# Interval Temporal Logics over Finite Linear Orders: the Complete Picture

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**Abstract.** Interval temporal logics provide a natural framework for temporal reasoning about interval structures over linearly ordered domains, where intervals are taken as the primitive ontological entities. In this paper, we identify all fragments of Halpern and Shoham's interval temporal logic HS whose finite satisfiability problem is decidable. We classify them in terms of both relative expressive power and complexity. We show that there are exactly 62 expressively-different decidable fragments, whose complexity ranges from NP-complete to non-primitive recursive (all other HS fragments have been already shown to be undecidable).

## 1 Introduction

Interval temporal logics provide a natural framework for temporal reasoning about interval structures over linearly (or partially) ordered domains. They take time intervals as the primitive ontological entities and define truth of formulas relative to time intervals, rather than time points. In the so-called *pure* (or *strict*) approach, which is the one we focus on in this paper, intervals with coincident endpoints are excluded from the semantics. Interval logic modalities correspond to various relations between pairs of intervals. In particular, the well-known logic HS, introduced by Halpern and Shoham in [14], features a set of modalities that make it possible to express all Allen's interval relations [1]. Interval-based formalisms have been extensively used in various areas of AI, such as, for instance, planning, theories of action and change, natural language processing, and constraint satisfaction. However, most of them make severe syntactic and semantic restrictions that considerably weaken their expressive power. Interval temporal logics relax these restrictions, thus allowing one to cope with much more complex application domains and scenarios. Unfortunately, many of them, including HS and the majority of its fragments, turn out to be undecidable (an up-to-date comprehensive survey can be found in [11]).

One of the few cases of a decidable interval logic with truly interval-based semantics, that is, not reducible to point-based semantics, is Propositional Neighborhood Logic, denoted by $A\overline{A}$. It is the fragment of HS with two modalities corresponding to Allen's relations *meets* and *met by* (the complete list of Allen's relations can be found in Fig. 1). $A\overline{A}$ has been intensively studied and its decidability has been proved with respect to various classes of interval structures (all, dense, and discrete linear orders, natural numbers, integers, rationals) [5, 12].

In this paper, we focus our attention on the class of all finite linear orders, that come into play in a variety of application domains. Consider, for instance, planning problems. They consist of finding a finite partially-ordered sequence of actions that, applied to an initial world state, leads to a final state (the goal), within a bounded amount of time, satisfying suitable conditions about which sequence of states the world must go through. We give a complete picture of HS fragments with respect to (un)decidability of their satisfiability problem over finite linear orders, reviewing known results and providing missing ones. In particular, we identify the set of all expressively-different decidable fragments, and we determine the exact complexity of each of them. We will denote HS fragments by the set of their modalities, in alphabetical order, and omitting those which are definable in terms of the others (in the considered fragment). As we will see, if we restrict our attention to decidable fragments, the only definable operators are $\langle L \rangle$ and $\langle \overline{L} \rangle$, corresponding to Allen's relations *after* and *before*, respectively: $\langle L \rangle$ can be defined as $\langle A \rangle \langle A \rangle$, and $\langle \overline{L} \rangle$ by $\langle \overline{A} \rangle \langle \overline{A} \rangle$. Moreover, thanks to the highly symmetrical structure of the class of decidable fragments, all decidability results for fragments involving modalities $\langle B \rangle$ and $\langle \overline{B} \rangle$ (for Allen's relations *starts* and *started by*) can be immediately transferred to mirror fragments involving modalities $\langle E \rangle$ and $\langle \overline{E} \rangle$ (for Allen's relations *finishes* and *finished by*). More precisely, each HS fragment in Fig. 2 can be transformed into its mirror image by reversing the time order and replacing $\langle A \rangle$ by $\langle \overline{A} \rangle$, $\langle \overline{A} \rangle$ by $\langle A \rangle$, $\langle L \rangle$ by $\langle \overline{L} \rangle$, $\langle \overline{L} \rangle$ by $\langle L \rangle$, $\langle B \rangle$ by $\langle E \rangle$, and $\langle \overline{B} \rangle$ by $\langle \overline{E} \rangle$. We will refer to the Hasse diagram obtained by replacing each fragment with its mirror image as the mirror diagram. Fig. 2 displays 35 different decidable fragments. If we pair them with the fragments in the mirror diagram, we obtain a total of 62 different decidable fragments (8 fragments belong to both diagrams).

Most of the results reported in this paper were already known: $B\overline{B}$ (and thus also its fragments B and $\overline{B}$) is NP-complete [13]; $A\overline{A}$ and all its fragments featuring at least one between $\langle A \rangle$ and $\langle \overline{A} \rangle$ are NEXPTIME-complete [5, 7]; AB, $AB\overline{B}$, and $ABB\overline{L}$ are EXPSPACE-complete [9, 18]; $A\overline{A}B$ $A\overline{A}\overline{B}$, and $A\overline{A}B\overline{B}$ are non-primitive recursive [17]. In this paper, we complete the picture by proving the following new results: *(i)* NP-completeness (in particular, NP-membership) of $B\overline{B}$ can be lifted to $B\overline{B}L\overline{L}$ and each of its (other) fragments; *(ii)* EXPSPACE-completeness (in particular, EXPSPACE-hardness) of AB can be adapted to prove that $A\overline{B}$ is EXPSPACE-complete as well; *(iii)* non-primitive recursiveness of $A\overline{A}B$ can be sharpened to both $\overline{A}B$ and $\overline{A}\overline{B}$; and *(iv)* results in [10] can be easily adapted to the case of finite linear orders, thus showing that the proposed classification of the considered fragments with respect to their expressive power is sound and complete. Pairing *(iv)* with already known undecidability results, we can conclude that the classification of HS fragments with respect to finite satisfiability is

