Fuzzy Halpern and Shoham's Interval Temporal Logics

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Abstract

The most representative interval temporal logic, called HS, was introduced by Halpern and Shoham in the nineties. Recently, HS has been proposed as a suitable formalism for modern artificial intelligence applications; however, when dealing with real-life data one is not always able to express temporal relations and propositional labels in a definite, crisp way. In this paper, following the seminal ideas of Fitting and Zadeh, we present a fuzzy generalization of HS, called FHS, that partially solves such problems of expressive power. We study FHS from both a theoretical and an application standpoint: first, we discuss its syntax, semantics, expressive power, and satisfiability problem; then, we define and solve the time series FHS finite model checking problem, to serve as the basis of future applications.

Keywords: fuzzy interval temporal logic, satisfiability, expressive power, finite model checking.
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1. Introduction

Temporal reasoning based on intervals has been deeply studied in the past years. Starting with Allen's interval algebra for existential reasoning about sets of events and their relative positions [1], later sharpened in several studies concerning fragments of the interval algebra with better computational properties [2], the focus has moved progressively toward the logical level, with the introduction of Halpern and Shoham's modal logic for temporal intervals [3], also called *HS*, and with the systematic study of classical problems of its fragments, including sub-logics in which the underlying temporal structure is constrained [4], the set of modal operators is restricted [5, 6], the semantics is softened to a reflexive one [7], the nesting of modal operators is reduced [8], or the propositional power of the languages is limited [9]; a common denominator to all such proposals is the *crisp* semantics of the languages. HS has been recently used in pure, bottom-up, machine learning applications, e.g., in [10, 11]. However, when HS is used to describe real data, as suggested in [12, 13], the need to generalize its syntax and semantics emerges in order to improve its ability of describing and working with concrete situations. For example, while from the point of view of a logician it is perfectly acceptable that a combination of symptoms such as *a period of fever, followed by a period of headache* is described by a crisp formula in which the two events (*fever* and *headache*) have precisely one point in common (the ending point of the first one, which is equal to the beginning point of the second one), from the point of view of a physician such a description may be too restrictive. In particular, *followed by* may be represented by the Allen's relation *meets* but also, to a certain extent, by the Allen's relation *overlaps* (provided that the overlapping period is not too long), and by the relation *later* (provided that the distance between the two events is not too long).

Propositional *many-valued* (or *fuzzy*) logics (from the early work of Łukasiewicz, Post, and Tarski) extend Boolean propositional logic by allowing more than two truth values [14]. Fuzzy modal logics, at least in the sense we think of them, were introduced by Fitting [15] and have enjoyed sustained attention in recent years [16, 17, 18, 19]. Fitting, in particular, gives a very general approach to fuzzy modal logic in which not only propositions, but also accessibility relations are not just true or false, but may take different truth values, and on a similar bases we introduce here a fuzzy version of HS. We start by defining the concept of fuzzy linear ordering, following Zadeh [20], Bodenhofer [21], Kundu [22], and Ovchinnikov [23]. Then we build, on a fuzzy linear ordering, the classical infrastructure of Allen's relations in terms of the fuzzy version of equality and linear ordering relations (unlike, and more generally than, [24], in which they are defined on a underlying crisp linear ordering as functions that depend on the distances between points), and, finally, we give a fuzzy semantics to formulæ of HS; the resulting logic is called *Fuzzy HS* (*FHS*, for short). Following Fitting, our approach is parameterized by a Heyting algebra (which generalizes the Boolean algebra of truth values), so that both the propositional values and Allen's relations are relativized to it.

The main motivation behind this work is twofold. On the one side, having a fuzzy generalization of Halpern and Shoham's interval temporal logic HS is natural and mathematically interesting. On the other side, we want to build up a theoretical basis for future applications. As a matter of fact, in the recent literature, interval temporal logic is being applied to an ever growing variety of contexts, including medical, environmental, and technological research (see, e.g., [11, 25, 26]). As it turns out, this flourish is primarily driven by *symbolic learning* algorithms, that is algorithms that are able to induce logical descriptions of data sets and are universally considered a cornerstone of modern (interpretable) artificial intelligence. Such algorithms are being progressively adapted from the propositional case to the (crisp) interval (modal) one, and then immediately applied to temporal data, and this phenomenon, although in a more limited way, has also occurred for point-based (crisp) temporal logics [27, 28]. In both cases, the ability of efficiently solving the finite model checking problem plays a pivotal role. Temporal data, in this context, is usually presented as (multivariate) time (or temporal) series, and emerge in many application contexts: the temporal history of some hospitalized patients can be described by the time series of the values of their temperature, blood pressure, and oxygenation; the pronunciation of a word in sign language can be described by the time series of the relative and absolute positions of the ten fingers w.r.t. some reference point; different sport activities can be distinguished by the time series of some relevant physical quantities; all active sensors during a flight give time-changing values that form a time series.

In this paper, therefore, we first define syntax and semantics of FHS, we study the *expressive power* of FHS in comparison with that of HS in terms of frame properties definability as well as in terms of inter-definability of modal operators, and we prove that its *satisfiability problem* is undecidable under the same hypothesis of HS, at least in the particular case in which the underlying algebra is a chain. Then, we define and solve the problem of finite temporal series fuzzy model checking in deterministic polynomial time, and we show how to systematically interpret temporal series as fuzzy interval temporal models.

This paper, which an extended and revised version of [29, 30], is organized as follows. In Section 2 we discuss some background work on fuzzy modal and temporal logic. In Section 3 we introduce FHS and we study its syntax, semantics, expressive power, and satisfiability problem. Then, in Section 4 we show how to interpret time series as fuzzy interval models, and we study the finite multivariate time series checking problem for FHS. Open problems and conclusions are in Section 5.

2. Fuzzy Modal and Temporal Logics: Related Work

Fitting introduces in a systematic way *fuzzy* modal logics in [15], and, since then, fuzzy modal logics have been studied by several authors (see, e.g., [16, 17, 18, 19, 31]). Fitting's approach, in particular, consists of defining a generalization of a Kripke frame relativized to an algebra $\mathcal{H}$, so that, given two worlds v, w, the value vRw for an accessibility relation R is a value in $\mathcal{H}$, instead of a Boolean value. As in classical fuzzy propositional logics, propositional truth values are relativized to $\mathcal{H}$ as well; in this paper, we follow a similar approach.

Temporal logics have already been studied from a fuzzy point of view, at least to some extent. For instance, in [32] a model for the representation and handling of fuzzy temporal references has been presented, and the concepts of date, time extent, and interval, according to the formalism of possibility theory have been introduced. In [33], the formalism of Fuzzy Linear Temporal Logic (FLTL) is defined as a generalization of propositional linear temporal logic with fuzzy temporal events and fuzzy temporal states; in the same paper, this logic was extended also to its branching version. In its linear version, FLTL is based on an absolute and linear time model with a continuous time domain, where fuzzy temporal events are defined as fuzzy numbers. Other attempts to work with non-Boolean temporal logics include a study on model checking fuzzy CTL formulæ [34].

Concerning the fuzzyfication of Allen's interval algebra, in [24] the authors define fuzzy Allen's relations from a fuzzy partition made by three possible fuzzy relations between dates (*approximately equal*, *clearly smaller*, and *clearly greater*), which are specific functions which take their values in the real interval $[0, 1]$. On the other hand, in [35], it is shown how temporal reasoning about fuzzy time intervals can be reduced to reasoning about linear constraints, without using any version of fuzzy Allen's relations. To the best of our knowledge, the only attempt to define a fuzzy version of HS is [36]; in it, a language that is suitable for representation of preferences has

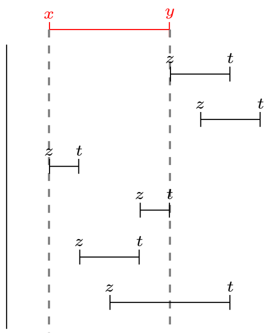
HS	Allen's relations	Graphical representation
$\langle A \rangle$	$[x, y]R_A[z, t] \Leftrightarrow y = z$	
$\langle L \rangle$	$[x, y]R_L[z, t] \Leftrightarrow y < z$	
$\langle B \rangle$	$[x, y]R_B[z, t] \Leftrightarrow x = z, t < y$	
$\langle E \rangle$	$[x, y]R_E[z, t] \Leftrightarrow y = t, x < z$	
$\langle D \rangle$	$[x, y]R_D[z, t] \Leftrightarrow x < z, t < y$	
$\langle O \rangle$	$[x, y]R_O[z, t] \Leftrightarrow x < z < y < t$	

Figure 1: Allen's interval relations and HS modalities.

been given, in which preference logic operators are interpreted with a *degree* α belonging to a finite subset of the real interval $[0, 1]$. More recently, in [37] it has been presented a fuzzy logic whose sentences are Boolean combinations of propositional variables and Allen's relations between temporal intervals. However, they are considered in their classical (crisp) definition and embedded in the propositional language, and the fuzziness is restricted only to the interpretation of the formulæ, which take values, again, in the real interval $[0, 1]$.

