Interval Temporal Logics over Strongly Discrete Linear Orders: the Complete Picture

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Interval temporal logics provide a general framework for temporal reasoning about interval structures over linearly ordered domains, where intervals are taken as the primitive ontological entities. In this paper, we identify all fragments of Halpern and Shoham's interval temporal logic HS with a decidable satisfiability problem over the class of strongly discrete linear orders. We classify them in terms of both their relative expressive power and their complexity. We show that there are exactly 44 expressively different decidable fragments, whose complexity ranges from NP to EXPSPACE. In addition, we identify some new undecidable fragments (all the remaining HS fragments were already known to be undecidable over strongly discrete linear orders). We conclude the paper by an analysis of the specific case of natural numbers, whose behavior slightly differs from that of the whole class of strongly discrete linear orders. The number of decidable fragments over $\mathbb{N}$ raises up to 47: three undecidable fragments become decidable with a non-primitive recursive complexity.

1 Introduction

Interval temporal logics provide a general framework for temporal reasoning about interval structures over linearly (or partially) ordered domains. They take time intervals as the primitive ontological entities and define truth of formulas relative to time intervals, rather than time points. Interval logic modalities correspond to various relations between pairs of intervals, with the exception of Venema's CDT and its fragments, that consider ternary relations [21]. In particular, Halpern and Shoham's modal logic of time intervals HS [15] features a set of modalities that makes it possible to express all Allen's interval relations [1] (see Table 1). Interval-based formalisms have been extensively used in many areas of computer science, such as, for instance, planning, natural language processing, constraint satisfaction, and verification of hardware and software systems. However, most of them impose severe syntactic and semantic restrictions that considerably weaken their expressive power. Interval temporal logics relax these restrictions, allowing one to cope with much more complex application domains and scenarios, Unfortunately, many of them, including HS and the majority of its fragments, turn out to be undecidable [4].

In this paper, we focus our attention on the class of strongly discrete linear orders, that is, of those linear structures characterized by the presence of finitely many points in between any two points. This class includes, for instance, $\mathbb{N}$, $\mathbb{Z}$, and finite linear orders. We give a complete classification of all HS fragments, reviewing known results and solving open problems. The aim of such a classification is twofold: on the one hand, we identify the subset of all expressively-different decidable fragments, thus marking the decidability border; on the other hand, we determine the exact complexity of each of them. As shown in Figure 1, $\mathsf{A\overline{A}B\overline{B}}$ (that features modal operators for Allen's relations *meets* and *started-by*, and their inverses) and its mirror image $\mathsf{A\overline{A}E\overline{E}}$ (that replaces relations *starts* and *started-by* by relations

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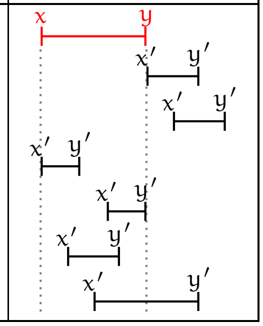
Relation	**Operator**	**Formal definition**	**Pictorial example**
meets	$\langle A\rangle$	$[x,y]R_A[x',y'] \Leftrightarrow y = x'$	
before	$\langle L\rangle$	$[x,y]R_L[x',y'] \Leftrightarrow y < x'$	
started-by	$\langle B\rangle$	$[x,y]R_B[x',y'] \Leftrightarrow x = x', y' < y$	
finished-by	$\langle E\rangle$	$[x,y]R_E[x',y'] \Leftrightarrow y = y', x < x'$	
contains	$\langle D\rangle$	$[x,y]R_D[x',y'] \Leftrightarrow x < x', y' < y$	
overlaps	$\langle O\rangle$	$[x,y]R_O[x',y'] \Leftrightarrow x < x' < y < y'$	

Table 1: Allen's interval relations and the corresponding HS modalities.

finishes and *finished-by*) are the minimal fragments including all decidable subsets of operators from the HS repository, for a total of 62 languages. Of those, 44 turn out to be decidable.

As a matter of fact, the status of various fragments was already known: *(i)* D, $\overline{\mathsf{D}}$, O, and $\overline{\mathsf{O}}$ have been shown to be undecidable in [6, 16]; *(ii)* BE, $\mathsf{B\overline{E}}$, $\mathsf{\overline{B}E}$, and $\overline{\mathsf{BE}}$ are undecidable, as each of them can define either $\langle O\rangle$ or $\langle D\rangle$, or one of their inverses; *(iii)* undecidability of $\mathsf{A\overline{A}B}$ and $\mathsf{A\overline{AB}}$ (resp., $\mathsf{A\overline{A}E}$ and $\mathsf{A\overline{AE}}$) can be proved as that of $\mathsf{A\overline{A}B\overline{B}}$ (resp., $\mathsf{A\overline{A}E\overline{E}}$) [18]; *(iv)* $\mathsf{AB\overline{BL}}$ (resp., $\mathsf{\overline{A}E\overline{E}L}$) is in EXPSPACE [10], and EXPSPACE-hardness already holds for AB and $\mathsf{A\overline{B}}$ (resp., $\mathsf{\overline{A}E}$ and $\overline{\mathsf{AE}}$) over finite linear orders [7]; *(v)* $\mathsf{A\overline{A}}$ (aka Propositional Neighborhood Logic) is in NEXPTIME [8, 13], and NEXPTIME-hardness already holds for A and $\overline{\mathsf{A}}$ [9]; *(vi)* $\mathsf{B\overline{B}}$ is NP-complete [14], and, obviously, NP-hardness already holds for B and $\overline{\mathsf{B}}$ (both include propositional logic); *(vii)* the relative expressive power of the HS fragments we are interested in is as shown in Figure 1 [7, 11].

In this paper, we complete the picture by proving the following new results: *(i)* the undecidability of $\mathsf{A\overline{A}B}$ (resp., $\mathsf{A\overline{A}E}$) and $\mathsf{A\overline{AB}}$ (resp., $\mathsf{A\overline{AE}}$) can be sharpened to $\mathsf{\overline{A}B}$ (resp., AE) and $\overline{\mathsf{AB}}$ (resp., $\mathsf{A\overline{E}}$), respectively (Section 3); *(ii)* the NP-completeness (in particular, NP-membership) of $\mathsf{B\overline{B}}$ can be extended to $\mathsf{B\overline{B}L\overline{L}}$ (Section 4). In addition, we analyze the behavior of the various fragments over $\mathbb{N}$ (Section 6). As $\mathbb{N}$-models are not left/right symmetric, reversing the time order and coherently replacing modalities (e.g., $\langle A\rangle$ by $\langle \overline{A}\rangle$) does not preserve, in general, the computational properties of a fragment. We show that: *(i)* $\mathsf{\overline{A}B}$ becomes decidable (which is a direct consequence of [18]), precisely, non-primitive recursive [7]; *(ii)* the same holds for $\overline{\mathsf{AB}}$ and $\mathsf{\overline{A}B\overline{B}}$, but, in these cases, the decidability proof for $\mathsf{A\overline{A}B\overline{B}}$ given in [18] must be suitably adapted; *(iii)* $\mathsf{\overline{A}BL}$, $\mathsf{\overline{AB}L}$, and $\mathsf{\overline{A}B\overline{B}L}$ remain undecidable, but original reductions must be suitably adapted. Thus, the number of decidable fragments over $\mathbb{N}$ raises up to 47, the three new decidable fragments being all non-primitive recursive. As a matter of fact, we can slightly generalize such a result, as the addition of finite linear orders (finite prefixes of $\mathbb{N}$) to $\mathbb{N}$ does not alter the decidability/undecidability/complexity picture. However, to keep presentation and proofs as simple as possible, we restrict our attention to $\mathbb{N}$-models only.

