The importance of the past in interval temporal logics: the case of Propositional Neighborhood Logic

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Abstract. The expedience of adding past operators to a (point-based) temporal logic has been largely discussed in the literature. Opponents argue that in various relevant cases such an addition does not involve any increase in expressiveness. Supporters reply that many statements are easier to express when past operators are included; moreover, also in the cases in which no expressive power is added, succinctness is achieved, that is, there are classes of properties that can be expressed by means of much shorter formulas. In the present contribution, we study in detail the effects of adding past operators to interval temporal logics. We focus our attention on the representative case of Propositional Neighborhood Logic, taking into consideration different temporal domains.

1 Introduction

The problem of reducing complex modal (temporal) logics, with many modalities, to simpler ones, possibly with just one modality, preserving their distinctive properties, is a well-known problem in modal (temporal) logic. One classical work in this area is that by Thomason [25,26], who has shown how to reduce propositional temporal logic to propositional modal logic preserving both the relation of logical consequence and that of derivability. The same problem can be viewed from the opposite side, looking at the possible advantages of adding a new modal (temporal) operator to a given modal (temporal) logic. This is our point of view here: we investigate the effects of adding past operators to interval temporal logics. We focus our attention on the representative case of the propositional interval logic of temporal neighborhood, taking into consideration different temporal domains.

The expedience of adding past operators to a (point-based) temporal logic has been largely discussed in the literature [9,10,11,14,15]. Opponents argue that in various relevant cases such an addition does not involve any increase in expressiveness. Supporters reply that many statements are easier to express when past operators are included (simplicity); moreover, also in the cases in which no expressive power is added, succinctness is achieved, that is, there are classes of properties that can be expressed by means of much shorter formulas.

In [11], Gabbay et al. prove the expressive completeness of Linear Time Temporal Logic (LTL for short) with respect to first-order logic over $\mathbb{N}$ (and beyond). Paired with Kamp's theorem, such a result shows that the addition of past modalities does not increase the expressive power of LTL. In a subsequent paper, Gabbay provides a translation algorithm to map formulas of LTL+Past into equivalent LTL-formulas, where the size of the resulting LTL-formula is assumed to possibly be non-elementary in the size of the original LTL+Past-formula [10]. As pointed out in [18], a more efficient translation algorithm can be obtained passing through (counter-free) Büchi automata. It consists of three main steps. First, it translates any given formula of LTL+Past into a corresponding Büchi automaton [15]. Then, it maps such an automaton into an equivalent deterministic Muller automaton [24]. Since the language it defines is star-free, it can be assumed to be a counter-free automaton. Finally, it transforms the counter-free Muller automaton into a formula of LTL [16]. Since each of these three steps possibly involves an exponential blowup, the size of the resulting LTL-formula is at most triply exponential in the size of the original LTL+Past-formula. In [14], Laroussinie et al. prove that LTL+Past can be exponentially more succinct than LTL, that is, there exists a family of formulas in LTL+Past, with size $\mathrm{O}(n)$, such that the size of the equivalent LTL-formulas is $\Omega(2^n)$.

In the following, we study what happens when past modalities are added to future-only interval temporal logics. Interval temporal logics are a family of modal logics for reasoning about relational interval structures over linear orders. The set of all possible binary relations between such intervals is known as the set of *Allen's interval relations* [1]. A distinct modal operator can be associated with each of them. While formulas of point-based temporal logics are evaluated at time points, formulas of interval temporal logics are evaluated at time intervals. This results in a substantially higher expressiveness and computational complexity of interval temporal logics as compared to point-based ones. Hence, it does not come as a surprise that, while decidability is a common feature of point-based temporal logics, undecidability dominates among interval-based ones [13,21,27].

For a long time, such a situation has discouraged the search for practical applications and further theoretical investigation on interval-based temporal logics. This bleak picture started lightening up in the last few years when various non-trivial decidable interval temporal logics have been identified [3,4,5,6,7,8,19,20]. (Un)decidability of an interval temporal logic depends on two main factors: (i) the set of its interval modalities, and (ii) the class of interval models (the linear order) over which it is interpreted. Gradually, it became evident that the trade-off between expressiveness and computational affordability in the family of interval temporal logics is rather subtle and sometimes unpredictable, with the border between decidability and undecidability cutting right across the core of that family. One real character is the interval temporal logic of the subinterval relation: it is PSPACE-complete when interpreted over dense linear orders [3,23], while it turns out to be undecidable when interpreted over finite or discrete linear orders [17].

A special position in the family of interval temporal logics is reserved to Propositional Neighborhood Logic, denoted $\mathsf{A}\overline{\mathsf{A}}$. $\mathsf{A}\overline{\mathsf{A}}$ features two modalities $\langle A \rangle$ and $\langle \overline{A} \rangle$ that make it possible to access intervals adjacent to the right (future) and to the left (past) of the current interval, respectively. By iterating the application of the modality $\langle A \rangle$ (resp., $\langle \overline{A} \rangle$), one can reach any interval in the future (resp., past) of the current one. In the following, we address various issues about the expressive power of $\mathsf{A}\overline{\mathsf{A}}$, of its future fragment A, and of the interval logic $\mathsf{A}\overline{\mathsf{L}}$ that one obtains by replacing the adjacent-to-the-left modality $\langle \overline{A} \rangle$ by the past modality $\langle \overline{L} \rangle$ (this latter modality makes it possible to access intervals in the past of the current one).

In Section 3, we show that, unlike what happens with LTL, where LTL and LTL+Past over $\mathbb{N}$ are expressively equivalent (even though LTL+Past is exponentially more succinct than LTL), the addition of past operators to A makes the resulting logic strictly more expressive than the original one. Moreover, we show that $\mathsf{A}\overline{\mathsf{A}}$ is in fact strictly more expressive than $\mathsf{A}\overline{\mathsf{L}}$. Then, in Section 4, we show that the satisfiability problem for $\mathsf{A}\overline{\mathsf{A}}$ over $\mathbb{Z}$ can actually be reduced to its satisfiability problem over $\mathbb{N}$. The proof turns out to be much more involved than the corresponding proof for LTL+Past. Finally, in Section 5, we show that, unlike A, $\mathsf{A}\overline{\mathsf{A}}$ is expressive enough to distinguish between the temporal domains $\mathbb{Q}$ and $\mathbb{R}$.

2 Preliminaries

In this section, we provide background knowledge about interval temporal logics (a comprehensive survey on interval temporal logics and duration calculi can be found in [12]). Moreover, we introduce basic concepts and notations that will be used in the following sections.

Let $\mathbb{D} = \langle D, < \rangle$ be a linearly-ordered domain. A strict interval over $\mathbb{D}$ (interval for short) is a pair $[i, j]$, with $i, j \in D$ and $i < j$. We denote by $\mathbb{I}(\mathbb{D})$ the set of all intervals over $\mathbb{D}$ (let us call it *interval structure*). Unlike the case of points, where there are three ordering relations only, namely, before, equal, and after, there exist thirteen different ordering relations between any pair of intervals in a linear order (Allen's interval relations): the six relations *meets*, *later*, *begins*, *ends*, *during*, and *overlaps* depicted in Table 1, their inverse relations, and the equality relation. We treat interval structures as Kripke structures and Allen's relations as accessibility relations in them, thus associating a modal operator $\langle X \rangle$ with each Allen's relation R_X. For each operator $\langle X \rangle$, its *transpose*, denoted $\langle \overline{X} \rangle$ (we assume $\langle \overline{\overline{X}} \rangle$ to be equal to $\langle X \rangle$), corresponds to the inverse relation $R_{\overline{X}}$ of R_X (that is, $R_{\overline{X}} = (R_X)^{-1}$).

The modal logic HS of Allen's interval relations has been investigated by Halpern and Shoham in [13]. The language of HS includes a set of proposition

$\langle A \rangle$	$[a,b]R_A[c,d] \Leftrightarrow b = c$
$\langle L \rangle$	$[a,b]R_L[c,d] \Leftrightarrow b < c$
$\langle B \rangle$	$[a,b]R_B[c,d] \Leftrightarrow a = c, d < b$
$\langle E \rangle$	$[a,b]R_E[c,d] \Leftrightarrow b = d, a < c$
$\langle D \rangle$	$[a,b]R_D[c,d] \Leftrightarrow a < c, d < b$
$\langle O \rangle$	$[a,b]R_O[c,d] \Leftrightarrow a < c < b < d$

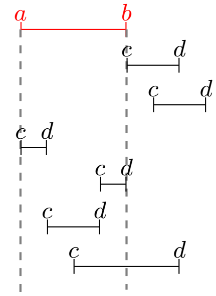

Table 1. Allen's interval relations and the corresponding modalities.

letters $\mathcal{AP}$, the classical propositional connectives $\neg$ and $\vee$, and a family of *interval modalities* of the form $\langle X \rangle$, one for each Allen's relation but equality. Formulas of HS are defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X \rangle \varphi$$

An *interval model* is a pair $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$, where $V : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{D})}$ is a *labeling function* assigning to each proposition letter the set of intervals over which it holds. The truth of a formula over an interval $[i,j]$ in an interval model M is defined by structural induction on formulas as follows:

- $M, [i,j] \Vdash p$ iff $[i,j] \in V(p)$, for all $p \in \mathcal{AP}$;
- $M, [i,j] \Vdash \neg\psi$ iff it is not the case that $M, [i,j] \Vdash \psi$;
- $M, [i,j] \Vdash \varphi \vee \psi$ iff $M, [i,j] \Vdash \varphi$ or $M, [i,j] \Vdash \psi$;
- $M, [i,j] \Vdash \langle X \rangle \psi$ iff there exists an interval $[h,k]$ such that $[i,j]\ R_X\ [h,k]$ and $M, [h,k] \Vdash \psi$, where R_X is the binary interval relation corresponding to the modal operator $\langle X \rangle$.

For the sake of brevity, given a model $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ over the set of proposition letters $\mathcal{AP}$, we call every interval $[i,j] \in V(p)$ a p-interval, for each $p \in \mathcal{AP}$.

The other Boolean connectives as well as the logical constants $\top$ and $\bot$ are defined as usual. Moreover, for each of the above-defined diamond modalities, the corresponding box modality is defined as the dual modality, e.g., $[A]\varphi \equiv \neg\langle A \rangle \neg\varphi$.

Satisfiability and *validity* of HS formulas are defined as usual. We say that an HS-formula φ is *satisfiable* if there exists an interval model M and an interval $[i,j]$ such that $M, [i,j] \Vdash \varphi$. Moreover, given a model $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ and an HS-formula φ, we say that M is a model for φ, denoted $M \models \varphi$, if there exists $[i,j] \in \mathbb{I}(\mathbb{D})$ such that $M, [i,j] \models \varphi$. Finally, we say that an HS-formula φ is *valid*, denoted $\models \varphi$, if it is true on every interval in every interval model. Two formulas φ and ψ are *equivalent*, denoted $\varphi \equiv \psi$, if $\models \varphi \leftrightarrow \psi$.

In the following, we will consider special classes of interval models obtained by substituting a specific temporal domain for $\mathbb{D}$. Formally, $\mathbb{N}$-models are pairs of the form $\langle \mathbb{I}(\mathbb{N}), V \rangle$, where $\mathbb{I}(\mathbb{N})$ is the set of all intervals over $\mathbb{N}$ and V is a mapping from $\mathcal{AP}$ to $2^{\mathbb{I}(\mathbb{N})}$. $\mathbb{Z}$-models (resp., $\mathbb{Q}$-models, $\mathbb{R}$-models) are obtained from $\mathbb{N}$-models by replacing $\mathbb{N}$ by $\mathbb{Z}$ (resp., $\mathbb{Q}$, $\mathbb{R}$).

With every subset $\mathcal{X} = \{\langle X_1\rangle, \ldots, \langle X_k\rangle\}$ of HS modalities we associate the *fragment* $\mathcal{F}_{\mathcal{X}}$ of HS, denoted $\mathsf{X_1 X_2 \ldots X_k}$, with formulas built on the same set of propositional letters $\mathcal{AP}$, but only featuring modalities from $\mathcal{X}$. As an example, $\mathsf{B\overline{B}}$ denotes the fragment involving the modalities $\langle B\rangle$ and $\langle \overline{B}\rangle$ only. In the following, we will restrict our attention to the fragments A, $\mathsf{A\overline{L}}$, and $\mathsf{A\overline{A}}$, the latter two being obtained from the first one by adding the past operators $\langle \overline{L}\rangle$ and $\langle \overline{A}\rangle$, respectively.