3. Fuzzy Interval Temporal Logic: Theory

3.1. Interval Temporal Logic: the Crisp Case

Let $\mathbb{D} = \langle D, \leq \rangle$ be a linearly ordered set, in which we assume that the equality is defined in the standard way, that is, for every x, y, it is the case that:

$$(x = y) \text{ iff } (x \leq y) \text{ and } (y \leq x),$$

and where $x < y$ is defined as a shortcut. An *interval* over $\mathbb{D}$ is an ordered pair $[x, y]$, where $x, y \in D$ and $x < y$. While in the original approach to interval temporal logic intervals with coincident endpoints were included in the semantics, in the recent literature they tend to be excluded except, for instance, in [38] where a two-sorted approach has been studied. If we exclude the identity relation, there are 12 different relations between two intervals in a linear order, often called *Allen's relations* [1]: the six relations R_A (adjacent to, or meets), R_L (later than), R_B (begins), R_E (ends), R_D (during), and R_O (overlaps), depicted in Fig. 1, together with their inverses $R_{\overline{X}} = (R_X)^{-1}$, for each $X \in \{A, L, B, E, D, O\}$. From now on, let $Op = \{A, L, B, E, D, O\}$ and let $Op^* = \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$. We interpret interval structures as Kripke structures, with Allen's relations playing the role of the accessibility relations. Thus, we associate a universal modality $[X]$ and an existential modality $\langle X \rangle$ with each Allen's relation R_X. For each $X \in Op$, the *inverse* of the modalities $[X]$ and $\langle X \rangle$ are the modalities $[\overline{X}]$ and $\langle \overline{X} \rangle$, corresponding to the inverse relation $R_{\overline{X}}$ of R_X; in the following, for $X \in Op$, we identify $\overline{\overline{X}}$ with X. Halpern and Shoham's logic, denoted HS [3], is a multi-modal logic with formulæ built from a finite, non-empty set Ap of atomic propositions (also referred to as propositional letters), the classical propositional connectives, and a modal operator for each Allen's relation, as follows:

$$\varphi ::= \bot \mid p \mid \neg\psi \mid \psi \vee \xi \mid \langle X \rangle \psi.$$

In the above grammar, $p \in Ap$ and $X \in Op^*$. The other propositional connectives and constants (e.g., $\rightarrow$, and $\top$), as well as the dual modalities (e.g., $[A]\varphi \equiv \neg\langle A \rangle \neg\varphi$), can be defined in the standard way. Let $\mathcal{L}_{HS}$ be the smallest set that contains all formulas generated by the above grammar. Given a formula of HS, its *inverse* formula is obtained by substituting every operator $\langle X \rangle$ with its inverse one $\langle \overline{X} \rangle$, and the other way around, for $X \in Op$, while its *symmetric* is obtained by substituting every operator $\langle X \rangle$ with its inverse one $\langle \overline{X} \rangle$, and the other way around, for $X \in \{A, L, O\}$, and every $\langle B \rangle$ (resp., $\langle \overline{B} \rangle$) with $\langle E \rangle$ (resp., $\langle \overline{E} \rangle$), and the other way around.

The semantics of HS is given in terms of *interval models* of the type:

$$M = \langle \mathbb{I}(\mathbb{D}), V \rangle,$$

where $\mathbb{D}$ is a linear order, $\mathbb{I}(\mathbb{D})$ is the set of all intervals over $\mathbb{D}$, and V is a *valuation function*:

$$V : Ap \mapsto 2^{\mathbb{I}(\mathbb{D})},$$

which assigns to each atomic proposition $p \in Ap$ the set of intervals $V(p)$ on which p holds. The *truth* of a formula φ on a given interval $[x, y]$ in an interval model M is defined by structural induction on formulæ as follows:

$$\begin{array}{lll}
M,[x,y] \Vdash p & \text{iff} & [x,y] \in V(p),\ \text{for } p \in Ap; \\
M,[x,y] \Vdash \neg\psi & \text{iff} & M,[x,y] \nVdash \psi; \\
M,[x,y] \Vdash \psi \vee \xi & \text{iff} & M,[x,y] \Vdash \psi \text{ or } M,[x,y] \Vdash \xi; \\
M,[x,y] \Vdash \langle X \rangle \psi & \text{iff} & M,[z,t] \Vdash \psi \text{ for a } [z,t] \text{ s.t.} \\
 & & [x,y] R_X [z,t], \text{for } X \in Op^*.
\end{array}$$

3.2. Interval Temporal Logic: the Fuzzy Case

Recall that a bounded distributive lattice is a set with internal operations $\cap$ (*meet*[1]) and $\cup$ (*join*), both commutative, associative, and connected by the absorption law, in which a partial order can be defined:

$$\alpha \preceq \beta \text{ iff } \alpha \cap \beta = \alpha \text{ iff } \alpha \cup \beta = \beta.$$

In what follows, we use $\bigcap$ (resp., $\bigcup$) to indicate the generalized $\cap$ (resp., $\cup$), and we assume them to have the lowest priority in algebraic expressions; moreover, we omit the quantification domains when it is clear from the context. The symbols 0 and 1 denote, respectively, least and the greatest elements of $\mathcal{H}$. Following Fitting [15], a formula of a fuzzy modal logic is evaluated in a Heyting Algebra. A *Heyting Algebra* is a structure:

$$\mathcal{H} = (H, \cap, \cup, \hookrightarrow, 0, 1),$$

where $(H, \cap, \cup, 0, 1)$ is a bounded distributive lattice with (non-empty) *domain* H, and in which $\hookrightarrow$, called the *Heyting implication*, is the right residual of $\cap$, i.e., it satisfies:

$$\alpha \cap \beta \preceq \gamma \quad \text{iff} \quad \alpha \preceq \beta \hookrightarrow \gamma.$$

This implies that $\alpha \hookrightarrow \beta$ is the greatest element γ such that $\alpha \cap \gamma \preceq \beta$ and this element is guaranteed to exists or, in other words, $\alpha \hookrightarrow \beta$ is the *relative pseudo-complement* of α with respect to β (see e.g. [39]). The Heyting implication can be equationally captured, e.g., by requiring that:

(i) $\alpha \hookrightarrow \alpha = 1$,
(ii) $(\alpha \hookrightarrow \beta) \cap \beta = \beta$,

[1] This is the classical nomenclature in lattice theory, and it should not be confused with Allen's relation *meets*, used in this paper.

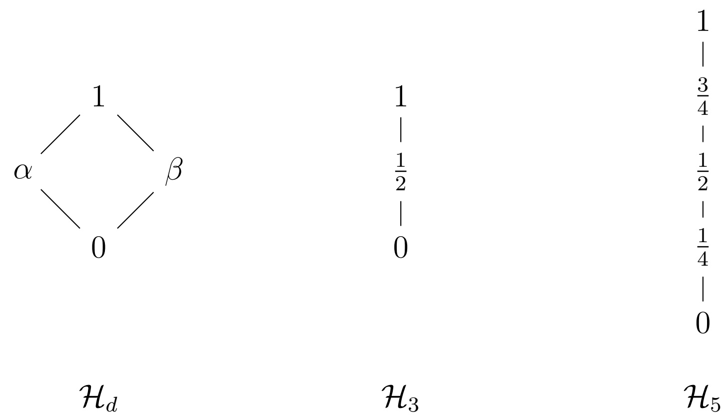


Figure 2: Graphical representation of three Heyting algebras used in this paper.

(iii) $\alpha \cap (\alpha \hookrightarrow \beta) = \alpha \cap \beta$,
(iv) $\alpha \hookrightarrow (\beta \cap \gamma) = (\alpha \hookrightarrow \beta) \cap (\alpha \hookrightarrow \gamma)$,
(v) $(\alpha \cup \beta) \hookrightarrow \gamma = (\alpha \hookrightarrow \gamma) \cap (\alpha \hookrightarrow \gamma)$,

for all $\alpha, \beta, \gamma \in H$. Hence the class of Heyting algebras forms a variety. A Heyting algebra is said to be *complete* if the underlying lattice is, i.e., for every subset $H' \subseteq H$, both its least upper bound $\bigcup H'$ and its greatest lower bound $\bigcap H'$ exist. This means that in complete Heyting algebras $\alpha \hookrightarrow \beta$ can be defined as:

$$\bigcup \{\gamma \mid \alpha \cap \gamma \preceq \beta\}.$$

It can be easily shown that every complete Heyting algebra is *join-infinite distributive* (see e.g. [39, Lemma 1.4]), that is, it holds that is such that for all subsets $H' \subseteq H$:

$$\alpha \cap \bigcup H' = \bigcup_{\beta \in H'} \alpha \cap \beta.$$

Finally, a Heyting algebra is said to be a *chain* if $\preceq$ is total. Typical realizations include the two-element Boolean algebra and the closed interval $[0, 1]$ in $\mathbb{R}$. From now on, we assume that $\mathcal{H}$ is a complete Heyting algebra; as we shall see later, such an assumption allows us to preserve some intuitive and useful properties of the crisp version of HS. In the rest of this paper, three specific Heyting algebras will be used in all proofs and examples, namely the four-elements Boolean algebra $\mathcal{H}_d$, also known as Belnap's diamond, the three-elements Heyting chain $\mathcal{H}_3$, and the five-elements Heyting chain $\mathcal{H}_5$, all represented in Fig. 2.