2 HS and its Fragments

Let $\mathbb{D} = \langle D, <\rangle$ be a *strongly discrete linearly ordered set*, that is, a linearly ordered set where for every pair x, y such that $x < y$, there exist at most finitely many $z_1, z_2, \ldots, z_n$ such that $x < z_1 < z_2 < \ldots < z_n < y$. According to the strict (or pure) approach, we exclude intervals with coincident endpoints (point-intervals) from the semantics. Then, an *interval* over $\mathbb{D}$ is defined as an ordered pair $[x, y]$, where

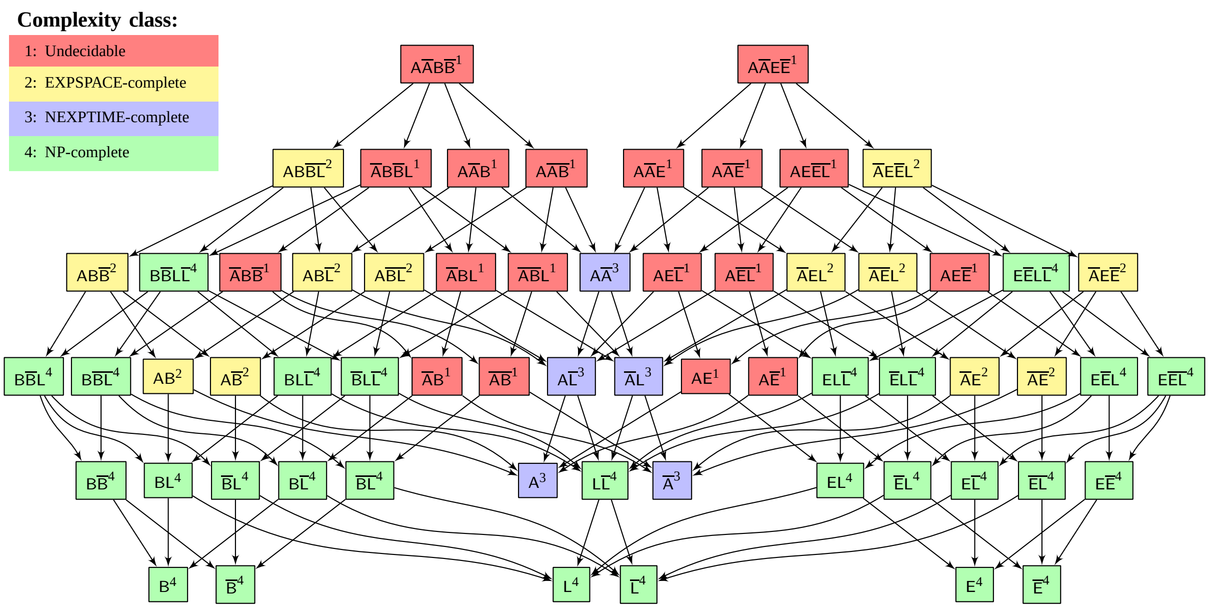


Figure 1: Hasse diagram of fragments of $A\overline{A}B\overline{B}$ and $A\overline{A}E\overline{E}$ over strongly discrete linear orders.

$x, y \in D$ and $x < y$. 12 different ordering relations (excluding equality) between any pair of intervals are possible, often called *Allen's relations* [1]: the six relations depicted in Table 1 and their inverses. We interpret interval structures as Kripke structures and Allen's relations as accessibility relations, thus associating a modality $\langle X \rangle$ with each Allen's relation R_X. For each modality $\langle X \rangle$, its *inverse* (or *transpose*), denoted by $\langle \overline{X} \rangle$, corresponds to the inverse relation $R_{\overline{X}}$ of R_X (that is, $R_{\overline{X}} = (R_X)^{-1}$). Halpern and Shoham's logic HS is a multi-modal logic whose formulas are built on a set $\mathcal{AP}$ of proposition letters, the boolean connectives $\vee$ and $\neg$, and a set of modalities, one for each Allen's relation. With every subset $\{R_{X_1}, \ldots, R_{X_k}\}$ of these relations, we associate the fragment $X_1X_2\ldots X_k$ of HS, whose formulas are defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X_1 \rangle \varphi \mid \ldots \mid \langle X_k \rangle \varphi.$$

The other boolean connectives can be viewed as abbreviations, and the dual operators $[X]$ are defined as usual ($[X]\varphi \equiv \neg\langle X \rangle \neg\varphi$). Given a formula φ, its *length* $|\varphi|$ is the number of its symbols.

The semantics of HS is given in terms of *interval models* $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$, where $\mathbb{I}(\mathbb{D})$ is the set of all intervals over $\mathbb{D}$. The *valuation function* $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$ assigns to every $p \in \mathcal{AP}$ the set of intervals $V(p)$ over which p holds. The *truth* of a formula over a given interval $[x, y]$ of an interval model M is defined by structural induction on formulas:

- $M, [x, y] \Vdash p$ iff $[x, y] \in V(p)$, for all $p \in \mathcal{AP}$;
- $M, [x, y] \Vdash \neg\psi$ iff it is not the case that $M, [x, y] \Vdash \psi$;
- $M, [x, y] \Vdash \varphi \vee \psi$ iff $M, [x, y] \Vdash \varphi$ or $M, [x, y] \Vdash \psi$;
- $M, [x, y] \Vdash \langle X \rangle \psi$ iff there exists an interval $[x', y']$ such that $[x, y] R_X [x', y']$ and $M, [x', y'] \Vdash \psi$, where R_X is the relation corresponding to $\langle X \rangle$.

An HS-formula ϕ is *valid*, denoted by $\Vdash \phi$, if it is true over every interval of every interval model.

In this paper, we study expressiveness and computational complexity of HS fragments over the class of strongly discrete linear orders. Given a fragment $\mathcal{F} = X_1X_2\ldots X_k$ and a modality $\langle X \rangle$, we write $\langle X \rangle \in \mathcal{F}$

if $X \in \{X_1, \ldots, X_k\}$. Given two fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if $\langle X \rangle \in \mathcal{F}_1$ implies $\langle X \rangle \in \mathcal{F}_2$, for every modality $\langle X \rangle$.

Definition 1. *We say that an* HS *modality* $\langle X \rangle$ *is* definable *in an* HS *fragment* $\mathcal{F}$ *if there exists a formula* $\psi(p) \in \mathcal{F}$ *such that* $\langle X \rangle p \leftrightarrow \psi(p)$ *is valid, for any fixed proposition letter* p. *In such a case, the equivalence* $\langle X \rangle p \equiv \psi(p)$ *is called an* inter-definability equation for $\langle X \rangle$ in $\mathcal{F}$.

Definition 2. *Let* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *be two* HS *fragments. We say that (i)* $\mathcal{F}_2$ *is* at least as expressive as $\mathcal{F}_1$ *(*$\mathcal{F}_1 \preceq \mathcal{F}_2$*) if modality* $\langle X \rangle \in \mathcal{F}_1$ *is definable in* $\mathcal{F}_2$*; (ii)* $\mathcal{F}_1$ *is* strictly less expressive *than* $\mathcal{F}_2$*, (*$\mathcal{F}_1 \prec \mathcal{F}_2$*) if* $\mathcal{F}_1 \preceq \mathcal{F}_2$*, but not* $\mathcal{F}_2 \preceq \mathcal{F}_1$*; (iii)* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are* equally expressive, *or* expressively equivalent *(*$\mathcal{F}_1 \equiv \mathcal{F}_2$*), if* $\mathcal{F}_1 \preceq \mathcal{F}_2$ *and* $\mathcal{F}_2 \preceq \mathcal{F}_1$*; (iv)* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are* expressively incomparable *(*$\mathcal{F}_1 \not\equiv \mathcal{F}_2$*) if neither* $\mathcal{F}_1 \preceq \mathcal{F}_2$ *nor* $\mathcal{F}_2 \preceq \mathcal{F}_1$.

We denote each HS fragment $\mathcal{F}$ by the list of its modalities in alphabetical order, omitting those modalities which are definable in terms of the others. As a matter of fact, in our setting, only modalities $\langle L \rangle$ and $\langle \overline{L} \rangle$ turn out to be definable in some fragments. Any fragment $\mathcal{F}$ can be transformed into its mirror image by reversing the time order and simultaneously replacing (each occurrence of) $\langle A \rangle$ by $\langle \overline{A} \rangle$, $\langle L \rangle$ by $\langle \overline{L} \rangle$, $\langle B \rangle$ by $\langle E \rangle$, and $\langle \overline{B} \rangle$ by $\langle \overline{E} \rangle$. In the considered class of linear orders, the mirroring operation can be applied to any given fragment preserving all its computational properties. Thus, all results given in this paper, except for the ones in Section 6, hold both for the considered fragments and their mirror images. On the contrary, when the considered class of models is not left/right symmetric, as it happens with $\mathbb{N}$, this is no longer true (see Section 6). The rest of the paper, with the exception of Section 6, is devoted to prove the following theorem.

Theorem 1. *The Hasse diagram in Figure 1 correctly shows all the decidable fragments of* HS *over the class of strongly discrete linear orders, their relative expressive power, and the precise complexity class of their satisfiability problem.*

3 Relative Expressive Power and Undecidability

The most basic definability results in HS are known since [15]: HS $\equiv \mathsf{A\overline{A}B\overline{B}E\overline{E}}$. Notice also that, when point-intervals are included in the semantics, Venema proved that HS $\equiv \mathsf{B\overline{B}E\overline{E}}$ [21]. In order to show non-definability of a given modality in a given fragment, we use the standard notion of bisimulation and the invariance of modal formulas with respect to bisimulations (see, e.g., [2]). In particular, we exploit the fact that, given a modal logic $\mathcal{F}$, any $\mathcal{F}$-bisimulation preserves the truth of all formulas in $\mathcal{F}$. Thus, in order to prove that a modality $\langle X \rangle$ is not definable in $\mathcal{F}$, it suffices to construct a pair of interval models M and M' and a $\mathcal{F}$-bisimulation between them, relating a pair of intervals $[x,y] \in M$ and $[x',y'] \in M'$, such that $M,[x,y] \Vdash \langle X \rangle p$ and $M',[x',y'] \not\Vdash \langle X \rangle p$.

