to decidability and showed that even the weakest natural extensions become undecidable, which in some cases is somewhat surprising, being in sharp contrast with the decidability of MPNL. Finally, (R)PNL and its metric version have been generalized to the spatial case [9, 4]. It is therefore natural to ask whether it is possible to generalize these logics by means of classical machinery, such as first order constructs, still keeping their good computational properties.

In this paper, we focus on a different extension of PNL, called *FORPNL* (*First Order RPNL*), obtained by generalizing propositional variables into first-order formulas. In the point-based case, the most prominent work concerning first-order temporal languages is the one by Hodkinson, Wolter and Zakharyaschev [15]. The authors show that first-order Linear Time Temporal Logic (LTL) with Since and Until, interpreted over discrete structures is already undecidable when only two distinct variables are allowed. The proof also applies for LTL with Next and Future only. But, unexpectedly, when one extends LTL with monadic first-order formulas (only one variable), the logic becomes decidable with temporal domains based on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ (in the last case the result holds only with finite first-order domains). We show here that for interval logics the situation is way worse. To this end, we consider the fragment of PNL, called *Right PNL* (*RPNL*), featuring only the modal operator corresponding to the Allen's relation *meets*; we prove that, independently from the properties of the underlying temporal order, the first-order extension of RPNL with only one variable over finite first-order domains is undecidable. This paper can be considered a first attempt of extending an interval-based temporal logic with truly first-order features (over the *first-order domain*), since previous work, such as ITL [19], only deal with first-order characteristics for the temporal domain. This

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also justify the choice for the name FORPNL: we want to keep the modal characteristics of the propositional logic, which allow one to move along the time domain only by means of the modal operators, and generalize the assertion over interval from propositional to first order. On the contrary, the cases of first order ITL [19] and NL [2] are different in this sense, since those languages include quantification over the temporal domain.

The paper is structured as follows. Section 2 introduces syntax and semantics of the logic we are interested in, namely FORPNL. Section 3 briefly reviews the state of the art on first order temporal logics. Next, in Section 4, we give the undecidability proof of FORPNL, before concluding.

2 First Order RPNL

At the propositional level, RPNL is built from a set $\mathcal{AP} = \{p, q, \ldots\}$ of propositional letters, the classical connectives $\vee, \neg$ (the remaining ones can be considered as abbreviations), and a modal operator $\Diamond$ which allows one to capture any right neighboring interval from the current one. Formulas are obtained from the grammar:

$$\varphi ::= \pi \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond\varphi.$$

where π is a pre-interpreted propositional letter that is true over all and only intervals of the type $[i, i]$, called point-intervals.

Given a linearly ordered domain $\mathbb{D} = \langle D, < \rangle$, a (*non-strict*) *interval* over $\mathbb{D}$ is any ordered pair $[i, j]$ such that $i \leq j$. An *interval structure* is a pair $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$, where $\mathbb{I}(\mathbb{D})$ is the set of all intervals over $\mathbb{D}$. An *interval model* is a tuple $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V \rangle$, where $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ is an interval structure and $V : \mathbb{I}(\mathbb{D}) \to 2^{\mathcal{AP}}$ is a valuation function assigning to every interval the set of propositional letters that hold over it. Given an interval model $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V \rangle$ and an interval $[i, j]$ over it, the semantics of RPNL-formulae is given by the clauses:

- $M, [i, j] \Vdash \pi$ iff $i = j$;
- $M, [i, j] \Vdash p$ iff $p \in V([i, j])$, for any $p \in \mathcal{AP}$;
- $M, [i, j] \Vdash \neg\psi$ iff it is not the case that $M, [i, j] \Vdash \psi$;
- $M, [i, j] \Vdash \psi \vee \tau$ iff $M, [i, j] \Vdash \psi$ or $M, [i, j] \Vdash \tau$;
- $M, [i, j] \Vdash \Diamond\psi$ iff there exists $h \geq j$ such that $M, [j, h] \Vdash \psi$.

A RPNL-formula φ is *satisfiable* if there exists a model M and an interval $[i, j]$ over it such that $M, [i, j] \Vdash \varphi$. The satisfiability problem for RPNL has been shown to be NEXPTIME-complete in [10].