There are several possible definitions of fuzzy linear orders. For example Zadeh [20], defines a similarity relation in a set, imposing that it is reflexive, symmetric, and transitive, as well as a notion of fuzzy ordering, with a form of antisymmetry and fuzzy versions of totality. Similarly, Bodenhofer [21] advocates for the use of similarity-based fuzzy orderings, in which the linearity is in a strong form; the same notion is also used in [22]. On the other hand, Ovchinnikov [23] proposes a notion of fuzzy ordering with a non-strict ordering relation. A common denominator to all such proposals is the definition of a very weak fuzzy version of the transitivity property, which allows one to obtain very general definitions. Among these previous works, the proposal that is most similar to ours is Zadeh's, which we modify to take into account both the fuzzy linear order and the fuzzy equality in the same structure, and we call it an *adequate* fuzzy strictly linearly ordered set. This is motivated by the specific requirements of the fuzzy interval temporal logic we are constructing: we consider fuzzy Allen's relations, where both (fuzzy) equality and (fuzzy) linear order are involved, and the intuitive interpretation of these accessibility relations need to be adequate with the idea of *fuzziness* we want to be reflected in this logic. In this sense, both the ordering and the equality need to be adequately formalized w.r.t. transitivity. The linear order, on the one hand, should be only weakly transitive, and the equality, on the other, should not be transitive at all (i.e., if x is *almost equal* to y, and y is *almost equal* to z, we cannot expect x to be *almost equal* to z). Nonetheless, to be congruent with our intuition of actual flows of time, a crisp linear order should always serve as basis of a fuzzy one; among others, real-world applications on time series in which, for example, different granularities are used for different temporal variables justify, at least in part, these choices. Thus, assuming that $\mathcal{H}$ is a complete Heyting algebra with domain H, as discussed above, we start with a domain D enriched with two functions:

$$\widetilde{<}, \widetilde{=} : D \times D \mapsto H$$

and we say that the structure $\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=} \rangle$ is a *adequate fuzzy strictly linearly ordered set* if it holds, for every x, y, and z:

1. $\widetilde{=}(x, y) = 1$ iff $x = y$,
2. $\widetilde{=}(x, y) = \widetilde{=}(y, x)$,

3. $\widetilde{<}(x,x) = 0$,
4. $\widetilde{<}(x,z) \succeq \widetilde{<}(x,y) \cap \widetilde{<}(y,z)$,
5. if $\widetilde{<}(x,y) \succ 0$ and $\widetilde{<}(y,z) \succ 0$ then $\widetilde{<}(x,z) \succ 0$,
6. if $\widetilde{<}(x,y) = 0$ and $\widetilde{<}(y,x) = 0$ then $\widetilde{=}(x,y) = 1$,
7. if $\widetilde{=}(x,y) \succ 0$ then $\widetilde{<}(x,y) \prec 1$.

As expected, the function $\widetilde{<}$ is asymmetric: if we had that both $\widetilde{<}(x,y)$ and $\widetilde{<}(y,x)$ were positive, then by transitivity $\widetilde{<}(x,x)$ would be also positive, which is in contradiction with the irreflexivity of $\widetilde{<}$ itself. Observe also that by irreflexivity of $\widetilde{<}$ and reflexivity of $\widetilde{=}$, one obtains that $\widetilde{<}(x,y) \succ 0$ implies that $\widetilde{=}(x,y) \prec 1$, that is, that $\widetilde{=}$ does not contradict $\widetilde{<}$. Moreover, it is worth to point out that 4 is the standard transitivity used in fuzzy orderings; a stronger version of it, such as, for example, imposing that $\widetilde{<}(x,z)$ is equal to the join of $\widetilde{<}(x,y)$ and $\widetilde{<}(y,z)$, would lead, in fact, to a system that can only be realized in a crisp ordering. Also, 4 and 5 are both needed to enforce a suitable version of transitivity for $\widetilde{<}$; 4, in particular, guarantees that the extent to which a point x is less than another point y is never less that the meet of the degrees to which any intervening point is greater than x and less than y. If the truth value algebra is a finite chain, then this meet is of course simply the minimum of the two values, but in non-chain truth-value algebras this meet might be 0, in which case 4 becomes vacuous. It is precisely the latter case that 5 is intended to address. In fact, condition 5 plays an essential role in ensuring that every adequate strict fuzzy linear order has an *underlying* crisp strict linear order, as we will now show. Given an adequate fuzzy strict linear order $\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=} \rangle$, we define the *crispification of* $\widetilde{\mathbb{D}}$ to be the crisp linear order $\mathbb{D} = \langle D, < \rangle$, where $x < y$ if and only if $\widetilde{<}(x,y) \neq 0$. It is easy to verify that $\mathbb{D} = \langle D, < \rangle$, so defined, is, in fact, a linear order. Indeed, condition 3 guarantees the irreflexivity of $<$, condition 5 ensures its transitivity while 1 and 6, together, provide for connectedness. Moreover, it should be noticed that axioms 4 and 5 are both independent. Let us prove, first, that $\{1, \ldots, 7\} \setminus \{5\}$ do not imply 5. Consider $\mathcal{H}_d$ as truth algebra and define $\widetilde{\mathbb{D}}_1 = \langle D_1, \widetilde{<}_1, \widetilde{=}_1 \rangle$ where $D_1 = \{u, v\}$, $\widetilde{<}_1(u,v) = \alpha$, $\widetilde{<}_1(v,u) = \beta$, $\widetilde{<}_1(u,u) = 0 = \widetilde{<}_1(v,v)$, $\widetilde{=}_1(u,u) = 1 = \widetilde{=}_1(v,v)$ and $\widetilde{=}_1(u,v) = 0 = \widetilde{=}_1(v,u)$; one can check that this structure satisfies all conditions except 5; in fact, $\widetilde{<}_1(u,v) \succ 0$ and $\widetilde{<}_1(v,u) \succ 0$ violate the intuition of linearity in the fuzzy setting. Conversely, to see that$\{1, \ldots, 7\} \setminus \{4\}$ do not imply 4, consider $\mathcal{H}_3$ as truth value algebra, and define the structure $\widetilde{\mathbb{D}}_2 = \langle D_2, \widetilde{<}_2, \widetilde{=}_2 \rangle$ where $D_2 = \{u, v, w\}$, $\widetilde{<}_1(u,v) = 1$, $\widetilde{<}_2(v,w) = 1$, $\widetilde{<}_2(u,w) = \frac{1}{2}$, $\widetilde{<}_2(x,y) = 0$ for all

$(x, y) \in D_2 \times D_2 \setminus \{(u, v), (v, w), (u, w)\}$, $\widetilde{=}_2(u, u) = \widetilde{=}_2(v, v) = \widetilde{=}_2(w, w) = 1$ and $\widetilde{=}_2(x, y) = 0$ for all $(x, y) \in D_2 \times D_2 \setminus \{(u, u), (v, v), (w, w)\}$. It is easy to see that this structure satisfies all conditions except 4; the fact that u is, in a way, less to the left of w than both u is to the left of v and v is to the left of w violates precisely the intuition 4 is designed to capture.

Under such premises we say that, given a set of propositional letters Ap and a complete Heyting algebra $\mathcal{H}$, a well-formed *fuzzy interval temporal logic* (FHS, for short) formula is obtained by the following grammar:

$$\varphi ::= \alpha \mid p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \langle X \rangle \varphi \mid [X]\varphi,$$

where (slightly abusing the notation) $\alpha \in H$, $p \in Ap$, and, as in the crisp case, $X \in Op^*$. We use $\neg\varphi$ to denote the formula $\varphi \to 0$. As we did in the crisp case, let $\widetilde{\mathcal{L}}_{FHS}$ be the smallest set that contains all formulas generated by above grammar. Given a fuzzy strictly linearly ordered set, we can now define the set of fuzzy strict intervals in $\widetilde{\mathbb{D}}$:

$$\mathbb{I}(\widetilde{\mathbb{D}}) = \{[x, y] \mid \widetilde{<}(x, y) \succ 0\}.$$

Generalizing classical Boolean evaluation, propositional letters are directly evaluated in the underlying algebra, by defining a fuzzy valuation function:

$$\widetilde{V} : Ap \times \mathbb{I}(\widetilde{\mathbb{D}}) \mapsto H,$$

that generalizes the crisp function V. It could be argued that in the natural fuzzy semantics of an interval temporal logic one could require that intervals are subject to a requirement of non-punctuality, that is, that intervals with too close endpoints are considered almost points from the ontological point of view (in algebraic terms). This requirement, however, can be embedded in the design of $\widetilde{V}$, which, while assigning the truth value to a propositional letter over an interval, could take into account the relative distance between its endpoints. On top of the fuzzyfication of valuations we need to define how accessibility relations behave in the fuzzy context. Unlike classical modal logic in interval temporal logic accessibility relations are not primitive, but they are defined over the underlying linear order. The natural definition of fuzzy Allen's relations, therefore, is obtained by generalizing the original, crisp definition, and substituting every $=$ with $\widetilde{=}$ and every $<$ with $\widetilde{<}$. Thus we have, for every $X \in Op^*$, we have:

$$\widetilde{R}_X : \mathbb{I}(\widetilde{\mathbb{D}}) \times \mathbb{I}(\widetilde{\mathbb{D}}) \to H$$

defined by:

$$\begin{array}{rcl}
\widetilde{R}_A([x,y],[z,t]) & = & \widetilde{=}(y,z); \\
\widetilde{R}_L([x,y],[z,t]) & = & \widetilde{<}(y,z); \\
\widetilde{R}_B([x,y],[z,t]) & = & \widetilde{=}(x,z) \cap \widetilde{<}(t,y); \\
\widetilde{R}_E([x,y],[z,t]) & = & \widetilde{<}(x,z) \cap \widetilde{=}(y,t); \\
\widetilde{R}_D([x,y],[z,t]) & = & \widetilde{<}(x,z) \cap \widetilde{<}(t,y); \\
\widetilde{R}_O([x,y],[z,t]) & = & \widetilde{<}(x,z) \cap \widetilde{<}(z,y) \cap \widetilde{<}(y,t),
\end{array}$$

and similarly for the inverse relations. Now, we say that an $\mathcal{H}$*-valued interval model* (or *fuzzy interval model*) is a tuple of the type:

$$\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V} \rangle,$$

where $\widetilde{\mathbb{D}}$ is a fuzzy strictly linearly ordered set and $\widetilde{V}$ is a fuzzy valuation function. We interpret an FHS formula in a fuzzy interval model $\widetilde{M}$ and an interval $[x,y]$ by inductively extending the valuation $\widetilde{V}$ of propositional letters to the whole language:

$$\widetilde{V} : \widetilde{\mathcal{L}}_{FHS} \times \mathbb{I}(\widetilde{D}) \rightarrow H,$$

defined by:

$$\begin{array}{rcl}
\widetilde{V}(\alpha,[x,y]) & = & \alpha; \\
\widetilde{V}(\varphi \wedge \psi,[x,y]) & = & \widetilde{V}(\varphi,[x,y]) \cap \widetilde{V}(\psi,[x,y]); \\
\widetilde{V}(\varphi \vee \psi,[x,y]) & = & \widetilde{V}(\varphi,[x,y]) \cup \widetilde{V}(\psi,[x,y]); \\
\widetilde{V}(\varphi \rightarrow \psi,[x,y]) & = & \widetilde{V}(\varphi,[x,y]) \hookrightarrow \widetilde{V}(\psi,[x,y]); \\
\widetilde{V}(\langle X \rangle \varphi,[x,y]) & = & \bigcup\{\widetilde{R}_X([x,y],[z,t]) \cap \widetilde{V}(\varphi,[z,t])\}; \\
\widetilde{V}([X]\varphi,[x,y]) & = & \bigcap\{\widetilde{R}_X([x,y],[z,t]) \hookrightarrow \widetilde{V}(\varphi,[z,t])\},
\end{array}$$

where $X \in Op^*$ and $[z,t]$ varies in $\mathbb{I}(\widetilde{\mathbb{D}})$. We say that a formula of FHS φ is α*-satisfied* at an interval $[x,y]$ in a fuzzy interval model $\widetilde{M}$ if and only if:

$$\widetilde{V}(\varphi,[x,y]) \succeq \alpha.$$

The formula φ is α*-satisfiable* if and only if there exists a fuzzy interval model and an interval in that model where it is α-satisfied. A formula is *satisfiable* if it is α-satisfiable for some $\alpha \in H$, $\alpha \neq 0$. A formula is α*-valid* if it is α-satisfied at every interval in every model, and *valid* if it is 1-valid. Observe that since a Heyting algebra, in general, does not encompass

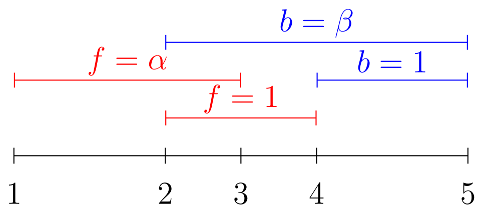


Figure 3: Example of fuzzy interval model and algebra. In non-shown intervals all propositional letters are evaluated to 0.

x	y	$\widetilde{=}(x,y)$, $\widetilde{=}(y,x)$	$\widetilde{<}(x,y)$	$\widetilde{<}(y,x)$
1	1	1	0	0
1	2	β	α	0
1	3	β	α	0
1	4	0	1	0
1	5	0	1	0
2	2	1	0	0
2	3	α	β	0
2	4	β	β	0
2	5	0	1	0
3	3	1	0	0
3	4	α	β	0
3	5	0	1	0
4	4	1	0	0
4	5	0	1	0
5	5	1	0	0

Table 1: Explicit $\widetilde{=}$ and $\widetilde{<}$ functions for the example in Fig. 3; one can check that all axioms are respected.

classical negation, and since our definition of satisfiability is graded, instead of absolute, then the usual duality of satisfiability and validity does not hold anymore.

To understand and visualize the practical expressive power of FHS in real-like situations, consider the following example. We want to model the medical data of a patient under observation, and keep trace of his/her *high fever* and *high blood pressure*. Observe that the degree of measure and evaluation of different variables may not be the same, and *high temperature*, represented by propositional variable f, may simply not be comparable with *high blood pressure*, represented by b (we use *high* in both cases to stress the fact that they are not always comparable to each other), this implies not always the truth algebra can be a Heyting chain. One requirement for FHS to be designed is that the truth value algebra used to fuzzify the interval relations and the one used to fuzzify the propositional letters are the same. Now, assume that we want to describe the above situation with a range of 5 points, as in Fig. 3. One simple way to do so is, therefore, to consider $\mathcal{H}_d$ as truth value algebra, and design explicit $\tilde{=}$ and $\tilde{<}$ functions in such a way to ensure that properties 1-7 are met, as in Tab. 1. In this example, then, $\tilde{=}(2,3) = \tilde{=}(3,4) = \alpha$ (which can be read as *not too equal*), *high fever* (f) on $[1,3]$ has also value α (here it should be read as *not too much, in terms of high fever*), while *high blood pressure* has value β (read, here, as *not too much, in terms of high blood pressure*); in such a situation, one may want to ask the question *to what degree high fever meets high blood pressure*, represented by formula $f \wedge \langle A \rangle b)$ on $[1,3]$. The answer, following the semantic rules, can be computed as:

$$
\begin{array}{rcl}
\widetilde{V}(f \wedge \langle A \rangle b, [1,3]) & = & \alpha \cap \widetilde{V}(\langle A \rangle b, [1,3]) \\
& = & \alpha \cap \bigcup\{\widetilde{R}_A([1,3],[2,5]) \cap \beta, \widetilde{R}_A([1,3],[4,5]) \cap 1\} \\
& = & \alpha \cap \bigcup\{\alpha \cap \beta, \alpha \cap 1\} \\
& = & \alpha \cap \alpha = \alpha
\end{array}
$$

that is, *not too much*.

3.3. Expressive Power of FHS

The inter-definability among different operators of HS has been studied at large, with the aim to determine which are, under given conditions, the expressively different fragments of HS. As a consequence, by joining the expressive power results in [3] and the inter-definability result that can be found in [5], one may obtain a rather complete picture of interesting validities of HS in the case of a general linear order. As we shall see, FHS is somewhat weaker than HS in terms of expressive power, but some familiar properties are preserved; of those, some depend on the general (Fitting-like) approach to the fuzzyfication of a modal logic, while some other are related to our particular choices on the properties of a fuzzy strictly linear order. From [3, 5], we know that:

- HS is a normal modal temporal logic, that is, it is closed under modus ponens, necessitation, and uniform substitution, and the following formulæ, and their inverse version, are valid for every $X \in Op^*$:
  - the K axioms: $[X](p \to q) \to ([X]p \to [X]q)$
  - the temporal axioms: $p \to [X]\langle \overline{X} \rangle p$

- the relations R_L, R_D, R_B, R_E, and their inverse ones, are all transitive, that is, the following formulæ, and their inverse versions, are valid:
  - $\langle D \rangle \langle D \rangle p \to \langle D \rangle p$
  - $\langle B \rangle \langle B \rangle p \to \langle B \rangle p$
  - $\langle L \rangle \langle L \rangle p \to \langle L \rangle p$
  - $\langle E \rangle \langle E \rangle p \to \langle E \rangle p$

- the following inter-definabilities hold, that is, the following formulæ, its inverse, and its symmetric version is valid:
  - $\langle L \rangle p \leftrightarrow \langle A \rangle \langle A \rangle p$
  - $\langle D \rangle p \leftrightarrow \langle B \rangle \langle E \rangle p$
  - $\langle D \rangle p \leftrightarrow \langle E \rangle \langle B \rangle p$

  as well as the following ones and their symmetric versions:
  - $\langle L \rangle p \leftrightarrow \langle \overline{B} \rangle [E] \langle \overline{B} \rangle \langle E \rangle p$
  - $\langle O \rangle p \leftrightarrow \langle E \rangle \langle \overline{B} \rangle p$

In the first group of properties, FHS retains the full power of HS, as the following theorem proves.

Theorem 1. *FHS is a normal temporal logic, that is, it is closed under modus ponens, necessitation, and uniform substitution, and the following formulæ, and their inverse ones, are valid in FHS for every $X \in Op^*$:*

- *the K axiom*

- $p \rightarrow [X]\langle\overline{X}\rangle p$

Proof. The fact that K holds for every relation is a consequence of a more general result proven in [16], and the fact that FHS is closed under is closed under modus ponens, necessitation, and uniform substitution, follows from the crisp case. Now, we prove that the temporal axiom still holds for every modal operator, using the fact that, in a Heyting algebra, $\alpha \hookrightarrow \beta = 1$ if and only if $\alpha \preceq \beta$. Therefore, we prove that for every $X \in Op^*$, it holds that $\widetilde{V}([X]\langle\overline{X}\rangle p, [x, y]) \succeq \widetilde{V}(p, [x, y])$ in every fuzzy model and interval. Indeed:

$$
\begin{aligned}
&\widetilde{V}([X]\langle\overline{X}\rangle p, [x, y]) = \\
&\bigcap \widetilde{R}_X([x, y], [z, t]) \hookrightarrow \bigcup \widetilde{R}_{\overline{X}}([z, t], [u, v]) \cap \widetilde{V}(p, [u, v]) \succeq \\
&\bigcap \widetilde{R}_X([x, y], [z, t]) \hookrightarrow (\widetilde{R}_{\overline{X}}([z, t], [x, y]) \cap \widetilde{V}(p, [x, y])) = \\
&\bigcap \widetilde{R}_X([x, y], [z, t]) \hookrightarrow (\widetilde{R}_X([x, y], [z, t]) \cap \widetilde{V}(p, [x, y])) \succeq \\
&\widetilde{V}(p, [x, y]).
\end{aligned}
$$

Observe, in particular, that the last inequality follows from the fact that, in a Heyting algebra, $\alpha \preceq (\beta \hookrightarrow (\beta \cap \alpha))$ (by the definition of $\hookrightarrow$ itself). $\square$

In terms of transitivity of operators, FHS turns out to be sensibly weaker than HS, as only $\langle D\rangle$ and its inverse can be proven to be transitive.