3 On the expressive power of A, $\mathsf{A\overline{L}}$, and $\mathsf{A\overline{A}}$ over $\mathbb{N}$

In this section, we first prove that $\mathsf{A\overline{A}}$ is strictly more expressive than A over $\mathbb{N}$. As a matter of fact, we show that $\mathsf{A\overline{L}}$ is strictly more expressive than A. Since $\langle \overline{L}\rangle$ can be easily defined in terms of $\langle \overline{A}\rangle$, the thesis immediately follows. Then, we show that $\mathsf{A\overline{A}}$ is in its turn strictly more expressive than $\mathsf{A\overline{L}}$ over $\mathbb{N}$. In addition, we show that both results actually hold for other meaningful temporal domains.

In order to show undefinability of a given modality in a certain interval logic, one can use a standard technique from modal logic based on the notion of *bisimulation* and the invariance of modal formulas with respect to bisimulations [2]. Let $\mathcal{F}$ be the considered interval logic. An $\mathcal{F}$-bisimulation between two interval models $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}'), V'\rangle$ over the set of proposition letters $\mathcal{AP}$ is a relation $Z \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ satisfying the following properties:

- *local condition*: pairs of Z-related intervals satisfy the same proposition letters over $\mathcal{AP}$;
- *forward condition*: if $([i,j],[i',j']) \in Z$ and $([i,j],[h,k]) \in R_X$ for some $\langle X\rangle \in \mathcal{F}$, then there exists $[h',k']$ such that $([i',j'],[h',k']) \in R_X$ and $([h,k],[h',k']) \in Z$;
- *backward condition*: if $([i,j],[i',j']) \in Z$ and $([i',j'],[h',k']) \in R_X$ for some $\langle X\rangle \in \mathcal{F}$, then there exists $[h,k]$ such that $([i,j], [h,k]) \in R_X$ and $([h,k], [h',k']) \in Z$.

Since any $\mathcal{F}$-bisimulation preserves the truth of *all* formulas in $\mathcal{F}$, in order to prove that an operator $\langle X\rangle$ is not definable in $\mathcal{F}$, it suffices to construct a pair of interval models M and M' and an $\mathcal{F}$-bisimulation between them such that $M,[i,j] \Vdash \langle X\rangle p$ and $M',[i',j'] \nVdash \langle X\rangle p$, for a pair of $\mathcal{F}$-bisimilar intervals $[i,j] \in M$ and $[i',j'] \in M'$.

Theorem 1. *The modality $\langle \overline{L}\rangle$ is not definable in A over $\mathbb{N}$.*

Proof. Let us consider the pair of interval models $M = \langle \mathbb{I}(\mathbb{N}), V\rangle$ and $M' = \langle \mathbb{I}(\mathbb{N}), V'\rangle$, over the set of proposition letters $\mathcal{AP} = \{p\}$, where $V(p) = V'(p) = \{[i, i+1] : i \geq 0\}$. Moreover, let $Z \subseteq \mathbb{I}(\mathbb{N}) \times \mathbb{I}(\mathbb{N})$ be the set $\{([i,j],[i,j]) : 0 \leq$

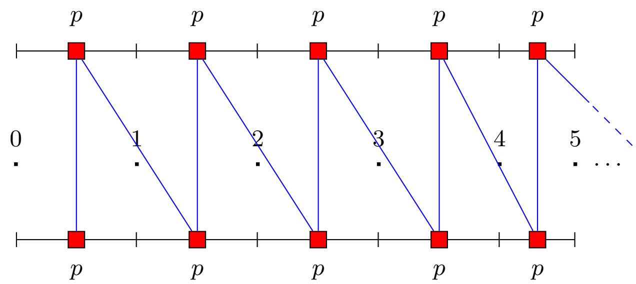

Fig. 1. The relation Z for Theorem 1.

$i < j\} \cup \{([i,j],[i+1,j+1]) : 0 \leq i < j\}$ (the relation Z is depicted in Figure 1).

We show that Z is an A-bisimulation. Checking that it satisfies the local condition is immediate. As for the forward condition, we must distinguish two cases. First, we must consider any pair of the form $([i,j],[i,j])$. In such a case, the $\langle A\rangle$-move from $[i,j]$ to $[j,k]$ in M can be simulated by the very same $\langle A\rangle$-move from $[i,j]$ to $[j,k]$ in M'. Notice that p-intervals come into play when $j = i+1$ or $k = j+1$ (or both). The second case is that of pairs of the form $([i,j),[i+1,j+1])$. In such a case, the $\langle A\rangle$-move from $[i,j]$ to $[j,k]$ in M can be simulated by the $\langle A\rangle$-move from $[i+1,j+1]$ to $[j+1,k+1]$ in M'. As in the previous case, p-intervals come into play when $j = i+1$ or $k = j+1$ (or both). Satisfaction of the backward condition can be checked in a very similar way.

To conclude the proof, it suffices to show that Z does not preserve the relation induced by the modality $\langle\overline{L}\rangle$. To this end, consider the pair $([1,2],[2,3]) \in Z$. We have that $M',[2,3] \models \langle\overline{L}\rangle p$, while $M,[1,2] \not\models \langle\overline{L}\rangle p$. $\square$

Corollary 1. *The modality $\langle\overline{A}\rangle$ is not definable in* A *over* $\mathbb{N}$.

Proof. As the modal operator $\langle\overline{L}\rangle$ is definable in any interval logic featuring the modal operator $\langle\overline{A}\rangle$ (for any fixed proposition letter p, it holds that $\langle\overline{L}\rangle p \equiv \langle\overline{A}\rangle\langle\overline{A}\rangle p$), the thesis immediately follows from Theorem 1. $\square$

It is worth pointing out that the bisimulation exploited in the proof of Theorem 1 still works if we replace $\mathbb{N}$ by $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$, thus showing that the modality $\langle\overline{A}\rangle$ is not definable in A over these temporal domains. Moreover, it can be easily adapted to work with the class of finite temporal domains, that is, finite prefixes of $\mathbb{N}$.

The next theorem shows that $\mathsf{A\overline{A}}$ is in fact strictly more expressive than $\mathsf{A\overline{L}}$ over $\mathbb{N}$. Since $\langle\overline{L}\rangle$ can be defined in terms of $\langle\overline{A}\rangle$, we have that $\mathsf{A\overline{A}}$ is at least as expressive as $\mathsf{A\overline{L}}$. In order to show that $\mathsf{A\overline{A}}$ is strictly more expressive than $\mathsf{A\overline{L}}$, we must show that $\langle\overline{A}\rangle$ is not definable in $\mathsf{A\overline{L}}$ over $\mathbb{N}$. To this end, we define an $\mathsf{A\overline{L}}$-bisimulation between a suitable pair of models.

Theorem 2. *The modality $\langle\overline{A}\rangle$ is not definable in* $\mathsf{A\overline{L}}$ *over* $\mathbb{N}$.

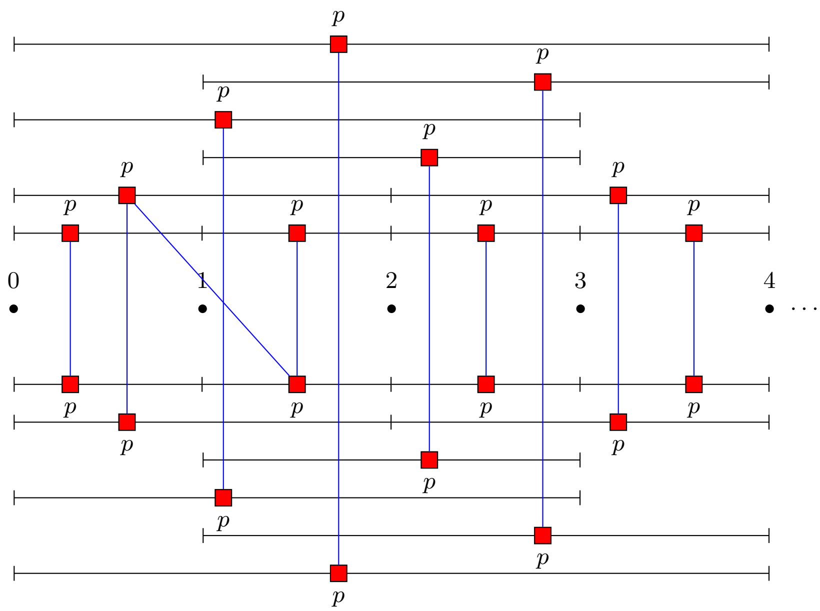

Fig. 2. The relation Z for Theorem 2.

Proof. Let us consider the pair of interval models $M = \langle \mathbb{I}(\mathbb{N}), V \rangle$ and $M' = \langle \mathbb{I}(\mathbb{N}), V' \rangle$, over the set of proposition letters $\mathcal{AP} = \{p\}$, where $V(p) = V'(p) = \mathbb{I}(\mathbb{N})$. Moreover, let $Z \subseteq \mathbb{I}(\mathbb{N}) \times \mathbb{I}(\mathbb{N})$ be the set $\{([i,j],[i,j]) : 0 \leq i < j\} \cup \{([0,2],[1,2])\}$ (the relation Z is depicted in Figure 2).

We show that Z is an $\mathsf{A}\overline{\mathsf{L}}$-bisimulation. Checking that it satisfies the local condition is trivial. As for the forward condition, let us first consider the pair of intervals $([0,2],[1,2]) \in Z$. Since there is not an interval $[i,j]$ such that $([0,2],[i,j]) \in R_{\overline{L}}$ in M, we only need to check the forward condition for the relation R_A. For this, it suffices to observe that the $\langle A \rangle$-move from $[0,2]$ to $[2,j]$ in M can be simulated by the very same $\langle A \rangle$-move from $[1,2]$ to $[2,j]$ in M', as $([2,j],[2,j]) \in Z$. Let us consider now pairs of intervals of the form $([i,j],[i,j]) \in Z$. It can be easily checked that, for every interval $[k,l]$ such that $([i,j],[k,l]) \in R_{\overline{L}}$ (resp., $([i,j],[k,l]) \in R_A$) in M, we have that $([i,j],[k,l]) \in R_{\overline{L}}$ (resp., $([i,j],[k,l]) \in R_A$) in M' and $([k,l],[k,l]) \in Z$. Satisfaction of the backward condition can be checked in a very similar way (in particular, notice that there is not an interval $[i,j]$ such that $([1,2],[i,j]) \in R_{\overline{L}}$ in M').

To conclude the proof, it suffices to show that Z does not preserve the relation induced by the modality $\langle \overline{A} \rangle$. To this end, consider the pair $([0,2],[1,2]) \in Z$. We have that $M',[1,2] \Vdash \langle \overline{A} \rangle \top$, while $M,[0,2] \nVdash \langle \overline{A} \rangle \top$. $\square$

From Theorem 2, it immediately follows that the two modalities $\langle \overline{A} \rangle$ and $\langle \overline{L} \rangle$ are not inter-definable: $\langle \overline{L} \rangle$ can be defined in terms of $\langle \overline{A} \rangle$, but not vice versa.

As it happened with Theorem 1, it is possible to generalize Theorem 2 to $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, as well as to the class of finite temporal domains. As a matter of fact, the bisimulation given in the proof of Theorem 2 still works if we replace $\mathbb{N}$ by any prefix of it. The undefinability of $\langle \overline{A} \rangle$ in $\mathsf{A}\overline{\mathsf{L}}$ over the class of finite temporal domains immediately follows. In order to lift Theorem 2 to the cases of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, a different bisimulation is needed. Let M and M' be two interval models, over the set of proposition letters $\mathcal{AP} = \{p\}$, defined as follows:

- $M = \langle \mathbb{I}(\mathbb{N}), V \rangle$, where $V(p) = \{[a, b] \in \mathbb{I}(\mathbb{N}) \mid b - a$ is an odd number$\}$
- $M' = \langle \mathbb{I}(\mathbb{N}), V' \rangle$, where $V'(p) = \{[a, b] \in \mathbb{I}(\mathbb{N}) \mid b - a$ is an odd number and $b \neq 0\}$.