We introduce now a first-order version of the logic RPNL, hereafter called *First Order RPNL* (*FORPNL*, for short). At the first-order level, propositional variables are generalized into *predicate symbols* $P, Q, \ldots$, each one of which has fixed arity. In addition, the language features a set of *individual variables* $x, y, \ldots$, a set of *individual constants* $a, b, \ldots$, and the *universal quantifier* $\forall x$ for each individual variable. Propositional variables can be viewed as 0-ary predicates. *Terms* $\tau_1, \tau_2, \ldots$ are either individual variables or individual constants. As standard, we have that $\exists x\varphi \equiv \neg\forall x\neg\varphi$. A *First Order Interval Model* is of the type $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathfrak{D}, \mathcal{I} \rangle$, where $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ is an interval structure as before, $\mathfrak{D}$ is the *first-order domain* of M, and $\mathcal{I}$ is a function associating each interval of $\mathbb{I}(\mathbb{D})$ with a first-order structure

$$\mathcal{I}([i, j]) = \langle \mathfrak{D}, P^{\mathcal{I}([i,j])}, Q^{\mathcal{I}([i,j])}, \ldots \rangle.$$

At each interval $[i, j]$, a predicate $P^{\mathcal{I}([i,j])}$ is a relation on $\mathfrak{D}$ of the same arity as P (for propositional variable, it is simply true or false). Finally, λ is an *assignment* function mapping terms into elements in $\mathfrak{D}$. Notice that we are assuming that constants are *rigid*, that is, a constant a refers to the same element of the first-order domain $\mathfrak{D}$ regardless of which is the current interval. The semantics of FORPNL is the following:

- $M, [i, j], \lambda \Vdash \pi$ iff $i = j$;
- $M, [i, j], \lambda \Vdash P(\tau_1, \ldots, \tau_n)$ iff $P^{\mathcal{I}([i,j])}(\lambda(\tau_1), \ldots, \lambda(\tau_n))$;
- $M, [i, j], \lambda \Vdash \neg\psi$ iff it is not the case that $M, [i, j], \lambda \Vdash \psi$;
- $M, [i, j], \lambda \Vdash \psi \vee \phi$ iff $M, [i, j], \lambda \Vdash \psi$ or $M, [i, j], \lambda \Vdash \phi$;
- $M, [i, j], \lambda \Vdash \forall x\psi$ iff $M, [i, j], \lambda' \Vdash \psi$ for any assignment λ' that differs from λ at most for the value of x;
- $M, [i, j], \lambda \Vdash \Diamond\psi$ iff there exists $h \geq j$ such that $M, [j, h], \lambda \Vdash \psi$.

Therefore, FORPNL is a *partial* first order generalization of the propositional logic RPNL: one is allowed to move along the time domain by using only the modal operator, and to assert over a specific interval by using first-order construct. Moreover, it can be considered as the *product* of First-Order Logic and RPNL [12], since the first-order part and the modal part may interact freely.

3 Is FORPNL Without Hopes?

In this section, we recall some well-known results in the literature, that makes the result presented in this paper somehow surprising. First of all, we know that among the maximal first order logic fragments that have been shown to be decidable we can find:

- two-variable first order logic [3];
- two-variable first order logic over ordered domains (specifically, the class of all linear orders, and all linear orders over $\mathbb{N}$) [20].

In the framework of temporal logics, as already mentioned above, it has been shown in [15] that extending LTL (with Since and Until, but the result also applies to the fragment with Future and Next only) with a first-order machinery with two distinct variables yields undecidability. To retrieve decidability one must restrict the language by allowing only one variable.

We want to prove here that in the interval-based case, the situation is way worse. RPNL represents one of the first, and most studied, case of decidable interval logics. It has been shown to be decidable [6]:

- in the class of all linearly ordered sets;
- in the class of all discrete linearly ordered sets;
- in the class of all dense linearly ordered sets;
- in the class of all finite linearly ordered sets;
- in the class of all linearly ordered sets based on $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$.

In despite of the generally good behaviour of RPNL (w.r.t. the problem of satisfiability) and of the possibility of extending the temporal (point-based) logic LTL with first-order constructs, as we will prove below, the combination of almost any first-order ingredient and of the interval-based frame results in undecidability.