Theorem 2. *The following formula and its inverse one are valid in FHS:*

- $\langle D\rangle\langle D\rangle p \rightarrow \langle D\rangle p.$

Conversely, the following formulæ, and their inverse ones, are not valid in FHS:

- $\langle B\rangle\langle B\rangle p \rightarrow \langle B\rangle p$

- $\langle L\rangle\langle L\rangle p \rightarrow \langle L\rangle p$

- $\langle E\rangle\langle E\rangle p \rightarrow \langle E\rangle p$

Proof. Let us start by proving that:

$$\langle D\rangle\langle D\rangle p \rightarrow \langle D\rangle p$$

is valid in FHS. First, observe that, given any fuzzy strictly linearly ordered set $\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=} \rangle$ and any three intervals $[x,y],[z,t],[u,v] \in \mathbb{I}(\widetilde{\mathbb{D}})$, it is the case that $\widetilde{R}_D([x,y],[z,t]) \wedge \widetilde{R}_D([z,t],[u,v]) = \widetilde{<}(x,z) \cap \widetilde{<}(t,y) \cap \widetilde{<}(z,u) \cap \widetilde{<}(v,t) \preceq \widetilde{<}(x,u) \cap \widetilde{<}(v,y) = \widetilde{R}_D([x,y],[u,v])$, where the inequality holds by the transitivity of $\widetilde{<}$ (property 4). Thus, we have that:

$$\begin{aligned}
&\widetilde{V}(\langle D\rangle\langle D\rangle p,[x,y]) = \\
&\bigcup \widetilde{R}_D([x,y],[z,t]) \cap \bigcup \widetilde{R}_D([z,t],[u,v]) \cap \widetilde{V}(p,[u,v]) = \\
&\bigcup\bigcup \widetilde{R}_D([x,y],[z,t]) \cap \widetilde{R}_D([z,t],[u,v]) \cap \widetilde{V}(p,[u,v]) \preceq \\
&\bigcup \widetilde{R}_D([x,y],[u,v]) \cap \widetilde{V}(p,[u,v]) = \\
&\widetilde{V}(\langle D\rangle p,[x,y])
\end{aligned}$$

Observe, in particular, that the second equality holds because $\mathcal{H}$ is jointly-infinite distributive. Now, let us prove that:

$$\langle B\rangle\langle B\rangle p \rightarrow \langle B\rangle p$$

is not valid in FHS. Consider a model $\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V} \rangle$ with domain $D = \{0,1,2,3,4,5\}$, and let $\mathcal{H} = \mathcal{H}_3$, with $\widetilde{=}(x,y) = \max\{0, 1 - \frac{1}{2}|x-y|\}$ and $\widetilde{<}(x,y) = \min\{1, \max\{\frac{1}{2}(y-x), 0\}\}$. It is easy to check that both $\widetilde{=}$ and $\widetilde{<}$, so defined, satisfy the required conditions. Further, consider the valuation $\widetilde{V}$ with $\widetilde{V}(p,[2,3]) = \frac{1}{2}$ and with $\widetilde{V}(p,[x,y]) = 0$ for every interval $[x,y] \neq [2,3]$. It is easy to see that the only significant intervals to be checked in order to obtain $\widetilde{V}(\langle B\rangle\langle B\rangle p,[0,5])$ are $[0,5]$, $[1,4]$ and $[2,3]$, hence:

$$\begin{aligned}
&\widetilde{V}(\langle B\rangle\langle B\rangle p,[0,5]) = \\
&\bigcup \widetilde{R}_B([0,5],[z,t]) \cap \bigcup \widetilde{R}_B([z,t],[u,v]) \cap \widetilde{V}(p,[u,v]) = \\
&\widetilde{R}_B([0,5],[1,4]) \cap \widetilde{R}_B([1,4],[2,3]) \cap \widetilde{V}(p,[2,3]) = \\
&\frac{1}{2} \cap \frac{1}{2} \cap \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
&\widetilde{V}(\langle B\rangle p,[0,5]) = \\
&\bigcup \widetilde{R}_B([0,5],[z,t]) \cap \widetilde{V}(p,[z,t]) = \\
&\widetilde{R}_B([0,5],[2,3]) \cap \widetilde{V}(p,[2,3]) = (0 \cap 1) \cap \frac{1}{2} = 0
\end{aligned}$$

Since $\widetilde{V}(\langle B\rangle\langle B\rangle p, [0,5]) \succ \widetilde{V}(\langle B\rangle p, [0,5])$, we have the result. The remaining two cases can be treated in a similar way. $\square$

Finally, most inter-definability among operators still hold in FHS, but in only one direction; unfortunately, this means that, unlike HS, no operator can be defined in terms of the others, which, in turn, means that all fragments of FHS are expressively different, and they may present different properties.

Theorem 3. *The following formulæ are valid in FHS:*

- $\langle L\rangle p \to \langle A\rangle\langle A\rangle p$
- $\langle D\rangle p \to \langle B\rangle\langle E\rangle p$
- $\langle D\rangle p \to \langle E\rangle\langle B\rangle p$
- $\langle O\rangle p \to \langle E\rangle\langle \overline{B}\rangle p$

Conversely, their right-to-left versions are not valid in FHS. Moreover the following formula and its right-to-left version are not valid in FHS either:

- $\langle L\rangle p \to \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$

Proof. First, let us focus on proving that:

$$\langle L\rangle\varphi \to \langle A\rangle\langle A\rangle\varphi$$

is valid in FHS. We begin by noticing that if $\widetilde{R}_L([x,y],[z,t]) = \widetilde{<}(y,z) \succ 0$, then it must be that $[y,z] \in \mathbb{I}(\widetilde{\mathbb{D}})$. So, $\widetilde{R}_A([x,y],[y,z]) = \widetilde{=}(y,y) = 1$ and $\widetilde{R}_A([y,z],[z,t]) = \widetilde{=}(z,z) = 1$. Therefore we have that:

$$\begin{aligned}
&\widetilde{V}(\langle L\rangle\phi, [x,y]) = \\
&\bigcup \widetilde{R}_L([x,y],[z,t]) \cap \widetilde{V}(\phi,[z,t]) \preceq \\
&\bigcup \widetilde{R}_A([x,y],[y,z]) \cap \widetilde{R}_A([y,z],[z,t]) \cap \widetilde{V}(\phi,[z,t]) \preceq \\
&\bigcup\bigcup \widetilde{R}_A([x,y],[u,v]) \cap \widetilde{R}_A([u,v],[z,t]) \cap \widetilde{V}(\phi,[z,t]) = \\
&\bigcup \widetilde{R}_A([x,y],[u,v]) \cap \bigcup \widetilde{R}_A([u,v],[z,t]) \cap \widetilde{V}(\phi,[z,t]) = \\
&\widetilde{V}(\langle A\rangle\langle A\rangle\phi, [x,y])
\end{aligned}$$

Observe, again, that the second equality holds because $\mathcal{H}$ is jointly-infinite distributive. Now, to prove the validity of:

$$\langle D\rangle p \to \langle B\rangle\langle E\rangle p$$

observe that if $\widetilde{R}_D([x,y],[z,t]) = \widetilde{<}(x,z) \cap \widetilde{<}(t,y) \succ 0$, then $\widetilde{<}(x,z) \succ 0$ and $\widetilde{<}(t,y) \succ 0$. Moreover, as $[z,t] \in \mathbb{I}(\widetilde{\mathbb{D}})$, we have $\widetilde{<}(z,t) \succ 0$. This implies, by using property 5 of $\widetilde{<}$, that $\widetilde{<}(x,t) \succ 0$, hence $[x,t] \in \mathbb{I}(\widetilde{\mathbb{D}})$. Now, thanks to the previous consideration, we obtain $\widetilde{R}_D([x,y],[z,t]) = \widetilde{R}_B([x,y],[x,t]) \cap \widetilde{R}_E([x,t],[z,t])$, and this leads to prove the desired validity by reasoning as above. The proof of the validity of $\langle D\rangle p \to \langle E\rangle\langle B\rangle p$ follows the same steps. Now, to prove the validity of

$$\langle O\rangle p \to \langle E\rangle\langle \overline{B}\rangle p$$

we have that $\widetilde{R}_O([x,y],[z,t]) \succ 0$ implies $\widetilde{<}(z,y) \succ 0$, that is, $[z,y] \in \mathbb{I}(\widetilde{\mathbb{D}})$. Then, $\widetilde{R}_O([x,y],[z,t]) \preceq \widetilde{R}_E([x,y],[z,y]) \cap \widetilde{R}_{\overline{O}}([z,y],[z,t])$, and we can reason as in previous cases. Finally, in order to prove that:

$$\langle A\rangle\langle A\rangle p \to \langle L\rangle p$$

is not valid, take a model $\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V}\rangle$ with domain $D = \{0,1,2,3,4\}$ and let $\mathcal{H} = \mathcal{H}_3$, with $\widetilde{=}(x,y) = \max\{0, 1 - \frac{1}{2}|x-y|\}$ and $\widetilde{<}(x,y) = \min\{1, \max\{\frac{1}{2}(y-x), 0\}\}$. Then, consider a valuation $\widetilde{V}$ such that $\widetilde{V}(p,[2,4]) = 1$ and $\widetilde{V}(p,[x,y]) = 0$ for every $[x,y] \neq [2,4]$. Then, the only interval $[x,y]$ such that $\widetilde{R}_L([0,2],[x,y]) \succ 0$ is $[3,4]$, and, in particular, $\widetilde{R}_L([0,2],[3,4]) = \widetilde{<}(2,3) = \frac{1}{2}$. So $\widetilde{V}(\langle L\rangle p,[0,2]) = \widetilde{R}_L([0,2],[3,4]) \cap \widetilde{V}(p,[3,4]) = \frac{1}{2} \cap 0 = 0$. On the other hand, it holds $\widetilde{V}(\langle A\rangle\langle A\rangle p,[0,2]) \succeq \widetilde{R}_A([0,2],[1,2]) \cap (\widetilde{R}_A([1,2],[2,4]) \cap \widetilde{V}(p,[2,4])) = \frac{1}{2} \cap 1 \cap 1 = \frac{1}{2}$. Therefore, $\widetilde{V}(\langle A\rangle\langle A\rangle \varphi \to \langle L\rangle\varphi,[0,2]) = \frac{1}{2} \hookrightarrow 0 = 0$, and so $\langle A\rangle\langle A\rangle\varphi \to \langle L\rangle\varphi$ is not valid. The other right-to-left implications can be shown not to be valid using similar arguments. To conclude, let us prove that neither:

$$\langle L\rangle p \to \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$$

nor

$$\langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p \to \langle L\rangle p$$

is valid, starting with the former one. Consider a model $\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V}\rangle$ with domain $D = \{0,3,5,6\}$ and let, again, $\mathcal{H} = \mathcal{H}_3$, with $\widetilde{=}(x,y) =$

