Decidable and Undecidable Fragments of Halpern and Shoham's Interval Temporal Logic: Towards a Complete Classification

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Abstract. Interval temporal logics are based on temporal structures where time intervals, rather than time instants, are the primitive ontological entities. They employ modal operators corresponding to various relations between intervals, known as Allen's relations. Technically, validity in interval temporal logics translates to dyadic second-order logic, thus explaining their complex computational behavior. The full modal logic of Allen's relations, called HS, has been proved to be undecidable by Halpern and Shoham under very weak assumptions on the class of interval structures, and this result was discouraging attempts for practical applications and further research in the field. A renewed interest has been recently stimulated by the discovery of interesting decidable fragments of HS. This paper contributes to the characterization of the boundary between decidability and undecidability of HS fragments. It summarizes known positive and negative results, it describes the main techniques applied so far in both directions, and it establishes a number of new undecidability results for relatively small fragments of HS.

1 Introduction

Interval temporal logics are based on interval structures over linearly ordered domains, where time intervals, rather than time instants, are the primitive ontological entities. The variety of relations between intervals in linear orders was first studied systematically by Allen [1], who explored their use in systems for time management and planning. Interval reasoning arises naturally in various other fields of artificial intelligence, such as theories of action and change, natural language analysis and processing, and constraint satisfaction problems. Temporal logics with interval-based semantics have also been proposed as a useful formalism for the specification and verification of hardware [21] and of real-time

** Guido Sciavicco was co-financed by the Spanish projects TIN 2006-15460-C04-01 and PET 2006_0406.

systems [11]. Thus, the relevance of interval temporal logics in many areas of artificial intelligence and computer science is nowadays widely recognized.

Interval temporal logics feature modal operators corresponding to various possible relations over intervals. A special role is played by the thirteen different binary relations (on linear orders) known as Allen's relations. In [15], Halpern and Shoham introduce a modal logic for reasoning about interval structures, called HS, with modal operators corresponding to Allen's interval relations. Formulas of HS are evaluated at intervals, i.e., pairs of points, and, consequently, they translate into binary relations in interval models. Accordingly, validity in HS translates to dyadic second-order logic, thus causing its complex and generally bad computational behavior, where undecidability is the common case and decidability is usually achieved by imposing severe restrictions on the interval-based semantics, which essentially reduce it to a point-based one. More precisely, HS turns out to be undecidable under very weak assumptions on the class of interval structures [15]: we get undecidability for any class of interval structures over linear orders that contains at least one linear order with an infinite ascending (or descending) chain, thus including all natural numerical time-flows $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$.

For a long time, such a sweeping undecidability result has discouraged attempts for practical applications and further research on interval logics. A renewed interest in the area has been recently stimulated by the discovery of some interesting decidable fragments of HS [3,4,5,6,7,9]. As an effect, the identification of expressive enough decidable fragments of HS has been added to the current research agenda for (interval) temporal logic. While the algebra of Allen's relations, the so-called Allen's Interval Algebra, has been extensively studied and completely classified from the point of view of computational complexity [17] (tractability/intractability of the consistency problem for fragments of Interval Algebra), the characterization of decidable/undecidable fragments of the modal logic of Allen's relations (HS) is considerably harder.

This paper aims at contributing to the identification of the boundary between decidability and undecidability of HS fragments. It summarizes known positive and negative results, it presents the main techniques so far exploited in both directions, and it establishes new undecidability results. Two important parameters of the proposed classification are the set of modalities of the fragment and the class of linearly ordered sets in which it is interpreted. We shall take into consideration the full set of modal operators corresponding to Allen's relations as defined in HS, apart for the trivial one corresponding to equality, plus two definable modalities, namely, those for the *proper during* relation and its inverse *proper contains* (the interval logic of the *proper during* relation has been recently shown to be decidable on dense orders [3]).

The paper is structured as follows. In the next section, we introduce the framework of interval-based temporal logic with unary modalities. In Section 3, we give an up-to-date survey of known decidable fragments. In Section 4, we first summarize known undecidability results and then we provide a number of new

Op.	Semantics
$\langle A\rangle$	$\mathbf{M},[a,b] \Vdash \langle A\rangle\phi \Leftrightarrow \exists c(b < c.\mathbf{M},[b,c] \Vdash \phi)$
$\langle L\rangle$	$\mathbf{M},[a,b] \Vdash \langle L\rangle\phi \Leftrightarrow \exists c,d(b < c < d.\mathbf{M},[c,d] \Vdash \phi)$
$\langle B\rangle$	$\mathbf{M},[a,b] \Vdash \langle B\rangle\phi \Leftrightarrow \exists c(a \leq c < b.\mathbf{M},[a,c] \Vdash \phi)$
$\langle E\rangle$	$\mathbf{M},[a,b] \Vdash \langle E\rangle\phi \Leftrightarrow \exists c(a < c \leq b.\mathbf{M},[c,b] \Vdash \phi)$
$\langle D\rangle$	$\mathbf{M},[a,b] \Vdash \langle D\rangle\phi \Leftrightarrow \exists c,d(a < c \leq d < b.\mathbf{M},[c,d] \Vdash \phi)$
$\langle O\rangle$	$\mathbf{M},[a,b] \Vdash \langle O\rangle\phi \Leftrightarrow \exists c,d(a < c \leq b < d.\mathbf{M},[c,d] \Vdash \phi)$
$\langle D\rangle_{\sqsubset}$	$\mathbf{M},[a,b] \Vdash \langle D\rangle_{\sqsubset}\phi \Leftrightarrow \exists c,d(a \leq c \leq d \leq b.\mathbf{M},[c,d] \Vdash \phi \wedge [c,d] \neq [a,b])$

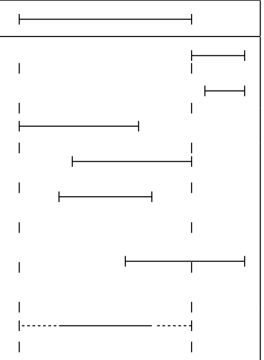

Fig. 1. Formal semantics for some interval operators.

undecidability results for other fragments of HS by reduction from the octant and the $\mathbb{N} \times \mathbb{N}$ tiling problems.

