A complete classification of the expressiveness of interval logics of Allen's relations over dense linear orders*

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1 Introduction

Interval reasoning (where time intervals, rather than time instants, are the primitive ontological entities) naturally arises in various fields of computer science and AI, ranging from hardware and real-time system verification to natural language processing, from constraint satisfaction to planning [2, 3, 6, 9–11]. The variety of binary relations between intervals in a linear order was first studied by Allen [2]. In [7], Halpern and Shoham introduced and systematically analyzed the (full) logic of Allen's relations, called HS, that features one modality for each Allen's relation. The undecidability of HS over most classes of linear orders motivated the search for (syntactic) HS fragments offering a good balance between expressiveness and decidability/complexity. A comparative analysis of the expressive power of the variety of HS fragments naturally sets the scene for such a search. This analysis is far from being trivial, because some HS modalities are definable in terms of others and such inter-definabilities may depend on the class of linear orders in which the logic is interpreted. Many classes of linear orders are of practical interest, including the class of all linear orders and the class of all dense (resp., discrete, finite) linear orders. In [5], Della Monica et al. gave a complete characterization of all expressively different subsets of HS modalities over all linear orders. Unfortunately, such a classification cannot be easily transferred to any other class of linear orders (proving a specific undefinability result amounts to providing a counterexample based on concrete linear orders belonging to the considered class). In this paper, we give a complete classification of the expressiveness of HS fragments over all *dense* linear orders. Undefinability results are essentially based on counterexamples referring to the linear order of $\mathbb{R}$. However, the proposed constructions can be modified to deal with specific sub-classes of the class of all dense linear orders, e.g., the linear order of $\mathbb{Q}$. As a final result, we show that there are exactly 966 expressively different HS fragments over (all) dense linear orders (over all linear orders, they are 1347), out of 4096 distinct subsets of HS modalities.

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HS modalities	Allen's relations
$\langle A \rangle$	$[a,b]R_A[c,d] \Leftrightarrow b = c$
$\langle L \rangle$	$[a,b]R_L[c,d] \Leftrightarrow b < c$
$\langle B \rangle$	$[a,b]R_B[c,d] \Leftrightarrow a = c, d < b$
$\langle E \rangle$	$[a,b]R_E[c,d] \Leftrightarrow b = d, a < c$
$\langle D \rangle$	$[a,b]R_D[c,d] \Leftrightarrow a < c, d < b$
$\langle O \rangle$	$[a,b]R_O[c,d] \Leftrightarrow a < c < b < d$

Graphical representation

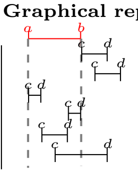


Fig. 1. Allen's interval relations and the corresponding HS modalities.

2 Preliminaries

Let $\mathbb{D} = \langle D, < \rangle$ be a linear order. An *interval* over $\mathbb{D}$ is an ordered pair $[a, b]$, where $a, b \in D$ and $a \leq b$. An interval is called a *point* (resp., *strict*) *interval* if $a = b$ (resp., $a < b$). In this paper, we restrict ourselves to strict intervals. If we exclude equality, there are 12 different relations between two strict intervals in a linear order, often called *Allen's relations* [2]: the six relations R_A, R_L, R_B, R_E, R_D, and R_O depicted in Figure 1 and the inverse ones, that is, $R_{\overline{X}} = (R_X)^{-1}$, for each $X \in \{A, L, B, E, D, O\}$. We treat interval structures as Kripke structures and Allen's relations as accessibility relations over them, thus associating a modality $\langle X \rangle$ with each Allen's relation R_X. For each $X \in \{A, L, B, E, D, O\}$, the *transpose* of modality $\langle X \rangle$ is modality $\langle \overline{X} \rangle$, corresponding to the inverse relation $R_{\overline{X}}$ of R_X.

HS is a multi-modal logic with formulae built from a finite, non-empty set $\mathcal{AP}$ of atomic propositions, the propositional connectives $\vee$ and $\neg$, and a modality for each Allen's relation [7] . With every subset $\{R_{X_1}, \ldots, R_{X_k}\}$ of these relations, we associate the fragment $\mathsf{X_1 X_2 \ldots X_k}$ of HS, whose formulae are defined by the grammar: $\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X_1 \rangle \varphi \mid \ldots \mid \langle X_k \rangle \varphi$, where $p \in \mathcal{AP}$. The other propositional connectives and constants (e.g., $\wedge$, $\rightarrow$, and $\top$) can be derived in the standard way, as well as the dual modalities (e.g., $[A]\varphi \equiv \neg\langle A \rangle \neg\varphi$).