Let the relation $Z \subseteq \mathbb{I}(\mathbb{N}) \times \mathbb{I}(\mathbb{N})$ be the union of the set $V(p) \times V'(p)$ and the set $(\mathbb{I}(\mathbb{N}) \setminus V(p)) \times (\mathbb{I}(\mathbb{N}) \setminus V'(p))$. It is not difficult to prove that Z is an $\mathsf{A}\overline{\mathsf{L}}$-bisimulation. By definition, Z relates pairs of intervals that either both satisfy p or both do not satisfy it. Thus the local condition trivially holds. To prove that the forward condition holds as well, consider a generic pair of Z-related intervals $([i, j], [i', j'])$. If we move from $[i, j]$ to $[k, l]$ in M by either an A- or an $\overline{\mathsf{L}}$-move, and $[k, l]$ satisfies p (resp., $\neg p$), it is always possible to make a corresponding move in M' that allows us to reach an interval $[k', l']$ satisfying p (resp., $\neg p$). The same argument can be used to check that Z satisfies the backward condition. To show that Z does not preserve the relation induced by the modality $\langle \overline{A} \rangle$ consider the pair of Z-related intervals $([0, 1], [0, 1])$. We have that $M, [0, 1] \Vdash \langle \overline{A} \rangle p$, while $M', [0, 1] \nVdash \langle \overline{A} \rangle p$, and thus the thesis.

4 On the satisfiability problem for $\mathsf{A}\overline{\mathsf{A}}$ over $\mathbb{N}$ and $\mathbb{Z}$

When past operators are added to a temporal logic, replacing $\mathbb{N}$ by $\mathbb{Z}$ is a natural step. In this section, we focus our attention on the satisfiability problem for $\mathsf{A}\overline{\mathsf{A}}$ over $\mathbb{N}$ and $\mathbb{Z}$. As a matter of fact, both cases have been already dealt with elsewhere. In [4], Bresolin et al. have shown the expressive completeness of $\mathsf{A}\overline{\mathsf{A}}$ with respect to the two-variable fragment of first-order logic for binary relational structures over various linearly-ordered domains $FO^2[<]$. Decidability (in NEXPTIME) of $\mathsf{A}\overline{\mathsf{A}}$ over the classes of all linear orders, well-orders, and finite linear orders, as well as over $\mathbb{N}$, immediately follows from the decidability results for $FO^2[<]$ over these classes of linear orders obtained by Otto in [22]. A proof of NEXPTIME-hardness, and thus the NEXPTIME-completeness of the satisfiability problem for $\mathsf{A}\overline{\mathsf{A}}$ over the considered (classes of) linear orders, can be found in [8]. NEXPTIME-completeness of the satisfiability problem for $\mathsf{A}\overline{\mathsf{A}}$ over $\mathbb{Z}$ has been proved by a model-theoretic argument in [5], where an optimal tableau system for $\mathsf{A}\overline{\mathsf{A}}$ over $\mathbb{Z}$ can also be found.

In the following, we prove that the satisfiability problem for $\mathsf{A}\overline{\mathsf{A}}$ over $\mathbb{Z}$ can actually be reduced to that over $\mathbb{N}$. The proof somehow resembles the one for (point-based) LTL+Past. However, interpreting formulas over time intervals, instead of over time points, introduces a number of additional complications. In particular, we must deal with intervals whose left endpoint is less than 0 and whose right endpoint is greater than it, and, among them, with intervals of the form $[-h, h]$.

Let φ be the $\mathsf{A}\overline{\mathsf{A}}$-formula to be checked for satisfiability over $\mathbb{Z}$. We show how to build an $\mathsf{A}\overline{\mathsf{A}}$-formula $\varphi_{\mathbb{Z}\to\mathbb{N}}$, to be interpreted over $\mathbb{N}$, such that φ is satisfiable over $\mathbb{Z}$ if and only if $\varphi_{\mathbb{Z}\to\mathbb{N}}$ is satisfiable over $\mathbb{N}$. As a preparation, we first define a suitable mapping μ of $\mathbb{Z}$-models into $\mathbb{N}$-models; then, we define a suitable rewriting τ of $\mathsf{A}\overline{\mathsf{A}}$-formulas over an extended vocabulary of proposition letters. Hereafter, we will denote a formula φ with proposition letters $p_1, \ldots, p_n$ by $\varphi(p_1, \ldots, p_n)$.

Definition 1 (the model mapping μ). *Let $\varphi(p_1, \ldots, p_n)$ be an $\mathsf{A}\overline{\mathsf{A}}$-formula and, for each p_i in $\{p_1, \ldots, p_n\}$, let $p_i^{++}, p_i^{-+}, p_i^{+-}, p_i^{--}$, and $p_i^{L_-^+}$ be five new (distinct) proposition letters. We map each $\mathbb{Z}$-model $M = \langle \mathbb{I}(\mathbb{Z}), V\rangle$, with $V : \{p_1, \ldots, p_n\} \mapsto 2^{\mathbb{I}(\mathbb{Z})}$, into an $\mathbb{N}$-model $\mu(M) = \langle \mathbb{I}(\mathbb{N}), V'\rangle$, with $V' : \{p_1^{++}, p_1^{-+}, p_1^{+-}, p_1^{--}, p_1^{L_-^+}, \ldots, p_n^{++}, p_n^{-+}, p_n^{+-}, p_n^{--}, p_n^{L_-^+}\} \mapsto 2^{\mathbb{I}(\mathbb{N})}$, that satisfies the following conditions: for every $p \in \{p_1, \ldots, p_n\}$ and every $h, k \in \mathbb{N}$, with $h < k$,*

- *$[h, k] \in V'(p^{++})$ if and only if $[h, k] \in V(p)$;*
- *$[h, k] \in V'(p^{-+})$ if and only if $[-h, k] \in V(p)$;*
- *$[h, k] \in V'(p^{+-})$ if and only if $[-k, h] \in V(p)$;*
- *$[h, k] \in V'(p^{--})$ if and only if $[-k, -h] \in V(p)$;*
- *$[h, k] \in V'(p^{L_-^+})$ if and only if $[-h, h] \in V(p)$.*

It can be easily checked that μ is a function, that is, for any $\mathbb{Z}$-model M over $p_1, \ldots, p_n$, the above conditions univocally identify a corresponding $\mathbb{N}$-model $\mu(M)$ over $p_1^{++}, p_1^{-+}, p_1^{+-}, p_1^{--}, p_1^{L_-^+}, \ldots, p_n^{++}, p_n^{-+}, p_n^{+-}, p_n^{--}, p_n^{L_-^+}$.

The mapping μ can be interpreted as follows. For any $p \in \{p_1, \ldots, p_n\}$ and any pair of natural numbers h, k, with $h < k$, four different p-intervals in M, namely, $[h, k], [-h, k]$, $[-k, h]$, and $[-k, -h]$, are accommodated by the same interval $[h, k]$ in $\mathbb{I}(\mathbb{N})$. To distinguish among them, we replace the proposition letter p by the four proposition letters p^{++}, p^{-+}, p^{+-}, and p^{--}. The case of $\mathbb{Z}$-intervals of the form $[0, k]$ (resp., $[-k, 0]$) deserves a closer look. The left endpoint of $[0, k]$, the number 0, can be equivalently viewed as a positive or a negative number, that is, for every natural number k, $[+0, k]$ and $[-0, k]$ denote the very same interval. Hence, for every $p \in \{p_1, \ldots, p_n\}$, the function μ constrains the truth value of proposition letters p^{++} and p^{-+} to be the same over all intervals of the form $[0, k]$. Similarly, for every natural number k, $[-k, -0]$ and $[-k, +0]$ denote the same interval, and thus the function μ constrains the truth value of proposition letters p^{--} and p^{+-} to be the same over all intervals of the form $[0, k]$.

These constraints can be formally stated in $\mathsf{A}\overline{\mathsf{A}}$ by means of the following formula:

$$[G]([\overline{A}]\bot \to (p^{++} \leftrightarrow p^{-+}) \wedge (p^{--} \leftrightarrow p^{+-}))$$

where $[G]$ is the *universal modality* that constrains its argument to hold everywhere, that is, for every model $M = \langle \mathbb{I}(\mathbb{N}), V\rangle$, every interval $[i, j] \in \mathbb{I}(\mathbb{N})$, and every $\mathsf{A\overline{A}}$-formula ψ, $M, [i,j] \models [G]\psi$ if and only if $M, [i',j'] \models \psi$ for every $[i',j'] \in \mathbb{I}(\mathbb{D})$. The universal modality $[G]$ can be defined in $\mathsf{A\overline{A}}$ over $\mathbb{N}$ as follows:

$$[G]\psi = [A][A][\overline{A}][\overline{A}]\psi$$

Let us denote by ψ_0 the following $\mathsf{A\overline{A}}$-formula that imposes the above constraints on all proposition letters $p \in \{p_1, \ldots, p_n\}$:

$$[G]\left(\bigwedge_{p\in\{p_1,\ldots,p_n\}} ([\overline{A}]\bot \to (p^{++} \leftrightarrow p^{-+}) \wedge (p^{--} \leftrightarrow p^{+-}))\right)$$

Proposition letters p^{++}, p^{-+}, p^{+-}, and p^{--} make it possible to deal with any kind of p-interval in $\mathbb{I}(\mathbb{Z})$, but those p-intervals of the form $[-h,h]$, as point-intervals of the form $[h,h]$ are not allowed. To cope with them, we make use of a distinct proposition letter $p^{L^+_-}$, that the function μ constrains to assume the same truth value over every pair of intervals with the same left endpoint (locality).

The locality of $p^{L^+_-}$ can be formally stated as follows: for every $h,k,l \in \mathbb{N}$, with both $h < k$ and $h < l$, and every $p \in \{p_1, \ldots, p_n\}$, $[h,k] \in V'(p^{L^+_-})$ if and only if $[h,l] \in V'(p^{L^+_-})$. Such a condition can be forced by means of the following $\mathsf{A\overline{A}}$-formula:

$$[G](p^{L^+_-} \leftrightarrow [\overline{A}][A]p^{L^+_-})$$

Let us denote by $\psi_{L^+_-}$ the following $\mathsf{A\overline{A}}$-formula that forces the locality of all proposition letters of the form $p^{L^+_-}$:

$$[G]\left(\bigwedge_{p\in\{p_1,\ldots,p_n\}} (p^{L^+_-} \leftrightarrow [\overline{A}][A]p^{L^+_-})\right)$$

The second step in preparation for the proof of the main result of this section is a suitable mapping of $\mathsf{A\overline{A}}$-formulas $\psi_{\mathbb{Z}}(p_1, \ldots, p_n)$, interpreted over $\mathbb{Z}$, into "equivalent" $\mathsf{A\overline{A}}$-formulas $\psi_{\mathbb{N}}(p_1^{++}, p_1^{-+}, p_1^{+-}, p_1^{--}, p_1^{L^+_-}, \ldots, p_n^{++}, p_n^{-+}, p_n^{+-}, p_n^{--}, p_n^{L^+_-})$, interpreted over $\mathbb{N}$.

Definition 2 (the formula mapping τ). *Let $\psi(p_1, \ldots, p_n)$ be an $\mathsf{A\overline{A}}$-formula (interpreted over $\mathbb{Z}$) and let $q \in \{++, -+, +-, --, L^+_-\}$ be a qualifier. We map every pair (ψ, q) into an $\mathsf{A\overline{A}}$-formula $\tau(\psi,q)(p_1^{++}, p_1^{-+}, p_1^{+-}, p_1^{--}, p_1^{L^+_-}, \ldots, p_n^{++}, p_n^{-+}, p_n^{+-}, p_n^{--}, p_n^{L^+_-})$ (interpreted over $\mathbb{N}$) by applying the set of translation rules given in Figure 3.*

By properly combining the model mapping μ and the formula mapping τ, we can prove the following technical lemma.

