Undecidability of the Logic of the *Overlap* Relation over Discrete Linear Orderings

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The ability of representing and reasoning about time intervals is requested in a variety of computer science fields, including natural language processing, constraint satisfaction problems, theories of action and change, temporal databases, specification and verification of concurrent and real-time systems. Unlike point-based ones, interval (linear) temporal logics assume time intervals as their primitive ontological entities. A systematic analysis of the variety of relations between intervals on linear orderings was first accomplished by Allen in a restricted algebraic setting [1], with the aim of exploiting interval reasoning in systems for time management and planning. The temporal logic counterpart of Allen's Interval Algebra is Halpern and Shoham's modal logic HS [8], which features a modal operator for each Allen's interval relation, except for the equality one, namely, "ends" E, "during" D, "begins" B, "overlaps" O, "meets" A, "later" L, and their inverses $\bar{\mathrm{E}}, \bar{\mathrm{D}}, \bar{\mathrm{B}}, \bar{\mathrm{O}}, \bar{\mathrm{A}}, \bar{\mathrm{L}}$. The validity/satisfiability problem for HS turns out to be highly undecidable under very weak assumptions on the class of interval structures over which its formulas are interpreted. In particular, it is undecidable when interpreted over any class of linearly-ordered structures that contains at least one linear ordering with an infinite ascending or descending chain, thus including $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$.

The bad computational behavior of HS motivates a systematic analysis of the large set of its fragments in the attempt of identifying the precise boundaries between decidability and undecidability and in the search for expressive enough, but decidable, fragments. A first step in this direction was done by Halpern and Shoham themselves in their original paper. They show that the negative complexity bounds for HS hold even if one restricts the logic to its ABE fragment and suggest to investigate weaker or incomparable meaningful fragments such as BE and $\mathrm{D}\bar{\mathrm{D}}$ (as a matter of fact, BE undecidability over dense linear orderings was proved by Lodaya almost ten years later [9] and the decidability of $\mathrm{D}\bar{\mathrm{D}}$ over $\mathbb{Q}$ has been just proved [10]). In the recent years, the identification of significant decidable fragments of HS, such as the logic of temporal neighborhood $\mathrm{A}\bar{\mathrm{A}}$ [5, 6, 7] and the logic of the subinterval relation D [4, 11], brought new interest about the investigation of HS fragments. A (still incomplete) classification of HS fragments with respect to decidability/undecidability can be found in [2]. Additional undecidability results are given in [3]. The case of one-modality fragments is particularly interesting. While undecidability dominates over the complete set of HS fragments, decidability is the rule in the subset of one-modality fragments. The decidability of B, $\bar{\mathrm{B}}$, E, $\bar{\mathrm{E}}$ can be easily shown by a reduction to point-based logics. The decidability of A, $\bar{\mathrm{A}}$, and thus that of L, $\bar{\mathrm{L}}$, can be proved by a model-theoretic argument (small model theorem). The decidability of D over dense linear orderings has been demonstrated by an application of the filtration technique and the proof can be adapted to the case of $\bar{\mathrm{D}}$ (the decision problem for D over finite/discrete linear orderings presents various technical subtleties and it is still open). In this work, we show that O ($\bar{\mathrm{O}}$ is completely symmetric) is an exception: O (resp., $\bar{\mathrm{O}}$), interpreted over discrete linear orderings, turns out to be undecidable.

The proof is based on a reduction from the (undecidable) *octant tiling problem*, which is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile $\mathcal{O} = \{(i, j) : i, j \in \mathbb{N} \wedge 0 \leqslant i \leqslant j\}$, respecting the color constraints between tiles that are vertically or horizontally adjacent in the octant. To this end, we build a formula $\phi_{\mathcal{T}}$ of O which is satisfiable if and only if $\mathcal{T}$ can tile the octant. Such a construction consists of three main steps: (i) the encoding of the octant by means of a suitable chain of intervals, called u-chain, (ii) the encoding of the above-neighbour relation by means of up_rel-intervals, and (iii) the encoding of the right-neighbour relation. In the following, we focus our attention on the first two (more difficult) steps.