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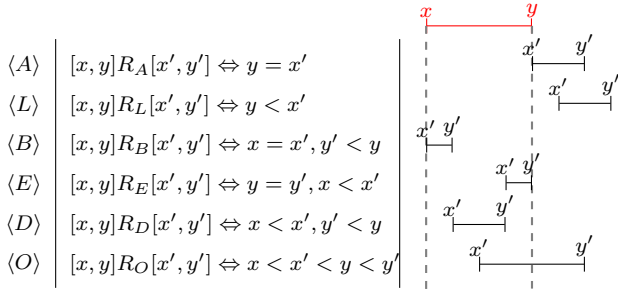[5] University of Murcia, Spain, *guido@um.es*

$$
\begin{array}{c|l}
\langle A\rangle & [x,y]R_A[x',y'] \Leftrightarrow y=x' \\
\langle L\rangle & [x,y]R_L[x',y'] \Leftrightarrow y<x' \\
\langle B\rangle & [x,y]R_B[x',y'] \Leftrightarrow x=x', y'<y \\
\langle E\rangle & [x,y]R_E[x',y'] \Leftrightarrow y=y', x<x' \\
\langle D\rangle & [x,y]R_D[x',y'] \Leftrightarrow x<x', y'<y \\
\langle O\rangle & [x,y]R_O[x',y'] \Leftrightarrow x<x'<y<y'
\end{array}
$$

**Figure 1.** Allen's interval relations and the corresponding HS modalities.

now complete. In particular, we would like to point out that fragments D and $\overline{D}$, and O and $\overline{O}$ have been shown to be indecidable in [16] and [6], respectively. Undecidability of any fragment including them immediately follows. Similarly, undecidability of any fragment including BE, B$\overline{E}$, $\overline{B}$E, or $\overline{BE}$ has been shown in [3].

## 2 Preliminaries

Let $\mathbb{D}=\langle D,<\rangle$ be a finite linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[x,y]$, where $x,y \in D$ and $x<y$ (*strict semantics*). There are 12 different non-trivial ordering relations (excluding equality) between any pair of intervals in a linear order, often called *Allen's relations* [1]: the six relations depicted in Fig. 1 and the inverse ones. We interpret interval structures as Kripke structures and Allen's relations as accessibility relations, thus associating a modality $\langle X\rangle$ with each Allen's relation $R_X$. For each operator $\langle X\rangle$, its *inverse* (or *transpose*), denoted by $\langle \overline{X}\rangle$, corresponds to the inverse relation $R_{\overline{X}}$ of $R_X$ (that is, $R_{\overline{X}}=(R_X)^{-1}$).

Halpern and Shoham's logic HS is a multi-modal logic with formulas built on a set $\mathcal{AP}$ of proposition letters, the boolean connectives $\vee$ and $\neg$, and a modality for each Allen's relation. We denote by $X_1 \ldots X_k$ the fragment of HS featuring a modality for each Allen's relation in the subset $\{R_{X_1}, \ldots, R_{X_k}\}$. Formulas of $X_1 \ldots X_k$ are defined by the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X_1\rangle\varphi \mid \ldots \mid \langle X_k\rangle\varphi.$$

The other boolean connectives can be viewed as abbreviations, and the dual operators $[X]$ are defined as usual, that is, $[X]\varphi \equiv \neg\langle X\rangle\neg\varphi$. Given a formula $\varphi$, its *length*, denoted by $|\varphi|$, is the number of its symbols. The semantics of HS is given in terms of *interval models* $M=\langle \mathbb{I}(\mathbb{D}),V\rangle$, where $\mathbb{I}(\mathbb{D})$ is the set of all intervals over $\mathbb{D}$ and $V: \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$ is a *valuation function* that assigns to every $p \in \mathcal{AP}$ the set of intervals $V(p)$ over which $p$ holds. The *truth* of a formula over a given interval $[x,y]$ in an interval model $M$ is defined by structural induction on formulas: (i) a proposition letter $p$ is true over an interval $[x,y]$ iff $[x,y] \in V(p)$; (ii) boolean connectives are dealt with in the standard way; (iii) for each modality $\langle X\rangle$, it holds that $M,[x,y] \Vdash \langle X\rangle\psi$ iff there exists an interval $[x',y']$ such that $[x,y]R_X[x',y']$ and $M,[x',y'] \Vdash \psi$, where $R_X$ is the relation corresponding to $\langle X\rangle$. An HS-formula $\phi$ is *valid*, denoted by $\Vdash \phi$, if it is true on every interval in every interval model.

## 3 Expressiveness and Undecidability

In this section, we study the expressive power of HS fragments over the class of finite linear orders. Given a fragment $\mathcal{F} = X_1 X_2 \ldots X_k$ and a modal operator $\langle X\rangle$, we write $\langle X\rangle \in \mathcal{F}$ if $X \in$

$\{X_1, \ldots, X_k\}$. Given two fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if $\langle X\rangle \in \mathcal{F}_1$ implies $\langle X\rangle \in \mathcal{F}_2$, for every modality $\langle X\rangle$.