4 Undecidability

As it becomes clear from the above, there are a number of possible parameters here. Beside the usual possible choices for the temporal domain, that is, discrete, dense, finite, bounded, unbounded, and so on, we can vary on the first order component by assuming that the first-order domain is finite, infinite, constant, variable, expanding, or assuming other specific properties for it (linearity, discreteness, denseness, and so on), and also by limiting the number of distinct variables in formulas. Since we are interested in tight undecidability results, in contrast with decidability results for first order point-based temporal logic, we focus our attention on very restrictive assumptions. In particular, assuming the temporal domain to be finite, the decidability result becomes really simple (although the complexity is the same as in the other cases, NEXPTIME, the constants hidden in the complexity function are low, and the idea under the model theoretic argument is easy to understand [10]). For these reasons, from now on, we assume that both D and $\mathfrak{D}$ are finite, and that our language has only one variable. Nevertheless, the results presented in this paper hold even over the class of all (resp., all dense, all discrete) linearly ordered sets, independently from the assumption on the first-order domain (infinite, expanding, dense, discrete, and so on). Moreover, in our construction there are neither free variables nor constants, so we omit the variable assignment λ.

We make use of the undecidability of the *Finite Tiling Problem* [16]. It is the problem of establishing whether, for a given set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$, there exists a finite rectangle $\mathcal{R} = [1, X] \times [1, Y] = \{(i, j) : i, j \in \mathbb{N}, 1 \leq i \leq X, \text{and } 1 \leq j \leq Y\}$ for some $X, Y \in \mathbb{N}$, such that $\mathcal{T}$ can correctly tile $\mathcal{R}$ with the entire border colored by the same designated color \$, also called *side color*. To be more precise, for every tile type $t_i \in \mathcal{T}$, let $right(t_i)$, $left(t_i)$, $up(t_i)$, and $down(t_i)$ be the colors of the corresponding sides of t_i. To solve the Finite Tiling Problem for $\mathcal{T}$ one must find two natural numbers X and Y, and a mapping $f : \mathcal{R} \to \mathcal{T}$ such that:

$$\begin{array}{ll} right(f(i,j)) = left(f(i+1,j)), & 0 \leq i < X, 0 \leq j \leq Y, \\ up(f(i,j)) = down(f(i,j+1)), & 0 \leq i \leq X, 0 \leq j < Y, \end{array}$$

and that satisfies, in addition, the following constraints:

$$\begin{array}{llll} left(f(0,j)) = \$ & \text{and} & right(f(X,j)) = \$, & 0 \leq j \leq Y, \\ down(f(i,0)) = \$ & \text{and} & up(f(i,Y)) = \$, & 0 \leq i \leq X. \end{array}$$

where \$ is the side color of $\mathcal{R}$.

In order to perform the reduction from the Finite Tiling Problem for the set of tiles $\mathcal{T} = \{t_1, \ldots, t_k\}$ to the satisfiability problem for FORPNL, we will make use of some special 0-ary predicate symbols, namely u, Id, $\mathsf{up\_rel}$, final, $\mathtt{t}_1, \mathtt{t}_2, \ldots, \mathtt{t}_k$. The reduction consists of three main steps:

1. the encoding of the rectangle by means of a suitable finite chain of so-called 'unit' intervals (u-*intervals*, for short);
2. the encoding of the 'above-neighbor' relation by means of a suitable family of so-called $\mathsf{up\_rel}$-*intervals*; and
3. the encoding of the 'right-neighbor' relation.

Here is a sketch of the encoding. First, we set our framework by forcing the existence of a unique finite chain of u-intervals on the linear ordering (u-*chain*, for short). The u-intervals are used as cells to arrange the tiling. In other words, they represent the parts of the plane that must be covered by tiles. Next, we define a chain of Id-intervals (Id-*chain*, for short), each of them representing a row of the rectangle. Any Id-interval consists of a sequence of u-intervals; each Id will contain exactly the same number of u-intervals. Then, we use $\mathsf{up\_rel}$ to encode the relation that connects each tile with its above neighbor in $\mathcal{R}$. Finally, we introduce a set of 0-ary predicate symbols $\mathtt{T} = \{\mathtt{t}_1, \mathtt{t}_2, \ldots, \mathtt{t}_k\}$ corresponding to the set of tile types $\mathcal{T} = \{t_1, t_2, \ldots, t_k\}$ and define a formula $\Phi_{\mathcal{T}}$ which is satisfiable if and only if there exists a finite rectangle $\mathcal{R}$ for some $X, Y \in \mathbb{N}$ and a proper tiling of $\mathcal{R}$ by $\mathcal{T}$, i.e., a tiling that satisfies the color constraints on the border tiles and between vertically- and horizontally-adjacent tiles.