$\max\{0, 1-\frac{1}{2}|x-y|\}$ and $\widetilde{<}(x,y) = \min\{1, \max\{\frac{1}{2}(y-x), 0\}\}$. Then, consider a valuation $\widetilde{V}$ such that $\widetilde{V}(p,[5,6]) = 1$ and $\widetilde{V}(p,[x,y]) = 0$ for every $[x,y] \neq [5,6]$. Then, the only interval $[x,y]$ such that $\widetilde{R}_L([0,3],[x,y]) \succ 0$ is $[5,6]$, and, in particular, $\widetilde{R}_L([0,3],[5,6]) = \widetilde{<}(3,5) = 1$. Then $\widetilde{V}(\langle L\rangle p,[0,3]) = \widetilde{R}_L([0,3],[5,6]) \cap \widetilde{V}(p,[5,6]) = 1$. On the other hand, the only intervals $[z,t]$ such that $\widetilde{R}_{\overline{B}}([0,3],[z,t]) \succ 0$ are $[0,5]$ and $[0,6]$, so, for $[z,t]$ varying in $\{[0,5],[0,6]\}$, it holds:

$$\begin{aligned}
&\widetilde{V}(\langle\overline{B}\rangle[E]\langle\overline{B}\rangle\langle E\rangle p,[0,3]) = \\
&\bigcup \widetilde{R}_{\overline{B}}([0,3],[z,t]) \cap \widetilde{V}([E]\langle\overline{B}\rangle\langle E\rangle p,[z,t])
\end{aligned}$$

The only intervals $[u,v]$ such that $\widetilde{R}_E([0,5],[x,y]) \succ 0$ are $[3,5]$, $[3,6]$, and $[5,6]$, so for $[u,v]$ varying in $\{[3,5],[3,6],[5,6]\}$, we have:

$$\begin{aligned}
&\widetilde{V}([E]\langle\overline{B}\rangle\langle E\rangle p,[0,5]) = \\
&\bigcap \widetilde{R}_E([0,5],[u,v]) \hookrightarrow \widetilde{V}(\langle\overline{B}\rangle\langle E\rangle p,[u,v]) \preceq \\
&\widetilde{R}_E([0,5],[3,6]) \hookrightarrow \widetilde{V}(\langle\overline{B}\rangle\langle E\rangle p,[3,6]) = \\
&\frac{1}{2} \hookrightarrow 0 = 0
\end{aligned}$$

Similarly, we have $\widetilde{V}([E]\langle\overline{B}\rangle\langle E\rangle p,[0,6]) = 0$; hence $\widetilde{V}(\langle\overline{B}\rangle[E]\langle\overline{B}\rangle\langle E\rangle p,[0,3]) = 0$. Therefore, $\widetilde{V}(\langle L\rangle p \rightarrow \langle\overline{B}\rangle[E]\langle\overline{B}\rangle\langle E\rangle p,[0,3]) = 1 \hookrightarrow 0 = 0$, proving that $\langle L\rangle p \rightarrow \langle\overline{B}\rangle[E]\langle\overline{B}\rangle\langle E\rangle p$ is not valid. In order to prove that $\langle\overline{B}\rangle[E]\langle\overline{B}\rangle\langle E\rangle p \rightarrow \langle L\rangle p$ is not valid either, take a model $\widetilde{M} = \langle\mathbb{I}(\widetilde{\mathbb{D}}),\widetilde{V}\rangle$ with domain $D = \{0,\frac{1}{2},1,3\}$ and set $\mathcal{H} = \mathcal{H}_5$. Define, as above, $\widetilde{=}(x,y) = \max\{0, 1-\frac{1}{2}|x-y|\}$ and $\widetilde{<}(x,y) = \min\{1,\max\{\frac{1}{2}(y-x),0\}\}$. Consider now a valuation $\widetilde{V}$ such that $\widetilde{V}(p,[x,y]) = 0$ for every $[x,y] \in \mathbb{I}(\widetilde{\mathbb{D}})$. This implies that $\widetilde{V}(\langle L\rangle p,[0,\frac{1}{2}]) = 0$. On the other hand, $\widetilde{R}_{\overline{B}},([0,\frac{1}{2}],[1,3]) = \frac{1}{2}$ and, for all $[x,y] \in \mathbb{I}(\widetilde{\mathbb{D}})$ it happens that $\widetilde{R}_E([1,3],\ [x,y]) = 0$. This implies that $\widetilde{V}(\langle\overline{B}\rangle[E]\langle\overline{B}\rangle\langle E\rangle p,[0,\frac{1}{2}]) \neq 0$ and, as recalled above, $\widetilde{V}(\langle L\rangle p,[0,\frac{1}{2}]) = 0$, proving that $\langle\overline{B}\rangle[E]\langle\overline{B}\rangle\langle E\rangle p \rightarrow \langle L\rangle p$ is not valid. $\square$

3.4. The Satisfiability Problem of FHS

In this section, we assume that $\mathcal{H}$ is a Heyting chain; this assumption will be briefly discussed at the end of the section. The satisfiability problem for crisp HS has been studied in a comprehensive way [4, 5, 6, 7, 8, 9, 40, 41]; its fuzzy counterpart asks the question of whether a given formula is satisfiable at any degree at all. In the crisp case, without syntactical or semantical restrictions, satisfiability for HS is generally undecidable, regardless the properties of the underlying linear order and, in addition, it remains undecidable for most syntactical fragments of HS. As we have seen in the previous section, the taxonomy of expressively different syntactical fragments of FHS is very different from that of HS; however, the negative computational properties still transfer from included to including fragments. Let us denote by $\mathrm{X_1 \ldots X_n}$ a syntactical fragment of HS, characterized by featuring, only, the modalities corresponding to the relations $R_{X_1}, \ldots, R_{X_n}$, and by $\mathrm{FX_1 \ldots X_n}$ its fuzzy counterparts. The previous observation implies that if any satisfiability problem is undecidable for a syntactical fragment $\mathrm{FX_1 \ldots X_n}$ of FHS under a certain hypothesis, then so is the same problem for full FHS under the same hypothesis. Let us consider the particular case of the fragment FO (i.e., the fuzzy counterpart of the HS fragment O, containing only the Allen relation O) of FHS, whose crisp counterpart has been studied in [42]. Observe that every formula of the fragment O, modulo writing 0 for $\perp$ and $\varphi \rightarrow 0$ for $\neg\varphi$, is, in fact, a formula of FO. Now, we want to show that a formula φ of the fragment O of HS is satisfiable if and only if its fuzzy counterpart is satisfiable in FHS.

Theorem 4. *Let φ be a formula of the fragment* O *of (crisp) HS. Then, φ is satisfiable in the class of all linear orders if and only if its fuzzy counterpart is α-satisfiable in the class of all fuzzy linear orders for some $\alpha \succ 0$, $\alpha \in H$.*