**Definition 1** *An HS modality $\langle X\rangle$ is* definable *in an HS* fragment $\mathcal{F}$, *denoted $\langle X\rangle \lhd \mathcal{F}$, if $\langle X\rangle p \equiv \psi(p)$ for some formula $\psi(p) \in \mathcal{F}$, for any fixed proposition letter $p$. The equivalence $\langle X\rangle p \equiv \psi(p)$ is called an* inter-definability equation *for $\langle X\rangle$ in $\mathcal{F}$.*

In [14], Halpern and Shoham show that, according to strict semantics, all HS modalities are definable in the fragment featuring the modalities $\langle A\rangle$, $\langle B\rangle$, and $\langle E\rangle$, and their transposes $\langle \overline{A}\rangle$, $\langle \overline{B}\rangle$, and $\langle \overline{E}\rangle$ (in case non-strict semantics is assumed, the four modalities $\langle B\rangle$, $\langle E\rangle$, $\langle \overline{B}\rangle$, and $\langle \overline{E}\rangle$ suffice, as shown in [21]). Given two HS fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we say that $\mathcal{F}_2$ is *at least as expressive as* $\mathcal{F}_1$ ($\mathcal{F}_1 \preceq \mathcal{F}_2$) if each operator $\langle X\rangle \in \mathcal{F}_1$ is definable in $\mathcal{F}_2$, and that $\mathcal{F}_1$ is *strictly less expressive than* $\mathcal{F}_2$, ($\mathcal{F}_1 \prec \mathcal{F}_2$), if $\mathcal{F}_1 \preceq \mathcal{F}_2$, but not $\mathcal{F}_2 \preceq \mathcal{F}_1$. Moreover, we say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are *expressively incomparable* ($\mathcal{F}_1 \not\equiv \mathcal{F}_2$), if neither $\mathcal{F}_1 \preceq \mathcal{F}_2$ nor $\mathcal{F}_2 \preceq \mathcal{F}_1$.

In order to show non-definability of a given modality in a certain fragment, we use the standard notion of *bisimulation*, and the invariance of modal formulas with respect to bisimulations (see, e.g., [2]). More precisely, we exploit the fact that any $\mathcal{F}$-bisimulation preserves the truth of *all* formulas in $\mathcal{F}$. Thus, to prove that a modality $\langle X\rangle$ is not definable in $\mathcal{F}$, it suffices to construct a pair of interval models $M$ and $M'$ and a $\mathcal{F}$-bisimulation between them, relating a pair of intervals $[a,b] \in M$ and $[a',b'] \in M'$, such that $M,[a,b] \Vdash \langle X\rangle p$, while $M',[a',b'] \not\Vdash \langle X\rangle p$.

To prove that Fig. 2 is sound and complete with respect to the class of finite linear orders (see Theorem 4 below), we focus our attention on A$\overline{A}$B$\overline{B}$ and its fragments showing that *(i)* each pair of fragments which are not related to each other in Fig. 2 are expressively incomparable; *(ii)* an edge from a fragment $\mathcal{F}_1$ to a fragment $\mathcal{F}_2$ means that $\mathcal{F}_2 \prec \mathcal{F}_1$; and *(iii)* each fragment which is displayed neither in Fig. 2 nor in the mirror diagram is undecidable. It can be easily shown that *(i)* and *(ii)* are immediate consequences of the following lemma.

**Lemma 2** $\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$ *and* $\langle \overline{L}\rangle p \equiv \langle \overline{A}\rangle\langle \overline{A}\rangle p$ *are all and only the inter-definability equations for* A$\overline{A}$B$\overline{B}$ *over finite linear orders.*

**Proof.** The soundness proof is straightforward. To prove that these equations are the only possible ones, for each operator $\langle X\rangle \in$ A$\overline{A}$B$\overline{B}$, we show that $\langle X\rangle$ is not definable in the maximal fragment of A$\overline{A}$B$\overline{B}$ not containing $\langle X\rangle$ itself. This amounts to prove that: *(1)* $\langle A\rangle \not\lhd \overline{A}$B$\overline{B}$L and $\langle \overline{A}\rangle \not\lhd$ AB$\overline{B}$L; *(2)* $\langle \overline{B}\rangle \not\lhd$ A$\overline{A}$B and $\langle B\rangle \not\lhd$ A$\overline{A}$$\overline{B}$; and *(3)* $\langle \overline{L}\rangle \not\lhd$ AB$\overline{B}$ and $\langle L\rangle \not\lhd$ $\overline{A}$B$\overline{B}$.

As for *(1)*, let $M_1=\langle \mathbb{I}(\mathbb{D}),V_1\rangle$ and $M_2=\langle \mathbb{I}(\mathbb{D}),V_2\rangle$ be two models based on $D=\{0,1,2\}$, with the usual ordering, $V_1$ and $V_2$ be such that $V_1(p)=\{[1,2]\}$ and $V_2(p)=\emptyset$, where $p$ is the only proposition letter in $\mathcal{AP}$, and $Z$ be a relation between (intervals of) $M_1$ and $M_2$ defined as $Z=\{([0,1],[0,1]),([0,2],[0,2])\}$. It can be easily shown that $Z$ is an $\overline{A}$B$\overline{B}$L-bisimulation. The local property trivially holds, since all $Z$-related intervals satisfy $\neg p$. As for forward and backward conditions, starting from interval $[0,1]$, modalities in $\overline{A}$B$\overline{B}$L only allows one to reach interval $[0,2]$ (and vice versa), that in both models satisfies $\neg p$. Hence, since $([0,1],[0,1]) \in Z$, it holds that $M_1,[0,1] \Vdash \psi$ iff $M_2,[0,1] \Vdash \psi$, for every $\psi \in \overline{A}$B$\overline{B}$L. However, $M_1,[0,1] \Vdash \langle A\rangle p$, but $M_2,[0,1] \Vdash \neg\langle A\rangle p$. Therefore, $\langle A\rangle \not\lhd \overline{A}$B$\overline{B}$L. A similar (reversed) argument works for $\langle \overline{A}\rangle$.