The proof exploits the fact that introducing first order constructs makes it possible to express properties of the type: "*if an interval satisfyes φ, then all its beginning intervals (resp., ending intervals, strict sub-intervals) do not stisfy ψ*", where the strict sub-intervals of an interval $[a, b]$ are all intervals $[c, d]$ such that $a < c < d < b$. In order to express such properties, we firstly define some kind of 'nominals' for each point of the temporal domain. Intuitively, we univocally identify each point i of the temporal domain with a non-empty set of constants that make a special predicate true in intervals starting from i. More formally, we force a predicate of the type $P(x)$ in such a way that if $P(x)$ is true, for some x, over an interval $[i, j]$, then it can be possibly true (for the same x) only over interval starting from i and it must be false over all intervals starting from some different point $h \neq i$. For example, given an interval $[i, j]$ that satisfies $P^{\mathcal{I}([i,j])}(a)$, for some constant a, we force $\neg P^{\mathcal{I}([h,k])}(a)$ to hold over each interval $[h, k]$, with $h \neq i$. To this end, we exploit the following formula:

$$\Box\Box(\exists x \Diamond P(x) \wedge \forall x (\Diamond P(x) \to \Box(\neg\pi \to \Box\neg P(x)))) \quad (1)$$

It is easy to verify the following lemma:

Lemma 1 *Let M be a FORPNL model and $[i, j]$ an interval over it. If $M, [i, j] \Vdash (1)$, then for each $h \in D$:*

1. *there exists a point $k > h$ such that $P^{\mathcal{I}([h,k])}(a)$ holds for some a,*
2. *for each a such that $P^{\mathcal{I}([h,k])}(a)$ holds, then $\neg P^{\mathcal{I}([l,m])}(a)$ holds, for each $l \neq h$.*

At this step, we can express properties about beginning intervals, ending intervals, or strict sub-intervals of a given interval, by exploiting such a notion of nominals, formalized in the above lemma. For example, it is easy to see that the following formula correctly defines the operator $[B^{\varphi}_{\psi}]$ (resp., $[E^{\varphi}_{\psi}]$, $[D^{\varphi}_{\psi}]$), expressing the property: "*if an interval satisfies the property φ, then each beginning interval (resp., ending interval, strict sub-interval) satisfies the property ψ*", thus 'simulating' the modal operator $[B]$ (resp., $[E]$, $[D]$) of the logic HS, corresponding to the Allen's relation *begins* (resp., *ends, during*):

$$[B^{\varphi}_{\psi}] \equiv \Box\Box\forall x(\Diamond(\varphi \wedge \Diamond P(x)) \to \Box(\Diamond(\neg\pi \wedge \Diamond P(x)) \to \psi))$$

$$[E^{\varphi}_{\psi}] \equiv \Box\Box\forall x(\Diamond(\varphi \wedge \Diamond P(x)) \to \Box(\neg\pi \to \Box(\Diamond P(x) \to \psi)))$$

$$[D^{\varphi}_{\psi}] \equiv \left\{ \begin{array}{l} \Box\Box\forall x(\Diamond(\varphi \wedge \Diamond P(x)) \to \\ \Box(\neg\pi \to \Box(\Diamond(\neg\pi \wedge \Diamond P(x)) \to \psi))) \end{array} \right.$$

Notice that we are not able to properly define the HS operators $[B]$, $[E]$, and $[D]$, since we cannot capture beginning, ending, and during intervals of the current one.

To define the u-chain we use the following formulae:

$$\Diamond(\neg\pi \wedge \mathsf{u}) \quad (2)$$

$$\Box\Box(\mathsf{u} \to (\neg\pi \wedge (\Diamond\mathsf{u} \vee \Box\pi))) \quad (3)$$

$$[B^{\mathsf{u}}_{\neg\mathsf{u}}] \wedge [B^{\mathsf{u}}_{\neg\pi\to\neg\Diamond\mathsf{u}}] \quad (4)$$

$$(1) \wedge (2) \wedge (3) \wedge (4) \quad (5)$$

Lemma 2 *Let $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathfrak{D}, \mathcal{I} \rangle$ be a FORPNL model based on a finite linearly ordered temporal domain and with a finite first-order domain, such that*

$$M, [i_0, j_0] \Vdash (5).$$

Then, there exists a finite sequence of points $j_0 < j_1 < \ldots < j_n$, with $n > 0$, such that:

1. $M, [j_l, j_{l+1}] \Vdash \mathsf{u}$ *for each* $0 \leq l \leq n-1$*;*
2. $M, [j', j''] \Vdash \mathsf{u}$ *holds for no other interval* $[i', j']$.