$\forall * \in \{++,-+,+-,--,L^{+}_{-}\}$ $\tau(p,*)=p^{*}$ $\tau(\neg\psi,*)=\neg\tau(\psi,*)$ $\tau(\psi_1 \vee \psi_2,*)=\tau(\psi_1,*)\vee\tau(\psi_2,*)$	$\tau(\langle A\rangle\psi,L^{+}_{-}) = \langle\overline{A}\rangle\langle A\rangle\tau(\psi,++)$ $\tau(\langle\overline{A}\rangle\psi,L^{+}_{-}) = \langle\overline{A}\rangle\langle A\rangle\tau(\psi,--)$
$\tau(\langle A\rangle\psi,-+) = \langle A\rangle\tau(\psi,++)$ $\tau(\langle\overline{A}\rangle\psi,-+) = \langle\overline{A}\rangle\langle A\rangle\tau(\psi,--)\vee$ $([\overline{A}]\bot\wedge(\tau(\psi,--)\vee$ $\langle A\rangle\langle\overline{A}\rangle\langle\overline{A}\rangle([\overline{A}]\bot\wedge\tau(\psi,--))\vee$ $\langle A\rangle\langle A\rangle\langle\overline{A}\rangle([\overline{A}]\bot\wedge\tau(\psi,--))))$	$\tau(\langle\overline{A}\rangle\psi,+-) = \langle A\rangle\tau(\psi,--)$ $\tau(\langle A\rangle\psi,+-) = \langle\overline{A}\rangle\langle A\rangle\tau(\psi,++)\vee$ $([\overline{A}]\bot\wedge(\tau(\psi,++)\vee$ $\langle A\rangle\langle\overline{A}\rangle\langle\overline{A}\rangle([\overline{A}]\bot\wedge\tau(\psi,++))\vee$ $\langle A\rangle\langle A\rangle\langle\overline{A}\rangle([\overline{A}]\bot\wedge\tau(\psi,++))))$
$\tau(\langle A\rangle\psi,++) = \langle A\rangle\tau(\psi,++)$ $\tau(\langle\overline{A}\rangle\psi,++) = \langle\overline{A}\rangle\tau(\psi,++)\vee\langle\overline{A}\rangle\tau(\psi,-+)\vee$ $\langle\overline{A}\rangle\langle A\rangle\tau(\psi,+-)\vee$ $(\langle\overline{A}\rangle\top\wedge\tau(\psi,L^{+}_{-}))\vee$ $([\overline{A}]\bot\wedge(\tau(\psi,+-)\vee$ $\langle A\rangle\langle\overline{A}\rangle\langle\overline{A}\rangle([\overline{A}]\bot\wedge\tau(\psi,+-))\vee$ $\langle A\rangle\langle A\rangle\langle\overline{A}\rangle([\overline{A}]\bot\wedge\tau(\psi,+-))))$	$\tau(\langle\overline{A}\rangle\psi,--) = \langle A\rangle\tau(\psi,--)$ $\tau(\langle A\rangle\psi,--) = \langle\overline{A}\rangle\tau(\psi,--)\vee\langle\overline{A}\rangle\tau(\psi,+-)\vee$ $\langle\overline{A}\rangle\langle A\rangle\tau(\psi,-+)\vee$ $(\langle\overline{A}\rangle\top\wedge\tau(\psi,L^{+}_{-}))\vee$ $([\overline{A}]\bot\wedge(\tau(\psi,-+)\vee$ $\langle A\rangle\langle\overline{A}\rangle\langle\overline{A}\rangle([\overline{A}]\bot\wedge\tau(\psi,-+))\vee$ $\langle A\rangle\langle A\rangle\langle\overline{A}\rangle([\overline{A}]\bot\wedge\tau(\psi,-+))))$

Fig. 3. The inductive definition of the translation function $\tau(\cdot,\cdot)$.

Lemma 1. *Let M be a $\mathbb{Z}$-model, $\psi(p_1,\ldots,p_n)$ be an $\mathsf{A}\overline{\mathsf{A}}$-formula, μ be the model mapping of Definition 1, and τ be the formula mapping of Definition 2. For every $h,k\in\mathbb{N}$, with $h<k$, it holds that:*

- $\mu(M),[h,k]\models\tau(\psi,++)$ *if and only if* $M,[h,k]\models\psi$;
- $\mu(M),[h,k]\models\tau(\psi,-+)$ *if and only if* $M,[-h,k]\models\psi$;
- $\mu(M),[h,k]\models\tau(\psi,+-)$ *if and only if* $M,[-k,h]\models\psi$;
- $\mu(M),[h,k]\models\tau(\psi,--)$ *if and only if* $M,[-k,-h]\models\psi$;
- $\mu(M),[h,k]\models\tau(\psi,L^{+}_{-})$ *if and only if* $M,[-h,h]\models\psi$.

Proof. The proof is by induction on the (structural) complexity of the formula.

The base case of proposition letters is trivial: it directly follows from the definition of the evaluation V' of $\mu(M)$. The cases of the Boolean connectives $\neg$ and $\vee$ immediately follow from the inductive hypothesis.

Let us consider now the case of the formula $\langle\overline{A}\rangle\psi$ (the case of the formula $\langle A\rangle\psi$ can be dealt with in a similar way).

We focus our attention on $\tau(\langle\overline{A}\rangle\psi,++)$. Let us assume that $M,[h,k]\models\langle\overline{A}\rangle\psi$ for some $h,k\in\mathbb{N}$. We must distinguish two cases: either $h>0$ or $h=0$.
Suppose that $h>0$. By definition, $M,[h,k]\models\langle\overline{A}\rangle\psi$ if and only if either (i) $M,[h',h]\models\psi$, for some $h'\geq 0$, or (ii) $M,[-h',h]\models\psi$, for some $-h<-h'<0$, or (iii) $M,[-h',h]\models\psi$, for some $-h'<-h$, or (iv) $M,[-h,h]\models\psi$. Let us consider each of them separately (a graphical account of them is given in Figure 4).

(i) By inductive hypothesis, $M,[h',h]\models\psi$ if and only if $\mu(M),[h',h]\models\tau(\psi,++)$. By definition of modality $\langle\overline{A}\rangle$, it immediately follows that $\mu(M),[h,k]\models\langle\overline{A}\rangle\tau(\psi,++)$. Then, by definition of $\tau(\langle\overline{A}\rangle\psi,++)$ (first disjunct), we get $\mu(M),[h,k]\models\tau(\langle\overline{A}\rangle\psi,++)$.

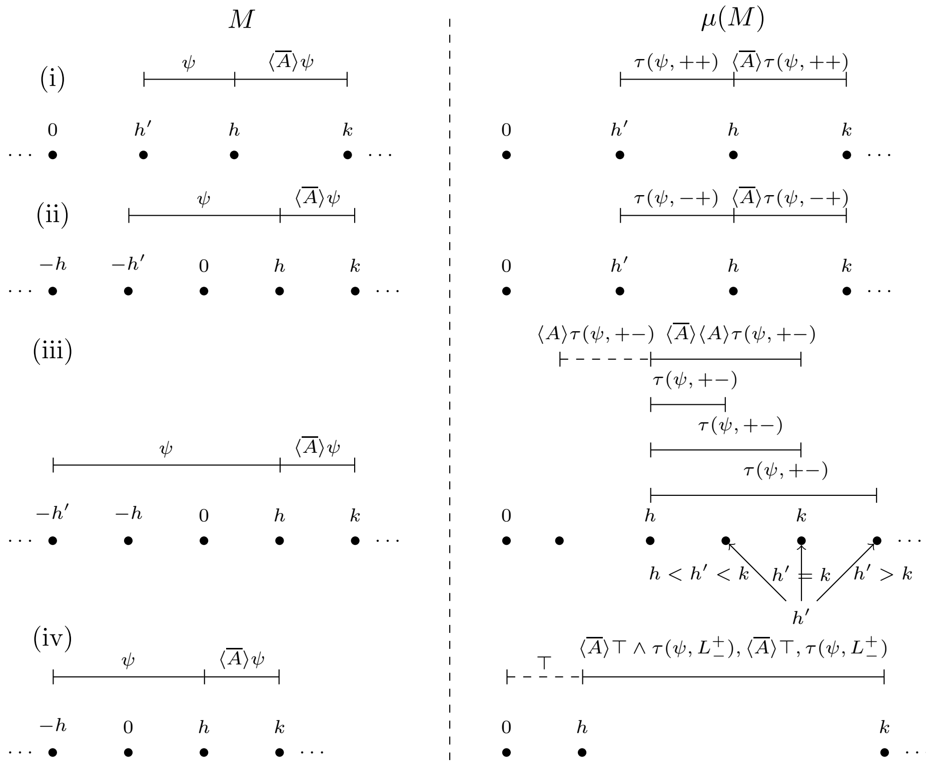

Fig. 4. A graphical account of the proof for $\tau(\langle \overline{A} \rangle \psi, ++)$ (when $h > 0$).

(ii) By inductive hypothesis, $M, [-h', h] \models \psi$, with $h' < h$, if and only if $\mu(M), [h', h] \models \tau(\psi, -+)$. By definition of modality $\langle \overline{A} \rangle$, it immediately follows that $\mu(M), [h, k] \models \langle \overline{A} \rangle \tau(\psi, -+)$, and then, by definition of $\tau(\langle \overline{A} \rangle \psi, ++)$ (second disjunct), we get $\mu(M), [h, k] \models \tau(\langle \overline{A} \rangle \psi, ++)$.

(iii) By inductive hypothesis, $M, [-h', h] \models \psi$, with $h' > h$, if and only if $\mu(M), [h, h'] \models \tau(\psi, +-)$. By hypothesis, $h > 0$, and thus, by definition of modalities $\langle A \rangle$ and $\langle \overline{A} \rangle$, $\mu(M), [h, h'] \models \tau(\psi, +-)$ implies that $\mu(M), [h, k] \models \langle \overline{A} \rangle \langle A \rangle \tau(\psi, +-)$. By definition of $\tau(\langle \overline{A} \rangle \psi, ++)$ (third disjunct), we get $\mu(M)$, $[h, k] \models \tau(\langle \overline{A} \rangle \psi, ++)$.

(iv) By inductive hypothesis, $M, [-h, h] \models \psi$ if and only if $\mu(M), [h, h'] \models \tau(\psi, L_{-}^{+})$ for all $h' > h$. In particular, $\mu(M), [h, k] \models \tau(\psi, L_{-}^{+})$. Moreover, by hypothesis, $h > 0$, and thus $\mu(M), [h, k] \models \langle \overline{A} \rangle \top$. From $\mu(M)$, $[h, k] \models \langle \overline{A} \rangle \top \wedge \tau(\psi, L_{-}^{+})$, by definition of $\tau(\langle \overline{A} \rangle \psi, ++)$ (fourth disjunct), we get $\mu(M), [h, k] \models \tau(\langle \overline{A} \rangle \psi, ++)$.

Let us consider now the case with $h = 0$. We must distinguish among three possible ways of satisfying $M, [h, k] \models \langle \overline{A} \rangle \psi$: either (i) $M, [-k, h] \models \psi$, or (ii) $M, [-h', h] \models \psi$, for some $-k < -h'$, or (iii) $M, [-h', h] \models \psi$, for some $-h' < -k$.

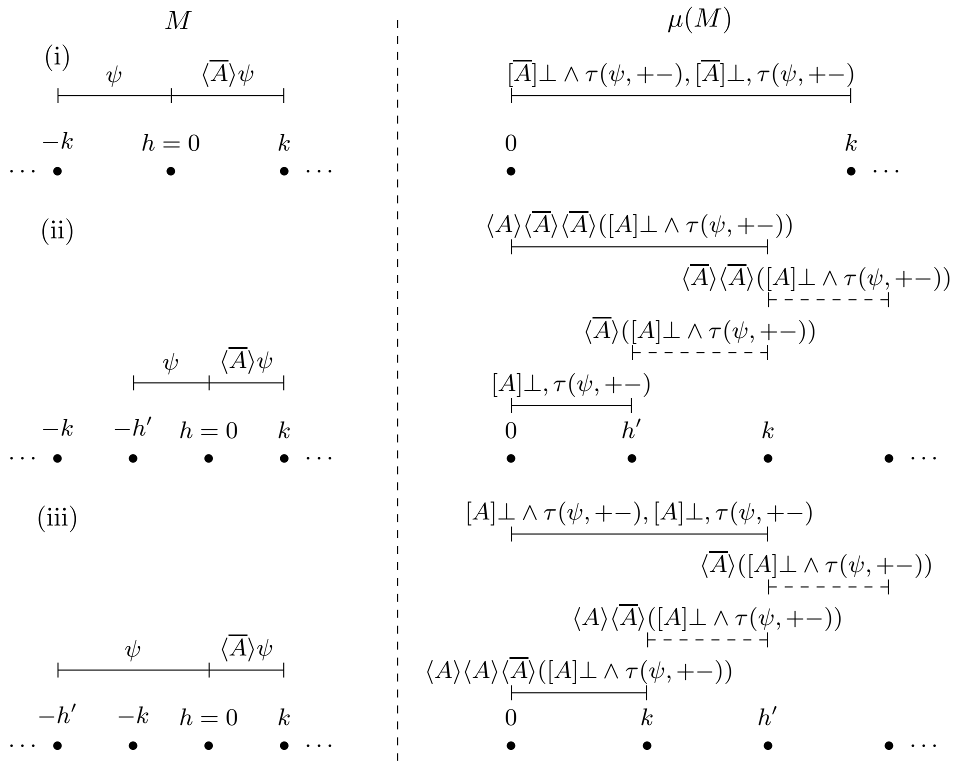

Fig. 5. A graphical account of the proof for $\tau(\langle\overline{A}\rangle\psi,++)$ (when $h=0$).