As for *(2)*, let $M_1$ and $M_2$ be defined as in case *(1)*, the only difference being that $V_1(p)=\{[0,2]\}$ and $V_2(p)=\emptyset$, and $Z=\{([0,1],[0,1]),([1,2],[1,2])\}$. It can be easily shown that $Z$ is an
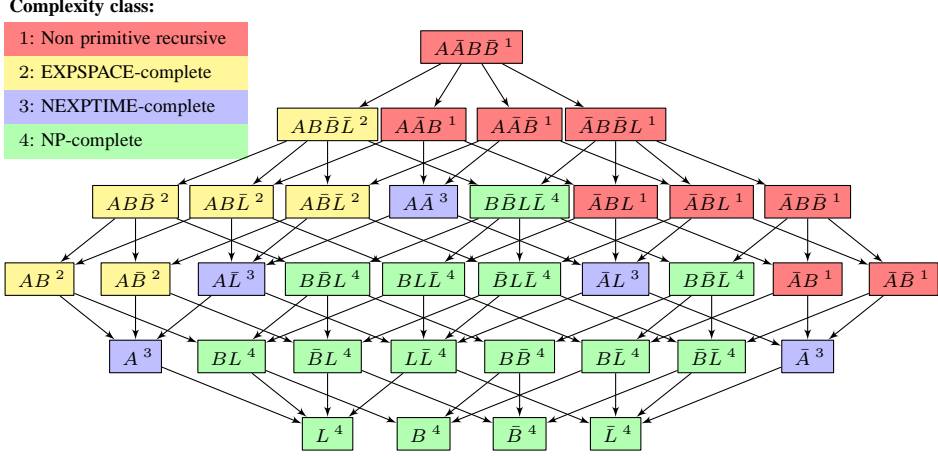
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**Figure 2.** Hasse diagram of all and only decidable fragments of HS over finite linear orders.

A$\overline{A}$B-bisimulation. The only interval that differentiates the two models (interval $[0,2]$) is not reachable from $[0,1]$ by using modalities in A$\overline{A}$B. Since $([0,1],[0,1]) \in Z$, $M_1, [0,1] \Vdash \langle\overline{B}\rangle p$, and $M_2, [0,1] \Vdash \neg\langle\overline{B}\rangle p$, we can conclude that $\langle\overline{B}\rangle \not\preceq$ A$\overline{A}$B. As before, a reversed argument works for $\langle B\rangle$.

As for *(3)*, let $M_1 = \langle\mathbb{I}(\mathbb{D}), V_1\rangle$ and $M_2 = \langle\mathbb{I}(\mathbb{D}), V_2\rangle$, where $D = \{0,1,2,3\}$, with the usual ordering, and $V_1$ and $V_2$ are such that $V_1(p) = \{[0,1]\}$ and $V_2(p) = \emptyset$. $Z = \{([2,3],[2,3])\}$ is an AB$\overline{B}$-bisimulation, as no interval is reachable from $[2,3]$. Since $M_1, [2,3] \Vdash \langle\overline{L}\rangle p$ and $M_2, [2,3] \Vdash \neg\langle\overline{L}\rangle p$, it follows that $\langle\overline{L}\rangle \not\preceq$ AB$\overline{B}$. A similar argument works for $\langle L\rangle \not\preceq \overline{A}B\overline{B}$. ∎

Property *(iii)* can be proved by pairing Lemma 2 with known undecidability results for HS fragments.

**Lemma 3** *Each HS fragment which is displayed neither in Fig. 2 nor in the mirror diagram is undecidable over finite linear orders.*

**Proof.** First, observe that, by Lemma 2, Fig. 2 contains all expressively-different fragments of HS featuring modalities from the set $\{\langle A\rangle, \langle\overline{A}\rangle, \langle B\rangle, \langle\overline{B}\rangle, \langle L\rangle, \langle\overline{L}\rangle\}$. Now, by contradiction, suppose that there exists a decidable fragment $\mathcal{F}$ which is not included in Fig. 2 or in the mirror diagram. By the previous observation, $\mathcal{F}$ must contain at least one modality from the set $\{\langle D\rangle, \langle\overline{D}\rangle, \langle O\rangle, \langle\overline{O}\rangle, \langle E\rangle, \langle\overline{E}\rangle\}$. If it contains one modality from the set $\{\langle D\rangle, \langle\overline{D}\rangle, \langle O\rangle, \langle\overline{O}\rangle\}$, then it is undecidable, since all HS fragments featuring one (and only one) of these modalities are already undecidable [6, 16]. Hence, $\mathcal{F}$ must contain at least one modality in the set $\{\langle E\rangle, \langle\overline{E}\rangle\}$. This prevents modalities $\langle B\rangle$ and $\langle\overline{B}\rangle$ to be included in $\mathcal{F}$, as they would immediately yield undecidability [3]. Then, it follows that $\mathcal{F}$ can contain only modalities from the set $\{\langle A\rangle, \langle\overline{A}\rangle, \langle E\rangle, \langle\overline{E}\rangle, \langle L\rangle, \langle\overline{L}\rangle\}$, and thus it must belong to the mirror diagram (contradiction). ∎

**Theorem 4** *The Hasse diagram in Fig. 2, together with the mirror diagram, displays all and only decidable fragments of HS over the class of finite linear orders, and their relative expressive power.*

## 4 NP-completeness

In this section, we prove that NP-completeness of B$\overline{B}$ [13] can be extended to B$\overline{B}$L$\overline{L}$. Since the satisfiability problem for proposi-

tional logic is itself NP-complete, B$\overline{B}$L$\overline{L}$ and its fragments are NP-hard. The core of this section is a membership proof, namely, NP-membership. By a model-theoretic argument, we show that finite satisfiability of a B$\overline{B}$L$\overline{L}$-formula $\varphi$ can be reduced to satisfiability in a model whose domain has a cardinality lower than a certain value which is polynomial in $|\varphi|$.