2 Interval Logics over Linearly Ordered Sets

Let $\mathbb{D} = \langle D, <\rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[a, b]$, where $a, b \in D$ and $a \leq b$. Intervals of the type $[a, a]$ are called *point intervals*; if these are excluded, the resulting semantics is called *strict interval semantics* (*non-strict* otherwise). In this paper, we take the more standard non-strict semantics as default. The language of a propositional interval logic consists of a set $\mathcal{AP}$ of propositional letters, any complete set of classical operators (such as $\vee$ and $\neg$), and a set of modal operators $\langle X_1\rangle,\ldots,\langle X_k\rangle$, each of them associated with a specific binary relation over intervals[5]. Formulas are defined by the following grammar:

$$\varphi ::= p \mid \pi \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X_1\rangle\varphi \mid \ldots \mid \langle X_k\rangle\varphi,$$

where π is a modal constant, true precisely at point intervals. We omit π when it is definable in the language or when the strict semantics is adopted.

The semantics of an interval-based temporal logic is given in terms of *interval models* $\mathbf{M} = \langle \mathbb{I}(\mathbb{D}), V\rangle$, where $\mathbb{I}(\mathbb{D})$ is the set of all intervals over $\mathbb{D}$ and the *valuation function* $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$ assigns to every $p \in \mathcal{AP}$ the set of intervals $V(p)$ over which it holds. The *truth of a formula over a given interval* $[a, b]$ in a model $\mathbf{M}$ is defined by structural induction on formulas:

- $\mathbf{M}, [a, b] \Vdash \pi$ iff $a = b$;
- $\mathbf{M}, [a, b] \Vdash p$ iff $[a, b] \in V(p)$, for all $p \in \mathcal{AP}$;
- $\mathbf{M}, [a, b] \Vdash \neg\psi$ iff it is not the case that $\mathbf{M}, [a, b] \Vdash \psi$;

[5] In this paper, we restrict our attention to *unary* modal operators only (decidability issues for *binary* modal operators are addressed in [16]).

- $\mathbf{M}, [a, b] \Vdash \varphi \vee \psi$ iff $\mathbf{M}, [a, b] \Vdash \varphi$ or $\mathbf{M}, [a, b] \Vdash \psi$;
- $\mathbf{M}, [a, b] \Vdash \langle X_i \rangle \psi$ iff there exists an interval $[c, d]$ such that $[a, b]\ R_{X_i}\ [c, d]$, and $\mathbf{M}, [c, d] \Vdash \psi$,

where R_{X_i} is the (binary) interval relation corresponding to the modal operator $\langle X_i \rangle$. In Figure 1 we list the most common unary interval operators and their semantics. Moreover, we denote by $\langle \overline{X} \rangle$ the transpose of each modal operator $\langle X \rangle$, which corresponds to the inverse of the relation R_X. Except for *proper during* and its inverse [3], these are precisely Allen's interval relations [1]. It is easy to show that some of these modal operators are definable in terms of others (some of these definitions do not work with the strict semantics), e.g., $\langle D \rangle p = \langle B \rangle \langle E \rangle p$, $\langle D \rangle_{\sqsubset} p = \langle B \rangle p \vee \langle E \rangle p \vee \langle B \rangle \langle E \rangle p$, $\langle A \rangle p = \langle E \rangle ([B]\bot \wedge \langle \overline{B} \rangle p) \vee ([E]\bot \wedge \langle \overline{B} \rangle p)$, $\langle L \rangle p = \langle A \rangle \langle A \rangle p$, $\langle O \rangle p = \langle E \rangle \langle \overline{B} \rangle p$, and likewise for their transposes. Moreover, the modal constant π is definable in most sufficiently rich languages, viz.:

$$\pi = [B]\bot = [E]\bot = [O]\bot = [\overline{O}]\bot = [D]_{\sqsubset}\bot. \qquad (1)$$

Thus, eventually, all operators corresponding to Allen's interval relations turn out to be definable in terms of $\langle B \rangle$,$\langle E \rangle$, and their transposes (as a matter of fact, $\langle A \rangle$ was included in the original formulation of HS; its definability in terms of the other operators was later shown in [23]).

Here we will consider all HS fragments and for that purpose we will assume all operators listed in Figure 1 (and their transposes) to be primitive in the language. In general, when referring to a specific fragment of HS, we name it by its modal operators. For example, the fragment featuring the operators $\langle B \rangle, \langle E \rangle$ will be denoted by BE.

Besides the usual $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Q}$, we introduce a suitable notation for some common classes of strict linear orders:

- Lin = the class of all linear orders;
- Fin = the class of all **finite** linear orders;
- Den = the class of all **dense** linear orders;
- Dis = the class of all **discrete** linear orders;
- Asc = the class of all linear orders **with an infinite ascending sequence**;
- Des = the class of all linear orders **with an infinite descending sequence**.

3 Decidable Fragments of HS

In this section, we briefly survey the maximal known decidable fragments of HS.

All early decidability results about interval logics were based on severe restrictions of the interval-based semantics, essentially reducing it to a point-based one. Such restrictions include *locality*, according to which all atomic propositions are point-wise and truth over an interval is defined as truth at its initial point, and *homogeneity*, according to which truth of a formula over an interval implies truth of that formula over every sub-interval. By imposing such constraints, decidability of HS can be proved by embedding it into linear temporal logic [21,23].

Decidability can also be achieved by constraining the class of temporal structures over which the logic is interpreted. This is the case with *split-structures*, where any interval can be "chopped" in at most one way. The decidability of various interval logics, including HS, interpreted over split-structures, has been proved by embedding them into first-order decidable theories of time granularity [20].

For some simple fragments of HS, like $\mathsf{B}\overline{\mathsf{B}}$ and $\mathsf{E}\overline{\mathsf{E}}$, decidability has been obtained without any semantic restriction by means of direct translation to the point-based semantics and reduction to decidability of respective point-based temporal logics [14]. In any of these logics, one of the endpoints of every interval related to the current one remains fixed, thereby reducing the interval-based semantics to the point-based one by mapping every interval of the generated sub-model to its non-fixed endpoint. Consequently, these fragments can be polynomially translated to the linear time Temporal Logic with Future and Past TL[F,P], thus proving that they are NP-complete when interpreted on the class of all linearly ordered sets or on any of $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ [12,14].