Proof. The left-to-right direction follows easily, by observing that any interval model $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ can be seen as a fuzzy interval model $\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V} \rangle$ with $\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=} \rangle$, where $\widetilde{<}(x,y) = 1$ if $x < y$ and $\widetilde{<}(x,y) = 0$, otherwise, $\widetilde{=}(x,y) = 1$ if $x = y$ and $\widetilde{=}(x,y) = 0$, otherwise, and $\widetilde{V}(p,[x,y]) = 1$ if $[x,y] \in V(p)$ and $\widetilde{V}(p,[x,y]) = 0$, otherwise. Conversely, suppose that, for some fuzzy model $\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V} \rangle$ with $\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=} \rangle$ based on a Heyting chain $\mathcal{H}$, and some $[x,y] \in \mathbb{I}(\widetilde{\mathbb{D}})$, we have $\widetilde{V}(\varphi,[x,y]) = \alpha \succ 0$. Consider the interval model $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ obtained from $\widetilde{M}$ by setting $\mathbb{D} = \langle D, < \rangle$ equal to the crispification of $\widetilde{\mathbb{D}}$ and $V(p) = \{[x,y] \in \mathbb{I}(\widetilde{\mathbb{D}}) \mid \widetilde{V}(p,[x,y]) \neq 0\}$. It is clear that $\mathbb{I}(\mathbb{D}) = \mathbb{I}(\widetilde{\mathbb{D}})$. Observe, now, that for all $[x,y],[z,t] \in \mathbb{I}(\mathbb{D})$ it is the case that $[x,y]R_O[z,t]$ if and only if $\widetilde{R}_O([x,y],[z,t]) \neq 0$. To see this, suppose that $[x,y]R_O[z,t]$. Then $x < z < y < t$ and so, by definition of $<$, we have $\widetilde{<}(x,z), \widetilde{<}(z,y), \widetilde{<}(y,t) \neq 0$ and, since $\mathcal{H}$ is a chain, $\widetilde{R}_O([x,y],[z,t]) = \widetilde{<}(x,z) \cap \widetilde{<}(z,y) \cap \widetilde{<}(y,t) \neq 0$. Conversely, suppose that $\widetilde{R}_O([x,y],[z,t]) \neq 0$, i.e., $\widetilde{<}(x,z) \cap \widetilde{<}(z,y) \cap \widetilde{<}(y,t) \neq 0$. This implies that $\widetilde{<}(x,z), \widetilde{<}(z,y), \widetilde{<}(y,t) \neq 0$. So, by definition, we must have $x < z < y < t$, and, hence, that $[x,y]R_O[z,t]$. Now, we can prove that, for every O-formula φ, and every interval $[x,y] \in \mathbb{I}(\mathbb{D})$, it is the case that $[x,y] \in V(\varphi)$ if and only if $\widetilde{V}(\varphi,[x,y]) \neq 0$. To see this, we proceed by structural induction on φ. The base case for propositional variables is immediate by the definition of V, and the case for $\perp$ is trivial. The inductive cases for φ of the form $\theta \wedge \psi$ or $\theta \vee \psi$ are straightforward. Suppose, now, that φ is of the form $\theta \to \psi$, and suppose that $\widetilde{V}(\theta \to \psi,[x,y]) = 0$. Then, $\widetilde{V}(\theta,[x,y]) \hookrightarrow \widetilde{V}(\psi,[x,y]) = 0$, and hence $\widetilde{V}(\theta,[x,y]) \neq 0$ while $\widetilde{V}(\psi,[x,y]) = 0$ (because we are assuming that $\mathcal{H}$ is a chain). By the inductive hypothesis, $[x,y] \in V(\theta)$ and $[x,y] \notin V(\psi)$, so $[x,y] \notin V(\theta \to \psi)$. Conversely, suppose that $\widetilde{V}(\theta \to \psi,[x,y]) = (\widetilde{V}(\theta,[x,y]) \hookrightarrow \widetilde{V}(\psi,[x,y])) \neq 0$. Then either $\widetilde{V}(\theta,[x,y]) = 0 = \widetilde{V}(\psi,[x,y])$, or $\widetilde{V}(\psi,[x,y]) \neq 0$. In both cases, the inductive hypothesis yields $[x,y] \in V(\theta \to \psi)$. Now, consider the case in which φ is of the form $\langle O\rangle\psi$, and suppose that $[x,y] \in V(\langle O\rangle\psi)$. So, there is an interval $[z,t] \in \mathbb{I}(\mathbb{D})$ such that $[x,y]R_O[z,t]$ and $[z,t] \in V(\psi)$. As we have proved before, we have that $\widetilde{R}_O([x,y],[z,t]) \neq 0$ and, by the inductive hypothesis, $\widetilde{V}(\psi,[z,t]) \neq 0$. Therefore $\widetilde{V}(\langle O\rangle\psi,[x,y]) = \bigcup\{\widetilde{R}_O([x,y],[z,t]) \cap \widetilde{V}(\varphi,[z,t]) \mid [z,t] \in \mathbb{I}(\widetilde{\mathbb{D}})\} \neq 0$. Conversely, suppose that $\widetilde{V}(\langle O\rangle\psi,[x,y]) = \bigcup\{\widetilde{R}_O([x,y],[z,t]) \cap \widetilde{V}(\varphi,[z,t]) \mid [z,t] \in \mathbb{I}(\widetilde{\mathbb{D}})\} \neq 0$. Then, there must be an interval $[z,t] \in \mathbb{I}(\widetilde{\mathbb{D}})$ such that $\widetilde{R}_O([x,y],[z,t]) \neq 0$ and $\widetilde{V}(\varphi,[z,t]) \neq 0$. As we have seen above, this means that $[x,y]R_O[z,t]$ and, by the inductive hypothesis, $[z,t] \in V(\psi)$, so $[x,y] \in V(\langle O\rangle\psi)$. The case for φ of the form $[O]\psi$ can be proved using a similar argumentation. $\square$

Corollary 1. *The satisfiability problem for FHS interpreted in the class of all fuzzy linear orders is undecidable.*

The satisfiability problem for HS and its fragments has been studied in particular classes of linearly ordered sets, such as the class of all finite linearly ordered sets, the class of all strongly discrete linearly ordered sets, and so on. For most of cases, the computational properties are unaffected by the additional conditions of the underlying linear order, at least for classes with at least one arbitrarily long linear order. The fragment O, among others, is undecidable in all such cases. The investigation of the satisfiability problem

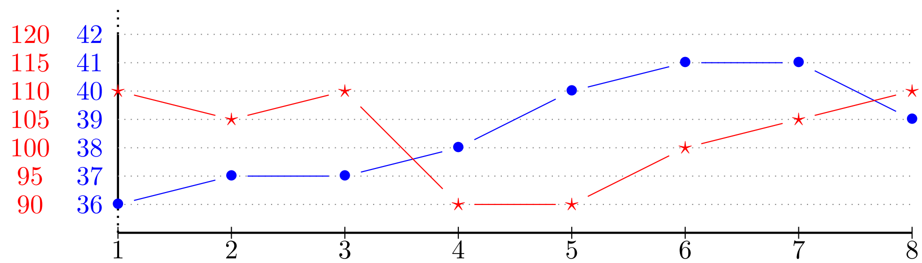


Figure 4: A multivariate time series with two variables.

for (fragments of) FHS w.r.t. different linear orders would require first to identify a set of axioms characterizing the fuzzy counterpart of each of these classes of linear orders (along the line of the set of axioms 1–7 provided here for the class of all linear orders). Another natural question is whether the assumption of the underlying algebra $\mathcal{H}$ being a chain can be relaxed w.r.t. the undecidability of the satisfiability problem of FHS, and if there are other interesting fragments of FHS for which an analogous of the above result holds.

4. Fuzzy Interval Temporal Logic: Applications

4.1. Multivariate Time Series as Fuzzy Interval Models

As we have shown in a previous example, fuzzy interval models emerge naturally from real-world applications, such as in the medical domain. More commonly, however, temporal data are represented in the form of time series, and, as it turns out, time series are, in a way, a systematic source of fuzzy interval models over which one may want to reason.

A *time series* (as in Fig. 4, for example) is a set of *temporal variables*, or *attributes*, $\mathcal{A} = \{A_1, \ldots, A_n\}$ that change over time, and it can be *univariate*, if $n = 1$, or *multivariate*, if $n > 1$. Each variable of a multivariate time series T is an ordered collection of N_T numerical or categorical values, instead of a single value, and it can be described as follows:

$$T = \left\{ \begin{array}{lcllll} A_1 & = & a_{1,1}, & a_{1,2}, & \ldots, & a_{1,N_T} \\ A_2 & = & a_{2,1}, & a_{2,2}, & \ldots, & a_{2,N_T} \\ & \vdots & & & & \\ A_n & = & a_{n,1}, & a_{n,2}, & \ldots, & a_{n,N_T}. \end{array} \right.$$

An univariate time series with categorical values is also known as a *time* (or *temporal*) *sequence*; we use the term *time series* to denote multivariate, mixed (numerical and categorical) set of temporal variables; the value of an attribute A at a point x is denoted by $A(x)$, and the domain of A by $dom(A)$. Categorical values are fairly uncommon in time series, and typical temporal datasets are usually numerical; here, we limit our attention to purely numerical cases. A time series is the direct temporal generalization of a static data, in which adimensional attributes become temporally ordered series of values. Suppose, for example, that we want to describe the clinical history of a patient: the set of numerical attributes may include *fever* and *blood pressure*, which in the atemporal case must be abstracted and described by a single value (e.g., their mean over the observation period) or a predetermined set of values (e.g., the mean, the standard deviation, the maximum, the minimum), while in the temporal case can be reported as they are sampled (e.g., the fever at time $1, 2, \ldots$).