As a preliminary step, we show that satisfiability of a B$\overline{B}$L$\overline{L}$-formula $\varphi$ in a finite model $M = \langle\mathbb{I}(\{0,\ldots,N\}), V\rangle$ can be reduced to satisfiability of the formula $\tau(\varphi) = \varphi \vee \langle\overline{B}\rangle\varphi \vee \langle L\rangle\varphi \vee \langle L\rangle\langle\overline{L}\rangle(\varphi \vee \langle\overline{B}\rangle\varphi)$ over the interval $[0,1]$, that is, $M, [x,y] \Vdash \varphi$ if and only if $M, [0,1] \Vdash \tau(\varphi)$. The reader can easily check that the transformation $\tau$ does not work whenever $N = 2$ (resp., $N = 3$) and $\varphi$ is satisfied by the interval $[1,2]$ (resp., by $[1,3]$). However, in both cases, by a bisimulation argument, we can prove that there exists a model $M' = \langle\mathbb{I}(\{0,\ldots,N'\}), V\rangle$, with $N' < N$, such that $M', [0,1] \Vdash \varphi$. Thus, we can safely restrict our attention to the problem of satisfiability over $[0,1]$ (*initial* satisfiability).

Given a B$\overline{B}$L$\overline{L}$-formula $\varphi$, let $Cl(\varphi)$ be the set of all its sub-formulas and of their negations, and let $M$ be a model such that $M, [0,1] \Vdash \varphi$. For each point $x$ of the domain of $M$, let $\mathcal{R}_L(x)$ (resp., $\mathcal{R}_{\overline{L}}(x)$) be the maximal subset of $Cl(\varphi)$ consisting of all and only $\langle L\rangle$-formulas (resp., $\langle\overline{L}\rangle$-formulas) and their negations that are satisfied over intervals ending (resp., beginning) at $x$. It can be easily checked that all intervals ending (resp., beginning) at the same point satisfy the same $\langle L\rangle$-formulas (resp., $\langle\overline{L}\rangle$-formulas) and their negations. Let $\mathcal{R}(x) = \mathcal{R}_L(x) \cup \mathcal{R}_{\overline{L}}(x)$. $\mathcal{R}(x)$ is consistent, as it cannot contain a formula and its negation. Now, let $\mathcal{R}$ be the subset of $Cl(\varphi)$ that contains all possible $\langle L\rangle$- and $\langle\overline{L}\rangle$-formula and their negations. $|\mathcal{R}|$ is polynomial (linear) in $|\varphi|$.

**Lemma 5** *Let $\varphi$ be a B$\overline{B}$L$\overline{L}$-formula. Then, $\varphi$ is initially satisfiable over a finite model if and only if it is initially satisfiable over a model $M = \langle\mathbb{I}(\{0,\ldots,N\}), V\rangle$, with $N \leq (m_L + 1) \cdot m_B + m_L + 2$, where $m_L = 2 \cdot |\mathcal{R}|$ and $m_B$ is the cardinality of the set of all $\langle B\rangle$- and $\langle\overline{B}\rangle$-formulas in $Cl(\varphi)$.*

**Proof.** One direction is trivial. As for the other, let us assume that $\varphi$ is initially satisfied over a finite model $M = \langle\mathbb{I}(\{0,\ldots,N\}), V\rangle$, with $N > (m_L+1)\cdot m_B+m_L+2$. For each $\psi \in Cl(\varphi)$ such that $\langle L\rangle\psi \in \mathcal{R}(x)$, for some $1 < x < N$, we choose an interval $[x_{max}^{\psi}, y_{max}^{\psi}]$ such that it satisfies $\psi$ and for each $z > x_{max}^{\psi}$ no interval starting at $z$

satisfies $\psi$. We collect all such points into a set (of *L-blocked* points) $Bl_L \subset \{0, \ldots, N\}$. Next, for each $\psi \in Cl(\varphi)$ such that $\langle \overline{L} \rangle \psi \in \mathcal{R}(x)$, for some $1 < x < N$, we choose an interval $[x_{min}^\psi, y_{min}^\psi]$ such that it satisfies $\psi$ and for each $z < y_{min}^\psi$ no interval ending at $z$ satisfies $\psi$. We collect all points $x_{min}^\psi, y_{min}^\psi$ into a set (of $\overline{L}$-*blocked* points) $Bl_{\bar{L}} \subset \{0, \ldots, N\}$. Let $Bl = Bl_L \cup Bl_{\bar{L}}$. It holds that $|Bl| \leq m_L$.

Now, let $Bl = \{x_1 < x_2 < \ldots < x_n\}$. For each $1 \leq i < n$, let $Bl_i = \{x | x_i < x < x_{i+1}\}$; moreover, let $Bl_0 = \{x | 0 < x < x_1\}$ and $Bl_n = \{x | x_n < x < N\}$. We prove that if $y, y' \in Bl_i$, for some $0 \leq i \leq n$, then $\mathcal{R}(y) = \mathcal{R}(y')$. Suppose, by contradiction, that this is not the case, that is, assume $\mathcal{R}(y) \neq \mathcal{R}(y')$. If $\langle L \rangle \psi \in \mathcal{R}(y)$ and $\langle L \rangle \psi \notin \mathcal{R}(y')$, then, by definition, $[L]\neg\psi \in \mathcal{R}(y')$. This implies that $y < y'$, as $\langle L \rangle$ is transitive. Now, consider the above-defined interval $[x_{max}^\psi, y_{max}^\psi]$. Two cases may arise: either $x_{max}^\psi < y$ or $x_{max}^\psi > y'$. In the former case, since $\langle L \rangle \psi \in \mathcal{R}(y)$, there must be an interval $[x'', y'']$, with $x'' > y$, that satisfies $\psi$, thus contradicting the definition of $x_{max}^\psi$. In the latter case, $[L]\neg\psi \notin \mathcal{R}(y')$, against the hypothesis. The case in which $\langle \overline{L} \rangle \psi \in \mathcal{R}(y)$ and $\langle \overline{L} \rangle \psi \notin \mathcal{R}(y')$ can be proved in a similar way.