Decidability results for fragments of HS with unrestricted interval-based semantics, non-reducible to point-based one, have been recently obtained by means of a translation method. This is the case with $\mathsf{A}\overline{\mathsf{A}}$, also known as *Propositional Neighborhood Logic* (PNL) [13][6]. In [6,7], decidability in NEXPTIME of $\mathsf{A}\overline{\mathsf{A}}$ has been proved by translation to the two-variable fragment of first-order logic with binary relations over linear domains $\mathrm{FO}^2[<]$ and reference to the NEXPTIME-complete decidability result for $\mathrm{FO}^2[<]$ by Otto [22] (for proof details and NEXPTIME-hardness, we refer the reader to [6,7]). Otto's results, and consequently the decidability of $\mathsf{A}\overline{\mathsf{A}}$, apply not only to the class of all linear orders, but also to some natural subclasses of it, such as the class of all well-founded linear orders, the class of all finite linear orders, and $\mathbb{N}$.

Finally, decidability of some fragments of HS has been demonstrated by taking advantage of the small model property with respect to suitable classes of satisfiability preserving *pseudo-models*. This method has been successfully applied to the logics of subintervals D and $\mathsf{D}_{\sqsubset}$, interpreted over dense linear orders [3,4,5], and to the logic $\mathsf{A}\overline{\mathsf{A}}$ (resp., A), interpreted over $\mathbb{Z}$ (resp., $\mathbb{N}$) [8,10]. In [3,4,5], Bresolin et al. make use of this technique to develop optimal tableau systems for D and $\mathsf{D}_{\sqsubset}$ that work in PSPACE. (NEXPTIME) tableau-based decision procedures for $\mathsf{A}\overline{\mathsf{A}}$ over $\mathbb{Z}$ and A over $\mathbb{N}$ have been developed in [8,10]. The tableau system for A over $\mathbb{N}$ has been recently generalized to the case of all linearly ordered domains [9].

4 Undecidable Fragments of HS

Undecidable fragments of HS are much more common than decidable ones. In the following, we first summarize some well-known undecidability results, which have been proved by means of a reduction from the non-halting problem for Turing Machines. Then, we recall recent undecidability results for 6 fragments

[6] Since L and $\overline{\mathsf{L}}$ are definable in $\mathsf{A}\overline{\mathsf{A}}$, decidability of this fragment actually implies decidability of $\mathsf{A}\overline{\mathsf{A}}\mathsf{L}\overline{\mathsf{L}}$.

of HS that properly extend $\mathsf{A}\overline{\mathsf{A}}$, namely, $\mathsf{A}\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{E}}$, $\mathsf{A}\overline{\mathsf{A}}\mathsf{E}\overline{\mathsf{B}}$, and $\mathsf{A}\overline{\mathsf{A}}\mathsf{D}^*$, where $\mathsf{D}^* \in \{\mathsf{D}, \overline{\mathsf{D}}, \mathsf{D}_{\sqsubset}, \overline{\mathsf{D}}_{\sqsubset}\}$, interpreted over any class of linear orders containing a linear order with an infinite chain, which have been obtained by means of an encoding from the octant tiling problem [7]. Next, we show that a similar reduction from the octant tiling problem can be exploited to prove the undecidability of other 24 fragments of HS, namely, $\mathsf{A}\mathsf{D}^*\mathsf{E}$, $\mathsf{A}\mathsf{D}^*\overline{\mathsf{E}}$, and $\mathsf{A}\mathsf{D}^*\overline{\mathsf{O}}$ (over any class of linear orders containing a linear order with an infinite ascending chain), $\overline{\mathsf{A}}\mathsf{D}^*\mathsf{B}$, $\overline{\mathsf{A}}\mathsf{D}^*\overline{\mathsf{B}}$, and $\overline{\mathsf{A}}\mathsf{D}^*\mathsf{O}$ (over any class of linear orders containing a linear order with an infinite descending chain). Finally, we take advantage of a reduction from the $\mathbb{N} \times \mathbb{N}$ tiling problem to prove the undecidability of $\mathsf{B}\overline{\mathsf{E}}$, $\overline{\mathsf{B}}\mathsf{E}$, and $\overline{\mathsf{B}}\overline{\mathsf{E}}$ over the appropriate classes of linear orders, thus improving the results for $\mathsf{A}\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{E}}$ and $\mathsf{A}\overline{\mathsf{A}}\mathsf{E}\overline{\mathsf{B}}$ given in [7].

4.1 Reduction from the Non-halting Problem

The undecidability of HS with respect to most classes of linear orders has been proved by means of a reduction from the non-halting problem for Turing Machines [15] (in fact, the reduction is to any of the fragments ABE and $\overline{\mathsf{A}}\mathsf{BE}$).

Theorem 1 (Halpern and Shoham [15]). *The satisfiability problem for* ABE *is undecidable in any class of linear orders that contains at least one linear order with an infinite ascending sequence (in particular, in* $\mathsf{Lin}, \mathsf{Den}, \mathsf{Dis}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, Asc, *and* $\mathbb{N}$*). Similarly, the satisfiability problem for* $\overline{\mathsf{A}}\mathsf{BE}$ *is undecidable in each of the classes* $\mathsf{Lin}, \mathsf{Den}, \mathsf{Dis}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, Des, *and* $\mathbb{Z}^-$.

The undecidability of the satisfiability problem for HS in all the classes Theorem 1 refers to immediately follows.

In [18], Lodaya shows that a suitable sharpening of the reduction technique from [15] can be exploited to prove the undecidability of the fragment BE over dense linear orders (thus strengthening Halpern and Shoham's result in this restricted setting). As a preliminary result, he proves that the logic with the binary *chop* operator C, that splits an interval in two parts (and is not definable in HS), and the modal constant π is undecidable by means of an adaptation of the proof for HS. Then, he shows that the operators $\langle B \rangle$ and $\langle E \rangle$, which can be easily defined in terms of C and π, suffice for undecidability. In [14] it was observed that this result actually applies to the class of all linear orders.

Theorem 2 (Lodaya, Goranko et al. [14,18]). *The satisfiability problem for* BE *is undecidable in the classes* Lin *and* Den.