Proof. If $M, [i_0, j_0] \Vdash (5)$, then, by (2), for some $j_1 > j_0$ the interval $[j_0, j_1]$ is a u-interval. By (3), j_1 starts a finite chain of u-intervals $[j_l, j_{l+1}]$, with $l \geq 0$. The satisfiability of (3) over finite temporal domains follows from the fact that the last point of the temporal domain satisfies $\Box\pi$. Now suppose, by contradiction, that for some interval $[j', j'']$, it is the case that $[j', j'']$ is a u-interval but $[j', j''] \neq [j_l, j_{l+1}]$ for all $l > 0$. Then either $j' = j_l$ for some l, contradicting the first conjunct of (4), or $j_l < j' < j_{l+1}$, contradicting the second conjunct of (4). ■

We now define the Id-chain with the following formulae:

$$\Diamond\mathsf{Id} \wedge \Box\Box((\Diamond\mathsf{Id} \rightarrow \Diamond\mathsf{u}) \wedge (\mathsf{Id} \rightarrow \neg\pi \wedge \neg\mathsf{u} \wedge (\Diamond\mathsf{Id} \vee \Box\pi))) \quad (6)$$

$$[B^{\mathsf{Id}}_{\neg\mathsf{Id}}] \wedge [B^{\mathsf{Id}}_{\neg\pi\rightarrow\neg\Diamond\mathsf{Id}}] \quad (7)$$

$$(6) \wedge (7) \quad (8)$$

Lemma 3 *Let* $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathfrak{D}, \mathcal{I} \rangle$ *be a FORPNL model based on a finite linearly ordered temporal domain and with a finite first-order domain, such that*

$$M, [i_0, j_0] \Vdash (5) \wedge (8).$$

Then, there exist a positive integer v *and a finite sequence of positive integers* $m_1, m_2, \ldots, m_v$ *and a finite sequence of points* $j_0^1 < j_1^1 < \ldots < j_{m_1}^1 = j_0^2 < \ldots < j_{m_2}^2 = \ldots = j_0^{v-1} < \ldots < j_{m_{v-1}}^{v-1} = j_0^v < \ldots < j_{m_v}^v$ *such that, for each* $1 \leq s \leq v$*, we have* $M, [j_0^s, j_{m_s}^s] \Vdash \mathsf{Id}$*, and no other interval satisfies* Id.

Proof. First of all, by Lemma 2, there is a finite sequence of points $j_0 < j_1 < \ldots < j_n$, with $n > 0$, defining a finite chain of u-intervals. By (6), j_0 starts a Id-interval, which must end at some $j_l > j_1$. By (6), each Id-interval is followed by another Id-interval, and each Id-interval must end at some j_l. Thus, every Id-interval spans several u-intervals, and there are finitely many Id-intervals. Let their number be v. Hence, the sequence $j_0 < j_1 < \ldots < j_n$ can be written as $j_0^1 < j_1^1 < \ldots < j_{m_1}^1 = j_0^2 < \ldots < j_{m_2}^2 = \ldots = j_0^{v-1} < \ldots < j_{m_{v-1}}^{v-1} = j_0^v < \ldots < j_{m_v}^v$, as required. We want to show that there are no other Id-interval beside those of the type $[j_0^s, j_{m_s}^s]$. This can be shown exactly as in Lemma 2, by using (7), joined with (1). ■