Since $N > (m_L + 1) \cdot m_B + m_L + 2$, by a simple combinatorial argument, we can conclude that there must be a set $Bl_i$ such that $|Bl_i| > m_B$. Let $\bar{x}$ be the least point in $Bl_i$. We prove that the model $M' = \langle \mathbb{I}(\{0, \ldots, N-1\}), V' \rangle$, obtained from $M$ by deleting $\bar{x}$ and by replacing $V$ by a suitable adaptation of it $V'$, is such that $M', [0, 1] \Vdash \varphi$. To this end, consider $M'' = \langle \mathbb{I}(\{0, \ldots, N-1\}), V'' \rangle$, where $V''$ is the projection of $V$ over the intervals that neither start nor end at $\bar{x}$. The replacement of $M$ by $M''$ does not affect satisfaction of box-formulas in $Cl(\varphi)$. The only possible problem is the existence of diamond-formulas which were satisfied in $M$ and are not satisfied anymore in $M''$.

Let $[x, y]$, with $y < \bar{x}$, be such that $M, [x, y] \Vdash \langle L \rangle \psi$. Since $M$ is a model of $\varphi$, then there exists an interval $[x', y']$, with $x' > y$, in $M$ that satisfies $\psi$. Now, by definition of $Bl$, there exists an interval $[x_{max}^\psi, y_{max}^\psi]$ such that $x_{max}^\psi, y_{max}^\psi \in Bl$, $[x_{max}^\psi, y_{max}^\psi]$ satisfies $\psi$, and $x_{max}^\psi \geq x'$. Therefore, $M'', [x, y] \Vdash \langle L \rangle \psi$. A symmetric argument can be applied to the case of $\langle \overline{L} \rangle \psi$. Thus, the removal of point $\bar{x}$ does not generate any problem with $\langle L \rangle$- or $\langle \overline{L} \rangle$-formulas.

Now, let $[y, x]$, with $x < \bar{x}$ (resp., $y < \bar{x} < x$), be such that $M, [y, x] \Vdash \langle \overline{B} \rangle \psi$ (resp., $M, [y, x] \Vdash \langle B \rangle \psi$), for some formula $\langle \overline{B} \rangle \psi \in Cl(\varphi)$ (resp., $\langle B \rangle \psi \in Cl(\varphi)$), and $[y, \bar{x}]$ is the only interval in $M$, starting at $y$, that satisfies $\psi$. Since $\bar{x}$ is the least point in $Bl_i$, $M, [y, x_i] \Vdash \langle \overline{B} \rangle \psi$ (resp., $M, [y, x_{i+1}] \Vdash \langle B \rangle \psi$) as well, by transitivity of $\langle \overline{B} \rangle$ (resp., $\langle B \rangle$).

Consider now the first $m_B$ successors of $\bar{x}$: $\bar{x} + 1, \ldots, \bar{x} + m_B$. Since $|Bl_i| > m_B$, all these points belong to $Bl_i$. We prove that there exists at least one point $\bar{x} + k$ among them that satisfies the following properties: *(a)* for every $\langle B \rangle \xi \in Cl(\varphi)$, if $M, [y, \bar{x}+k+1] \Vdash \langle B \rangle \xi$, then $M, [y, \bar{x}+k] \Vdash \langle B \rangle \xi$, and *(b)* for every $\langle \overline{B} \rangle \zeta \in Cl(\varphi)$, if $M, [y, \bar{x}+k-1] \Vdash \langle \overline{B} \rangle \zeta$, then $M, [y, \bar{x}+k] \Vdash \langle \overline{B} \rangle \zeta$. To this end, it suffices to observe that, by transitivity of $\langle B \rangle$, if $M, [y, \bar{x}+k+1] \Vdash \langle B \rangle \xi$, then $M, [y, x'] \Vdash \langle B \rangle \xi$ for every $x' \geq \bar{x} + k + 1$. Hence, if $\bar{x} + k$ does not satisfy property *(a)* for $\langle B \rangle \xi$, then all its successors are forced to satisfy it for $\langle B \rangle \xi$. Symmetrically, by transitivity of $\langle \overline{B} \rangle$, if $M, [y, \bar{x} + k - 1] \Vdash \langle \overline{B} \rangle \zeta$, but $M, [y, \bar{x} + k] \nVdash \langle \overline{B} \rangle \zeta$, then $M, [y, x'] \nVdash \langle \overline{B} \rangle \zeta$ for every $x' \geq \bar{x} + k$. Hence, all successors of $\bar{x} + k$ trivially satisfy property *(b)* for $\langle \overline{B} \rangle \zeta$. Since the number of $\langle B \rangle$- and $\langle \overline{B} \rangle$-formulas is limited by $m_B$, a point with the required properties can always be found.

We fix the defect by defining the labeling $V'$ as follows: for every proposition letter $p$ and $1 \leq t \leq k$, we put $[y, \bar{x} + t] \in V'(p)$ if and only if $[y, \bar{x} + t - 1] \in V(p)$; the labeling of the other intervals remain unchanged. From the definition of the set $Bl$, it easily follows that such a change in the labeling does not introduce new defects of any kind.