Let us consider each of them separately (a graphical account of them is given in Figure 5).

(i) By inductive hypothesis, $M,[-k,h] \models \psi$ if and only if $\mu(M),[h,k] \models \tau(\psi,+-)$. Since, by hypothesis, $h=0$, it also holds that $\mu(M),[h,k] \models [\overline{A}]\bot$. By definition of $\tau(\langle\overline{A}\rangle\psi,++)$ (fifth disjunct), we get $\mu(M),[h,k] \models \tau(\langle\overline{A}\rangle\psi,++)$.

(ii) By inductive hypothesis, $M,[-h',h] \models \psi$ if and only if $\mu(M),[h,h'] \models \tau(\psi,+-)$. By definition of modalities $\langle A\rangle$ and $\langle\overline{A}\rangle$, from $\mu(M),[h,h'] \models \tau(\psi,+-)$ and $-k < -h'$, it follows that $\mu(M),[h,k] \models \langle A\rangle\langle\overline{A}\rangle\langle\overline{A}\rangle([\overline{A}]\bot \wedge \tau(\psi,+-))$. As in the previous case, it also holds that $\mu(M),[h,k] \models [\overline{A}]\bot$. Hence, by definition of $\tau(\langle\overline{A}\rangle\psi,++)$ (fifth disjunct), we get $\mu(M),[h,k] \models \tau(\langle\overline{A}\rangle\psi,++)$.

(iii) By inductive hypothesis, $M,[-h',h] \models \psi$ if and only if $\mu(M),[h,h'] \models \tau(\psi,+-)$. By definition of modalities $\langle A\rangle$ and $\langle\overline{A}\rangle$, from $\mu(M),[h,h'] \models \tau(\psi,+-)$ and $-h' < -k$, it follows that $\mu(M),[h,k] \models \langle A\rangle\langle A\rangle\langle\overline{A}\rangle([\overline{A}]\bot \wedge \tau(\psi,+-))$. As in the previous cases, it also holds that $\mu(M),[h,k] \models\models [\overline{A}]\bot$. Hence, by definition of $\tau(\langle\overline{A}\rangle\psi,++)$ (fifth disjunct), we get $\mu(M),[h,k] \models \tau(\langle\overline{A}\rangle\psi,++)$.

The proof of the opposite implication is quite similar, and thus omitted.

The proof for the case $\tau(\langle\overline{A}\rangle\psi, -+)$ is essentially the same, and the proofs for the cases $\tau(\langle\overline{A}\rangle\psi, +-)$ and $\tau(\langle\overline{A}\rangle\psi, --)$ are straightforward. Hence, we leave them to the reader.

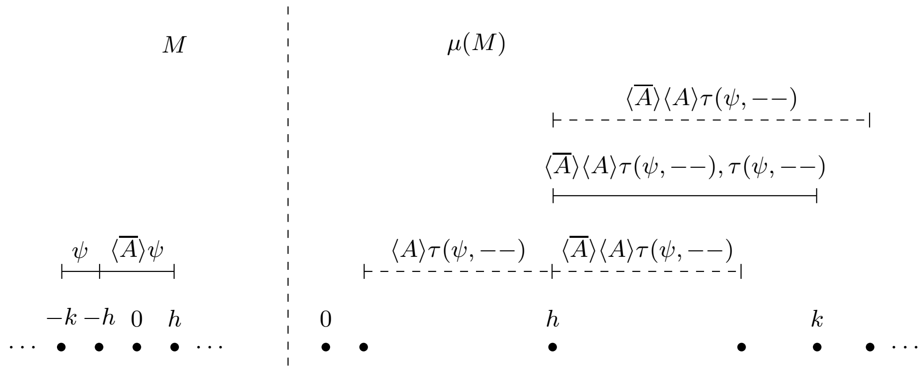

Fig. 6. A graphical account of the proof for $\tau(\langle\overline{A}\rangle\psi, L_-^+)$.

Let us consider now the case of $\tau(\langle\overline{A}\rangle\psi, L_-^+)$ (a graphical account of the argument is given in Figure 6). Assume that $M, [-h, h] \models \langle\overline{A}\rangle\psi$ for some $h(> 0)$. By definition, $M, [-h, h] \models \langle\overline{A}\rangle\psi$ if and only if $M, [-k, -h] \models \psi$, for some k. By inductive hypothesis, $M, [-k, -h] \models \psi$ if and only if $\mu(M), [h, k] \models \tau(\psi, --)$. By definition of modalities $\langle A\rangle, \langle\overline{A}\rangle$, $\mu(M), [h, k] \models \tau(\psi, --)$ implies $\mu(M), [h, h'] \models \langle\overline{A}\rangle\langle A\rangle\tau(\psi, --)$ for all $h' > h$. Hence, by definition of $\tau(\langle\overline{A}\rangle\psi, L_-^+)$, we get $\mu(M), [h, h'] \models \tau(\langle\overline{A}\rangle\psi, L_-^+)$ for all $h' > h$. Once more, the proof of the opposite implication is quite similar, and thus omitted. $\square$

We are now ready to prove the main result of the section, that easily follows from Lemma 1.

Theorem 3. *There is a log-space reduction from the satisfiability problem for* $\mathsf{A\overline{A}}$ *over* $\mathbb{Z}$ *to the satisfiability problem for* $\mathsf{A\overline{A}}$ *over* $\mathbb{N}$.

Proof. Let $\varphi(p_1, \ldots, p_n)$ be the $\mathsf{A\overline{A}}$-formula to be checked for satisfiability, and let $\varphi_{\mathbb{Z}\to\mathbb{N}}(p_1^{++}, p_1^{-+}, p_1^{+-}, p_1^{--}, p_1^{L_-^+}, \ldots, p_n^{++}, p_n^{-+}, p_n^{+-}, p_n^{--}, p_n^{L_-^+})$ be the following $\mathsf{A\overline{A}}$-formula:

$$\psi_0 \wedge \psi_{L_-^+} \wedge \tau(\varphi, ++)$$

We claim that $\varphi(p_1, \ldots, p_n)$ is satisfiable over $\mathbb{Z}$ if and only if $\varphi_{\mathbb{Z}\to\mathbb{N}}(p_1^{++}, p_1^{-+}, p_1^{+-}, p_1^{--}, p_1^{L_-^+}, \ldots, p_n^{++}, p_n^{-+}, p_n^{+-}, p_n^{--}, p_n^{L_-^+})$ is satisfiable over $\mathbb{N}$.

Suppose that $M, [h, k] \models \varphi(p_1, \ldots, p_n)$, for some $\mathbb{Z}$-model M and some interval $[h, k]$. It immediately follows that $\mu(M), [h, k] \models \varphi_{\mathbb{Z}\to\mathbb{N}}(p_1^{++}, p_1^{-+}, p_1^{+-}, p_1^{--},$

$p_1^{L_-^+}, \ldots, p_n^{++}, p_n^{-+}, p_n^{+-}, p_n^{--}, p_n^{L_-^+})$. The truth of the first and the second conjunct follows from Definition 1; the truth of the third conjunct directly follows from Lemma 1. The opposite implication can be proved in a similar way.

We now prove that the size of $\varphi_{\mathbb{Z}\to\mathbb{N}}(p_1^{++}, p_1^{-+}, p_1^{+-}, p_1^{--}, p_1^{L_-^+}, \ldots, p_n^{++}, p_n^{-+}, p_n^{+-}, p_n^{--}, p_n^{L_-^+})$ is polynomial (linear) in the size of $\varphi(p_1, \ldots, p_n)$. As usual, we assume formulas to be represented by directed acyclic graphs (DAGs). For each sub-formula of $\varphi(p_1, \ldots, p_n)$, there is a node in the DAG that represents it. Distinct nodes in the DAG for $\varphi(p_1, \ldots, p_n)$ correspond to distinct sub-formulas of it, that is, a sub-formula with multiple occurrences in $\varphi(p_1, \ldots, p_n)$ is associated with a single node of the DAG. The size of the formula $\varphi(p_1, \ldots, p_n)$, denoted $|\varphi(p_1, \ldots, p_n)|$, is the size of the DAG that represents it, measured by the number of its nodes.
Let us start with the first conjunct ψ_0. It can be easily checked that $|\psi_0|$ is $\mathrm{O}(n)$. For each $1 \leq i \leq n$, let us consider the sub-formula $(p_i^{++} \leftrightarrow p_i^{-+}) \wedge (p_i^{--} \leftrightarrow p_i^{+-})$. We first rewrite it as $(\neg p_i^{++} \vee p_i^{-+}) \wedge (\neg p_i^{-+} \vee p_i^{++}) \wedge (\neg p_i^{--} \vee p_i^{+-}) \wedge (\neg p_i^{+-} \vee p_i^{--})$. Then, as $\psi_1 \wedge \psi_2$ can be viewed as a shorthand for $\neg(\neg\psi_1 \vee \neg\psi_2)$, we can further rewrite it as $\neg(\neg(\neg p_i^{++} \vee p_i^{-+}) \vee \neg(\neg p_i^{-+} \vee p_i^{++}) \vee \neg(\neg p_i^{--} \vee p_i^{+-}) \vee \neg(\neg p_i^{+-} \vee p_i^{--}))$. Hence, 20 nodes (distinct sub-formulas) are needed to represent it. As any sub-formula of the form $[\overline{A}]\bot \rightarrow \theta$ can be rewritten as $\langle\overline{A}\rangle(\neg p_1^{++} \vee p_1^{++}) \vee \theta$, for each $1 \leq i \leq n$, at most $4 + 1 + 20$ nodes are needed to represent the formula $[\overline{A}]\bot \rightarrow (p_i^{++} \leftrightarrow p_i^{-+}) \wedge (p_i^{--} \leftrightarrow p_i^{+-})$. In fact, 2 nodes, instead of 4, suffice to represent once and for all $\langle\overline{A}\rangle(\neg p_1^{++} \vee p_1^{++})$ (to expand the logical constant $\top$, instead of introducing a fresh proposition letter, we make use of the proposition letter p_1^{++}), one for the sub-formula $\langle\overline{A}\rangle(\neg p_1^{++} \vee p_1^{++})$ and one for the sub-formula $\neg p_1^{++} \vee p_1^{++}$. Hence, the total number of nodes needed to represent the formula:

$$\neg\langle A\rangle\langle A\rangle\langle\overline{A}\rangle\langle\overline{A}\rangle \bigvee_{p\in\{p_1,\ldots,p_n\}} \neg([\overline{A}]\bot \rightarrow (p^{++} \leftrightarrow p^{-+}) \wedge (p^{--} \leftrightarrow p^{+-}))$$

is $(21 \cdot n + 2) + n + (n - 1) + 5 = 23 \cdot n + 6$.
In a similar way, we can prove that $|\psi_{L_-^+}|$ is $\mathrm{O}(n)$. First, we rewrite $\psi_{L_-^+}$ as follows:

$$\neg\langle A\rangle\langle A\rangle\langle\overline{A}\rangle\langle\overline{A}\rangle \bigvee_{p\in\{p_1,\ldots,p_n\}} (\neg(\neg p^{L_-^+} \vee \neg\langle\overline{A}\rangle\langle A\rangle\neg p^{L_-^+}) \vee \neg(\langle\overline{A}\rangle\langle A\rangle\neg p^{L_-^+} \vee p^{L_-^+}))$$