4.2 Reduction from the Octant Tiling Problem

The undecidability of a number of HS fragments has been proved by using variations of a reduction from the *unbounded tiling problem* for the second octant $\mathcal{O}$ of the integer plane. This is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile $\mathcal{O} = \{(i, j) : i, j \in \mathbb{N} \wedge 0 \leq i \leq j\}$. This problem can be shown to be undecidable by a simple application of the König's Lemma in the same way as it was used in [2] to show the undecidability of the $\mathbb{N} \times \mathbb{N}$ tiling problem from that of $\mathbb{Z} \times \mathbb{Z}$ one. For every tile type $t_i \in \mathcal{T}$, let $right(t_i)$, $left(t_i)$, $up(t_i)$, and $down(t_i)$ be the colors of the corresponding sides of t_i. To solve the problem, one must find a function $f : \mathcal{O} \to \mathcal{T}$ such that

$$right(f(n, m)) = left(f(n + 1, m))$$

and

$$up(f(n, m)) = down(f(n, m + 1)).$$

In [7], a reduction from the unbounded tiling problem for the second octant $\mathcal{O}$ of the integer plane has been applied to prove the undecidability of the extensions of $\mathsf{A}\overline{\mathsf{A}}$ with any of the operators $\langle D\rangle$, $\langle \overline{D}\rangle$, $\langle D\rangle_{\sqsubset}$, and $\langle \overline{D}\rangle_{\sqsubset}$, or with the pairs of operators $\langle B\rangle\langle \overline{E}\rangle$ or $\langle \overline{B}\rangle\langle E\rangle$, interpreted in any class of linear orders containing a linear order with an infinite (ascending or descending) chain.

Theorem 3 (Bresolin et al. [7]). *The satisfiability problem for each of the fragments* $\mathsf{A}\overline{\mathsf{A}}\mathsf{D}^*$, $\mathsf{A}\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{E}}$, *and* $\mathsf{A}\overline{\mathsf{A}}\mathsf{E}\overline{\mathsf{B}}$ *is undecidable in each of the classes* Lin, $\mathsf{Den}, \mathsf{Dis}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, Des, Asc, $\mathbb{N}$, *and* $\mathbb{Z}^-$.

In the following, we will show that similar reductions can be exploited to prove the undecidability of other meaningful fragments of HS.

Theorem 4. *The satisfiability problem for each of the fragments* $\mathsf{AD}^*\mathsf{E}$, $\mathsf{AD}^*\overline{\mathsf{E}}$, *and* $\mathsf{AD}^*\overline{\mathsf{O}}$ *is undecidable in any class of linear orders containing a linear order with an infinite ascending chain. Likewise, the satisfiability problem for the fragments* $\overline{\mathsf{A}}\mathsf{D}^*\mathsf{B}$, $\overline{\mathsf{A}}\mathsf{D}^*\overline{\mathsf{B}}$, *and* $\overline{\mathsf{A}}\mathsf{D}^*\mathsf{O}$ *is undecidable in any class of linear orders containing a linear order with an infinite descending chain.*

We give the details of the proof for the case ADE; the other cases are quite similar. We consider a signature containing, inter alia, the special propositional letters u, tile, Id, $\mathsf{t_1}, \ldots, \mathsf{t_k}$, bb, be, eb, and corr.

Unit-intervals. We set our framework by forcing the existence of a unique infinite chain of so-called *unit-intervals* (for short, u-*intervals*) on the linear order, which covers an initial segment of the model. These u-intervals will be labeled by the propositional variable u. They will be used as cells to arrange the tiling. First of all, we define an *always in the future* modality which captures future intervals only:

$$[G]p = p \wedge [A]p \wedge [A][A]p.$$

Then, u-*intervals* can be encoded as follows:

$$B_1 = \neg\mathsf{u} \wedge \langle A\rangle\mathsf{u} \wedge [G](\mathsf{u} \to (\neg\pi \wedge \langle A\rangle\mathsf{u} \wedge \neg\langle D\rangle\mathsf{u} \wedge \neg\langle D\rangle\langle A\rangle\mathsf{u})),$$
$$B_2 = [G] \bigwedge_{p \in \mathcal{AP}}((p \vee \langle A\rangle p) \to \langle A\rangle\mathsf{u}).$$

Formula B_2 restricts our domain of 'legitimate intervals' to those composed of u-intervals, while B_1 guarantees the existence of an infinite sequence of consecutive u-intervals, thus implying the following lemma.

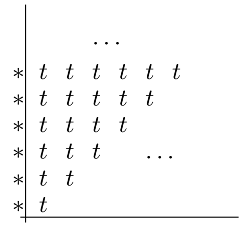

Fig. 2. A schema of the encoding (we abbreviate `tile` as t).

Lemma 1. *Suppose that* $\mathbf{M}, [a, b] \Vdash B_1$*. Then, there exists an infinite sequence of points* $b_0 < b_1 < \ldots$ *in* $\mathbf{M}$*, such that* $b_0 = b$*, for each* i*,* $\mathbf{M}, [b_i, b_{i+1}] \Vdash \mathsf{u}$*, and no other interval* $[c, d]$*, with* $c \neq d$*, in* $\mathbf{M}$ *satisfies* u*, unless* $c > b_i$ *for every* $i \in \mathbb{N}$*, or* $c < b$*.*

Encoding a tile. Every u-interval will represent either a tile or a special marker, denoted by $*$, that identifies the border between two ld-intervals (ld-intervals represent the rows of the tiling and will be defined later). Formally, we put:

$$B_3 = [G](\mathsf{u} \leftrightarrow (* \vee \mathsf{tile})) \wedge [G](* \rightarrow \neg\mathsf{tile}) \wedge [G]\neg(* \wedge \langle A\rangle *),$$
$$B_4 = [G](\mathsf{tile} \leftrightarrow (\bigvee_{i=1}^{k} \mathtt{t_i} \wedge \bigwedge_{i,j=1, i\neq j}^{k} \neg(\mathtt{t_i} \wedge \mathtt{t_j}))).$$

If a tile is placed on a u-interval $[a, b]$, we call a and b respectively the *beginning point* and the *ending point* of that tile.