For a fragment $\mathcal{F} = \mathsf{X_1 X_2 \ldots X_k}$ and a modality $\langle X \rangle$, we write $\langle X \rangle \in \mathcal{F}$ if $X \in \{X_1, \ldots, X_k\}$. Given two fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if $\langle X \rangle \in \mathcal{F}_1$ implies $\langle X \rangle \in \mathcal{F}_2$, for every modality $\langle X \rangle$. Finally, for a fragment $\mathcal{F} = \mathsf{X_1 X_2 \ldots X_k}$ and a formula φ, we write $\varphi \in \mathcal{F}$, or, equivalently, we say that φ is an $\mathcal{F}$-formula, meaning that φ belongs to the language of $\mathcal{F}$.

The (strict) semantics of HS is given in terms of *interval models* $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$, where $\mathbb{D}$ is a linear order, $\mathbb{I}(\mathbb{D})$ is the set of all (strict) intervals over $\mathbb{D}$, and V is a *valuation function* $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$, which assigns to every atomic proposition $p \in \mathcal{AP}$ the set of intervals $V(p)$ on which p holds. The *truth* of a formula on a given interval $[a, b]$ in an interval model M is defined as standard for the Boolean operators and propositions. The semantics of the modalities are defined as follows:

- $M, [a, b] \Vdash \langle X \rangle \psi$ iff there exists an interval $[c, d]$ such that $[a, b] R_X [c, d]$ and $M, [c, d] \Vdash \psi$, for each modality $\langle X \rangle$.

For every $p \in \mathcal{AP}$ and $[a, b] \in \mathbb{I}(\mathbb{D})$, we say that $[a, b]$ is a *p-interval* if $M, [a, b] \Vdash p$. By $M, [a, b] \nVdash \psi$, we mean that it is not the case that $M, [a, b] \Vdash \psi$. Formulae of HS can be interpreted in several interesting classes of interval models over linear orders (in short, classes of linear orders) such as the classes of all, dense, and discrete linear orders.

The following definition formalizes the notion of definability of modalities in terms of others.

Definition 1 (Inter-definability). *A modality $\langle X \rangle$ of* HS *is* definable *in an* HS *fragment $\mathcal{F}$ relative to a class $\mathcal{C}$ of linear orders, denoted $\langle X \rangle \lhd_{\mathcal{C}} \mathcal{F}$, if $\langle X \rangle p \equiv_{\mathcal{C}} \psi$ for some $\mathcal{F}$-formula ψ over the atomic proposition p, for some $p \in \mathcal{AP}$. In such a case, the equivalence $\langle X \rangle p \equiv_{\mathcal{C}} \psi$ is called an* inter-definability equation *(or simply* inter-definability*) for $\langle X \rangle$ in $\mathcal{F}$ relative to $\mathcal{C}$. We write $\langle X \rangle \not\lhd_{\mathcal{C}} \mathcal{F}$ if $\langle X \rangle$ is not definable in $\mathcal{F}$ over $\mathcal{C}$.*

Notice that smaller classes of linear orders inherit the inter-definabilities holding for larger classes of linear orders. Formally, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are classes of linear orders such that $\mathcal{C}_1 \subset \mathcal{C}_2$, then all inter-definabilities holding for $\mathcal{C}_2$ are also valid for $\mathcal{C}_1$. However, more inter-definabilities can possibly hold for $\mathcal{C}_1$. On the other hand, undefinability results for $\mathcal{C}_1$ hold also for $\mathcal{C}_2$. In the rest of the paper, we will omit the class of linear orders when it is clear from the context (e.g., we will simply say $\langle X \rangle p \equiv \psi$ and $\langle X \rangle \lhd \mathcal{F}$ instead of $\langle X \rangle p \equiv_{\mathcal{C}} \psi$ and $\langle X \rangle \lhd_{\mathcal{C}} \mathcal{F}$, respectively).