The above lemma guarantees the existence of an Id-chain. Now, we want to force the propositional letter up_rel to correctly encode the relation that connects pairs of tiles of the rectangle $\mathcal{R}$ that are vertically adjacent. Formally, we define two u-intervals $[j_l, j_{l+1}]$ and $[j_{l'}, j_{l'+1}]$ to be *above-connected* if and only if $[j_{l+1}, j_{l'}]$ is a up_rel-interval. At the same time, we want to make sure that each Id-interval spans the same number of tile-intervals. Intuitively, these two properties can be guaranteed by assuring that each u-interval of a Id-interval is connected with exactly one u-interval of the next Id-interval and with exactly one Id-interval of the previous level. To this end, firstly we suitably label u-intervals belonging to the last Id-interval with the propositional letter final. Then, we constraint each u-interval not belonging to the last Id-interval to be connected to at least one u-interval in the future (formula (10)) and at least one interval in the past (formula (16)) by means of a up_rel-interval. In order to guarantee the correct correspondence between u-intervals of consecutive Id-intervals and to guarantee that each u-interval is connected with at most one u-interval in the future and at most one u-interval in the past, we force the condition that no up_rel-interval is a biginning interval (resp., ending interval, strict sub-interval) of any other up_rel-interval. Finally, to guarantee that up_rel-intervals connect u-intervals belonging to consecutive Id-intervals, we have to make sure that no Id-interval is a biginning interval (resp., ending interval, strict sub-interval, strict super-interval) of a up_rel-interval.

$$\Box\Box(\mathsf{u} \wedge \Box\Box\neg\mathsf{Id} \leftrightarrow \mathsf{final}) \quad (9)$$

$$\Box\Box(\mathsf{u} \rightarrow (\neg\mathsf{final} \leftrightarrow \Diamond\mathsf{up\_rel})) \quad (10)$$

$$\Box\Box(\mathsf{up\_rel} \rightarrow \neg\mathsf{Id} \wedge \neg\pi \wedge \neg\mathsf{u} \wedge \Diamond\mathsf{u})) \quad (11)$$

$$\neg\Diamond\mathsf{up\_rel} \wedge \Box\Box(\Diamond\mathsf{up\_rel} \rightarrow \Diamond\mathsf{u}) \quad (12)$$

$$[B^{\mathsf{up\_rel}}_{\neg\mathsf{up\_rel}}] \wedge [E^{\mathsf{up\_rel}}_{\neg\mathsf{up\_rel}}] \wedge [D^{\mathsf{up\_rel}}_{\neg\mathsf{up\_rel}}] \quad (13)$$

$$[B^{\mathsf{up\_rel}}_{\neg\mathsf{Id}}] \wedge [E^{\mathsf{up\_rel}}_{\neg\mathsf{Id}}] \wedge [D^{\mathsf{up\_rel}}_{\neg\mathsf{Id}}] \quad (14)$$

$$[D^{\mathsf{Id}}_{\neg\mathsf{up\_rel}}] \quad (15)$$

$$\forall x(\Diamond(\mathsf{Id} \wedge \Diamond(\Diamond\mathsf{u} \wedge \Diamond P(x))) \rightarrow \Diamond\Diamond(\mathsf{up\_rel} \wedge \Diamond P(x))) \quad (16)$$

$$(9) \wedge (10) \wedge (11) \wedge (12) \wedge (13) \wedge (14) \wedge (15) \wedge (16) \quad (17)$$

Lemma 4 *Let* $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathfrak{D}, \mathcal{I} \rangle$ *be a FORPNL model based on a finite linearly ordered temporal domain and with a finite first-order domain, such that*

$$M, [i_0, j_0] \Vdash (5) \wedge (8) \wedge (17).$$

Then, we have that, for each $0 < s < v$ *and each* $0 \leq l < m_s$*,* $M, [j_{l+1}^s, j_l^{s+1}] \Vdash$ up_rel*, and no other interval satisfies* up_rel*. Moreover, we have that for each* $1 \leq s, s' \leq v$*,* $m_s = m_{s'}$*.*