For each $1 \leq i \leq n$, we need 6 nodes to represent $\neg p^{L_-^+} \vee \neg\langle\overline{A}\rangle\langle A\rangle\neg p^{L_-^+}$, and only 1 additional node to represent $\langle\overline{A}\rangle\langle A\rangle\neg p^{L_-^+} \vee p^{L_-^+}$, as all its proper sub-formulas are sub-formulas of $\neg p^{L_-^+} \vee \neg\langle\overline{A}\rangle\langle A\rangle\neg p^{L_-^+}$ as well. Three further nodes are needed to represent $\neg(\neg p^{L_-^+} \vee \neg\langle\overline{A}\rangle\langle A\rangle\neg p^{L_-^+}) \vee \neg(\langle\overline{A}\rangle\langle A\rangle\neg p^{L_-^+} \vee p^{L_-^+})$. Hence, 10 nodes are needed for any such formula. The total number of nodes of the DAG for $\psi_{L_-^+}$ is thus $10 \cdot n + (n - 1) + 5 = 11 \cdot n + 4$.
Let us consider now the third conjunct $\tau(\varphi, {++})$. To give an upper bound to its size, we proceed as follows. First, we create a $++$ (resp., $--, -+, +-, L_-^+$) copy

of the DAG for the input formula $\varphi(p_1,\ldots,p_n)$ by replacing each node labeled with θ by a node labeled with $\tau(\theta,++)$ (resp., $\tau(\theta,--), \tau(\theta,-+), \tau(\theta,+-), \tau(\theta, L^{+}_{-})$). Then, we observe that each node belonging to one of these 5 DAGs may occur at most once in the DAG for the output formula $\varphi_{\mathbb{Z}\to\mathbb{N}}(p_1^{++}, p_1^{-+}, p_1^{+-}, p_1^{--}, p_1^{L^{+}_{-}}, \ldots, p_n^{++}, p_n^{-+}, p_n^{+-}, p_n^{--}, p_n^{L^{+}_{-}})$, and, at worst, it may contribute 25 nodes (sub-formulas), the worst cases being those of the formulas $\tau(\langle\overline{A}\rangle\psi,++)$ and $\tau(\langle A\rangle\psi,--)$[4]. For instance, $\tau(\langle\overline{A}\rangle\psi,++)$ can be rewritten as $\langle\overline{A}\rangle\tau(\psi,++)\vee \langle\overline{A}\rangle\tau(\psi,-+)\vee\langle\overline{A}\rangle\langle A\rangle\tau(\psi,+-)\vee\neg(\neg\langle\overline{A}\rangle\top\vee\neg\tau(\psi,L^{+}_{-}))\vee\neg(\langle\overline{A}\rangle\top\vee\neg(\tau(\psi,+-)\vee \langle A\rangle\langle\overline{A}\rangle\langle\overline{A}\rangle\neg(\langle\overline{A}\rangle\top\vee\neg\tau(\psi,+-))\vee\langle A\rangle\langle A\rangle\langle\overline{A}\rangle\neg(\langle\overline{A}\rangle\top\vee\neg\tau(\psi,+-))))$. It can be easily checked that, if we ignore nodes (sub-formulas) of the forms $\tau(\psi,++), \tau(\psi, -+), \tau(\psi,+-)$, and $\tau(\psi,L^{+}_{-})$, whose contribution will be computed separately, and the node (sub-formula) $\langle\overline{A}\rangle\top$, that has been already introduced by the translation of ψ_0, it features 25 distinct nodes (sub-formulas). A rough approximation of the size of the translation of the third conjunct is thus provided by the following inequality: $|\tau(\varphi,++)|\leq 25\cdot 5\cdot|\varphi(p_1,\ldots,p_n)| = 125\cdot|\varphi(p_1,\ldots,p_n)|$. Since $|\varphi(p_1,\ldots,p_n)|$ is $\Omega(n)$, we can conclude that $|\varphi_{\mathbb{Z}\to\mathbb{N}}(p_1^{++}, p_1^{-+}, p_1^{+-}, p_1^{--}, p_1^{L^{+}_{-}}, \ldots, p_n^{++}, p_n^{-+}, p_n^{+-}, p_n^{--}, p_n^{L^{+}_{-}})|$ is $\mathrm{O}(|\varphi(p_1,\ldots,p_n)|)$. $\square$

It is worth pointing out that, without loss of generality, in the proof of Theorem 3, we have assumed that, whenever an $\mathsf{A\overline{A}}$-formula φ is satisfiable over $\mathbb{Z}$, then there exist a model M and an interval $[h,k]$, with $h,k\geq 0$, such that $M,[h,k]\models\varphi$. In $\mathbb{Z}$, for every $\mathsf{A\overline{A}}$-formula and every model M, it indeed holds that $M,[h,k]\models\varphi$, for some ordered pair of integers h,k, if and only if for every ordered pair of integers h',k', there exists a model M' such that $M',[h',k']\models\varphi$.

5 On the separation of $\mathbb{Q}$ and $\mathbb{R}$ in $\mathsf{A\overline{A}}$ (and not in A)

In this section, we show that $\mathsf{A\overline{A}}$ is expressive enough to separate $\mathbb{Q}$ and $\mathbb{R}$. More precisely, we prove that if an $\mathsf{A\overline{A}}$-formula is satisfiable over $\mathbb{R}$, then it is satisfiable over $\mathbb{Q}$ as well, but the vice versa does not hold, that is, there exist $\mathsf{A\overline{A}}$-formulas which are satisfiable over $\mathbb{Q}$ and unsatisfiable over $\mathbb{R}$. To emphasize the role of the modality $\langle\overline{A}\rangle$ in such a separation result, we then show that this is not the case with A: whenever an A-formula is satisfiable over $\mathbb{R}$, then it is satisfiable over $\mathbb{Q}$ as well (the proof is basically the same as that for $\mathsf{A\overline{A}}$), and, vice versa, if an A-formula is satisfiable over $\mathbb{Q}$, then it is also satisfiable over $\mathbb{R}$.

To start with, we introduce some preliminary notions and results. Let φ be an $\mathsf{A\overline{A}}$-formula to be checked for satisfiability. We define the *closure* $\mathrm{CL}(\varphi)$ of φ as the set of all sub-formulas of φ and of their negations (we identify $\neg\neg\psi$ with ψ, $\neg\langle A\rangle\psi$ with $[A]\neg\psi$, and $\neg\langle\overline{A}\rangle\psi$ with $[\overline{A}]\neg\psi$). Among the formulas in $\mathrm{CL}(\varphi)$, a special role is played by temporal formulas. We define the set of *temporal*

[4] It is worth pointing out that the choice of using DAGs to represent formulas plays a crucial role here to guarantee that the size of the output formula is polynomial (linear) in the size of the input one.

formulas of φ as the set $\mathrm{TF}(\varphi) = \{\langle A\rangle\psi, [A]\psi, \langle\overline{A}\rangle\psi, [\overline{A}]\psi \in \mathrm{CL}(\varphi)\}$. A *maximal set of requests* for φ is a set $S \subseteq \mathrm{TF}(\varphi)$ that satisfies the following conditions: (i) for every $\langle A\rangle\psi \in \mathrm{TF}(\varphi)$, $\langle A\rangle\psi \in S$ if and only if $[A]\neg\psi \notin S$, and (ii) for every $\langle\overline{A}\rangle\psi \in \mathrm{TF}(\varphi)$, $\langle\overline{A}\rangle\psi \in S$ if and only if $[\overline{A}]\neg\psi \notin S$. We define a *$\varphi$-atom* as a set $A \subseteq \mathrm{CL}(\varphi)$ such that (i) for every $\psi \in \mathrm{CL}(\varphi)$, $\psi \in A$ if and only if $\neg\psi \notin A$, and (ii) for every $\psi_1 \vee \psi_2 \in \mathrm{CL}(\varphi)$, $\psi_1 \vee \psi_2 \in A$ iff $\psi_1 \in A$ or $\psi_2 \in A$. Let us denote by $\mathcal{A}_\varphi$ the set of all φ-atoms. We connect atoms by a *binary relation* LR_φ such that for every pair of atoms $A_1, A_2 \in \mathcal{A}_\varphi$, $A_1\ LR_\varphi\ A_2$ if and only if (i) for every $[A]\psi \in \mathrm{CL}(\varphi)$, if $[A]\psi \in A_1$, then $\psi \in A_2$, and (ii) for every $[\overline{A}]\psi \in \mathrm{CL}(\varphi)$, if $[\overline{A}]\psi \in A_2$, then $\psi \in A_1$.

We now introduce a suitable labeling of interval structures based on φ-atoms that will play an important role in the following proofs. We define a *φ-labeled interval structure* (LIS for short) as a pair $L = \langle \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$, where $\mathbb{I}(\mathbb{D})$ is an interval structure and $\mathcal{L} : \mathbb{I}(\mathbb{D}) \to \mathcal{A}_\varphi$ is a labeling function such that, for every pair of neighboring intervals $[i, j], [j, k] \in \mathbb{I}(\mathbb{D})$, $\mathcal{L}([i, j])\ LR_\varphi\ \mathcal{L}([j, k])$. If we interpret $\mathcal{L}$ as a valuation function, LISs can be viewed as *candidate models* for φ: the truth of formulas devoid of temporal operators follows from the definition of φ-atom, and universal temporal conditions, imposed by $[A]/[\overline{A}]$ operators, are forced by the relation LR_φ.

To turn a LIS into a *model* for φ, we must also guarantee the satisfaction of existential temporal conditions, imposed by $\langle A\rangle/\langle\overline{A}\rangle$ operators. To this end, we introduce the notion of fulfilling LIS. We say that a LIS $L = \langle \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ is *fulfilling* if and only if (i) for every $\langle A\rangle\psi \in \mathrm{TF}(\varphi)$ and every $[i, j] \in \mathbb{I}(\mathbb{D})$, if $\langle A\rangle\psi \in \mathcal{L}([i, j])$, then there exists $k > j$ such that $\psi \in \mathcal{L}([j, k])$ and (ii) for every $\langle\overline{A}\rangle\psi \in \mathrm{TF}(\varphi)$ and every $[i, j] \in \mathbb{I}(\mathbb{D})$, if $\langle\overline{A}\rangle\psi \in \mathcal{L}([i, j])$, then there exists $k < i$ such that $\psi \in \mathcal{L}([k, i])$.

The next theorem proves that for any $\mathsf{A}\overline{\mathsf{A}}$-formula φ and any linearly-ordered domain $\mathbb{D}$, the satisfiability of φ is equivalent to the existence of a fulfilling LIS with an interval labeled by φ.

Theorem 4. *An $\mathsf{A}\overline{\mathsf{A}}$-formula φ is satisfiable over a linearly-ordered domain $\mathbb{D}$ if and only if there exists a fulfilling* LIS $L = \langle \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ *such that $\varphi \in \mathcal{L}([i, j])$ for some $[i, j] \in \mathbb{I}(\mathbb{D})$.*

The implication from left to right is straightforward; the opposite implication is proved by induction on the structure of the formula [6]. It is worth pointing out that the statement of Theorem 4 is parametric in $\mathbb{D}$, that is, it holds whatever linearly-ordered domain we take as $\mathbb{D}$. On the basis of Theorem 4, from now on, we say that a fulfilling LIS $\mathbf{L} = \langle \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ *satisfies* φ if and only if there exists an interval $[i, j] \in \mathbb{I}(\mathbb{D})$ such that $\varphi \in \mathcal{L}([i, j])$.

Finally, we associate with each point the set of its temporal requests. Formally, given a LIS $L = \langle \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ and a point $i \in D$, we define the set of *temporal requests of i* as the set $\mathrm{REQ}^L(i) = \{\langle A\rangle\xi \in \mathrm{TF}(\varphi) : \exists\ i' \in D$ such that $\langle A\rangle\xi \in \mathcal{L}([i', i])\} \cup \{[A]\xi \in \mathrm{TF}(\varphi) : \exists\ i' \in D$ such that $[A]\xi \in \mathcal{L}([i', i])\} \cup \{\langle\overline{A}\rangle\xi \in \mathrm{TF}(\varphi) : \exists\ i' \in D$ such that $\langle\overline{A}\rangle\xi \in \mathcal{L}([i, i'])\} \cup \{[\overline{A}]\xi \in \mathrm{TF}(\varphi) : \exists\ i' \in D$ such that $[\overline{A}]\xi \in \mathcal{L}([i, i'])\}$. For the sake of simplicity, we will write $\mathrm{REQ}(i)$ for $\mathrm{REQ}^L(i)$ whenever the LIS L we refer to is evident from the context.

We are now ready to prove our first result.

Theorem 5. *Let φ be an $\mathsf{A}\overline{\mathsf{A}}$-formula. It holds that if φ is satisfiable over $\mathbb{R}$, then it is satisfiable over $\mathbb{Q}$.*

Proof. By Theorem 4 (left-to-right direction), from the satisfiability of φ over $\mathbb{R}$, we can infer the existence of a fulfilling LIS $L = \langle \mathbb{I}(\mathbb{R}), \mathcal{L} \rangle$ that satisfies φ.