Let T be a multivariate time series of length N_T, defined over $\mathcal{A} = \{A_1, \ldots, A_n\}$. Let us denote by D_T the set $\{1, \ldots, N_T\}$, and by $\mathbb{D}_T$ the corresponding linearly ordered set. Then let us fix:

$$\mathcal{H} = (\{\alpha \mid \alpha \in \mathbb{R}, 0 \leq \alpha \leq 1\}, \min, \max, \hookrightarrow, 0, 1),$$

where $\hookrightarrow$ becomes:

$$\alpha \hookrightarrow \beta = \begin{cases} 1 & \text{if } \alpha \preceq \beta; \\ \beta & \text{otherwise.} \end{cases}$$

To use $\mathbb{D}_T$ as a fuzzy strictly linearly ordered set, we first fix a nonzero parameter $h \in \mathbb{N}$ (which can be thought of as an *horizon*), and, then, define two parametric relations $\widetilde{=}$ and $\widetilde{<}$, as follows:

$$\widetilde{=}_h(x, y) = \begin{cases} 0 & \text{if } \ |y - x| \geq h; \\ \frac{h - |x-y|}{h} & \text{if } \ |y - x| < h, \end{cases}$$

and

$$\widetilde{<}_h(x, y) = \begin{cases} 0 & \text{if } \ y < x; \\ 1 & \text{if } \ y - x > h; \\ \frac{y-x}{h} & \text{if } \ 0 \leq y - x \leq h. \end{cases}$$

It is immediate to see that they satisfy 1-7, and that if $h = 1$ our definition immediately reduces to the crisp definitions of $=$ and $<$. Therefore, $\widetilde{\mathbb{D}_T} =$

$\langle D_T, \widetilde{<}_h, \widetilde{=}_h \rangle$ is a fuzzy strictly linearly ordered set. Now, we define a set of *decisions* $\mathcal{S}$ as:

$$\mathcal{S} = \{A \bowtie a \mid A \in \mathcal{A}, a \in dom(A), \bowtie \in \{\leq, <, =, >, \geq\}\},$$

and we set $\mathcal{S}$ as our set of propositional letters. Elements of $\mathcal{S}$ can be evaluated via a suitable evaluation function $\widetilde{V}$ which can take into account a wide range of semantical restrictions, depending on the particular application. A relatively natural way to interpret decisions over intervals is to assign to $A \bowtie a$ on an interval $[x, y]$ the truth value corresponding to the fraction of points that satisfy $A \bowtie a$ in the interval over the cardinality of the interval itself, that is:

$$\widetilde{V}_T(A \bowtie a, [x, y]) = \frac{\mid \{z \in \mathbb{N} \mid x \leq z \leq y, A(z) \bowtie a\} \mid}{\mid \{z \in \mathbb{N} \mid x \leq z \leq y\} \mid}.$$

In this way, given a time series T, its corresponding fuzzy interval model is uniquely defined:

$$\widetilde{T} = \langle \mathbb{I}(\widetilde{\mathbb{D}_T}), \widetilde{V}_T \rangle.$$

4.2. Fuzzy Multivariate Time Series Checking

The simplest and most natural problem in which one may be interested at this point is the *multivariate time series checking problem* (for us, *MTSMC* problem), that is, given a time series T and its corresponding fuzzy interval model $\widetilde{T}$, an interval $[x, y]$, and a FHS formula φ, the problem of establishing the truth value $\widetilde{V}_T(\varphi, [x, y])$. In its original formulation, *model checking* is the problem of verifying if a given formula is satisfied by a given model [43]. Usually, the model is the abstract representation of a system, in which the relevant properties become propositional letters, where the formula is written in a temporal logic and represents an interesting property. The prevailing adopted ontology for both the model and the logic is point-based: systems are represented in such a way that each state is a vertex on a Kripke model, atomic properties are descriptions of states, and the underlying logic is a point-based temporal logic, often LTL or CTL [44, 45]. The problem of checking *finite, linear*, and *fully represented* interval models (FIMC problem) against HS formulæ was formulated in [13], and its *infinite, periodical* generalization was presented in [46] for a fragment of HS. In the crisp case, the FIMC problem [13] presents itself in two variants, that is, for *sparse* and *non-sparse* models. In short, sparse models are a degenerate models whose representation is exponentially small than their informative content, and must be pre-processed before model checking can be efficiently applied on them. However, (fuzzy) interval models that emerge from time series do not present the sparseness problem, allowing us to define, and (polynomially) solve, the MTSMC problem (as shown above) which can be considered, in a way, a particular case of the more general fuzzy interval model checking one. Even in this reduced form, MTSMC has several potential applications, such as in the context of medical research (e.g., medical data that can be checked to make sure that some constraint is actually met), environmental sustainability research (e.g., environmental data can be checked to find out if certain phenomena behave as hypothesized), or technological research (e.g., flight data can be checked to guarantee suitable safety conditions). Clearly, our algorithm for MTSMC can be used to model check generic models as well, at the price of an exponential complexity.

As it turns out, a time series checking algorithm is not difficult to design. Let $\widetilde{T} = \langle \mathbb{I}(\widetilde{\mathbb{D}_T}), \widetilde{V}_T \rangle$ be a model that corresponds to a times series T, φ a formula of FHS, and $[x, y]$ an interval in $\mathbb{I}(\widetilde{\mathbb{D}_T})$. Algorithm 1 is the adaptation of Emerson and Clarke's classical CTL algorithm [47] to the interval, fuzzy case, and it returns the value $\widetilde{V}_T(\varphi, [x, y])$. In Algorithm 1, we use the symbol $\circ \in \{\vee, \wedge, \rightarrow\}$ to denote a logical symbol, and the symbol $\bullet$ to denote its algebraic corresponding one (resp., $\max, \min, \hookrightarrow$, as we have defined them above). Unlike the crisp case, every sub-formula must be checked on every interval, because, in the fuzzy case, any two intervals may be related by any relation $\widetilde{R}_X$. The auxiliary data structure L can be thought of as a hash table indexed by three elements, namely ψ, x, y, that is, a sub-formula, and two points. Accessing L may be considered to have constant time complexity. Formulæ, classically represented as binary trees, can be pre-processed in order to identify repeating sub-formulæ, so that the main cycle of Algorithm 1 can be implemented in an efficient way. It is worth observing that such a solution implicitly assumes that N_T is a reasonably low value; for very high values of N_T, a different solution should be designed for the dimension of L to be manageable. Let us now discuss the computational complexity of this approach.

Theorem 5. *The MTSMC problem can be solved in polynomial time by a deterministic algorithm.*

Proof. Let $\widetilde{T} = \langle \mathbb{I}(\widetilde{\mathbb{D}_T}), \widetilde{V}_T \rangle$ be a model based on n temporal attributes, each

Algorithm 1 Fuzzy multivariate time series checking algorithm.

> **function** $\textsc{Check}(\widetilde{T}, \varphi, [x,y])$
> **for** $\psi \in sub(\varphi)$ in increasing length order **do**
>   **if** $\psi = \alpha$, with $\alpha \in H$ **then**
>   **for** $[z,t] \in \mathbb{I}(\widetilde{\mathbb{D}_T})$ **do**
> $L(\psi, [z,t]) = \alpha$
>   **if** $\psi = A \bowtie a$ **then**
>   **for** $[z,t] \in \mathbb{I}(\widetilde{\mathbb{D}_T})$ **do**
> $L(\psi, [z,t]) = \widetilde{V}_T(A \bowtie a, [z,t])$
>   **if** $\psi = \tau \circ \xi$ **then**
>   **for** $[z,t] \in \mathbb{I}(\widetilde{\mathbb{D}_T})$ **do**
> $L(\psi, [z,t]) = L(\tau, [z,t]) \bullet L(\xi, [z,t])$
>   **if** $\psi = \langle X \rangle \tau$ **then**
>   **for** $[z,t] \in \mathbb{I}(\widetilde{\mathbb{D}_T})$ **do**
> $s = 0$
>   **for** $[v,w] \in \mathbb{I}(\widetilde{\mathbb{D}_T})$ **do**
> $s' \leftarrow \min(\widetilde{R}_X([z,t],[v,w]), L(\tau,[v,w]))$
> $s \leftarrow \max(s, s')$
> $L(\psi, [z,t]) = s$
>   **if** $\psi = [X]\tau$ **then**
>   **for** $[z,t] \in \mathbb{I}(\widetilde{\mathbb{D}_T})$ **do**
> $s = 1$
>   **for** $[v,w] \in \mathbb{I}(\widetilde{\mathbb{D}_T})$ **do**
> $s' \leftarrow \widetilde{R}_X([z,t],[v,w]) \hookrightarrow L(\tau,[v,w])$
> $s \leftarrow \min(s, s')$
> $L(\psi, [z,t]) = s$
> **return** $L(\varphi, [x,y])$

with N_T distinct points, and let $|\varphi|$ be the length of the input formula φ. Since the length of the time series grows as time passes, while its width remains unchanged as well as the property to check, we can assume that $n, |\varphi| = o(N_T)$, that is, that there are much less temporal variables and much less sub-formulæ than there are distinct points. Thus, we can express the size of the input as $O(N_T)$. Also, we can assume that $\min, \max$, and $\hookrightarrow$ can be computed in constant time w.r.t. N_T, and that each call to $\widetilde{V}_T()$ takes time $O(N_T)$ (in the worst case scenario, in fact, each call requires exploring an interval with $O(N_T)$ points). The most external cycle is executed $O(N_T)$ times. In the worst-case scenario, during each execution ψ is a modal formula. Since there are $O(N_T^2)$ intervals in $\widetilde{T}$, the complexity of the modal case is $O(N_T^4)$. Therefore, the entire algorithm runs in $O(N_T^5)$. $\square$

5. Conclusions and Future Work

Temporal reasoning is a major challenge in artificial intelligence and computer science. Halpern and Shoham's interval temporal logic HS is a milestone in the field of temporal logics, due to its naturalness and expressive power. In this paper, we proposed FHS, which is the first pure fuzzy version of HS, following the original work by Fitting on fuzzy modal logics, that is, including a fuzzyfication of both interval-interval relations and propositional letters. We studied the behaviour of FHS in terms of satisfiability problem and in terms of expressive power. Then, we proposed a fuzzy adaptation of the finite interval temporal logic model checking problem to the case of multivariate temporal series, and we studied its computational complexity, proving that it can be solved in polynomial time.

In terms of future work, three problems clearly arise. First, from the theoretical point of view, a stronger result is needed to prove the undecidability of FHS in the more general algebraic setting of unrestricted Heyting algebras. Second, from the application point of view, an adaptation of current (modern) symbolic learning algorithms to the fuzzy interval case (as it has been proposed, at the propositional level, in [48]) would be the natural next step. Third, for both theoretical and practical reasons, it would be interesting to understand if the fuzzy finite interval temporal checking problem can be efficiently solved in the sparse case as well.

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