It is known from [7] that, in the strict semantics, all HS modalities are definable in the fragment containing modalities $\langle A \rangle$, $\langle B \rangle$, and $\langle E \rangle$, and their transposes $\langle \overline{A} \rangle$, $\langle \overline{B} \rangle$, and $\langle \overline{E} \rangle$. In this paper, we compare and classify the expressiveness of all HS fragments relative to the class of all dense linear orders. Formally, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be any pair of such fragments. We say that $\mathcal{F}_2$ is *at least as expressive as* $\mathcal{F}_1$ (denoted $\mathcal{F}_1 \preceq \mathcal{F}_2$) if each modality $\langle X \rangle \in \mathcal{F}_1$ is definable in $\mathcal{F}_2$. The notions of being *strictly less expressive* ($\mathcal{F}_1 \prec \mathcal{F}_2$), *equally expressive* ($\mathcal{F}_1 \equiv \mathcal{F}_2$), and *expressively incomparable* ($\mathcal{F}_1 \not\equiv \mathcal{F}_2$) are defined accordingly. Now, it is possible to define the notion of optimal inter-definability, as follows.

Definition 2 (Optimal inter-definability). *A definability $\langle X \rangle \lhd \mathcal{F}$ is said* optimal *if $\langle X \rangle \not\lhd \mathcal{F}'$ for any fragment $\mathcal{F}'$ such that $\mathcal{F}' \prec \mathcal{F}$.*

In order to show non-definability of a given modality in an HS fragment, we use a standard technique in modal logic, based on the notion of *bisimulation* and the invariance of modal formulae with respect to bisimulations (see, e.g., [4, 8]). The important property of bisimulations used here is that any $\mathcal{F}$-bisimulation preserves the truth of *all* formulae in $\mathcal{F}$, that is, if $([a, b], [a', b']) \in Z$ and Z is an $\mathcal{F}$-bisimulation, then $[a, b]$ and $[a', b']$ satisfy exactly the same formulae in $\mathcal{F}$. Thus, in order to prove that a modality $\langle X \rangle$ is *not* definable in $\mathcal{F}$, it suffices to construct a pair of interval models $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}'), V' \rangle$, and an $\mathcal{F}$-bisimulation Z between them, relating a pair of intervals $[a, b] \in \mathbb{I}(\mathbb{D})$ and $[a', b'] \in \mathbb{I}(\mathbb{D}')$, such that $M, [a, b] \Vdash \langle X \rangle p$, while $M', [a', b'] \nVdash \langle X \rangle p$. In this case, we say that Z *breaks* $\langle X \rangle$.

The problem. As we already pointed out, every subset of the set of the 12 modalities corresponding to Allen's relations gives rise to a logic, namely, a fragment of HS. There are 2^{12} (the cardinality of the powerset of the set of modalities) such fragments. Due to possible inter-definabilities of modalities in terms of other ones, not all these fragments are expressively different. The problem we consider here is the problem of obtaining a complete classification of all HS fragments with respect to their expressive power over the class of (all) dense linear orders. In other words, given two HS fragments $\mathcal{F}_1$, $\mathcal{F}_2$, we want to be able to decide how they relate to each other with respect to expressiveness (that is,

$\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$	$\langle L\rangle \lhd \mathsf{A}$
$\langle \overline{L}\rangle p \equiv \langle \overline{A}\rangle\langle \overline{A}\rangle p$	$\langle \overline{L}\rangle \lhd \overline{\mathsf{A}}$
$\langle O\rangle p \equiv \langle E\rangle\langle \overline{B}\rangle p$	$\langle O\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$
$\langle \overline{O}\rangle p \equiv \langle B\rangle\langle \overline{E}\rangle p$	$\langle \overline{O}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$
$\langle D\rangle p \equiv \langle E\rangle\langle B\rangle p$	$\langle D\rangle \lhd \mathsf{BE}$
$\langle \overline{D}\rangle p \equiv \langle \overline{E}\rangle\langle \overline{B}\rangle p$	$\langle \overline{D}\rangle \lhd \overline{\mathsf{BE}}$
$\langle L\rangle p \equiv \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$	$\langle L\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$
$\langle \overline{L}\rangle p \equiv \langle \overline{E}\rangle[B]\langle \overline{E}\rangle\langle B\rangle p$	$\langle \overline{L}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$

(a) The complete set of optimal inter-definabilities for the class of all linear orders.

