Expressiveness of the Interval Logics of Allen's Relations on the Class of all Linear Orders: Complete Classification

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Interval reasoning naturally arises in various fields of computer science, including system verification, planning, natural language analysis and processing, and constraint satisfaction problems. Interval temporal logics formalize reasoning about interval structures over ordered domains, where time intervals, rather than time instants, are the primitive ontological entities. The variety of binary relations between intervals in linear orders was first studied systematically by Allen [1], who explored their use in systems for time management and planning. The modal logic featuring modal operators corresponding to Allen's interval relations was introduced by Halpern and Shoham in [7]; we hereafter call that logic HS.

In [7], it was shown that the satisfiability problem for HS is undecidable in all natural classes of linear orders. For a long time, these sweeping undecidability results have discouraged attempts for practical applications of interval logics. A renewed interest in the area has recently been stimulated by the discovery of several interesting decidable fragments of HS [3, 4, 5, 8, 9]. In that context, and for the purpose of identifying expressive interval logics for various intended applications, the comparative analysis of the expressiveness of the variety of interval logics is a major research problem in the area. In particular, the important problem arises to analyze the mutual definabilities among the modal operators of the logic HS and to classify the fragments of HS with respect to their expressiveness.

In the present paper we address and solve that problem, by identifying a complete set of inter-definability formulae among the modal operators of HS and thus providing a complete classification of all fragments of HS with respect to their expressiveness for the *strict* semantics (excl. point intervals) over the class of all linear orders. Using that result we have found that there are exactly 1347 expressively different such fragments out of $2^{12} = 4096$ sets of modal operators in HS.

The choice of strict semantics, excluding point intervals, instead of including them (non-strict semantics), conforms to the definition of interval adopted by Allen in [1]. It has at least two strong motivations. First, a number of representation paradoxes arise when the non-strict semantics is adopted, due to the presence of point intervals, as pointed out in [1]. Second, when point intervals are included, there seems to be no intuitive semantics for interval relations that makes them both pairwise disjoint and jointly exhaustive.

Definition 1. *A modal operator $\langle X \rangle$ of HS is* definable *in an HS-fragment $\mathcal{F}$, denoted $\langle X \rangle \lhd \mathcal{F}$, if $\langle X \rangle p \equiv \psi$ for some formula $\psi = \psi(p) \in \mathcal{F}$, for any fixed propositional variable p. In such a case, the equivalence $\langle X \rangle p \equiv \psi$ is called an* inter-definability equation for $\langle X \rangle$ in $\mathcal{F}$.

It is known from [7] that, in the strict semantics, all modal operators in HS are definable in the fragment containing the modalities $\langle A \rangle$, $\langle B \rangle$, and $\langle E \rangle$, and their transposes $\langle \overline{A} \rangle$, $\langle \overline{B} \rangle$, and $\langle \overline{E} \rangle$ (in the non-strict semantics, the four modalities $\langle B \rangle$, $\langle E \rangle$, $\langle \overline{B} \rangle$, and $\langle \overline{E} \rangle$ suffice, as shown in [10]).

In this paper, we compare and classify the expressiveness of all fragments of HS on the class of all interval structures over linear orders. Formally, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be any pair of such fragments. We say that:
- $\mathcal{F}_2$ is *at least as expressive as* $\mathcal{F}_1$, denoted $\mathcal{F}_1 \preceq \mathcal{F}_2$, if every operator $\langle X \rangle \in \mathcal{F}_1$ is definable in $\mathcal{F}_2$.
- $\mathcal{F}_1$ is *strictly less expressive* than $\mathcal{F}_2$, denoted $\mathcal{F}_1 \prec \mathcal{F}_2$, if $\mathcal{F}_1 \preceq \mathcal{F}_2$ but not $\mathcal{F}_2 \preceq \mathcal{F}_1$.
- $\mathcal{F}_1$ and $\mathcal{F}_2$ are *equally expressive* (or, *expressively equivalent*), denoted $\mathcal{F}_1 \equiv \mathcal{F}_2$, if $\mathcal{F}_1 \preceq \mathcal{F}_2$ and $\mathcal{F}_2 \preceq \mathcal{F}_1$.
- $\mathcal{F}_1$ and $\mathcal{F}_2$ are *expressively incomparable*, denoted $\mathcal{F}_1 \not\equiv \mathcal{F}_2$, if neither $\mathcal{F}_1 \preceq \mathcal{F}_2$ nor $\mathcal{F}_2 \preceq \mathcal{F}_1$.

In order to show non-definability of a given modal operator in a given fragment, we use a standard technique in modal logic, based on the notion of *bisimulation* and the invariance of modal formulae with respect to bisimulations (see, e.g., [2]). Let $\mathcal{F}$ be an HS-fragment. An $\mathcal{F}$-bisimulation between two interval models $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}'), V' \rangle$ over $\mathcal{AP}$ is a relation $Z \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ satisfying the following properties:
- *local condition*: Z-related intervals satisfy the same propositional letters over $\mathcal{AP}$;
- *forward condition*: if $([a, b], [a', b']) \in Z$ and $([a, b], [c, d]) \in R_X$ for some $\langle X \rangle \in \mathcal{F}$, then there exists $[c', d']$ such that $([a', b'], [c', d']) \in R_X$ and $([c, d], [c', d']) \in Z$;
- *backward condition*: likewise, but from M' to M.

The important property of bisimulations used here is that any $\mathcal{F}$-bisimulation preserves the truth of *all* formulae in $\mathcal{F}$. Thus, in order to prove that an operator $\langle X \rangle$ is not definable in $\mathcal{F}$, it suffices to construct a pair of interval models M and M' and a $\mathcal{F}$-bisimulation between them, relating a pair of intervals $[a, b] \in M$ and $[a', b'] \in M'$, such that $M, [a, b] \Vdash \langle X \rangle p$, while $M', [a', b'] \nVdash \langle X \rangle p$.