Encoding rows of the tiling. An ld-interval (or just ld) is an interval consisting of a finite sequence of at least two u-subintervals. Each ld represents a row (level) of the tiling of $\mathcal{O}$. The first u-subinterval in an ld is a $*$-interval and every following u-subinterval is the encoding of a tile (see Figure 2). The ld-intervals representing the bottom-up consecutive levels of the tiling of $\mathcal{O}$ are arranged one after another in a chain. The first ld is composed by a single tile. To prevent the existence of interleaving sequences of ld-intervals, we do not allow occurrences of $*$-subintervals inside an ld. These conditions are imposed by the following formulas:

$$B_5 = [G]((\mathsf{ld} \rightarrow (\neg\mathsf{u} \wedge \langle A\rangle\mathsf{ld} \wedge \neg\langle D\rangle\langle A\rangle\mathsf{ld}))) \wedge [G]((\langle A\rangle\mathsf{ld} \leftrightarrow \langle A\rangle *),$$
$$B_6 = \langle A\rangle(* \wedge \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle *)),$$
$$B_7 = B_1 \wedge B_2 \wedge B_3 \wedge B_4 \wedge B_5 \wedge B_6.$$

Lemma 2. *Let* $\mathbf{M}, [a, b] \Vdash B_7$*. Then, there is a sequence of points* $b = b_1^0 < b_1^1 < \ldots b_1^{k_1} = b_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$*, such that* $k_1 = 2$ *and for every* j*:*

1. $\mathbf{M}, [b_j^0, b_j^{k_j}] \Vdash \mathsf{ld}$ *and no other interval* $[c, d]$*, with* $c \neq d$*, in* $\mathbf{M}$ *is an* ld*-interval, unless possibly for* $c > b_j^{k_j}$ *for every* $j \in \mathbb{N}$*, or* $c < b$*;*
2. $\mathbf{M}, [b_j^0, b_j^1] \Vdash *$ *and no other interval* $[c, d]$*, with* $c \neq d$*, in* $\mathbf{M}$ *is a* $*$*-interval, unless possibly for* $c > b_j^{k_j}$ *for every* $j \in \mathbb{N}$*, or* $c < b$*;*

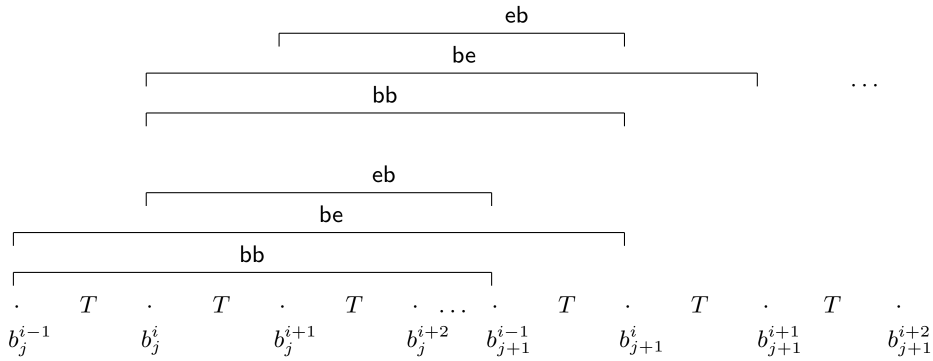

Fig. 3. A representation of bb, be, and eb-intervals.

3. *for every i such that $0 < i < k_j$, $\mathbf{M}, [b_j^i, b_j^{i+1}] \Vdash$ tile, and no other interval $[c, d]$, with $c \neq d$, in $\mathbf{M}$ is a tile-interval, unless possibly for $c > b_j^{k_j}$ for every $j \in \mathbb{N}$, or $c < b$.*

Definition 1. *Let $\mathbf{M}, [a, b] \Vdash B_7$ and $b_1^0 < b_1^1 < \ldots b_1^{k_1} = b_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 \ldots$ be the sequence of points whose existence is guaranteed by Lemma 2. For any j, the interval $[b_j^0, b_j^{k_j}]$ is the* j-th Id-interval of the sequence *and, for any $i \geq 1$, the interval $[b_j^i, b_j^{i+1}]$ is the* i-th tile of the Id-interval $[b_j^0, b_j^{k_j}]$.

Corresponding tiles. So far we have that, given a starting interval, the formula B_7 forces the underlying linearly ordered set to be, in the future of the current interval, a sequence of Id's, the first one of which containing exactly one tile. Now, we want to make sure that each tile at a certain *level* in $\mathcal{O}$ (i.e., Id) always has its corresponding tile at the immediate upper level. To this end, we will take advantage of some auxiliary propositional variables, namely, bb, which is to connect the beginning point of a tile to the beginning point of the corresponding tile above, be, which is to connect the beginning point of a tile to the ending point of the corresponding tile above, and eb, which is to connect the ending point of a tile to the beginning point of the corresponding tile above. If an interval is labeled with any of bb, eb, or be, we call it a *corresponding interval*, abbreviated corr-*interval*. A pictorial representation is given in Figure 3. The next formulas force corr-intervals to respect suitable properties so that all models satisfying them encode a correct tiling.

$B_8 = \ [G]((\mathsf{bb} \vee \mathsf{be} \vee \mathsf{eb}) \leftrightarrow \mathsf{corr}),$
$B_9 = \ [G]\neg(\mathsf{corr} \wedge \mathsf{Id}),$
$B_{10} = [G]((\mathsf{corr} \rightarrow \neg\langle D\rangle \mathsf{Id}) \wedge (\mathsf{Id} \rightarrow \neg\langle D\rangle \mathsf{corr})),$
$B_{11} = [G]((\mathsf{corr} \rightarrow \neg\langle A\rangle \mathsf{Id}) \wedge (\langle A\rangle(\mathsf{bb} \vee \mathsf{be}) \rightarrow \neg\langle A\rangle \mathsf{Id})),$
$B_{12} = B_8 \wedge B_9 \wedge B_{10} \wedge B_{11}.$

Lemma 3. *Let $\mathbf{M}, [a, b] \Vdash B_7 \wedge B_{12}$. Then, no* Id*-interval in* $\mathbf{M}$ *coincides with a* corr*-interval, nor is properly contained in a* corr*-interval, nor a* corr*-interval is properly contained in an* Id*-interval, unless it is an* eb*-interval beginning an* Id.

The next set of formulas guarantees that the corr-intervals satisfy the respective correspondences.