Proof. Consider any u-interval $[j_l^s, j_{l+1}^s]$ not belonging to the last Id-interval. Formula (10) makes sure that j_{l+1}^s starts a up_rel-interval, which cannot be point-interval and must end at some point of the type $j_{l'}^{s'} > j_{l+2}^s$. First of all, observe that $j_{l'}^{s'} \geq j_0^{s+1}$, otherwise we would have a contradiction with (15). Similarly, we have that $j_{l'}^{s'} < j_{m_{s+1}}^{s+1}$, in order to avoid a contradiction with (14). Now, suppose by contradiction that $[j_0^s, j_1^s]$ is above-connected with $[j_l^{s+1}, j_{l+1}^{s+1}]$, with $l > 0$, for some s. By (16), there must be an up_rel-interval ending in j_0^{s+1} and starting from a point $j_{l'}^s$, with $l' > 0$. It must also be $l' > 1$, otherwise there would be two different up_rel-intervals starting at the same point j_1^s, contradicting the first conjunct of (13). So, it is the case that the up_rel-interval $[j_{l'}^s, j_0^{s+1}]$ is a strict sub-interval of the up_rel-interval $[j_1^s, j_l^{s+1}]$, contradicting the third conjunt of (13). By applying a similar argument, and assuming that up to a given l, $[j_l^s, j_{l+1}^s]$ is above-connected to $[j_l^{s+1}, j_{l+1}^{s+1}]$, it is easy to show also that $[j_{l+1}^s, j_{l+2}^s]$ (if any) is above-connected to $[j_{l+1}^{s+1}, j_{l+2}^{s+2}]$. From (13) it follows that each u-interval can be connected with at most one u-interval in the future and at most one in the past, so we can conclude that for each $0 \leq s, s' \leq v$, $m_s = m_{s'}$. ■

Finally, we can force all tile-matching conditions to be respected, by using the following formulae, where $\mathtt{T_r}$ (resp., $\mathtt{T_l}$, $\mathtt{T_u}$, $\mathtt{T_d}$) is the subset of $\mathtt{T}$ containing all tiles having the right (resp., left, up, down) side colored with \$.

$$\Box\Box\left(\mathsf{u} \rightarrow \bigvee_{\mathtt{t_q}\in\mathtt{T}} \mathtt{t_q} \wedge \bigwedge_{\mathtt{t_q}\neq\mathtt{t_{q'}}} \neg(\mathtt{t_q}\wedge\mathtt{t_{q'}})\right) \tag{18}$$

$$\Box\Box\left(\bigvee_{\mathtt{t_q}\in\mathtt{T}} \mathtt{t_q} \rightarrow \mathsf{u}\right) \tag{19}$$

$$\Box\Box\left(\bigvee_{\mathtt{t_q}\in\mathtt{T}} \mathtt{t_q} \rightarrow \left(\neg(\Diamond\mathsf{Id}\vee\Box\pi) \rightarrow \bigvee_{right(t_q)=left(t_{q'})} \Diamond\mathtt{t}'_{\mathtt{q}}\right)\right) \tag{20}$$

$$\Box\Box\left(\bigvee_{\mathtt{t_q}\in\mathtt{T}} \mathtt{t_q} \rightarrow \left(\Diamond\mathsf{up\_rel} \rightarrow \bigvee_{up(t_q)=down(t_{q'})} \Diamond(\mathsf{up\_rel}\wedge\Diamond\mathtt{t}'_{\mathtt{q}})\right)\right) \tag{21}$$

$$\Box\Box\left(\Diamond\mathsf{Id} \rightarrow \left(\Diamond\bigvee_{\mathtt{t_q}\in\mathtt{T_l}} \mathtt{t_q}\right) \wedge \left(\mathsf{u} \rightarrow \bigvee_{\mathtt{t_q}\in\mathtt{T_r}} \mathtt{t_q}\right)\right) \tag{22}$$

$$\exists x\left(\Diamond(\mathsf{Id}\wedge\Diamond P(x)) \rightarrow \Box\Box\left(\mathsf{u}\wedge\Diamond\Diamond P(x) \rightarrow \bigvee_{\mathtt{t_q}\in\mathtt{T_d}} \mathtt{t_q}\right)\right) \tag{23}$$

$$\Box\Box\left(\mathsf{u}\wedge\mathsf{final} \rightarrow \bigvee_{\mathtt{t_q}\in\mathtt{T_u}} \mathtt{t_q}\right) \tag{24}$$

$$(18)\wedge(19)\wedge(20)\wedge(21)\wedge(22)\wedge(23)\wedge(24) \tag{25}$$

Theorem 5 *Given any finite set of tiles $\mathcal{T}$ and a side color \$, the formula*

$$\Phi_{\mathcal{T}} := (5)\wedge(8)\wedge(17)\wedge(25)$$

is satisfiable in a finite linearly ordered temporal domain and finite first-order domain if and only if $\mathcal{T}$ can tile a finite rectangle $\mathcal{R}$, for some $X, Y \in \mathbb{N}$, with side color \$.