$\langle L\rangle p \equiv \langle O\rangle(\langle O\rangle\top \wedge [O]\langle D\rangle\langle O\rangle p)$	$\langle L\rangle \lhd \mathsf{DO}$
$\langle \overline{L}\rangle p \equiv \langle \overline{O}\rangle(\langle \overline{O}\rangle\top \wedge [\overline{O}]\langle D\rangle\langle \overline{O}\rangle p)$	$\langle \overline{L}\rangle \lhd \mathsf{D}\overline{\mathsf{O}}$
$\langle L\rangle p \equiv \langle \overline{B}\rangle[D]\langle \overline{B}\rangle\langle D\rangle\langle \overline{B}\rangle p$	$\langle L\rangle \lhd \overline{\mathsf{B}}\mathsf{D}$
$\langle \overline{L}\rangle p \equiv \langle \overline{E}\rangle[D]\langle \overline{E}\rangle\langle D\rangle\langle \overline{E}\rangle p$	$\langle \overline{L}\rangle \lhd \overline{\mathsf{E}}\mathsf{D}$
$\langle L\rangle p \equiv \langle O\rangle[E]\langle O\rangle\langle O\rangle p$	$\langle L\rangle \lhd \mathsf{EO}$
$\langle \overline{L}\rangle p \equiv \langle \overline{O}\rangle[B]\langle \overline{O}\rangle\langle \overline{O}\rangle p$	$\langle \overline{L}\rangle \lhd \mathsf{B}\overline{\mathsf{O}}$
$\langle L\rangle p \equiv \langle O\rangle(\langle O\rangle\top \wedge [O]\langle B\rangle\langle O\rangle\langle O\rangle p)$	$\langle L\rangle \lhd \mathsf{BO}$
$\langle \overline{L}\rangle p \equiv \langle \overline{O}\rangle(\langle \overline{O}\rangle\top \wedge [\overline{O}]\langle E\rangle\langle \overline{O}\rangle\langle \overline{O}\rangle p)$	$\langle \overline{L}\rangle \lhd \mathsf{E}\overline{\mathsf{O}}$
$\langle L\rangle p \equiv \langle O\rangle[O][\overline{L}]\langle O\rangle\langle O\rangle p$	$\langle L\rangle \lhd \overline{\mathsf{L}}\mathsf{O}$
$\langle \overline{L}\rangle p \equiv \langle \overline{O}\rangle[\overline{O}][L]\langle \overline{O}\rangle\langle \overline{O}\rangle p$	$\langle \overline{L}\rangle \lhd \mathsf{L}\overline{\mathsf{O}}$

(b) A set of inter-definabilities for $\langle L\rangle$ and $\langle \overline{L}\rangle$ over the class of all dense linear orders.

Table 1.

whether $\mathcal{F}_1$ is strictly less expressive than $\mathcal{F}_2$, $\mathcal{F}_1$ is strictly more expressive than $\mathcal{F}_2$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are expressively equivalent, or $\mathcal{F}_1$ and $\mathcal{F}_2$ are incomparable).

In order to do so, all we need to do is to provide the complete set of optimal inter-definabilities between HS modalities. Indeed, provided with such a set, it is immediate to decide which relation exists between any two given fragments with respect to their expressive power.

The class of all linear orders. The problem we address in this paper has been solved for the class of all linear orders in [5], where the complete set of optimal inter-definabilities in Table 1a has been identified.

All the bisimulations used in [5] to solve the problem for the class of all linear orders are based on dense structures, apart from those for $\langle L\rangle$ and $\langle \overline{L}\rangle$, which are based on discrete structures. As a consequence, the above results for all modalities but $\langle L\rangle$ and $\langle \overline{L}\rangle$ immediately extend to all classes of dense linear orders. In what follows, we identify a new set of optimal inter-definabilities holding for $\langle L\rangle$ and $\langle \overline{L}\rangle$ over classes of dense linear orders, and we prove it to be complete (for the modalities $\langle L\rangle$ and $\langle \overline{L}\rangle$).