In the following, we focus our attention on fragments of $\mathsf{A\overline{A}B\overline{B}}$ and of its symmetric language, $\mathsf{A\overline{A}E\overline{E}}$, in order to prove that Figure 1 is sound and complete for the class of all strongly discrete linear orders, in the following sense. In Figure 1 we depicted the graph whose set of nodes is given by the set of expressively different fragments of $\mathsf{A\overline{A}B\overline{B}}$ and $\mathsf{A\overline{A}E\overline{E}}$ (including $\mathsf{A\overline{A}B\overline{B}}$ and $\mathsf{A\overline{A}E\overline{E}}$ themselves). Nodes are partitioned with respect to complexity of their satisfiability problem: nodes corresponding to undecidable fragments are identified by a red rectangle and by the superscript 1, while nodes corresponding to fragments that are EXPSPACE-complete (resp., NEXPTIME-complete, NP-complete) are identified by a yellow rectangle and by the superscript 2 (resp., blue rectangle/superscript 3, green rectangle/superscript 4). Furthermore, all fragments of HS that does not appear in the picture are undecidable. The arcs of the graph represent the relative expressive power of two fragments: if two nodes, corresponding to the fragments $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, are connected by an arrow going from $\mathcal{F}_1$ to $\mathcal{F}_2$, then we have $\mathcal{F}_2 \prec \mathcal{F}_1$ ($\mathcal{F}_2$ is strictly less expressive than $\mathcal{F}_1$); if two fragments $\mathcal{F}_2$ and $\mathcal{F}_1$ are not connected by any path, then we have $\mathcal{F}_1 \not\equiv \mathcal{F}_2$ (they are expressively incomparable). Thus, to show that Figure 1 is sound and complete, we need to prove: *(i)* every pair of fragments which are not related to each other in the picture displays two expressively incomparable fragments; *(ii)* every fragment $\mathcal{F}_1$ connected by a directed arrow to a fragment $\mathcal{F}_2$ is strictly more expressive than $\mathcal{F}_2$; and *(iii)* the complexity of the satisfiability problem of considered fragments is correctly depicted by the picture. One can easily convince him/herself that *(i)* and *(ii)* are direct consequences of the following lemma, which has been proved in [7], and whose proof makes use of bisimulations based on finite linearly ordered sets; as the class of all strongly discrete linearly ordered sets includes the finite ones too, all results immediately apply.

Lemma 1 ([7]). *The only definability equations for the* HS *fragment* $\mathsf{A}\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}$, *over the class of all strongly discrete linear orders, are* $\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$ *and* $\langle \overline{L}\rangle p \equiv \langle \overline{A}\rangle\langle \overline{A}\rangle p$.

It remains to be shown point *(iii)*. The rest of the section is devoted to prove the undecidability of all fragments marked as undecidable in Figure 1. Those fragments that are not in the picture have already been proved undecidable in the class of all strongly discrete linearly ordered sets (see [16, 6]). As a consequence, we have that Figure 1 depicts all decidable fragments of HS over the class of all strongly discrete linear orders. Point *(iii)* above will then be completed in the next sections with the exact complexity characterization of all such decidable fragments.

The undecidability result we give here presents some similarities to those in [7, 18]. Nevertheless, its adaptation is not trivial. From [18, 19], we know that there is a reduction from the satisfiability problem for $\mathsf{A}\overline{\mathsf{A}}\mathsf{B}$ and $\mathsf{A}\overline{\mathsf{A}\mathsf{B}}$ to the structural termination problem for a lossy counter automata, which is known to be undecidable [17]. Here, by partly exploiting some of the basic concepts of such a reduction, we focus on the non-emptiness problem for incrementing counter automata over infinite words, which, again, is known to be undecidable [12]. Incrementing counter automata can be considered a variant of lossy counter automata in which faulty transitions increase the values instead of decrementing them; a comprehensive survey on faulty machines and the complexity (and decidability/undecidability) of various problems associated with such machines can be found in [3]. Formally, an *incrementing counter automaton* is a tuple of the form $\mathcal{A} = (\Sigma, Q, q_0, k, \Delta, F)$, where Σ is a finite alphabet, Q is a finite set of control *states*, $q_0 \in Q$ is the initial state, k is the number of *counters* $c_1, \ldots, c_k$ (whose values range over $\mathbb{N}$), Δ is a *transition relation*, and $F \subseteq Q$ is the subset of final states. The relation Δ is defined as a subset of $Q \times (\Sigma \uplus \{\epsilon\}) \times L \times Q$, where ϵ denotes the *empty transition*, and L is the *instruction set* $L = \{inc, dec, ifz\} \times \{1, \ldots, k\}$. A *configuration* of $\mathcal{A}$ is a pair $(q, \overline{c})$, where $q \in Q$ and $\overline{c}$ is the vector of counter values. Standard transitions of error-free counter automata are defines as usual: $(q, \overline{c}) \xrightarrow{l,a} (q', \overline{c}')$, where $a \in (\Sigma \uplus \{\epsilon\})$, $l \in L$, and if $l = (inc, i)$ (resp., $l = (dec, i)$, $l = (ifz, i)$) then the counter c_i is incremented by 1 (resp., decremented by 1, required to be 0). Instead, a run of an incrementing counter automaton consists of *incrementing transitions* of the form $(q, \overline{c}) \xrightarrow{l,a}_\dagger (q', \overline{c}')$, which means that there exists $\overline{c}_\dagger$ and $\overline{c}'_\dagger$ such that $\overline{c} \leqslant \overline{c}_\dagger$, $(q, \overline{c}_\dagger) \xrightarrow{l,a} (q', \overline{c}'_\dagger)$, and $\overline{c}'_\dagger \leqslant \overline{c}'$, and, therefore, counters may have been increased nondeterministically before or after the transition by an arbitrary natural number. An infinite run on $\mathcal{A}$ is said to be *accepting* for an infinite word $w \in \Sigma^\omega$ if and only if it passes by a final state in F infinitely often. Given an automaton $\mathcal{A}$, the *non-emptiness problem over infinite words* consists of deciding if there exists at least one infinite word accepted by $\mathcal{A}$, and it is undecidable [12]. We will show now that this problem can be reduced to the satisfiability problem for the fragments $\overline{\mathsf{A}}\mathsf{B}$, $\overline{\mathsf{A}\mathsf{B}}$, $\mathsf{A}\mathsf{E}$, and $\mathsf{A}\overline{\mathsf{E}}$, thus proving their undecidability. For the sake of simplicity, we will show this result only for the fragment $\mathsf{A}\mathsf{E}$; notice that in the class of all strongly discrete linearly ordered sets, this fragment is in

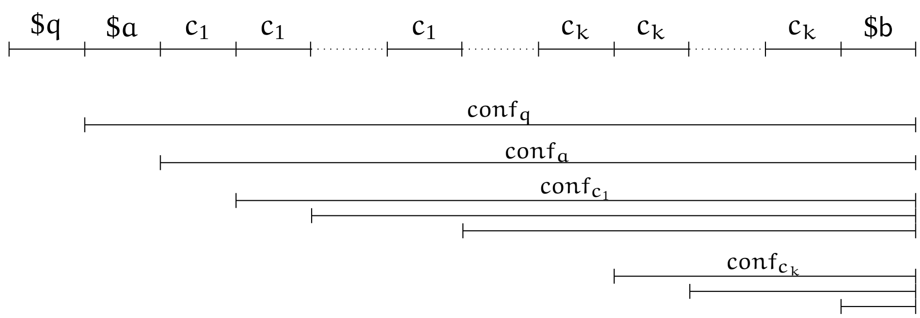


Figure 2: Encoding of a configuration of an incrementing counter automata in AE.

fact symmetric to $\overline{\mathsf{A}}\mathsf{B}$, thus the result trivially holds also for the latter fragment. Moreover, adapting it to $\mathsf{A}\overline{\mathsf{E}}$ (and therefore, by symmetry, to $\overline{\mathsf{AB}}$) is straightforward. In Section 6, on the other hand, we will show that when we focus our attention on the class of models based on $\mathbb{N}$, the situation is slightly different, as the symmetry does not hold anymore.