We show that, making use of such a LIS, one can produce a fulfilling LIS $L' = \langle \mathbb{I}(\mathbb{Q}), \mathcal{L}' \rangle$ that satisfies φ. More precisely, we show that L' can be obtained as the limit of an infinite sequence of finite LIS $L_0, L_1, \ldots$ over (finite suborders of) $\mathbb{Q}$ starting from a suitable LIS L_0. We show now how to build L_0 and how to expand L_i into L_{i+1}, for each $i \geq 0$.

The initial LIS L_0 is a pair $\langle \{[i,j]\}, \mathcal{L}_0 \rangle$, where $i, j \in \mathbb{Q}$ and $\varphi \in \mathcal{L}_0([i,j])$. Since L satisfies φ, there exist $h, k \in \mathbb{R}$ such that $\varphi \in \mathcal{L}([h,k])$. To mimic h, k in L_0, we choose $i, j \in \mathbb{Q}$ such that the function $f_0 : \{i, j\} \to \mathbb{R}$ with $f_0(i) = h$ and $f_0(j) = k$ is (strictly) monotone. Moreover, we put $\mathcal{L}_0([i,j]) = \mathcal{L}([f_0(i), f_0(j)])$. Finally, if there exists $\langle A \rangle \psi \in \mathrm{REQ}(f_0(j))$ (resp., $\langle \overline{A} \rangle \psi \in \mathrm{REQ}(f_0(i))$), we insert i (resp., j) in a queue Q_0 of pending requests.

Let $L_i = \langle \mathbb{I}(\mathbb{D}_i), \mathcal{L}_i \rangle$, where $\mathbb{D}_i$ is a finite linear order, be the i-the LIS in the sequence, and let l be the first element of Q_i. Three alternative cases must be taken into account.

Let $\langle A \rangle \psi \in \mathrm{REQ}(f_i(l))$ be such that there is not $m \in D_i$, with $m > l$, such that $\psi \in \mathcal{L}_i([l,m])$ (future pending request). The LIS L_{i+i} can be obtained from L_i as follows. First, we expand D_i into D_{i+1} by adding a point m (resp., n) such that $m < h$ (resp., $n > h$) for each $h \in D_i$, and by adding, for any pair of consecutive points $h, k \in D_i$, a point p, with $h < p < k$, the existence of such a point being guaranteed by the density of $\mathbb{Q}$. Then, we replace the queue Q_i by a queue Q_{i+1}, which is obtained from Q_i by inserting all these additional points. Next, we replace f_i by a mapping f_{i+1}. For each $h \in D_i$, we simply put $f_{i+1}(h) = f_i(h)$. The case of points in $D_{i+1} \setminus D_i$ is more complex. As a preliminary step, we observe that, since L is a fulfilling LIS, there exists $o \in \mathbb{R}$ such that $f_i(l) < o$ and $\psi \in \mathcal{L}([f_i(l), o])$. For each $p \in D_{i+1} \setminus D_i$, we must distinguish among three cases. If p is less than the least point h in D_i, we put $f_{i+1}(p) = q$ for some $q < f_i(h)$. If p is greater than the greatest point h in D_i, then if $o > f_i(h)$, we put $f_{i+1}(p) = o$; otherwise, we put $f_{i+1}(p) = q$ for some $q > f_i(h)$. Finally, if $h < p < k$ for a pair of consecutive points $h, k \in D_i$, then if $f_i(h) < o < f_i(k)$, we put $f_{i+1}(p) = o$; otherwise, we put $f_{i+1}(p) = q$ for some $f_i(h) < q < f_i(k)$. The existence of such a point q is guaranteed by the left unboundeness (resp., right unboundeness, density) of $\mathbb{R}$, respectively. Finally, for each interval $[h,k] \in \mathbb{I}(\mathbb{D}_{i+1}) \setminus \mathbb{I}(\mathbb{D}_i)$, we put $\mathcal{L}_{i+1}([h,k]) = \mathcal{L}([f_{i+1}(h), f_{i+1}(k)])$, and we let $\mathcal{L}_{i+1}([h,k]) = \mathcal{L}_i([h,k])$ for each pair $[h,k] \in \mathbb{I}(\mathbb{D}_i)$.

The case in which there exists no such a formula $\langle A \rangle \psi \in \mathrm{REQ}(f_i(l))$, but there exists a formula $\langle \overline{A} \rangle \psi \in \mathrm{REQ}(f_i(l))$ for which there is not $m \in D_i$, with $m < l$, such that $\psi \in \mathcal{L}_i([m,l])$ (past pending request), can be dealt with in a very similar way (the construction is completely symmetric).

If l has neither future pending requests nor past pending requests, we remove it from Q_{i+1}, and, to build L_{i+i}, we basically apply the above-described expansion strategy to L_i, the only difference being that there are no constraints on f_{i+1} apart from that of preserving monotonicity.

We define L' as the (component-wise) infinite union $\cup_{i \geq 0} L_i$.

By Theorem 4 (right-to-left direction), the existence of a fulfilling LIS $L' = \langle \mathbb{I}(\mathbb{Q}), \mathcal{L}' \rangle$ that satisfies φ implies the satisfiability of φ over $\mathbb{Q}$. $\square$

We now show that the opposite implication does not hold. Let θ be the $\mathsf{A}\overline{\mathsf{A}}$-formula $p \wedge \langle A \rangle \langle A \rangle q \wedge [G]((p \rightarrow \langle A \rangle p) \wedge (q \rightarrow \langle \overline{A} \rangle q) \wedge (p \rightarrow [A]([\overline{A}]p \wedge [\overline{A}][\overline{A}]p)) \wedge (q \rightarrow [\overline{A}]([A]q \wedge [A][A]q)) \wedge \neg(p \wedge q) \wedge (\neg p \wedge \neg q \rightarrow \langle \overline{A} \rangle p \wedge \langle A \rangle q))$. We can prove the following theorem.

Theorem 6. *The $\mathsf{A}\overline{\mathsf{A}}$-formula θ is satisfiable over $\mathbb{Q}$, but it is not satisfiable over $\mathbb{R}$.*

Proof. We first show that θ is satisfiable over $\mathbb{Q}$ by exhibiting a model $M = \langle \mathbb{I}(\mathbb{Q}), V \rangle$ for it. Let $\mathcal{AP} = \{p, q\}$ and let $r \in \mathbb{R} \setminus \mathbb{Q}$, say, $r = \sqrt{2}$. For every interval $[q, q'] \in \mathbb{I}(\mathbb{Q})$, we define V as follows:

$$V(p) = \{[i, j] : j < r\}$$

$$V(q) = \{[i, j] : i > r\}$$

It can be easily checked that M satisfies θ. A graphical account of the model is given in Figure 7.

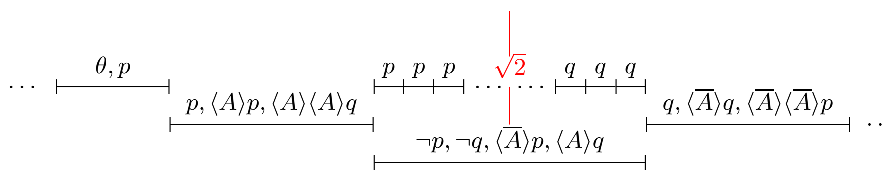

Fig. 7. A model for the $\mathsf{A}\overline{\mathsf{A}}$-formula θ over $\mathbb{Q}$.

We now prove that θ is not satisfiable over $\mathbb{R}$. Suppose, by contradiction, that there exists an $\mathbb{R}$-model $M = \langle \mathbb{I}(\mathbb{R}), V \rangle$ for it. Let S_p^M and S_q^M be two subsets of $\mathbb{R}$ that respectively collect all points which are right endpoints of intervals where p holds and all points which are left endpoints of intervals where q holds. Formally, let $S_p^M = \{r \in \mathbb{R} \ : \ \exists r' \in \mathbb{R}$ such that $r' < r$ and $[r', r] \in V(p)\}$ and $S_q^M = \{r \in \mathbb{R} \ : \ \exists r' \in \mathbb{R}$ such that $r' > r$ and $[r, r'] \in V(q)\}$.

The two conjuncts p and $\langle A \rangle \langle A \rangle q$ respectively force S_p^M and S_q^M to be non-empty. We now show that $S_p^M \cap S_q^M = \emptyset$. By contradiction, suppose that there exists $\overline{r} \in S_p^M \cap S_q^M$. By $p \rightarrow \langle A \rangle p$, it follows that there exists $\overline{r}'$, with $\overline{r} < \overline{r}'$, such that $[\overline{r}, \overline{r}'] \in V(p)$. Moreover, by $q \rightarrow [\overline{A}]([A]q \wedge [A][A]q)$, $[\overline{r}, \overline{r}'] \in V(q)$. The contradiction immediately follows from $\neg(p \wedge q)$. Next, we show that $\mathbb{R} = S_p^M \cup S_q^M$. By the conjunct $\neg(p \wedge q)$, for any interval $[r, r'] \in \mathbb{I}(\mathbb{R})$, we have that one, and only one, of the following cases applies: (i) $[r, r'] \in V(p)$ and

$[r, r'] \notin V(q)$, (ii) $[r, r'] \in V(q)$ and $[r, r'] \notin V(p)$, and (iii) $[r, r'] \notin V(p)$ and $[r, r'] \notin V(q)$. In case (i), by definition, $r' \in S_p^M$ and, by $p \rightarrow [A]([\overline{A}]p \wedge [\overline{A}][\overline{A}]p))$, $r \in S_p^M$ as well. By the same argument, with the obvious replacements, in case (ii), we can conclude that both $r \in S_q^M$ and $r' \in S_q^M$. In case (iii), from $\neg p \wedge \neg q \rightarrow \langle\overline{A}\rangle p \wedge \langle A\rangle q)$, it immediately follows that $r \in S_p^M$ and $r' \in S_q^M$. Finally, we prove that for every $r_p \in S_p^M$ and $r_q \in S_q^M$, $r_p < r_q$. By contradiction, suppose that there exist $r_p \in S_p^M$ and $r_q \in S_q^M$, $r_q < r_p$ (r_q cannot be equal to r_p, as $S_p^M \cap S_q^M = \emptyset$). By $q \rightarrow [\overline{A}]([A]q \wedge [A][A]q)$, $[r_q, r_p] \in V(q)$, and, by $p \rightarrow [A]([\overline{A}]p \wedge [\overline{A}][\overline{A}]p)$, $[r_q, r_p] \in V(p)$. Again, the contradiction immediately follows from $\neg(p \wedge q)$.

To summarize, we have that S_p^M and S_q^M define an *ordered* partition of $\mathbb{R}$. Since $\mathbb{R}$ is Dedekind-complete, it immediately follows that both $sup\,(S_p^M)$ and $inf\,(S_q^M)$ exist, and $sup\,(S_p^M) = inf\,(S_q^M)$. Let $\overline{r} = sup\,(S_p^M) = inf\,(S_q^M)$. Let us take now an interval of the form $[r, \overline{r}]$, for some $r < \overline{r}$ in $\mathbb{R}$. We show that there is no way to consistently define the truth value of p and q over $[r, \overline{r}]$. Four different cases must be considered:

– $[r, \overline{r}] \in V(p)$ and $[r, \overline{r}] \in V(q)$. By the conjunct $\neg(p \wedge q)$, this cannot be the case.
– $[r, \overline{r}] \in V(p)$ and $[r, \overline{r}] \notin V(q)$. By $p \rightarrow \langle A\rangle p$, it follows that there exists $r' > \overline{r}$ such that $[\overline{r}, r'] \in V(p)$, which contradicts the fact that $\overline{r} = sup\,(S_p^M)$.
– $[r, \overline{r}] \notin V(p)$ and $[r, \overline{r}] \in V(q)$. Since $r < \overline{r}$, this contradicts the fact that $\overline{r} = inf\,(S_q^M)$.
– $[r, \overline{r}] \notin V(p)$ and $[r, \overline{r}] \notin V(q)$. By $\neg p \wedge \neg q \rightarrow \langle\overline{A}\rangle p \wedge \langle A\rangle q$, it follows that there exists $r' > \overline{r}$ such that $[\overline{r}, r'] \in V(q)$. Hence, by $q \rightarrow \langle\overline{A}\rangle q$, there exists $r'' < \overline{r}$ such that $[r'', \overline{r}] \in V(q)$, which contradicts the fact that $\overline{r} = inf\,(S_q^M)$. □

In the following, we show that if we remove the past operator $\langle\overline{A}\rangle$ from $\mathsf{A\overline{A}}$, the ability of separating $\mathbb{Q}$ and $\mathbb{R}$ is lost.