3 The class of all dense linear orders

From now on, we focus our attention on the class of all dense linear orders, and we provide bisimulations based on $\mathbb{R}$. However, it is possible to extend our results to sub-classes of the class of all dense linear orders (that might not include $\mathbb{R}$), by providing bisimulations based on different (suitable) dense linear orders. In what follows, we first prove that Table 1b depicts a set of inter-definabilities for the operators $\langle L\rangle$ and $\langle \overline{L}\rangle$ (Lemma 1). Then, we show that the union of all equations for $\langle L\rangle$ and $\langle \overline{L}\rangle$ shown in Table 1a and Table 1b constitutes the complete set of optimal inter-definabilities for those operators (Theorem 1).

Lemma 1. *Table 1b depicts a set of inter-definabilities for the operators $\langle L\rangle$ and $\langle \overline{L}\rangle$.*

The rest of the paper is devoted to establishing our main result, that is, to prove that Table 1a and Table 1b depict a complete set of optimal inter-definabilities for the operator $\langle L\rangle$. This means that we cannot define $\langle L\rangle$ by means of any other optimal equation. It is immediate to verify, by symmetry, that the same result holds for the operator $\langle \overline{L}\rangle$.

As a first step, we need to identify all maximal HS fragments not containing, as definable (according to the inter-definabilities of Table 1a and Table 1b), the operator $\langle L\rangle$. Given the large number of inter-definabilities, it is not immediate to detect all such fragments. For this purpose, we used the algorithm presented in [1]. The algorithm, run on the list of inter-definabilities in Table 1a and Table 1b, and on modality $\langle L\rangle$ as input parameters, returned the three maximal fragments $\mathsf{O\overline{BEDO}}$, $\mathsf{BED\overline{ALEDO}}$, and $\mathsf{B\overline{ALBEDO}}$. In the light of the inter-definabilities in Table 1a, we can replace these three fragments with equivalent fragments featuring the smallest set of modalities, namely, $\mathsf{O\overline{BEO}}$, $\mathsf{BE\overline{AED}}$, and $\mathsf{B\overline{ABE}}$, respectively. Now, in order to establish the optimality of the set of inter-definabilities, for each such fragment $\mathcal{F}$, we provide an $\mathcal{F}$-bisimulation that breaks $\langle L\rangle$. (Due to lack of space, we only provide the proof for the first of the three bisimulations presented in what follows. For the other proofs, we refer the interested reader to [1].) In what follows, thanks to the next proposition, in our proofs we can safely assume that for each interval $[a,b]$ and Allen's relation R_X, there exists an interval $[c,d]$ such that $[a,b]R_X[c,d]$.

Proposition 1. *Let $\mathbb{D}$ be a dense linear order without least and greatest elements, and let $[a,b]\in\mathbb{I}(\mathbb{D})$. Then, there exists an interval $[c,d]\in\mathbb{I}(\mathbb{D})$ such that $[a,b]R_X[c,d]$, for each $X\in\{A,L,B,E,D,O,\overline{A},\overline{L},\overline{B},\overline{E},\overline{D},\overline{O}\}$.*

An $\mathsf{O\overline{BEO}}$-bisimulation that breaks $\langle L\rangle$. Consider the two interval models M and M', defined as $M=M'=\langle\mathbb{I}(\mathbb{R}),V\rangle$, where $V(p)=\{[-a,a]\mid a\in\mathbb{R}\}$ (observe that no interval $[c,d]$, with $c\geq 0$, satisfies p). Moreover, let $Z=\{([a,b],[a',b'])\mid -a\sim b \text{ and } -a'\sim b' \text{ for some } \sim\in\{<,=,>\}\}$ (see Fig. 2a).

Lemma 2. *Z is a $\mathsf{O\overline{BEO}}$-bisimulation.*

Proof. **Local condition.** Consider a pair $([a,b],[a',b'])$ of Z-related intervals. The following chain of double implications hold: $M,[a,b]\Vdash p$ iff $-a=b$ iff (by the definition of Z) $-a'=b'$ iff $M,[a',b']\Vdash p$.

Forward condition. Consider the three intervals $[a,b]$, $[a',b']$, and $[c,d]$ such that $[a,b]Z[a',b']$ and $[a,b]R_X[c,d]$ for some $X\in\{O,\overline{B},\overline{E},\overline{O}\}$. We need to exhibit an interval $[c',d']$ such that $[a',b']R_X[c',d']$ and $[c,d]Z[c',d']$. We distinguish three cases.