$B_{13} = [G](\langle A\rangle \mathsf{tile} \leftrightarrow \langle A\rangle \mathsf{bb})$,
$B_{14} = [A](\langle A\rangle(\mathsf{tile} \wedge \langle A\rangle \mathsf{tile}) \leftrightarrow \langle E\rangle \mathsf{bb})$,
$B_{15} = [G](\langle A\rangle \mathsf{tile} \leftrightarrow \langle A\rangle \mathsf{be})$,
$B_{16} = [A]((\langle E\rangle \mathsf{tile} \wedge \langle A\rangle \mathsf{tile}) \leftrightarrow \langle E\rangle \mathsf{be})$,
$B_{17} = [G](\mathsf{u} \rightarrow (\mathsf{tile} \leftrightarrow \langle A\rangle \mathsf{eb}))$,
$B_{18} = [A](\langle A\rangle(\mathsf{tile} \wedge \langle A\rangle \mathsf{tile}) \leftrightarrow \langle E\rangle \mathsf{eb})$,
$B_{19} = B_{13} \wedge B_{14} \wedge B_{15} \wedge B_{16} \wedge B_{17} \wedge B_{18}$.

Lemma 4. *Let* $\mathbf{M}, [a,b] \Vdash B_7 \wedge B_{12} \wedge B_{19}$ *and let* $b_1^0 < b_1^1 < b_1^2 = b_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$ *be the sequence of points whose existence is guaranteed by Lemma 2. Then, for every* $i \geq 0, j \geq 1$*:*

1. b_j^i *is the beginning point of a* bb *and a* be *iff* $1 \leq i \leq k_j - 1$.
2. b_j^i *is the beginning point of an* eb *iff* $2 \leq i \leq k_j$.
3. b_j^i *is the ending point of a* bb *and an* eb *iff* $1 \leq i \leq k_j - 2$.
4. b_j^i *is the ending point of a* be *iff* $2 \leq i \leq k_j - 1$.

Definition 2. *Given two* tile*-intervals* $[c,d]$ *and* $[e,f]$ *in a model* $\mathbf{M}$*, we say that* $[c,d]$ corresponds to $[e,f]$ *if* $\mathbf{M}, [c,e] \Vdash \mathsf{bb}$ *and* $\mathbf{M}, [c,f] \Vdash \mathsf{be}$ *and* $\mathbf{M}, [d,e] \Vdash \mathsf{eb}$.

The following formulas state the basic relationships between the three types of correspondence:

$B_{20} = [G] \bigwedge_{c,c' \in \{\mathsf{bb},\mathsf{eb},\mathsf{be}\}, c \neq c'} \neg(c \wedge c')$,
$B_{21} = [G](\mathsf{bb} \rightarrow \neg\langle D\rangle \mathsf{bb} \wedge \neg\langle D\rangle \mathsf{eb} \wedge \neg\langle D\rangle \mathsf{be})$,
$B_{22} = [G](\mathsf{eb} \rightarrow \neg\langle D\rangle \mathsf{bb} \wedge \neg\langle D\rangle \mathsf{eb} \wedge \neg\langle D\rangle \mathsf{be})$,
$B_{23} = [G](\mathsf{be} \rightarrow \langle D\rangle \mathsf{eb} \wedge \neg\langle D\rangle \mathsf{bb} \wedge \neg\langle D\rangle \mathsf{be})$,
$B_{24} = B_{20} \wedge B_{21} \wedge B_{22} \wedge B_{23}$.

Lemma 5. *Let* $\mathbf{M}, [a,b] \Vdash B_7 \wedge B_{12} \wedge B_{19} \wedge B_{24}$*. Then, for any* $j \geq 1$ *and* $i \geq 1$*:*

1. *the* i*-th tile of the* j*-th* ld*-interval corresponds to the* i*-th tile of the* $j+1$*-th* ld*-interval.*
2. *there are exactly* j *tiles in the* j*-th* ld*-interval.*
3. *no tile of the* j*-th* ld*-interval corresponds to the last tile of the* $j+1$*-th* ld*-interval.*

Encoding the tiling problem. We are now ready to show how to encode the octant tiling problem. Let $\phi_{\mathcal{T}}$ be the conjunction of B_7, B_{12}, B_{19}, B_{24}, B_{25}, and B_{26}, where B_{25} and B_{26} are the following two formulas:

$B_{25} = [G]((\mathsf{tile} \wedge \langle A\rangle \mathsf{tile}) \rightarrow \bigvee_{right(t_i)=left(t_j)} (\mathtt{t_i} \wedge \langle A\rangle \mathtt{t_j}))$,
$B_{26} = [G](\langle A\rangle \mathsf{tile} \rightarrow \bigvee_{up(t_i)=down(t_j)} (\langle A\rangle \mathtt{t_i} \wedge \langle A\rangle(\mathsf{bb} \wedge \langle A\rangle \mathtt{t_j})))$.

Lemma 6. *Given any finite set of tiles* $\mathcal{T}$*, the formula* $\Phi_{\mathcal{T}}$ *is satisfiable if and only if* $\mathcal{T}$ *can tile the second octant* $\mathcal{O}$*.*

As the model construction in the above proof can be carried out on any linear ordering containing an infinite ascending chain of points, Theorem 4 for the logic ADE immediately follows.

As for the other logics considered in the first half of Theorem 4, it suffices to modify the formulas involving $\langle D\rangle$ (see [7]) and the formulas B_{14}, B_{16}, and B_{18}, which involve $\langle E\rangle$. As an example, in the case of the logic $\mathsf{AD\overline{O}}$, formulas B_{14}, B_{16}, and B_{18} must be replaced with the following ones:

$$B'_{14} = [G](\langle A\rangle(\mathsf{tile} \wedge \langle A\rangle\mathsf{tile}) \leftrightarrow \langle A\rangle(\mathsf{tile} \wedge \langle \overline{O}\rangle\mathsf{bb})),$$
$$B'_{16} = [G](\langle A\rangle(\mathsf{tile} \wedge \langle A\rangle\mathsf{tile}) \leftrightarrow \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle\langle \overline{O}\rangle\mathsf{be})),$$
$$B'_{18} = [G](\langle A\rangle(\mathsf{tile} \wedge \langle A\rangle\mathsf{tile}) \leftrightarrow \langle A\rangle(\mathsf{tile} \wedge \langle \overline{O}\rangle\mathsf{eb})).$$

In the cases of the fragments where A is replaced with $\overline{\mathsf{A}}$ and E (resp., $\overline{\mathsf{E}}$) is replaced with B (resp., $\overline{\mathsf{B}}$), the proof is perfectly symmetric and it takes advantage of the existence of an infinite descending sequence.