Lemma 2. *There exists a reduction from the non-emptiness infinite problem for incrementing counter automata to the satisfiability problem for* AE *in the class of strongly discrete linear orders.*

Proof. To prove this result, given an automaton $\mathcal{A} = (\Sigma, Q, q_0, k, \Delta, F)$, we need to produce a formula $\varphi_{\mathcal{A}}$ that it is satisfiable in the class of all strongly discrete linearly ordered sets if and only if there exists at least one infinite word on the alphabet Σ accepted by $\mathcal{A}$. Let us assume that $|Q| = \mu$, $|\Sigma| = \nu$, $|F| = \eta$, $|C| = k$ (where C denotes the set of counters), and that: *(i)* there are μ proposition letters $q_0, q_1, \ldots, q_\mu$, each of them corresponding to a state of the automaton (q_0 corresponds to the initial one); *(ii)* similarly, $a_1, \ldots, a_\nu$ correspond to alphabet symbols; and *(iii)* $c_1, \ldots, c_k$ correspond to counter elements. Moreover, $\$q$ (resp., $\$a$, $\$c$) are proposition letters that are true if and only if at least one q_i (resp., a_i, c_i) is true, and are used to simplify formulas. Finally, let $conf$ be a proposition letter used to denote a configuration. Since in the strongly discrete case we can univocally identify intervals of length one, that is, of the type $[x, x+1]$, by means of the formula $[E]\bot$, we will use them to encode the elements of a configuration. A configuration will be encoded by a (non-unit) interval $[x, x+s]$ labeled by $conf$, and whose unit intervals are labeled as follows: $[x, x+1]$ will be labeled by a state in Q, $[x+1, x+2]$ by a letter in Σ, and every other unit interval will be labeled by a counter proposition letter but the last one; for technical reasons, the last unit interval $[x+s-1, x+s]$ of every configuration will be labeled by a special proposition letter $\$b$. Figure 2 depicts (part of) the encoding of a configuration. Given a configuration interval $[x, x+s]$, we guarantee that it contains exactly one state and one alphabet letter, and the number of unit intervals labeled with c_i in $[x, x+s]$ corresponds to the value of the counter c_i in that configuration. The first configuration will contain no counter proposition letters, as the in the non-emptiness problem we can always assume all the counter values to start at 0 ($\overline{c} = \overline{0}$). The *universal* modality $[U]$ can defined in our language as follows: $[U]\varphi = \varphi \wedge [A]\varphi \wedge [A][A]\varphi$. Now, let us start by making sure that proposition letters that denote counter values, states, and elements in Σ are correctly placed:

$$[U](\$q \leftrightarrow \bigvee_{i=0}^{\mu} q_i) \wedge [U](\$a \leftrightarrow \bigvee_{i=1}^{\nu} a_i) \wedge [U](\$c \leftrightarrow \bigvee_{i=1}^{k} c_i) \qquad \textit{placeholders are correctly set} \qquad (1)$$

$$[U]([E]\bot \leftrightarrow \$q \vee \$a \vee \$c \vee \$b) \qquad \textit{placeholders are unit intervals} \qquad (2)$$

$$[U](\bigwedge_{i\neq j}(q_i \to \neg q_j) \wedge \bigwedge_{i\neq j}(a_i \to \neg a_j) \wedge \bigwedge_{i\neq j}(c_i \to \neg c_j)) \quad \textit{exactly one state, letter, counter} \tag{3}$$

$$[U] \bigwedge_{p\in\{q,a,c,b\}} (\$p \to \neg \bigvee_{p'\neq p} \$p') \quad \textit{exactly one placeholder per unit interval} \tag{4}$$

After that, we encode a sequence of configurations as a (unique) infinite chain that starts at the ending point of the interval where $\varphi_{\mathcal{A}}$ is evaluated, making sure that the first configuration has the counters set to zero. In order to guarantee the uniqueness of the chain and to force configurations to neither contain nor overlap to each other, we use the proposition letter $conf'$, that we make true over all and only suffixes of a configuration:

$$\langle A\rangle(conf \wedge \langle E\rangle\langle E\rangle\top \wedge [E][E][E]\bot) \quad \textit{first configuration has two internal points} \tag{5}$$

$$[U](conf \to \langle A\rangle conf \wedge [E]\neg conf \wedge \langle E\rangle\langle E\rangle\top) \quad conf\textit{s form a chain with space for state and letter} \tag{6}$$

$$[U](conf \to [E]conf') \wedge [U](conf' \to \neg conf) \quad conf\textit{s are finished by } conf' \textit{ which is not } conf \tag{7}$$

$$[U]\big((\langle A\rangle conf' \to \neg conf) \wedge (conf' \to \langle A\rangle conf \wedge \neg\langle E\rangle conf)\big) \quad \textit{properties of } conf \textit{ and } conf' \tag{8}$$

At this point, we are able to force configuration to have the right structure, that is, making sure that configuration starts with a state, which is followed by a letter, in turn, possibly followed by counters proposition letters, while the last unit interval of a configuration is labeled by \$b. Moreover, the first configuration start with q_0. The proposition letters $conf_q$, $conf_a$, and $conf_{c_i}$ (one for each type of counter) are used in the following set of formulas for technical reasons. In fact, modalities $\langle A\rangle$ and $\langle E\rangle$ do not allow us, in general, to refer to the subinterval of a given interval. To overcome this problem, we label the suffix of a configuration interval starting immediately after an interval labeled with a given q (resp., a, c_i), with $conf_q$ (resp., $conf_a$, $conf_{c_i}$). This allows us to have indirect access to the components of a configuration by means of modality $\langle E\rangle$. For example, using such a technique, we can force that every configuration has at most one state and one alphabet symbol. Also, proposition letter \$b plays a central role here; we use it to guarantee that also the last c_i of each configuration can actually be associated with its corresponding $conf_{c_i}$.

$$\langle A\rangle q_0 \wedge [U](\langle A\rangle conf \leftrightarrow \langle A\rangle \$q) \quad \textit{the first configuration starts with } q_0 \tag{9}$$

$$[U]((\$q \to \langle A\rangle\$a) \wedge (\$a \vee \$c \to \langle A\rangle(\$c \vee \$b)) \wedge (\$b \to \langle A\rangle\$q)) \quad \textit{all } conf\textit{s have the right structure} \tag{10}$$

$$[U](\$q \to [A](conf' \to conf_q)) \wedge [U](\$a \wedge [A](conf' \to conf_a)) \quad conf_q \textit{ and } conf_a \textit{ are set} \tag{11}$$

$$[U]\neg(conf_q \wedge \langle E\rangle conf_q) \wedge [U]\neg(conf_a \wedge \langle E\rangle conf_a) \quad \textit{properties of } conf_q \textit{ and } conf_a \tag{12}$$

$$[U](\bigwedge_{i=1}^{k}(c_i \to [A](conf' \to conf_{c_i}))) \quad conf_{c_i} \textit{ is set for each counter} \tag{13}$$

A similar technical solution is used in the following set of formulas, where we introduce the proposition letters c_{dec} and c_{new}, together with the corresponding auxiliary proposition letters $conf_{dec}$ and $conf_{new}$. In particular, c_{dec}, that labels at most one counter element c_i of a given configuration, makes it possible to ensure that the i-th counter will be decreased by 1 as an effect of the next transition (when Δ contains such a command). The increasing of the i-th counter by 1 is encoded by means of the proposition letter c_{new}, that, possibly, labels a (unique) c_i added after the last transition (again, when Δ requires so):

$$[U]\Big(\bigwedge_{l\in\{new,dec\}} (c_l \rightarrow (\$c \wedge [A](conf' \rightarrow conf_l)))\Big) \quad \textit{when } c_l \textit{ then } conf_l \tag{14}$$

$$[U]\Big(\bigwedge_{l\in\{new,dec\}} ((\$c \wedge \langle A\rangle conf_l) \rightarrow c_l)\Big) \quad \textit{when } conf_l \textit{ then } c_l \tag{15}$$

$$[U]\Big(\bigwedge_{l\in\{new,dec\}} (conf_l \rightarrow \neg\langle E\rangle conf_l)\Big) \quad conf_{new} \textit{ and } conf_{dec} \textit{ are unique inside a } conf \tag{16}$$