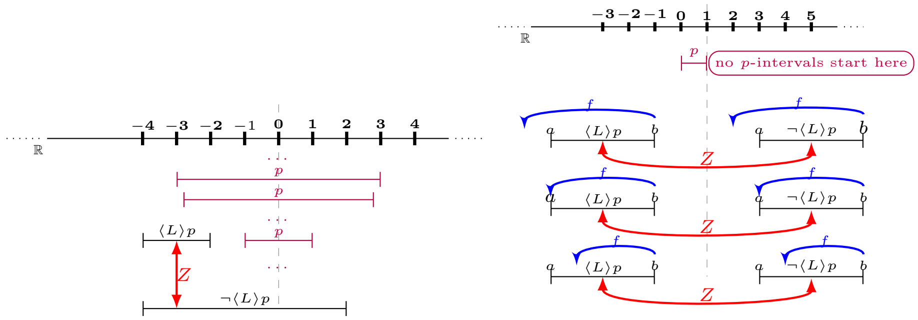
- If $-a>b$ and $-a'>b'$, then, as a preliminary step, we show that the following facts hold: (*i*) $a<0$ and $a'<0$; (*ii*) $|a|>|b|$ and $|a'|>|b'|$. We only show the proofs for $a<0$ and $|a|>|b|$ and we omit the ones for $a'<0$ and $|a'|>|b|$, which are analogous. As for the former claim above, it is enough to observe that, if $a\geq 0$, then $a\geq 0\geq -a>b$, which implies $b<a$, leading to a contradiction with the fact that $[a,b]$ is an interval (thus $a<b$). Notice that, as an immediate consequence, we have that $|a|=-a$ holds. As for the latter claim above, firstly we suppose, by contradiction, that $|a|=|b|$ holds. Then, $-a=|a|=|b|$ holds and this implies either $b=-a$, contradicting the hypothesis that $-a>b$, or $b=a$, contradicting the fact that $[a,b]$ is an interval. Secondly, we suppose, again by contradiction, that $|a|<|b|$ holds. Then, by the former claim, we have that $0<-a=|a|<|b|$ holds, which implies $b\neq 0$. Now, we show that both $b<0$ and $b>0$ lead to a contradiction. If $b<0$, then $|b|=-b$, and thus it holds $-a<-b$, which amounts to $a>b$, contradicting the fact that $[a,b]$ is an interval. If $b>0$, then $|b|=b$, and thus it holds $-a<b$, which

contradicts the hypothesis that $-a > b$. This proves the two claims above. Now, we distinguish the following sub-cases.

- If $X = O$, then $[c, d]$ is such that $a < c < b < d$. We distinguish the following cases.
  - ∗ If $-c > d$, then take some c' such that $a' < c' < -|b'| < 0$ (notice also that $c' < -|b'| \leq b'$ trivially holds), and d' such that $b' < d' < |c'| = -c'$ (the existence of such points c', d' is guaranteed by the density of $\mathbb{R}$). The interval $[c', d']$ is such that $[a', b']R_O[c', d']$ and $[c, d]Z[c', d']$.
  - ∗ If $-c = d$, then take some c' such that $a' < c' < -|b'| < 0$, and $d' = -c'$ (the existence of such a point c' is guaranteed by the density of $\mathbb{R}$). The interval $[c', d']$ is such that $[a', b']R_O[c', d']$ and $[c, d]Z[c', d']$.
  - ∗ If $-c < d$, then take c' such that $a' < c' < -|b'| < 0$, and any $d' > -c'$ (the existence of such a point c' is guaranteed by the density of $\mathbb{R}$). The interval $[c', d']$ is such that $[a', b']R_O[c', d']$ and $[c, d]Z[c', d']$.
- If $X = \overline{B}$, then $[c, d]$ is such that $a = c < b < d$. We distinguish the cases below.
  - ∗ If $-c > d$, then take $c' = a'$ and d' such that $b' < d' < -a' = -c'$ (the existence of such a point d' is guaranteed by the density of $\mathbb{R}$). The interval $[c', d']$ is such that $[a', b']R_{\overline{B}}[c', d']$ and $[c, d]Z[c', d']$.
  - ∗ If $-c = d$, then take $c' = a'$ and $d' = -c'(= -a' > b')$. The interval $[c', d']$ is such that $[a', b']R_{\overline{B}}[c', d']$ and $[c, d]Z[c', d']$.
  - ∗ If $-c < d$, then take $c' = a'$ and any $d' > -c'(= -a' > b')$. The interval $[c', d']$ is such that $[a', b']R_{\overline{B}}[c', d']$ and $[c, d]Z[c', d']$.
- If $X = \overline{E}$, then $[c, d]$ is such that $c < a < b = d$. Notice that $|c| = -c > -a = |a|$ holds, because $c < a < 0$. Thus $-c > -a > b = d$ also holds. Then, take $d' = b'$ and any $c' < a'$. We have that $-c' > -a' > b' = d'$. The interval $[c', d']$ is therefore such that $[a', b']R_{\overline{E}}[c', d']$ and $[c, d]Z[c', d']$.
- If $X = \overline{O}$, then $[c, d]$ is such that $c < a < d < b$. Notice that $|c| = -c > -a = |a|$ holds, because $c < a < 0$. Thus $-c > -a > b > d$ also holds. Then, take some d' such that $a' < d' < b'$ and any $c' < a'$ (the existence of such a point d' is guaranteed by the density of $\mathbb{R}$). Thus, it holds $-c' > -a' > b' > d'$. The interval $[c', d']$ is therefore such that $[a', b']R_{\overline{O}}[c', d']$ and $[c, d]Z[c', d']$.