Theorem 7. *Let φ be an A-formula. It holds that if φ is satisfiable over $\mathbb{R}$, then it is satisfiable over $\mathbb{Q}$.*

Proof. Since A is a proper fragment of $\mathsf{A\overline{A}}$, the thesis immediately follows from Theorem 5. □

Theorem 8. *Let φ be an A-formula. It holds that if φ is satisfiable over $\mathbb{Q}$, then it is satisfiable over $\mathbb{R}$.*

Proof. By Theorem 4 (left-to-right direction), from the satisfiability of φ over $\mathbb{Q}$, we can infer the existence of a fulfilling LIS $L = \langle\mathbb{I}(\mathbb{Q}), \mathcal{L}\rangle$ that satisfies φ. A fulfilling LIS $L' = \langle\mathbb{I}(\mathbb{R}), \mathcal{L}'\rangle$ that satisfies φ can be built as follows.

First, for every pair of points $i, j \in \mathbb{Q}$, we put $\mathcal{L}'([i, j]) = \mathcal{L}([i, j])$. Notice that, since L satisfies φ, there exists an interval $[i, j] \in \mathbb{I}(\mathbb{R})$ such that $\varphi \in \mathcal{L}'([i, j])$, and thus L' satisfies φ.

Let us define now the labeling of those intervals whose left or right endpoints belong to $\mathbb{R} \setminus \mathbb{Q}$. We observe that, for any point $i \in \mathbb{R} \setminus \mathbb{Q}$ and any $\epsilon > 0$, $\mathbb{Q}$ is dense over $[i, i+\epsilon]$. Hence, there exists an infinite descending sequence of rational numbers $\mathcal{S}_i = i_1 > i_2 > i_3 > \ldots$ such that $\mathrm{REQ}(i_1) = \mathrm{REQ}(i_2) = \mathrm{REQ}(i_3) = \ldots = R_i$, and for every $\epsilon > 0$, there exists an index l such that all elements i_m of $\mathcal{S}_i$ with $m \geq l$ belong to $[i, i+\epsilon]$. We put $REQ(i) = R_i$.

We first show how to fulfil all requests in $REQ(i)$. Let i_n be an arbitrary element of $\mathcal{S}_i$. For every $h \in \mathbb{Q}$, with $h > i_n$, we put $\mathcal{L}'([i,h]) = \mathcal{L}([i_n, h])$. Each request in $\mathrm{REQ}(i_n)$ is obviously fulfilled in L, as L is a fulfilling LIS. Then, it immediately follows that each request in $\mathrm{REQ}(i)$ is fulfilled in L'.

To complete the construction of L', for every $i \in \mathbb{R} \setminus \mathbb{Q}$, we need to properly define the labeling of the intervals of the form $[i,j]$ for (i) each $j \in \mathbb{R} \setminus \mathbb{Q}$, with $j > i_n$, (ii) each $j \in \mathbb{Q}$, with $i < j \leq i_n$, and (iii) each $j \in \mathbb{R} \setminus \mathbb{Q}$, with $i < j \leq i_n$. As for case (i), let $j \in \mathbb{R} \setminus \mathbb{Q}$, with $j > i_n$, and let $REQ(j) = R_j$. By definition, there exists an infinite descending sequence of rational numbers $\mathcal{S}_j = j_1 > j_2 > j_3 > \ldots$ such that $\mathrm{REQ}(j_1) = \mathrm{REQ}(j_2) = \mathrm{REQ}(j_3) = \ldots = R_j$, and for every $\epsilon > 0$, there exists an index o such that all elements j_p of $\mathcal{S}_j$ with $p \geq o$ belong to $[j, j+\epsilon]$. Let j_q be an arbitrary element of such a sequence. We put $\mathcal{L}'([i,j]) = \mathcal{L}([i_n, j_q])$. Let us consider now case (ii). For every $j \in \mathbb{Q}$, with $i < j \leq i_n$, there exists an index o, with $o > n$, such that $i_o \in \mathcal{S}_i$ and $i_o < j$. We put $\mathcal{L}'([i,j]) = \mathcal{L}([i_o, j])$. Finally, let us consider case (iii). Let $j \in \mathbb{R} \setminus \mathbb{Q}$, with $i < j \leq i_n$, and let $REQ(j) = R_j$. We choose an element j_q in $\mathcal{S}_j$ as in case (i), and we choose an element $i_o \in \mathcal{S}_i$, with $i_o < j$, as in case (ii). We put $\mathcal{L}'([i,j]) = \mathcal{L}([i_o, j_q])$.

By Theorem 4 (right-to-left direction), the existence of a fulfilling LIS $L' = \langle \mathbb{I}(\mathbb{R}), \mathcal{L}' \rangle$ that satisfies φ implies the satisfiability of φ over $\mathbb{R}$. $\square$

We believe it useful to explain why the construction given in the proof of Theorem 8 does not work in the case of $\mathsf{A}\overline{\mathsf{A}}$. The reason is that, in order to properly associate a set of requests $\mathrm{REQ}(i)$ with a point $i \in \mathbb{R} \setminus \mathbb{Q}$ when dealing with $\mathsf{A}\overline{\mathsf{A}}$, we must constrain $\mathrm{REQ}(i)$ to be consistent with both the infinite descending sequence of rational numbers to the right of it of Theorem 8 and a corresponding infinite ascending sequence of rational numbers to the left of it. Unfortunately, there are cases in which there is no way to jointly satisfy these constraints. One of these cases is given in the proof of Theorem 6.

We conclude the section by pointing out the differences between the results given in this section and those reported in the previous one. On the one hand, it can be easily shown that $\mathsf{A}\overline{\mathsf{A}}$ can separate $\mathbb{N}$ and $\mathbb{Z}$. Consider, for instance, the $\mathsf{A}\overline{\mathsf{A}}$-formula $p \wedge [G](p \rightarrow \langle \overline{A} \rangle p)$, which forces the existence of an infinite-to-the-left sequence of intervals over which p holds. This formula is satisfiable over $\mathbb{Z}$, but it is not satisfiable over $\mathbb{N}$. On the other hand, from the fact that satisfiability of an $\mathsf{A}\overline{\mathsf{A}}$ formula over $\mathbb{R}$ implies its satisfiability over $\mathbb{Q}$, we cannot conclude that there is a way to reduce the satisfiability problem for $\mathsf{A}\overline{\mathsf{A}}$ over $\mathbb{R}$ to its satisfiability problem over $\mathbb{Q}$. To this end, we should be able to provide a characterization of the class of $\mathbb{Q}$-models corresponding to $\mathbb{R}$-models in $\mathsf{A}\overline{\mathsf{A}}$, as we did in Section 4 in the case of $\mathbb{Z}$-models and $\mathbb{N}$-models, and we are not (a simple game-theoretic argument can be used to prove it).

References

1. J. Allen. Maintaining knowledge about temporal intervals. *Communications of the ACM*, 26(11):832–843, 1983.
2. P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2002.
3. D. Bresolin, V. Goranko, A. Montanari, and P. Sala. Tableaux for logics of subinterval structures over dense orderings. *Journal of Logic and Computation*, 20(1):133–166, 2010.
4. D. Bresolin, V. Goranko, A. Montanari, and G. Sciavicco. Propositional interval neighborhood logics: Expressiveness, decidability, and undecidable extensions. *Annals of Pure and Applied Logic*, 161(3):289–304, 2009.
5. D. Bresolin, A. Montanari, and P. Sala. An optimal tableau-based decision algorithm for Propositional Neighborhood Logic. In *Proc. of the 24th Int. Symposium on Theoretical Aspects of Computer Science*, volume 4393 of *LNCS*, pages 549–560. Springer, 2007.
6. D. Bresolin, A. Montanari, P. Sala, and Sciavicco G. Optimal tableau systems for propositional neighborhood logic over all, dense, and discrete linear orders. In *Proc. of the 20th International Conference on Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX)*, volume 6793 of *LNAI*, pages 73–87. Springer, 2011.
7. D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco. What's decidable about Halpern and Shoham's interval logic? The maximal fragment $\mathsf{AB\overline{BL}}$. In *Proc. of the 26th Symposium on Logic in Computer Science*, pages 387–396. IEEE Computer Society, 2011.
8. D. Bresolin, A. Montanari, and G. Sciavicco. An optimal decision procedure for Right Propositional Neighborhood Logic. *Journal of Automated Reasoning*, 38(1-3):173–199, 2007.
9. E. A. Emerson. Temporal and modal logic. In *Handbook of Theoretical Computer Science, Volume B: Formal Models and Semantics*, pages 995–1072. Elsevier and MIT Press, 1990.
10. D. M. Gabbay. The declarative past and imperative future: Executable temporal logic for interactive systems. In B. Banieqbal, H. Barringer, and A. Pnueli, editors, *Temporal Logic in Specification*, volume 398 of *LNCS*, pages 409–448. Springer, 1987.
11. D. M. Gabbay, A. Pnueli, S. Shelah, and J. Stavi. On the temporal basis of fairness. In *Proc. of the 7th Annual ACM Symposium on Principles of Programming Languages*, pages 163–173, 1980.
12. V. Goranko, A. Montanari, and G. Sciavicco. A road map of interval temporal logics and duration calculi. *Journal of Applied Non-Classical Logics*, 14(1–2):9–54, 2004.
13. J. Halpern and Y. Shoham. A propositional modal logic of time intervals. *Journal of the ACM*, 38(4):935–962, 1991.
14. F. Laroussinie, N. Markey, and P. Schnoebelen. Temporal logic with forgettable past. In *Proc. of the 17th Symposium on Logic in Computer Science*, pages 383–392. IEEE Computer Society, 2002.

15. O. Lichtenstein, A. Pnueli, and L. D. Zuck. The glory of the past. In R. Parikh, editor, *Logic of Programs*, volume 193 of *LNCS*, pages 196–218. Springer, 1985.
16. O. Maler and A. Pnueli. Tight bounds on the complexity of cascaded decomposition of automata. In *Proc. of the 31st Annual Symposium on Foundations of Computer Science, Volume II*, pages 672–682. IEEE Computer Society, 1990.
17. J. Marcinkowski and J. Michaliszyn. The ultimate undecidability result for the Halpern-Shoham logic. In *Proc. of the 26th Symposium on Logic in Computer Science*, pages 377–386. IEEE Computer Society, 2011.
18. N. Markey. Temporal logic with past is exponentially more succinct. *Bulletin of the EATCS*, 79:122–128, 2003.
19. A. Montanari, G. Puppis, and P. Sala. A decidable spatial logic with cone-shaped cardinal directions. In *Proc. of the 18th EACSL Annual Conference on Computer Science Logic*, volume 5771 of *LNCS*, pages 394–408. Springer, 2009.
20. A. Montanari, G. Puppis, and P. Sala. Maximal decidable fragments of halpern and shoham's modal logic of intervals. In *Proc. of the 37th International Colloquium on Automata, Languages and Programming - Part II*, volume 6199 of *LNCS*, pages 345–356, 2010.
21. B. Moszkowski. *Reasoning about digital circuits*. Tech. rep. stan-cs-83-970, Dept. of Computer Science, Stanford University, Stanford, CA, 1983.
22. M. Otto. Two variable first-order logic over ordered domains. *Journal of Symbolic Logic*, 66(2):685–702, 2001.
23. I. Shapirovsky. On PSPACE-decidability in Transitive Modal Logic. In *Advances in Modal Logic*, volume 5, pages 269–287. King's College Publications, 2005.
24. W. Thomas. Automata on infinite objects. In *Handbook of Theoretical Computer Science, Volume B: Formal Models and Semantics*, pages 133–192. Elsevier and MIT Press, 1990.
25. S. K. Thomason. Reduction of tense logic to modal logic I. *Journal of Symbolic Logic*, 39(3):549–551, 1974.
26. S. K. Thomason. Reduction of tense logic to modal logic II. *Theoria*, 41(3):154–169, 1975.
27. Y. Venema. A modal logic for chopping intervals. *Journal of Logic and Computation*, 1(4):453–476, 1991.