The following set of formulas axiomatizes the properties of a proposition letter $corr$ (and $corr_{conf}$), which will be used to maintain counters' values across single transitions. Two $corr$-intervals might start at the same point (but not end at the same point), and this represents the faulty behavior of $\mathcal{A}$ that can increment (but not decrease) the value of counters non-deterministically:

$$[A](\langle A\rangle c_{new} \rightarrow \neg\langle E\rangle corr) \quad \textit{new counters do not correspond to anyone} \tag{17}$$

$$[U]((\$c \wedge \neg c_{dec}) \rightarrow \langle A\rangle corr) \quad \textit{non } dec \textit{ counters correspond to someone} \tag{18}$$

$$[U]((\$q \vee \$a \vee c_{dec}) \rightarrow [A]\neg corr) \quad q\textit{s, } a\textit{s, and } dec \textit{ counters do not correspond to anyone} \tag{19}$$

$$[U](([E]\bot \wedge \langle A\rangle corr) \rightarrow \$c) \quad corr \textit{ always starts with a counter} \tag{20}$$

$$[U](corr \rightarrow [E]corr' \wedge \langle A\rangle \$c) \wedge [U](\langle A\rangle conf \rightarrow [A](corr' \rightarrow corr_{conf})) \quad \textit{properties of } corr \textit{ and } corr_{conf} \tag{21}$$

$$[U](\langle A\rangle corr_{conf} \rightarrow \langle A\rangle conf) \quad \textit{more properties } corr_{conf} \tag{22}$$

$$[U]\neg(corr_{conf} \wedge \langle E\rangle corr_{conf}) \wedge [U](corr \rightarrow \langle E\rangle corr_{conf}) \quad \textit{each } conf \textit{ has a unique } corr_{conf} \tag{23}$$

$$[U](\bigwedge_{i=1}^{k}(c_i \rightarrow [A](corr \rightarrow \langle A\rangle c_i))) \quad \textit{each } corr \textit{ corresponds to some counter} \tag{24}$$

$$[U](corr \rightarrow \neg\langle E\rangle corr) \quad corr\textit{s are unique for each counter} \tag{25}$$

We formalize the transition relation Δ by making sure that if the automata has modified q into q' by reading a and moving from one configuration to another one, then the instruction l of some transition $(q, a, l, q') \in \Delta$ must have been applied.

$$\bigvee_{(q,a,(inc,i),q')\in\Delta} \Big(\langle A\rangle(q \wedge \langle A\rangle a) \wedge \langle A\rangle(conf \wedge \langle A\rangle q' \wedge \langle A\rangle(conf \wedge \langle E\rangle(conf_{c_i} \wedge conf_{new}))))\Big) \quad \textit{constraining instruction } (inc, i) \tag{26}$$

$$\bigvee_{(q,a,(dec,i),q')\in\Delta} \Big(\langle A\rangle(q \wedge \langle A\rangle a) \wedge \langle A\rangle(conf \wedge \langle A\rangle q' \wedge \langle E\rangle(conf_{c_i} \wedge conf_{dec}))\Big) \quad \textit{constraining instruction } (dec, i) \tag{27}$$

$$\bigvee_{(q,a,(ifz,i),q')\in\Delta} \Big(\langle A\rangle(q \wedge \langle A\rangle a) \wedge \langle A\rangle(conf \wedge \langle A\rangle q' \wedge [E]\neg conf_{c_i})\Big) \quad \textit{constraining instruction } (ifz, i) \tag{28}$$

$$[U]\Big(\langle A\rangle conf \rightarrow \big((26) \vee (27) \vee (28)\big)\Big) \quad \textit{global constraining} \tag{29}$$

Finally, we define $\varphi_{\mathcal{A}}$ as the conjunction of all above formulas plus the requirement that the infinite computation passes through a final state infinitely often:

$$\varphi_{\mathcal{A}} = (1) \wedge \ldots \wedge (25) \wedge (29) \wedge [A]\langle A\rangle\langle A\rangle \bigvee_{q_f \in F} q_f$$

It is straightforward to prove that $\varphi_{\mathcal{A}}$ is satisfiable if and only if $\mathcal{A}$ accepts at least one infinite word. $\square$

4 NP-Completeness

In this section, we prove that NP-completeness of $\mathsf{B}\overline{\mathsf{B}}$, shown in [14], can be extended to $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$. Since the satisfiability problem for propositional logic is NP-complete, that for every proper fragment of $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ including it is at least NP-hard. Unlike all other cases, the core of this section is a membership proof (namely, NP-membership): by a model-theoretic argument, it shows that satisfiability of $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$-formulas can be reduced to satisfiability in a periodic model where the lengths of prefixes and periods have a bound which is polynomial in the length of the original formula.

For the sake of simplicity, we consider the case of $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ interpreted over $\mathbb{N}$. The generalization to the whole class of strongly discrete linear orders is straightforward. Moreover, it can be shown that satisfiability of a $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$-formula φ over $\mathbb{N}$ can be reduced to satisfiability of the formula $\tau(\varphi) = \varphi \vee \langle\overline{B}\rangle\varphi \vee \langle L\rangle\varphi \vee \langle L\rangle\langle\overline{L}\rangle(\varphi \vee \langle\overline{B}\rangle\varphi)$ over the interval $[0,1]$, that is, $M,[x,y] \Vdash \varphi$ for some $[x,y]$ if and only $M,[0,1] \Vdash \tau(\varphi)$. Thus, we can safely restrict our attention to the problem of satisfiability over $[0,1]$ (*initial* satisfiability). As a preliminary step, we introduce some useful notation and notions, including the definition of periodic model.

Definition 3. *An interval model* $M = \langle \mathbb{I}(\mathbb{N}), V\rangle$ *is* ultimately periodic with prefix Pre and period Per *if, for every interval* $[x,y] \in \mathbb{I}(\mathbb{N})$ *and every proposition letter* $p \in AP$, *(i) if* $x \geqslant Pre$, *then* $[x,y] \in V(p)$ *iff* $[x+Per, y+Per] \in V(p)$ *and (ii) if* $y \geqslant Pre$, *then* $[x,y] \in V(p)$ *iff* $[x, y+Per] \in V(p)$.

Now, consider a $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$-formula φ, and define $Cl(\varphi)$ as the set of all its subformulas and their negations. Let M be a model such that $M,[0,1] \Vdash \varphi$. For each point x of the model, we can identify the maximal subset $\mathcal{R}_L(x)$ (resp., $\mathcal{R}_{\overline{L}}(x)$) of $Cl(\varphi)$ consisting of all and only $\langle L\rangle$-formulas (resp., $\langle\overline{L}\rangle$-formulas) and their negations that are satisfied over intervals ending (resp., beginning) at x. Notice that all intervals ending (resp., beginning) at the same point satisfy the same $\langle L\rangle$-formulas (resp., $\langle\overline{L}\rangle$-formulas). Let $\mathcal{R}(x) = \mathcal{R}_L(x) \cup \mathcal{R}_{\overline{L}}(x)$. $\mathcal{R}(x)$ must be consistent, as it cannot contain a formula and its negation. Now, let $\mathcal{R}$ be the subset of $Cl(\varphi)$ that contains all possible $\langle L\rangle$- and $\langle\overline{L}\rangle$-formula. It is easy to see that $|\mathcal{R}| \leqslant 2|\varphi|$. In the following we need also to compare intervals with respect to satisfiability of $\langle B\rangle$- and $\langle\overline{B}\rangle$-formulas. Given a model M, we say that two intervals $[x,y]$ and $[x',y']$ are B-equivalent (denoted $[x,y] \equiv_B [x',y']$) when, for every $\langle B\rangle\psi \in Cl(\varphi)$, $M,[x,y] \Vdash \langle B\rangle\psi$ iff $M,[x',y'] \Vdash \langle B\rangle\psi$ and, for every $\langle\overline{B}\rangle\psi \in Cl(\varphi)$, $M,[x,y] \Vdash \langle\overline{B}\rangle\psi$ iff $M,[x',y'] \Vdash \langle\overline{B}\rangle\psi$. We fix m_B to be the number of all $\langle B\rangle$- and $\langle\overline{B}\rangle$-formulas in $Cl(\varphi)$. To prove that satisfiability problem for $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ is in NP we first prove that every satisfiable formula φ has an ultimately periodic model, and then we show how to contract such model to a smaller one whose prefix and period are polynomial in the length of φ.

Lemma 3. *Let* φ *be a* $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$*-formula and* $M = \langle \mathbb{I}(\mathbb{N}), V\rangle$ *be a model such that* $M,[0,1] \Vdash \varphi$. *Then, there exists an ultimately periodic model* $M^* = \langle \mathbb{I}(\mathbb{N}), V^*\rangle$ *that satisfies* φ.