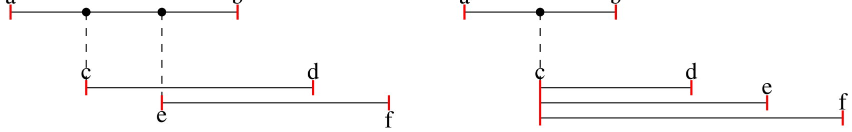

Fig. 1. An inconsistent scenario where the u-interval $[a, b]$ has length greater than 2 and there exist two begin_u-intervals starting inside it which overlap (left) and the correct scenario where the u-interval $[a, b]$ has length equal to 2 and all begin_u-intervals starting inside it do not overlap (right).

To encode the octant, we force the existence of a unique chain of intervals labeled by the propositional letter u (u-intervals). u-intervals are partitioned in two sets: the set of u-intervals encoding tiles and the set of u-intervals acting as separators between consecutive pairs of rows of the octant. The former ones are labeled by tile, the latter ones by $*$. The main problem with this construction is to specify how to reach, from a given u-interval, the next (adjacent) one. We solve this problem by building a chain of adjacent u-intervals, each of them of length 2 (the discrete nature of the linear ordering plays an important role here). More precisely, to constrain the length of the u-intervals, we first force each inner point of every u-interval to be the starting point of infinitely many intervals labeled by the propositional letter $\mathsf{begin_u}$ and then we constrain each $\mathsf{begin_u}$-interval to not overlap any other $\mathsf{begin_u}$-interval starting inside the same u-interval. In this way, we constrain each u-interval to have exactly one inner point (Figure 1). Moreover, to connect consecutive pairs of u-intervals (and making them adjacent), we take advantage of an auxiliary chain of k-intervals of length 2, such that the endpoints of each k-interval are the (unique) inner points of two consecutive u-intervals (Figure 2).

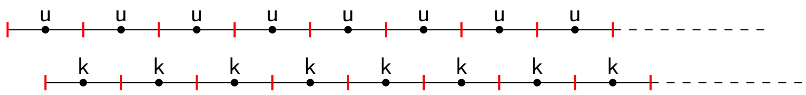

Fig. 2. u-intervals are adjacent and each pair of consecutive u-intervals is connected by a k-interval.

Let us consider now the encoding of the above-neighbour relation. The above-neighbour relation connects each tile in the octant with the tile immediately above it. If a tile t is connected to the tile t′ through the above-neighbour relation, then we simply say that t is above-connected to t′. To model such a relation, we use intervals labeled by up_rel, that is, up_rel-intervals connect pairs of tile-intervals encoding pairs of above-connected tiles of the octant. The structure of such a relation is shown in Figure 3. We distinguish between *backward* and *forward* rows of the octant using the propositional letters b and f. Moreover, we constrain each up_rel-interval starting from a backward (resp., forward) row of the octant to not overlap any other up_rel-interval starting from a backward (resp., forward) row of the octant. In such a way, we encode the correspondence between tiles of consecutive rows of the plane induced by the above-neighbour relation.

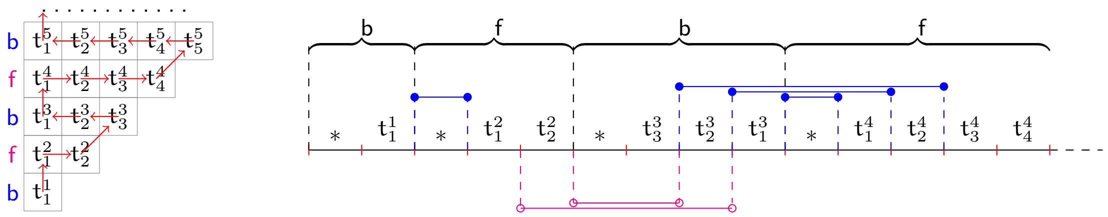

Fig. 3. up_rel-intervals starting from backward (resp., forward) rows of the octant do not overlap.

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