– If $-a = b$ and $-a' = b'$, then we have that $a < 0$ (resp., $a' < 0$) and $b > 0$ (resp., $b' > 0$). Indeed, if $a \geq 0$ held, then $b = -a \leq 0 \leq a$ would also hold, contradicting the fact that $[a, b]$ is an interval (and thus $b > a$). From $a < 0$ and $-a = b$, it immediately follows that $b > 0$. The facts that $a' < 0$ and $b' > 0$ can be shown analogously. Notice also that, from $-a = b$ and $-a' = b'$, it follows that $|a| = |b|$ and $|a'| = |b'|$. Now, we distinguish the following sub-cases.

- If $X = O$, then $[c, d]$ is such that $a < c < b < d$. Notice that $-c \leq |c| < |a| = |b| = b < d$ holds. Then, take $c' = 0$ and any $d' > b'(> 0)$. We have that $-c' < d'$. The interval $[c', d']$ is such that $[a', b']R_O[c', d']$ and $[c, d]Z[c', d']$.
- If $X = \overline{B}$, then $[c, d]$ is such that $a = c < b < d$. Notice that $-c \leq |c| = |a| = |b| = b < d$ holds. Then, take $c' = a'$ and any $d' > b'$. We have that $-c' = -a' = b' < d'$. The interval $[c', d']$ is such that $[a', b']R_{\overline{B}}[c', d']$ and $[c, d]Z[c', d']$.
- If $X = \overline{E}$, then $[c, d]$ is such that $c < a < b = d$. Notice that $|c| = -c > -a = |a|$ holds, because $c < a < 0$. Thus $-c > -a = b = d$ also holds. Then, take $d' = b'$



(a) $\mathsf{O\overline{B}\overline{E}\overline{O}}$-bisimulation.

(b) $\mathsf{BE\overline{A}\overline{E}\overline{D}}$-bisimulation.

Fig. 2.

and any $c' < a'$. We have that $-c' > -a' = b' = d'$. The interval $[c', d']$ is such that $[a', b']R_{\overline{E}}[c', d']$ and $[c, d]Z[c', d']$.
  - If $X = \overline{O}$, then $[c, d]$ is such that $c < a < d < b$. Notice that $|c| = -c > -a = |a|$ holds, because $c < a < 0$. Thus $-c > -a = b > d$ also holds. Then, take $d' = 0$ and any $c' < a'(< 0)$. We have that $-c' > d'$. The interval $[c', d']$ is such that $[a', b']R_{\overline{O}}[c', d']$ and $[c, d]Z[c', d']$.
- If $-a < b$ and $-a' < b'$, then the proof proceeds symmetrically to the case when $-a > b$ and $-a' > b'$. More precisely, the argument used there for modalities $\langle\overline{O}\rangle$ and $\langle\overline{E}\rangle$ applies now to modalities $\langle O\rangle$ and $\langle\overline{B}\rangle$, and vice versa. The details are omitted.