By iterating such a procedure, we obtain the required model $M'$. ∎

Since $m_L$ and $m_B$ are both polynomial in $|\varphi|$, we can state the following theorem.

**Theorem 6** *The finite satisfiability problem for* B$\overline{B}$L$\overline{L}$ *and all its sub-fragments is NP-complete.*

## 5 NEXPTIME-completeness

As we pointed out in Section 1, the subset of NEXPTIME-complete fragments has been already studied in its full detail. NEXPTIME-membership of A$\overline{A}$ has been shown in [5]. NEXPTIME-hardness of A, given in [8], holds also for finite satisfiability, and it can be easily adapted to the case of $\overline{A}$. NEXPTIME-hardness of any fragment containing $\langle A \rangle$ or $\langle \overline{A} \rangle$ immediately follows.

**Theorem 7** *The finite satisfiability problem for* A$\overline{A}$, A$\overline{L}$, $\overline{A}$L, A, *and* $\overline{A}$ *is NEXPTIME-complete.*

## 6 EXPSPACE-completeness

In this section, we study the computational complexity of AB$\overline{B}$L and of its subfragments. EXPSPACE-membership for AB$\overline{B}$L has been shown in [9]. EXPSPACE-hardness holds for AB, as proved in [18]. In the following, we show that the reduction used in [18] works also in the finite case, and it can be adapted to A$\overline{B}$. EXPSPACE-hardness follows from a reduction of the $2^n$-*corridor tiling problem*, which is known to be EXPSPACE-complete [15, Section 5.5]. Formally, an instance of the exponential-corridor tiling problem is a tuple $\mathcal{T} = (T, t_0, t_1, T_L, T_R, C_H, C_V, n)$ consisting of a finite set $T$ of tiles, two tiles $t_0, t_1 \in T$, a set of left tiles $T_L \subseteq T$, a set of right tiles $T_R \subseteq T$, two binary relations $C_H$ and $C_V$ over $T$, and a positive natural number $n$. The problem amounts to deciding whether there exists a positive natural number $l$ and a tiling $f : \{0, \ldots, 2^n - 1\} \times \{0, \ldots, l - 1\} \to T$ of the corridor of width $2^n$ and height $l$, that associates the tile $t_0$ to $(0, 0)$, the tile $t_1$ to $(0, l - 1)$, a tile in $T_L$ (resp., $T_R$) with the first (resp., last) tile of every row of the corridor and that respects the following horizontal and vertical constraints $C_H$ and $C_V$: *(i)* for every $x < 2^n - 1$ and every $y < l$, we have $f(x, y)\ C_H\ f(x + 1, y)$; and *(ii)* for every $x < 2^n$ and every $y < l - 1$, we have $f(x, y)\ C_V\ f(x, y + 1)$.

**Lemma 8** *There exists a polynomial-time reduction from the $2^n$-corridor tiling problem to the satisfiability problem for* A$\overline{B}$ *over finite linear orders.*

**Proof.** Consider an instance $\mathcal{T} = (T, t_0, t_1, T_L, T_R, C_H, C_V, n)$ of the $2^n$-corridor tiling problem, where $T = \{t_0, t_1, \ldots, t_k\}$. We guarantee the existence of a tiling function $f : \{0, \ldots, 2^n - 1\} \times \{0, \ldots, l - 1\} \to T$ that satisfies $\mathcal{T}$ by means of a suitable A$\overline{B}$-formula whose size is polynomial in $|\mathcal{T}|$. We use $k + 1$ proposition letters $t_0, t_1, \ldots, t_k$ to represent the tiles from $T$, $n$ proposition letters $x_0, \ldots, x_{n-1}$ to represent the binary expansion of the $x$-coordinate of a point in the corridor, and one propositional letter $c$ to identify those intervals that correspond to points $p = (x, y)$ of the corridor of width $2^n$ and height $l$. Such a correspondence is obtained by ensuring that we interpret those proposition letters over intervals

of the type $[x + 2^n y, x + 2^n y + 1]$. The valuation function $V$ of the model of the formula is then related to the tiling function $f$ as follows: for each point $p = (x, y) \in \{0, \ldots, 2^n - 1\} \times \{0, \ldots, l-1\}$ and each tile $t_i \in T$, if $f(p) = t_i$, then $[x + 2^n y, x + 2^n y + 1] \in V(\{c, t_i, x_{j_1}, \ldots, x_{j_h}\})$, where $\{j_1, \ldots, j_h\} \subseteq \{0, \ldots, n-1\}$ and $x = \sum_{j \in \{j_1, \ldots, j_h\}} 2^j$. Let the *universal* modal operator $[U]$ be defined as $[U]\varphi = \varphi \wedge [A]\varphi \wedge [A][A]\varphi$. First, we associate the proposition letter $c$ with all and only the intervals of the form $[x + 2^n y, x + 2^n y + 1]$:

$$\varphi_c = c \wedge [U]((c \wedge \langle A \rangle \top) \to \langle A \rangle c) \wedge [U]\neg \langle \overline{B} \rangle c.$$

The tiling function $f$ is represented by associating a unique proposition letter $t_i$ with each $c$-labeled interval:

$$\varphi_f = [U]\Big(c \to \bigvee_{0 \leq i \leq k} t_i\Big) \wedge [U]\Big(c \to \bigwedge_{0 \leq i < j \leq k} \neg(t_i \wedge t_j)\Big).$$