Proof. Let $M = \langle \mathbb{I}(\mathbb{N}), V\rangle$ be a model such that $M,[0,1] \Vdash \varphi$. If M is not ultimately periodic, we turn it into an ultimately periodic one as follows. First of all, by the transitivity of $\langle L\rangle$ and $\langle\overline{L}\rangle$, it is easy to see that there must exists a point $\overline{x} > 1$ such that $\mathcal{R}(y) = \mathcal{R}(\overline{x})$ for every $y \geqslant \overline{x}$. We fix the prefix Pre of the model to be equal to $\overline{x}$. Then, to define the periodic part of the model we choose a $\mathrm{Per} > m_B$ respecting the following properties: *(i)* for every point $x \leqslant \overline{x}$ and every formula $\langle L\rangle\psi \in \mathcal{R}(x)$ there exists an interval $[x_\psi, y_\psi]$ such that $M, [x_\psi, y_\psi] \Vdash \psi$ and $x < x_\psi < y_\psi < \mathrm{Pre} + \mathrm{Per}$; *(ii)* for every interval $[x, y]$ such that $x < \mathrm{Pre}$ and $y \geqslant \mathrm{Pre} + \mathrm{Per}$ and for every formula $\langle\overline{B}\rangle\psi$ such that $M, [x, y] \Vdash \langle\overline{B}\rangle\psi$ there exists an interval $[x, y_\psi]$ such that $[x, y] \equiv_B [x, y_\psi]$, $M, [x, y_\psi] \Vdash \psi$ and $\mathrm{Pre} \leqslant y_\psi < \mathrm{Pre} + \mathrm{Per}$. The transitivity of $\langle B\rangle$ and $\langle\overline{B}\rangle$ guarantees that such a Per can be found. To guarantee periodicity of the model, we must enforce the following additional property: *(iii)* for every interval $[x, y]$ such that $\mathrm{Pre} \leqslant x < \mathrm{Pre} + \mathrm{Per}$ and $y \geqslant \mathrm{Pre} + 2\mathrm{Per}$ and for every formula $\langle\overline{B}\rangle\psi$ such that $M, [x, y] \Vdash \langle\overline{B}\rangle\psi$ there exists an interval $[x, y_\psi]$ such that $[x, y] \equiv_B [x, y_\psi]$, $M, [x, y_\psi] \Vdash \psi$ and $y_\psi < \mathrm{Pre} + 2\mathrm{Per}$. If this not the case, we can modify the valuation V to guarantee that property *(iii)* holds as follows. Let $[x, y]$ be an interval that does not respect *(iii)*: we choose a finite set of "witness points" $\{y_1 < \ldots < y_k\}$ such that for every interval $[x, y']$ and every formula $\langle B\rangle\psi$, if $M, [x, y'] \Vdash \langle B\rangle\psi$ then there exists a witness point $x < y_i < y'$ such that $M, [x, y_i] \Vdash \psi$, and for every formula $\langle\overline{B}\rangle\psi$, if $M, [x, y'] \Vdash \langle\overline{B}\rangle\psi$ then there exists a witness point y_j such that $M, [x, y_j] \Vdash \psi$ and either $y_j > y'$ or $[x, y_j] \equiv_B [x, y']$. By the transitivity of $\langle B\rangle$ and $\langle\overline{B}\rangle$, and by the fact that the number of $\langle B\rangle$- and $\langle\overline{B}\rangle$-formulas in $Cl(\varphi)$ is bounded, it is easy to see that the number of witness points is less or equal to m_B. Now, we concentrate our attention only on those witness points $\{y_j < \ldots < y_k\}$ that are greater than $\mathrm{Pre} + \mathrm{Per}$, and we turn V into a new valuation V' where all intervals starting in x respect *(iii)* as follows: *(1)* for every $x < y' \leqslant \mathrm{Pre} + \mathrm{Per}$, we put $[x, y'] \in V'(p)$ iff $[x, y'] \in V(p)$; *(2)* for every $j \leqslant i \leqslant k$, we put $[x, \mathrm{Pre} + \mathrm{Per} + i] \in V'(p)$ iff $[x, y_i] \in V(p)$; *(3)* for every $x + \mathrm{Pre} + \mathrm{Per} + k < y' \leqslant y_k$, we put $[x, y'] \in V'(p)$ iff $[x, y_k] \in V(p)$; *(4)* the valuation of all other intervals is unchanged. Note that after this procedure no other interval $[x, y']$ starting at x can falsify property *(iii)*. By repeating the above procedure a sufficient number of times we can obtain a model for the formula respecting all the required properties.

We are now ready to build the required ultimately periodic model $M^* = \langle \mathbb{I}(\mathbb{N}), V^*\rangle$. First we define the valuation function V^* for some of the intervals in the prefix and in the first occurrence of the period: *(1)* for every $p \in AP$ and for every $[x, y]$ such that $y < \mathrm{Pre} + \mathrm{Per}$, $[x, y] \in V^*(p)$ iff $[x, y] \in V'(p)$; *(2)* for every $p \in AP$ and for every $[x, y]$ such that $\mathrm{Pre} \leqslant x < \mathrm{Pre} + \mathrm{Per}$ and $y \leqslant x + \mathrm{Per}$, $[x, y] \in V^*(p)$ iff $[x, y] \in V'(p)$. Then, we extend V^* to cover the entire model: *(1)* for every $p \in AP$ and for every $[x, y]$ such that $x < \mathrm{Pre}$ and $y \geqslant \mathrm{Pre} + \mathrm{Per}$, $[x, y] \in V^*(p)$ iff $[x, y - \mathrm{Per}] \in V^*(p)$; *(2)* for every $p \in AP$ and for every $[x, y]$ such that $\mathrm{Pre} \leqslant x < \mathrm{Pre} + \mathrm{Per}$ and $y > x + \mathrm{Per}$, $[x, y] \in V^*(p)$ iff $[x, y - \mathrm{Per}] \in V^*(p)$; *(3)* for every $p \in AP$ and for every $[x, y]$ such that $x \geqslant \mathrm{Pre} + \mathrm{Per}$, $[x, y] \in V^*(p)$ iff $[x - \mathrm{Per}, y - \mathrm{Per}] \in V^*(p)$. It is straightforward to prove that $M^*, [0, 1] \Vdash \varphi$ and thus that M^* is the ultimately periodic model we were looking for. □

By applying a point-elimination technique similar to the one used in [7] to prove NP-membership of $\mathsf{B\overline{B}L\overline{L}}$ over finite linear orders, we can reduce the length of the prefix and period of an ultimately periodic model to a dimension polynomial in the length of φ, as proved in the following lemma.

Lemma 4. *Let φ be a $\mathsf{B\overline{B}L\overline{L}}$-formula. Then, φ is initially satisfiable over the natural numbers if and only if it is initially satisfiable over an ultimately periodic model model $M = \langle \mathbb{I}(\mathbb{N}), V\rangle$ with prefix* Pre *and period* Per *such that* $\mathrm{Pre} + \mathrm{Per} \leqslant (m_L + 2)m_B + m_L + 4$, *where* $m_L = 2|\mathcal{R}|$.