Backward condition. Since the relation Z is symmetric, the forward condition implies the backward condition, as follows. Consider a pair $([a, b], [a', b'])$ of Z-related intervals and an interval $[c', d']$ such that $[a', b']R_X[c', d']$, for some $X \in \{O, \overline{B}, \overline{E}, \overline{O}\}$. We need to find an interval $[c, d]$ such that $[a, b]R_X[c, d]$ and $[c, d]Z[c', d']$. By symmetry, $([a', b'], [a, b]) \in Z$, as well. By the forward condition, we know that for every interval $[c', d']$ such that $[a', b']R_X[c', d']$, for some $X \in \{O, \overline{B}, \overline{E}, \overline{O}\}$, there exists an interval $[c, d]$ such that $[a, b]R_X[c, d]$ and $[c', d']Z[c, d]$. By symmetry $[c, d]Z[c', d']$ also holds, hence the backward condition is fulfilled, too. $\square$

Corollary 1. *The modality $\langle L\rangle$ is not definable in the fragment* $\mathsf{O\overline{B}\overline{E}\overline{O}}$ *(and in any of its sub-fragments) over the class of all dense linear orders.*

A $\mathsf{BE\overline{A}\overline{E}\overline{D}}$-bisimulation that breaks $\langle L\rangle$. In order to define a $\mathsf{BE\overline{A}\overline{E}\overline{D}}$-bisimulation that breaks $\langle L\rangle$, we will make use of the function $f : \mathbb{R} \to \{x \in \mathbb{R} \mid x < 1\}$, defined as: $f(x) = x - 1$ if $x \leq 1$ and $f(x) = 1 - \frac{1}{x}$ otherwise.

Lemma 3. *f is a monotonically increasing bijection from $\mathbb{R}$ to $\{x \in \mathbb{R} \mid x < 1\}$ such that $f(x) < x$ for every $x \in \mathbb{R}$.*

The bisimulation that breaks $\langle L\rangle$ is defined as follows. We consider two interval models M and M', defined as $M = M' = \langle \mathbb{I}(\mathbb{R}), V\rangle$, where $V(p) = \{[a, b] \mid a = f(b)\}$ and let $Z = \{([a, b], [a', b']) \mid a \sim f(b), a' \sim f(b') \text{ for some } \sim \in \{<, =, >\}\}$ (see Fig. 2b).

Lemma 4. *Z is a $\mathsf{BE\overline{A}\overline{E}\overline{D}}$-bisimulation.*

Corollary 2. *The modality $\langle L\rangle$ is not definable in the fragment $\mathsf{BE\overline{A}\overline{E}\overline{D}}$ (and in any of its sub-fragments) over the class of all dense linear orders.*

A $\mathsf{B\overline{A}\overline{B}\overline{E}}$-bisimulation that breaks $\langle L\rangle$. Consider the two interval models M and M', defined as $M = \langle \mathbb{I}(\mathbb{R}), V\rangle$ and $M' = \langle \mathbb{I}(\mathbb{R}), V'\rangle$, respectively, where $V(p) = \{[a,b] \mid a,b \in \mathbb{Q} \text{ or } a,b \in \mathbb{R}\setminus\mathbb{Q}\}$ and $V'(p) = \{[a',b'] \mid a' \leq 0 \text{ and } (a',b' \in \mathbb{Q} \text{ or } a',b' \in \mathbb{R}\setminus\mathbb{Q})\}$. Moreover, let $Z = \{([a,b],[a',b']) \mid a' \leq -1 \text{ and } M,[a,b] \Vdash p \text{ iff } M',[a',b'] \Vdash p\}$.

Lemma 5. *Z is a $\mathsf{B\overline{A}\overline{B}\overline{E}}$-bisimulation.*

Corollary 3. *The modality $\langle L\rangle$ is not definable in the fragment $\mathsf{B\overline{A}\overline{B}\overline{E}}$ (and in any of its sub-fragments) over classes of dense linear orders.*

Theorem 1. *Table 1a and Table 1b depict a complete set of optimal inter-definabilities for the modality $\langle L\rangle$.*

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