4.3 Reduction from the $\mathbb{N} \times \mathbb{N}$ Tiling Problem

In this section, we strengthen some of the results of Theorem 3 by showing that the satisfiability problem for the fragments $\overline{\mathsf{B}}\mathsf{E}$, $\mathsf{B}\overline{\mathsf{E}}$, and $\overline{\mathsf{B}}\overline{\mathsf{E}}$ is undecidable (the case of BE was already dealt with by Theorem 2 for the classes Lin and Den). The proof is based on a reduction from the $\mathbb{N} \times \mathbb{N}$ tiling problem, which is a non-trivial adaptation of the reduction from the same problem provided by Marx and Reynolds to prove the undecidability of Compass Logic [19].

Theorem 5. *The satisfiability problem for* $\overline{\mathsf{B}}\mathsf{E}$ *(respectively,* $\mathsf{B}\overline{\mathsf{E}}$*) is undecidable in any class of linear orders that contains a linear order with an infinite ascending (respectively, descending) chain. The satisfiability problem for* $\overline{\mathsf{B}}\overline{\mathsf{E}}$ *is undecidable in any class of linear orders that contains a linear order with an infinite chain indexed by the integers.*

The encoding of the quadrant $\mathbb{N} \times \mathbb{N}$ is close to that given in [19] (it is based on a suitable enumeration of its elements). From such a work, we also borrow the set of propositional variables p, q, $\mathsf{right}, \mathsf{left}, \mathsf{above}, \mathsf{floor}$, and wall used in the proof.

Hereafter, we restrict ourselves to the easiest case of $\overline{\mathsf{B}}\mathsf{E}$ (however, the proof can be adapted to the other two fragments). The operators of $\overline{\mathsf{B}}\mathsf{E}$ can be naturally mapped into those of Compass Logic as follows: if $\mathbf{M}, [a, b] \Vdash \langle\overline{B}\rangle\psi$, then $\mathbf{M}, [a, c] \Vdash \psi$ for some $c > b$ and thus $\langle\overline{B}\rangle$ corresponds to $\diamondsuit$ in Compass Logic, and if $\mathbf{M}, [a, b] \Vdash \langle E\rangle\psi$, then $\mathbf{M}, [c, b] \Vdash \psi$ for some $a < c \leq b$ and thus $\langle E\rangle$ corresponds to $\diamondsuit$.

First, we define the *always in the future* operator $[G]$:

$$[G]\varphi = \varphi \wedge [E]\varphi \wedge [\overline{B}](\varphi \wedge [E]\varphi).$$

The properties of p and q, that respectively encode the elements of the quadrant and the successor relation over them (with respect to the given enumeration), are expressed by the following formulas:

$N_1 = \mathsf{p}$,
$N_2 = [G](\mathsf{p} \rightarrow [\overline{B}]\neg\mathsf{p})$,
$N_3 = [G](\mathsf{p} \rightarrow [E]\neg\mathsf{p})$,
$N_4 = [G](\langle\overline{B}\rangle\mathsf{p} \rightarrow [E]\neg\mathsf{p})$,
$N_5 = [G](\mathsf{p} \rightarrow [E]([\overline{B}]\neg\mathsf{p})$,
$N_6 = [G](\mathsf{q} \rightarrow [\overline{B}]\neg\mathsf{q})$,
$N_7 = [G](\mathsf{p} \rightarrow \langle\overline{B}\rangle\mathsf{q})$,
$N_8 = [G](\mathsf{q} \rightarrow \langle E\rangle\mathsf{p})$,
$N_9 = [G](\langle\overline{B}\rangle\mathsf{q} \rightarrow [E]\neg\mathsf{p})$.

As an immediate consequence from N_1-N_9, we have:

$$N_{10} = \ [G](\mathsf{q} \rightarrow [\overline{B}]\neg\mathsf{p}).$$

The above formulas state that both p and q are injective functions, that is, if $\mathbf{M}, [a, b] \Vdash \mathsf{p}$, then for each $c \neq b$ $\mathbf{M}, [a, c] \Vdash \neg\mathsf{p}$ and for each $d \neq a$ $\mathbf{M}, [d, b] \Vdash \neg\mathsf{p}$, and similarly for q, that p-intervals cannot be subintervals of p-intervals (and they do not overlap), that q and p have the same domain and range, that is, $\mathbf{M}, [a, b] \Vdash \mathsf{p}$ if and only if there exists $c > b$ such that $\mathbf{M}, [a, c] \Vdash \mathsf{q}$ and $\mathbf{M}, [a, b] \Vdash \mathsf{p}$ if and only if there exists $c < a$ such that $\mathbf{M}, [c, b] \Vdash \mathsf{q}$, and, finally, that a p-interval cannot be a subinterval of a q-interval.

Lemma 7. *For every model* $\mathbf{M}$ *and every interval* $[a, b]$ *such that* $\mathbf{M}, [a, b] \Vdash N_1 \wedge \ldots \wedge N_9$ *there exists a sequence of intervals* $[a, b] = [a_0, b_0], [a_1, b_1], \ldots$ *such that, for every* $n \geq 0$*: (1)* $b_n \leq a_{n+1}$*; (2)* $\mathbf{M}, [a_n, b_n] \Vdash \mathsf{p}$*; (3)* $\mathbf{M}, [a_n, b_{n+1}] \Vdash \mathsf{q}$*; (4) if* $\mathbf{M}, [a', b'] \Vdash \mathsf{p}$ *and* $b_0 \leq b' < b_n$*, then there exists* $m < n$ *such that* $[a', b'] = [a_m, b_m]$*.*

Lemma 7 corresponds to Claim 5.2, Section 5.4 in [19]. To prove Claim 5.3, we translate formulas A_6-A_{18} in [19] to the language $\overline{\mathrm{B}}\mathrm{E}$. For a given formula φ, let $F(\varphi)$ be the conjunction of the following formulas:

$[G](\varphi \rightarrow [\overline{B}]\neg\varphi)$,
$[G](\varphi \rightarrow [E]\neg\varphi)$,
$[G](\mathsf{p} \rightarrow \langle\overline{B}\rangle\varphi)$,
$[G](\varphi \rightarrow \langle E\rangle\mathsf{p})$,
$[G](\mathsf{q} \rightarrow [E](\langle\overline{B}\rangle\varphi \rightarrow \mathsf{p})$.