Proof. By Lemma 3 we can assume that φ is initially satisfied over an ultimately periodic model $M = \langle \mathbb{I}(\mathbb{N}), V\rangle$. If $\mathrm{Pre} + \mathrm{Per} > (m_L + 2)m_B + m_L + 4$, we proceed as follows. Consider all points $1 < x < \mathrm{Pre} + 2\mathrm{Per}$: for each $\psi \in Cl(\varphi)$ such that $\langle L\rangle\psi \in \mathcal{R}(x)$ for some x, choose a point $1 < x^\psi_{max} \leqslant \mathrm{Pre} + \mathrm{Per}$ and a point $y^\psi_{max} < \mathrm{Pre} + 2\mathrm{Per}$ such that the interval $[x^\psi_{max}, y^\psi_{max}]$ satisfies ψ and that for each $x^{\psi}_{max} < x \leqslant \mathrm{Pre}+\mathrm{Per}$ no interval starting at x satisfies ψ. Collect all such points into a set (of L-*blocked* points) $Bl_L \subset \{0,\ldots,\mathrm{Pre}+2\mathrm{Per}\}$. Then, for each $\psi \in Cl(\varphi)$ such that $\langle\overline{L}\rangle\psi \in \mathcal{R}(x)$ for some x, choose an interval $[x^{\psi}_{min}, y^{\psi}_{min}]$ satisfying ψ and such that for each $y < y^{\psi}_{min}$ no interval ending at y satisfies ψ. Put all point $x^{\psi}_{min}, y^{\psi}_{min}$ into a set (of $\overline{L}$-*blocked* points) $Bl_{\overline{L}} \subset \{0,\ldots,\mathrm{Pre}\}$. Define $Bl = Bl_L \cup Bl_{\overline{L}} \cup \{\mathrm{Pre},\mathrm{Pre}+\mathrm{Per}\}$. Obviously, $|Bl| \leqslant m_L+2$. Now, suppose $Bl = \{x_1 < x_2 < \ldots < x_n\}$. For each $0 < i < n$, call $Bl_i = \{x | x_i < x < x_{i+1}\}$; similarly, let $Bl_0 = \{x | 0 < x < x_1\}$ and $Bl_n = \{x | x_n < x < \mathrm{Pre}+2\mathrm{Per}\}$. We prove that if $y, y' \in Bl_i$, for some i, then $\mathcal{R}(y) = \mathcal{R}(y')$. Proceed by contradiction, that is, assume $R(y) \neq R(y')$. By the definition of ultimately periodic model, this implies that at least one between y and y' must belong to the prefix of M. If $\langle L\rangle\psi \in R(y)$ and $\langle L\rangle\psi \notin R(y')$, then, by definition, $[L]\neg\psi \in R(y')$. This implies that $y < y'$, as $\langle L\rangle$ is transitive, and hence that $y < \mathrm{Pre}$. Now, consider the interval $[x^{\psi}_{max}, y^{\psi}_{max}]$ defined above. Two cases may arise: either $x^{\psi}_{max} < y$, or $x^{\psi}_{max} > y'$. In the former case, since $\langle L\rangle\psi \in R(y)$, there must exists an interval $[x'', y'']$ satisfying ψ and such that $x^{\psi}_{max} < x'' \leqslant y'$, in contradiction with the definition of x^{ψ}_{max}. In the latter case, we have $[L]\neg\psi \notin R(y')$, in contradiction with the hypothesis. The cases in which $\langle\overline{L}\rangle\psi \in R(y)$ and $\langle\overline{L}\rangle\psi \notin R(y')$ can be proved in a similar way. Since we assumed that $\mathrm{Pre}+\mathrm{Per} > (m_L+2)m_B + m_L + 4$, by a simple combinatorial argument there must exist a set Bl_i, for some $x_{i+1} \leqslant \mathrm{Pre}+\mathrm{Per}$, such that $|Bl_i| > m_B$: let $\overline{x}$ be the first point in such a Bl_i. We now prove that the model $M' = \langle \mathbb{I}(\mathbb{N}\setminus\{\overline{x}\}), V'\rangle$, where $\overline{x}$ has been eliminated and where V' is a suitable adaptation of V, is such that $M', [0,1] \Vdash \varphi$. Consider $M'' = \langle \mathbb{I}(\mathbb{N}\setminus\{\overline{x}\}), V''\rangle$, where V'' is the projection of V over the intervals that do not start nor end with $\overline{x}$. The satisfaction of box-formulas (from $Cl(\varphi)$) has not been affected anywhere in the model, by definition. The only potential problem is the presence of some diamond-formulas which were satisfied in M and are not satisfied anymore in M''. Let $[x,y]$, where $y < \overline{x}$, such that $M,[x,y] \Vdash \langle L\rangle\psi$. By definition of Bl, there exists an interval $[x^{\psi}_{max}, y^{\psi}_{max}]$ satisfying ψ and such that $x^{\psi}_{max}, y^{\psi}_{max} \in Bl$, $x^{\psi}_{max} \leqslant \mathrm{Pre}+\mathrm{Per}$, and that there exists no interval $[x', y']$ satisfying ψ, with $x^{\psi}_{max} < x \leqslant \mathrm{Pre}+\mathrm{Per}$. Then, either $x^{\psi}_{max} > y$ or there exists an interval $[x', y']$ such that $M,[x',y'] \Vdash \psi$ and $x' > \mathrm{Pre}+\mathrm{Per}$. Therefore, $M'',[x,y] \Vdash \langle L\rangle\psi$. The same argument, in a symmetric way, applies to the case of $\langle\overline{L}\rangle\psi$, and thus, diamond-formulas of the type $\langle L\rangle\vartheta$ or $\langle\overline{L}\rangle\vartheta$ never generate problems after the elimination. Assume now that, for some $y < x < \overline{x}$ (resp., $y < \overline{x} < x$) and some formula $\langle\overline{B}\rangle\psi \in Cl(\varphi)$ ($\langle B\rangle\psi \in Cl(\varphi)$), it is the case that $M,[y,x] \Vdash \langle\overline{B}\rangle\psi$ (resp., $M,[y,x] \Vdash \langle B\rangle\psi$) and that $[y,\overline{x}]$ was the only interval starting at y (in M) that satisfied ψ. Since $\overline{x}$ is the first point in Bl_i, we have that $M,[y,x_i] \Vdash \langle\overline{B}\rangle\psi$ (resp., $M,[y,x_{i+1}] \Vdash \langle B\rangle\psi$) by transitivity of $\langle\overline{B}\rangle$ (resp., $\langle B\rangle$). Consider now the first m_B successors of $\overline{x}$: $\overline{x}+1,\ldots\overline{x}+m_B$. Since $|Bl_i| > m_B$, we have that all those points belong to Bl_i. It is possible to prove that there exist at least one point $\overline{x}+k$ that satisfies the following properties: *(i)* for every $\langle B\rangle\xi \in Cl(\varphi)$, if $M,[y,\overline{x}+k+1] \Vdash \langle B\rangle\xi$, then $M,[y,\overline{x}+k] \Vdash \langle B\rangle\xi$, and *(ii)* for every $\langle\overline{B}\rangle\zeta \in Cl(\varphi)$, if $M,[y,\overline{x}+k-1] \Vdash \langle\overline{B}\rangle\zeta$, then $M,[y,\overline{x}+k] \Vdash \langle\overline{B}\rangle\zeta$. One can convince himself that this is the case by observing that, by the transitivity of $\langle B\rangle$, if $M,[y,\overline{x}+k+1] \Vdash \langle B\rangle\xi$ then $M,[y,x'] \Vdash \langle B\rangle\xi$ for every $x' \geqslant \overline{x}+k+1$. Hence, if $\overline{x}+k$ does not respect property *(i)* for ξ, all its successors are forced to respect it for $\langle B\rangle\xi$. Symmetrically, by the transitivity of $\langle\overline{B}\rangle$, if $M,[y,\overline{x}+k-1] \Vdash \langle\overline{B}\rangle\zeta$ but $M,[y,\overline{x}+k] \nVdash \langle\overline{B}\rangle\zeta$, then $M,[y,x'] \nVdash \langle\overline{B}\rangle\zeta$ for every $x' \geqslant \overline{x}+k$. Hence, all successors of $\overline{x}+k$ trivially respect property *(ii)* for $\langle\overline{B}\rangle\zeta$. Since the number of $\langle B\rangle$- and $\langle\overline{B}\rangle$-formulas is limited by m_B, a point with the required properties can always be found. We fix the defect by defining the labeling V' as follows: we put $[y,\overline{x}+t] \in V'(p)$ if and only if $[y,\overline{x}+t-1] \in V(p)$, for every proposition letter p and $1 \leqslant t \leqslant k$. The labeling of the other intervals remain unchanged. From the definition of the set Bl, it follows that this change of labeling does not introduce new defects of any kind. By iterating the above described procedure, we obtain a model $\overline{M} = \langle \mathbb{I}(\mathbb{N}), \overline{V}\rangle$ where $\mathrm{Pre}+\mathrm{Per} \leqslant (m_L+2)m_B + m_L + 4$. However, since we modified only the finite portion of the model included between 0 and $Pre+2Per$, to conclude the proof we must propagate the changes to the remaining infinite suffix. We do so as in the proof of the previous lemma, and build an ultimately periodic model $M^* = \langle \mathbb{I}(\mathbb{N}), V^* \rangle$ as follows: *(i)* for every $p \in AP$ and for every $[x,y]$ such that $y \leqslant Pre+Per$, $[x,y] \in V^*(p)$ iff $[x,y] \in \overline{V}(p)$; *(ii)* for every $p \in AP$ and for every $[x,y]$ such that $Pre < x \leqslant Pre+Per$ and $y \leqslant x+Per$, $[x,y] \in V^*(p)$ iff $[x,y] \in \overline{V}(p)$; *(iii)* for every $p \in AP$ and for every $[x,y]$ such that $x \leqslant Pre$ and $y > Pre+Per$, $[x,y] \in V^*(p)$ iff $[x,y-Per] \in V^*(p)$; *(iv)* for every $p \in AP$ and for every $[x,y]$ such that $Pre < x \leqslant Pre+Per$ and $y > x+Per$, $[x,y] \in V^*(p)$ iff $[x,y-Per] \in V^*(p)$; *(v)* for every $p \in AP$ and for every $[x,y]$ such that $x \geqslant Pre+Per$, $[x,y] \in V^*(p)$ iff $[x-Per,y-Per] \in V^*(p)$. This concludes the proof. $\square$