Proof. (Only if:): Suppose that $M, [i_0, j_0] \Vdash \Phi_{\mathcal{T}}$. Then, by Lemma 3, there is a sequence of points $j_0 = j_0^1 < j_1^1 < \ldots < j_{m_1}^1 = j_0^2 < \ldots < j_{m_2}^2 = \ldots = j_0^{v-1} < \ldots < j_{m_{v-1}}^{v-1} = j_0^v < \ldots < j_{m_v}^v = j_n$, and by Lemma 4, for each $1 \leq s, s' \leq v$, $m_s = m_{s'}$. We put $X = m_s$ and $Y = v$. For all l, s, where $0 \leq l \leq X-1$, $1 \leq s \leq Y$, define $f(l,s) = t_q$ if and only if $M, [j_l^s, j_{l+1}^s] \Vdash \mathtt{t_q}$. From Lemma 2, 3, and 4 it follows that the function $f : \mathcal{R} \rightarrow \mathcal{T}$ defines a correct tiling of $\mathcal{R}$, where X and Y are defined as above.

(If:) Let $f : \mathcal{R} \mapsto \mathcal{T}$ be a correct tiling function of the rectangle $\mathcal{R} = [1, X] \times [1, Y]$ for some X, Y, and a given border color \$. For convenience, we will identify the tile-variables $\mathtt{t}_1, \mathtt{t}_2, \ldots \in \mathtt{T}$ with their corresponding tiles $t_1, t_2, \ldots \in \mathcal{T}$. We will show that there exist a model M and an interval $[i_0, j_0]$ such that $M, [i_0, j_0] \Vdash \Phi_{\mathcal{T}}$. Let $D = \mathfrak{D} = \mathbb{N}\mid_{X\cdot Y+1}$, and let M the FORPNL model built over these two domains. We want to build an interpretation $\mathcal{I}$ in such a way that $M, [0,1] \Vdash \Phi_{\mathcal{T}}$. Then, we put

$$\mathsf{u}^{\mathcal{I}([i,i+1])} \quad \forall i.0 < i < X\cdot Y,$$

to guarantee that (5) is satisfied. Now, in order to satisfy the remaining part of $\Phi_{\mathcal{T}}$ on $[0,1]$, it suffices to define the valuation for the remaining propositional letters and the predicate symbol P:

$$\begin{array}{rl} P^{\mathcal{I}([i,j])}(i) & \forall i,j > 0 \\ \mathsf{Id}^{\mathcal{I}([i\cdot X+1,(i+1)\cdot X+1])} & \forall i.0 \leq i \leq Y-1 \\ \mathsf{up\_rel}^{\mathcal{I}([i,i+X-1])} & \forall i.2 \leq i \leq X\cdot(Y-1)+1 \\ \mathsf{final}^{\mathcal{I}([i,i+1])} & \forall i.X\cdot(Y-1)+1 \leq i \leq X\cdot Y \end{array}$$

Finally, we evaluate the tile-variables as follows. For each $\mathtt{t} \in \mathtt{T}$:

$$\mathtt{t_q}^{\mathcal{I}([i,i+1])} \Leftrightarrow f(l,s) = t_q \quad \forall i = X\cdot(s-1)+l.$$

■

5 Conclusions

Temporal logic has found numerous applications in computer science, ranging from the traditional and well-developed fields of program specification and verification, temporal databases, and distributed multi-agent systems, to more recent uses in knowledge representation and reasoning. This is true both at the propositional and first-order level. In the interval-based temporal logic world, undecidability is the rule and decidability the exception. Propositional Neighborhood Logic is one of the first examples of properly interval-based temporal logics shown to be decidable. Recently, it has also been extended with a sort of metric features that allow one to constrain the length of an interval (over natural numbers), without losing decidability. On the line of [17], here we have shown that yet another classical extension for temporal logics, obtained by generalizing propositional letters into first-order formulas, oversteps the barrier of decidability, even in a very restrictive case such as that of monadic first order formulas with finite domains. At a first glance this result may appear discouraging, concerning our aim of finding decidable first-order interval temporal logics. Nevertheless, it should be pointed out that the modal constant π plays an important role in the reduction. Thus, it could be worth considering the satisfiability problem for the language devoid of such an operator, as well as the satisfiability problem for FORPNL restricted with some natural syntactic rule that constrain the relationship between the modal and the first-order components.

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