In order to classify all fragments of HS with respect to their expressiveness, it suffices to identify all definabilities of modal operators $\langle X \rangle$ in fragments $\mathcal{F}$, where $\langle X \rangle \notin \mathcal{F}$.

$\langle \mathsf{L} \rangle p \equiv \langle \mathsf{A} \rangle \langle \mathsf{A} \rangle p$	$\langle \mathsf{L} \rangle \lhd \mathsf{A}$
$\langle \overline{\mathsf{L}} \rangle p \equiv \langle \overline{\mathsf{A}} \rangle \langle \overline{\mathsf{A}} \rangle p$	$\langle \overline{\mathsf{L}} \rangle \lhd \overline{\mathsf{A}}$
$\langle \mathsf{O} \rangle p \equiv \langle \mathsf{E} \rangle \langle \overline{\mathsf{B}} \rangle p$	$\langle \mathsf{O} \rangle \lhd \overline{\mathsf{B}}\mathsf{E}$
$\langle \overline{\mathsf{O}} \rangle p \equiv \langle \mathsf{B} \rangle \langle \overline{\mathsf{E}} \rangle p$	$\langle \overline{\mathsf{O}} \rangle \lhd \mathsf{B}\overline{\mathsf{E}}$
$\langle \mathsf{D} \rangle p \equiv \langle \mathsf{E} \rangle \langle \mathsf{B} \rangle p$	$\langle \mathsf{D} \rangle \lhd \mathsf{BE}$
$\langle \overline{\mathsf{D}} \rangle p \equiv \langle \overline{\mathsf{E}} \rangle \langle \overline{\mathsf{B}} \rangle p$	$\langle \overline{\mathsf{D}} \rangle \lhd \overline{\mathsf{B}}\overline{\mathsf{E}}$
$\langle \mathsf{L} \rangle p \equiv \langle \overline{\mathsf{B}} \rangle [\mathsf{E}] \langle \overline{\mathsf{B}} \rangle \langle \mathsf{E} \rangle p$	$\langle \mathsf{L} \rangle \lhd \overline{\mathsf{B}}\mathsf{E}$
$\langle \overline{\mathsf{L}} \rangle p \equiv \langle \overline{\mathsf{E}} \rangle [\mathsf{B}] \langle \overline{\mathsf{E}} \rangle \langle \mathsf{B} \rangle p$	$\langle \overline{\mathsf{L}} \rangle \lhd \mathsf{B}\overline{\mathsf{E}}$

Table 1. The complete set of inter-definability equations

A definability $\langle \mathsf{X} \rangle \lhd \mathcal{F}$ is *optimal* if $\langle \mathsf{X} \rangle \not\lhd \mathcal{F}'$ for any fragment $\mathcal{F}'$ such that $\mathcal{F}' \prec \mathcal{F}$. A set of such definabilities is optimal if it consists of optimal definabilities.

The main result of the paper is the following theorem. Details about the proof can be found in [6]

Theorem 1. *The set of inter-definability equations given in Table 1 is sound, complete, and optimal.*

Most of the equations in Table 1 are known from [7], except the definability $\langle \mathsf{L} \rangle \lhd \overline{\mathsf{B}}\mathsf{E}$ and its symmetric, $\langle \overline{\mathsf{L}} \rangle \lhd \mathsf{B}\overline{\mathsf{E}}$, which are new.

While proving the soundness of the given set of inter-definability equations is quite immediate, proving completeness is the hard task; optimality will be established together with it. The completeness proof is organized as follows. For each HS operator $\langle \mathsf{X} \rangle$, we show that $\langle \mathsf{X} \rangle$ is not definable in any fragment of HS that does not contain as definable (according to Table 1) all operators of some of the fragments in which $\langle \mathsf{X} \rangle$ is definable (according to Table 1). More formally, for each HS operator $\langle \mathsf{X} \rangle$, the proof consists of the following steps:

1. using Table 1, find all fragments $\mathcal{F}_i$ such that $\langle \mathsf{X} \rangle \lhd \mathcal{F}_i$;
2. identify the list $\mathcal{M}_1, \ldots, \mathcal{M}_m$ of all $\subseteq$-maximal fragments of HS that contain neither the operator $\langle \mathsf{X} \rangle$ nor any of the fragments $\mathcal{F}_i$ identified by the previous step;
3. for each fragment $\mathcal{M}_i$, with $i \in \{1, \ldots, m\}$, provide a bisimulation for $\mathcal{M}_i$ which is not a bisimulation for X.

We have used the equations in Table 1 as the basis of a simple program that identifies and counts all expressively different fragments of HS with respect to the strict semantics. Using that program, we have found that, under our assumptions (strict semantics, over the class of all linear orders) there are exactly 1347 genuine, that is, expressively different, fragments out of $2^{12} = 4096$ different subsets of HS-operators.

To sum up, in this paper, we have obtained a sound, complete, and optimal set of inter-definability equations among all modal operators in HS, thus providing a characterization of the relative expressive power of all interval logics definable as fragments of HS. Such a classification has a number of important applications. As an example, it allows one to properly identify the (small) set of HS fragments for which the decidability of the satisfiability problem is still an open problem.

It should be emphasized that the set of inter-definability equations listed in Table 1 and the resulting classification do not apply if the non-strict semantics is considered. Also, if the semantics is restricted to specific classes of linear orders, the completeness of the set of equations in Table 1 is no longer guaranteed. The classification of the expressiveness of HS fragments with respect to the non-strict semantics, as well as over specific classes of linear orders, is currently under investigation and will be reported in a forthcoming publication.

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