The above formulas state that φ is an injective function, that the domain of p is included in the domain of φ, that the range of φ is included in the range of p, and that the domain of φ is included in the domain of p (that is, the domain of p and that of φ coincide).

Formulas A_6-A_{18} can be encoded as follows:

$$\begin{aligned}
A_6 &= F(\mathsf{right}),\\
A_7 &= F(\mathsf{above}),\\
A_8 &= [G](\langle\overline{B}\rangle\mathsf{right} \rightarrow [E]\neg\mathsf{right}),\\
A_9 &= [G](\mathsf{right} \rightarrow \langle\overline{B}\rangle\mathsf{above}),\\
A_{10} &= [G](\mathsf{right} \rightarrow [\overline{B}](\langle\overline{B}\rangle\mathsf{above} \rightarrow [E]\neg\mathsf{p})),
\end{aligned}$$

which impose that both right and above are total injective functions from p-intervals to p-intervals, that right is strictly monotone, and that above is the composition of right and q, and:

$$\begin{aligned}
A_{11} &= \mathsf{floor} \wedge \mathsf{wall},\\
A_{12} &= [\overline{B}]\neg(\mathsf{floor} \wedge \mathsf{wall}) \wedge [E]\neg(\mathsf{floor} \wedge \mathsf{wall}) \wedge [\overline{B}][E]\neg(\mathsf{floor} \wedge \mathsf{wall}),\\
A_{13} &= [G]((\mathsf{floor} \vee \mathsf{wall}) \rightarrow \mathsf{p}),\\
A_{14} &= [G](\mathsf{wall} \rightarrow [\overline{B}](\mathsf{q} \rightarrow [E](\mathsf{p} \rightarrow \mathsf{floor}))),\\
A_{15} &= [G](\mathsf{wall} \rightarrow \langle\overline{B}\rangle(\mathsf{above} \wedge \langle E\rangle\mathsf{wall})),\\
A_{16} &= [G]((\mathsf{p} \wedge \neg\mathsf{wall}) \rightarrow [\overline{B}](\mathsf{above} \rightarrow [E]\neg\mathsf{wall})),\\
A_{17} &= [G](\mathsf{right} \rightarrow [E]\neg\mathsf{wall}),\\
A_{18} &= [\overline{B}](\langle E\rangle(\mathsf{p} \wedge \neg\mathsf{wall}) \rightarrow \mathsf{right} \vee \langle E\rangle(\mathsf{right} \wedge \langle E\rangle(\mathsf{p} \wedge \neg\mathsf{wall}))),
\end{aligned}$$

which state the properties of floor and wall. Intuitively, we have the following properties: the initial interval is labeled with floor and wall and this is not the case with any other interval; both floor and wall are p-intervals; the successor of a wall is a floor; above every wall there is a wall, and, with the exception of the initial interval, every wall is above a wall; right never goes to the wall, and every non-wall p-interval has a p-interval on the left.

Finally, let $\phi_{\mathcal{T}}$ be the conjunction of formulas N_1-N_9, A_6-A_{18}, and A_{19}-A_{22} below:

$$\begin{aligned}
A_{19} &= [G](\mathsf{p} \leftrightarrow \textstyle\bigvee_{i=1}^{k} \mathtt{t_i}),\\
A_{20} &= [G] \textstyle\bigwedge_{i \neq j} \neg(\mathtt{t_i} \wedge \mathtt{t_j}),\\
A_{21} &= \textstyle\bigwedge_{up(t_i) \neq down(t_j)} [G]\neg(\mathtt{t_i} \wedge \langle\overline{B}\rangle(\mathsf{above} \wedge \langle E\rangle\mathtt{t_j})),\\
A_{22} &= \textstyle\bigwedge_{right(t_i) \neq left(t_j)} [G]\neg(\mathtt{t_i} \wedge \langle\overline{B}\rangle(\mathsf{right} \wedge \langle E\rangle\mathtt{t_j})).
\end{aligned}$$

The proof of the next lemma repeats, *mutatis mutandis*, the one in [19].

Lemma 8. *A set of tiles $\mathcal{T}$ can tile $\mathbb{N} \times \mathbb{N}$ if and only if $\phi_{\mathcal{T}}$ is satisfiable.*

This concludes the proof of Theorem 5 for the case $\overline{\mathsf{B}}\mathsf{E}$. A similar construction can be carried out for the logics $\overline{\mathsf{BE}}$ and $\mathsf{B}\overline{\mathsf{E}}$. As for $\overline{\mathsf{BE}}$, it suffices to replace the first quadrant with the second one, where the operator $\langle\overline{B}\rangle$ corresponds to the operator $\diamondsuit$ and the operator $\langle\overline{E}\rangle$ corresponds to the operator $\diamondsuit$ of Compass Logic. As for $\mathsf{B}\overline{\mathsf{E}}$, the construction of the model is obtained in the third quadrant instead of the second one.

5 Concluding Remarks

In this paper, we have taken into consideration the variety of HS fragments that can be obtained by choosing suitable subsets of the set of the twelve basic modal operators (corresponding to Allen's relations) extended with two additional operators for subintervals. We have focused our attention on the problem of classifying them with respect to decidability/undecidability (first raised by Halpern and Shoham in [15], Problem 3). Besides a summary of the state of the art, we have given a number of new undecidability results based on suitable reductions from tiling problems.

The proposed classification is naturally related to definability/undefinability relations among operators. Known definability relations reduce the number of fragments from over 16 thousands to less than 5 thousands, and the results reported in this paper cover more than half of these cases. Our study not only makes a substantial contribution to the complete solution of the classification problem inherited from [15], but it also suggests some directions to explore in the search of other decidable interval logics.

It is worth pointing out that all undecidability results reported here hinge on the existence of an infinite ascending/descending chain of intervals. Decidability problems for interval logics over finite interval structures are still largely unexplored. Some positive results for PNL can be found in [6,7,8,9,10].

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