5 NEXPTIME- and EXPSPACE-Completeness

As pointed out in the introduction, NEXPTIME-complete and EXPSPACE-complete decidable fragments are already known. Let us briefly summarize here the situation. NEXPTIME-membership of $\mathsf{A}\overline{\mathsf{A}}$ has been proved in [5]. NEXPTIME-hardness of A, shown in [9], holds also for the class of strongly discrete linear orders, and it can be easily adapted to the case of $\overline{\mathsf{A}}$, thus proving NEXPTIME-hardness of any fragment including $\langle A \rangle$ or $\langle \overline{A} \rangle$. As for EXPSPACE-complete fragments, we know from [10] that $\mathsf{AB}\overline{\mathsf{BL}}$ is EXPSPACE-complete. Hardness for this class is claimed in the same paper for the fragments $\mathsf{AB}\overline{\mathsf{B}}$ and AB. This can be proved by a reduction from the exponential-corridor tiling problem, which is known to be EXPSPACE-complete [20]. In [7], it has been proved that this reduction can be modified in a suitable way to cover $\mathsf{A}\overline{\mathsf{B}}$, and both reductions for AB and $\mathsf{A}\overline{\mathsf{B}}$ immediately apply to the case of strongly discrete linearly ordered sets. Given a tuple $\mathcal{T} = (T, t_\perp, t_\top, H, V, n)$ consisting of a finite set T of tiles, a bottom tile $t_\perp \in T$, a top tile $t_\top \in T$, two binary relations H and V over T (specifying the horizontal and vertical constraints), the problem consists in deciding whether there exists a tiling function f from a discrete corridor of exponential height in n to T that associates the tile $t_\perp$ (resp., $t_\top$) with the bottom (resp., top) row of the corridor and that respects the horizontal and vertical constraints H and V.The reduction exploits the correspondence between the points inside the corridor and the intervals of the model, and $|T|$ proposition letters to represent the tiling function f. The coordinates of each row of the corridor are represented in binary by means of additional proposition letters. Modalities allow one to enforce the local constrains over the tiling function f.

6 Decidability and Complexity over $\mathbb{N}$

In this last section, we focus our attention on the domain of natural numbers. As already pointed in the introduction, the asymmetry of $\mathbb{N}$-models, which are left-bounded and right-unbounded, introduces an asymmetry in the computational behavior of (some of) the fragments of $\mathsf{A}\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}$ and its mirror image $\mathsf{A}\overline{\mathsf{A}}\mathsf{E}\overline{\mathsf{E}}$. More precisely, such an asymmetry of $\mathbb{N}$-models has the following consequences: *(i)* $\overline{\mathsf{A}}\mathsf{B}$, but not AE, becomes decidable (non-primitive recursive) [18]; *(ii)* $\overline{\mathsf{AB}}$ and $\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}$, but not $\mathsf{A}\overline{\mathsf{E}}$ nor $\mathsf{AE}\overline{\mathsf{E}}$, become decidable (this can be shown by a suitable adaptation of the argument given in [18]); *(iii)* $\overline{\mathsf{A}}\mathsf{BL}$ and $\overline{\mathsf{AB}}\mathsf{L}$ remain undecidable, but the undecidability proof given in [18] must be suitably adapted.

Theorem 2. *The Hasse diagram in Figure 3 correctly shows all the decidable fragments of* HS *over* $\mathbb{N}$*, their relative expressive power, and the precise complexity class of their satisfiability problem.*

The main ingredients of the decidability proof for $\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}$ (and thus for $\overline{\mathsf{AB}}$ and $\overline{\mathsf{A}}\mathsf{B}$) can be summarized as follows. Let φ be a satisfiable $\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}$-formula and let $M = \langle \mathbb{I}(\mathbb{N}), V \rangle$ be a model such that $M, [x_\varphi, y_\varphi] \Vdash$

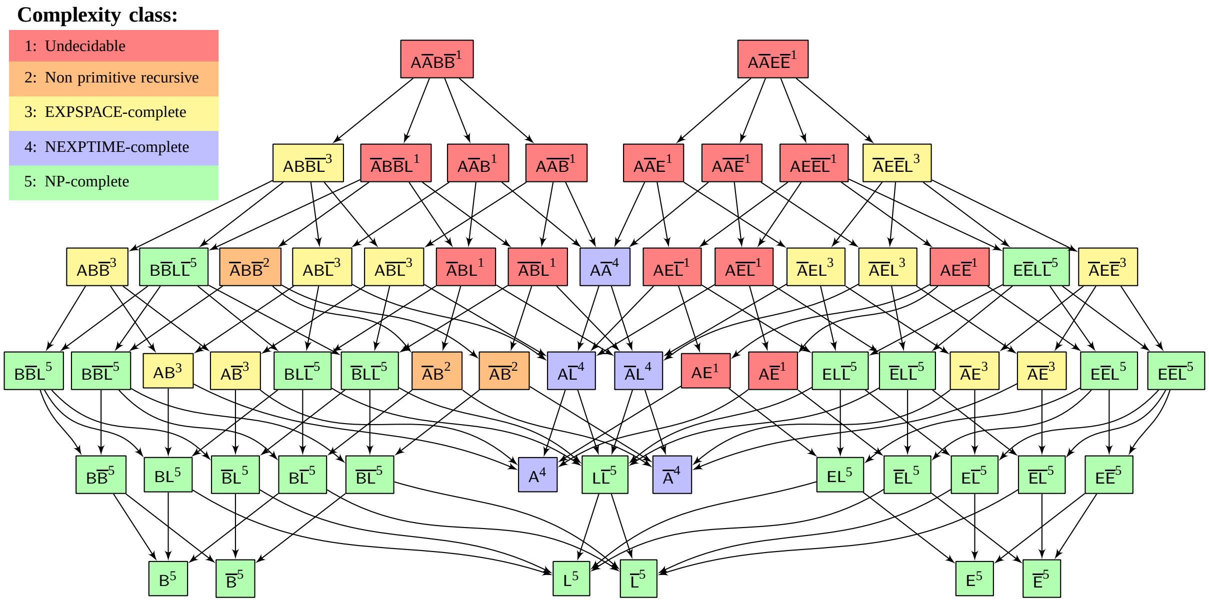


Figure 3: Hasse diagram of all fragments of $A\overline{A}B\overline{B}$ and $A\overline{A}E\overline{E}$ over the natural numbers.

φ for some interval $[x_\varphi, y_\varphi]$. It can be easily checked that modalities $\langle\overline{A}\rangle$, $\langle B\rangle$, and $\langle\overline{B}\rangle$ do not allow one to access any interval $[x, y]$, with $x > x_\varphi$, starting from interval $[x_\varphi, y_\varphi]$. Hence, the valuation of such intervals can be safely ignored, as it does not affect the truth value of the formula.

By exploiting such a limitation of $\overline{A}B\overline{B}$ modalities, we can reduce the search for a model of φ to the set of ultimately periodic models only, as it is possible to prove that for each satisfiable $\overline{A}B\overline{B}$-formula there exist an ultimately periodic model $M^* = \langle \mathbb{I}(\mathbb{N}), V^*\rangle$ and an interval $[x_\varphi, y_\varphi]$ such that $M, [x_\varphi, y_\varphi] \Vdash \varphi$, $y_\varphi < Pre$, and $Per \leqslant m_B$ (it can be easily shown that the length of the period is bounded by the number m_B of all $\langle B\rangle$- and $\langle\overline{B}\rangle$-formulas in $Cl(\varphi)$).

We can exploit the algorithm for satisfiability checking of $A\overline{A}B\overline{B}$ formulas over finite linear orders given in [18] to guess the non-periodic part of the model. Then, the algorithm for satisfiability checking of $\overline{A}B\overline{B}$ formulas can be exploited to check whether the guessed prefix can be extended to a complete model over $\mathbb{I}(\mathbb{N})$ by guessing the valuation of intervals $[x, y]$ such that $x < Pre$ and $Pre \leqslant y \leqslant Pre + Per$.

To prove termination of the algorithm, it suffices to observe that if the guessed prefix is not *minimal* in the sense of [18], we can shrink it into a smaller one that satisfies the minimality condition (see Proposition 2 and Figure 3 in [18]). Since the number of minimal prefix models is bounded, and the length of the period is bounded as well, we can conclude that the satisfiability problem for $\overline{A}B\overline{B}$ over $\mathbb{N}$ is decidable. Non-primitive recursiveness has been already shown in [7].

In a very similar way, it is not difficult to adapt the reduction given in [18] to prove the undecidability of $\overline{A}BL$ and $\overline{A}B\overline{L}$ over $\mathbb{N}$. In this case, we reduce the structural termination problem for lossy counter automata [17] to the satisfiability problem for $\overline{A}BL$ and for $\overline{A}B\overline{L}$. Since the universal modality $[U]$ can be expressed in $\overline{A}BL$ and $\overline{A}B\overline{L}$ as $[U]\varphi = \varphi \wedge [L]([\overline{A}]\varphi \wedge [\overline{A}][\overline{A}]\varphi)$, one can repeat the entire construction developed in [18] to encode an infinite computation of the lossy counter automata, using $\langle L\rangle$ to impose the required properties on final states.

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