Next, we associate a subset of the proposition letters $x_0, \ldots, x_{n-1}$, that encodes the binary expansion of $x$, with each interval of the form $[x + 2^n y, x + 2^n y + m]$. Such a labeling can be enforced by the conjunction $\varphi_x$ of the following three formulas:

$$\varphi_x^1 = \Big(\bigwedge_{0 \leq i < n} \neg x_i\Big), \quad \varphi_x^2 = [U]\Big(c \to \varphi_{inc}^0\Big),$$

$$\varphi_x^3 = [U]\Big(\bigwedge_{0 \leq i < n} (x_i \leftrightarrow [\overline{B}]x_i) \wedge (\neg x_i \leftrightarrow [\overline{B}]\neg x_i)\Big).$$

where $\varphi_{inc}^i$ is defined as $\top$ when $i = n$, and as

$$\Big(x_i \wedge \langle A \rangle(c \wedge \neg x_i) \wedge \varphi_{inc}^{i+1}\Big) \vee \Big(\neg x_i \wedge \langle A \rangle(c \wedge x_i) \wedge \varphi_{eq}^{i+1}\Big),$$

otherwise. Similarly, $\varphi_{eq}^i$ is defined as $\top$ when $i = n$, and as

$$\Big((x_i \wedge \langle A \rangle(c \wedge x_i)) \vee (\neg x_i \wedge \langle A \rangle(c \wedge \neg x_i))\Big) \wedge \varphi_{eq}^{i+1},$$

otherwise. Finally, we establish a correspondence between intervals that represent vertically adjacent tiles by setting the proposition letter $co$:

$$\varphi_{cs} = [U](co \to \varphi_{eq}^0) \wedge [U]((c \wedge \langle \overline{B} \rangle \varphi_{eq}^0) \to \langle \overline{B} \rangle co) \wedge$$
$$[U]\neg(\varphi_{eq}^0 \wedge \langle \overline{B} \rangle co).$$

To conclude the proof, we must enforce the horizontal and vertical constraints $C_H$ and $C_V$ and the constraints on the border of the corridor. This can be done by means of the following formulas (remember that, by definition of tiling, $t_0, t_1 \in T$ and $T_L, T_R \subseteq T$):

$$\varphi_{01} = t_0 \wedge \langle A \rangle \langle A \rangle \Big(c \wedge \bigwedge_{0 \leq i < n} \neg x_i \wedge t_1 \wedge \neg \langle \overline{B} \rangle co\Big)$$

$$\varphi_L = [U]\Big(c \wedge \bigwedge_{0 \leq i < n} \neg x_i \to \bigvee_{t_L \in T_L} t_L\Big),$$

$$\varphi_R = [U]\Big(c \wedge \bigwedge_{0 \leq i < n} x_i \to \bigvee_{t_R \in T_R} t_R\Big),$$

$$\varphi_H = [U]\bigwedge_{0 \leq i \leq k} \Big((t_i \wedge \langle A \rangle \top) \to \bigvee_{(t_i, t_j) \in C_H} \langle A \rangle t_j\Big),$$

$$\varphi_V = [U]\bigwedge_{0 \leq i \leq k} \Big(t_i \to [\overline{B}](co \to \bigvee_{(t_i, t_j) \in C_V} \langle A \rangle t_j)\Big).$$

The formula $\varphi_{\mathcal{T}} = \varphi_c \wedge \varphi_f \wedge \varphi_x \wedge \varphi_{cs} \wedge \varphi_{01} \wedge \varphi_L \wedge \varphi_R \wedge \varphi_H \wedge \varphi_V$ is of polynomial size w.r.t. $|\mathcal{T}|$ and is satisfiable if and only if $\mathcal{T}$ is a positive instance of the $2^n$-corridor tiling problem. ∎

**Theorem 9** *The finite satisfiability problem for* $AB\overline{B}\overline{L}$, $AB\overline{B}$, $AB$, $A\overline{B}$, $AB\overline{L}$, *and* $A\overline{B}\overline{L}$ *is EXPSPACE-complete.*

## 7 Non-Primitive Recursiveness

In this last section, we focus our attention on the remaining fragments. It will turn out that, although decidable, they are of non-primitive recursive complexity. From [17, 19], we know that there is a reduction from the finite satisfiability problem for $A\overline{A}B$ and $A\overline{A}\overline{B}$ to the so-called *reachability* problem for a *lossy (Minsky) counter machine*, which is known to be non-primitive recursive [20]. Here, we prove that such a reduction can be adapted to the cases of $\overline{A}B$ and $\overline{A}\overline{B}$, completing the picture. Due to space constraints, we limit ourselves to sketch the proof for the case of $\overline{A}B$, referring the interested reader to [19] for details.

A *lossy counter machine* is a triple of the form $\mathcal{A} = (Q, k, \Delta)$, where $Q$ is a finite set of control states, $k$ is the number of counters (whose values range over $\mathbb{N}$), and $\Delta$ is a function that maps each state $q \in Q$ to a transition rule having one of the following forms: *(i)* if $\mathcal{A}$ is at state $q$, then increase the counter $i$ and move to state $q'$; *(ii)* if $\mathcal{A}$ is at state $q$, then check the value of counter $i$: if it is equal to 0, then move to state $q'$, otherwise, decrement the counter $i$ and move to state $q'''$. In addition, from each configuration $(q, \bar{z}) \in Q \times \mathbb{N}^k$, a lossy counter machine can non-deterministically activate an internal (lossy) transition and move to a configuration $(q, \bar{z}')$, with $\bar{z}' \leq \bar{z}$ (componentwise). The *reachability* problem for a lossy counter machine $\mathcal{A}$ consists of deciding whether or not there is a computation that takes $\mathcal{A}$ from a given configuration $(q_{source}, \bar{z}_{source})$ to a given configuration $(q_{target}, \bar{z}_{target})$. As shown in [19], we can always assume $\bar{z}_{source} = \bar{z}_{target} = \bar{0}$, and that $q_{target}$ is a sink state, namely, the only state accessible from it is $q_{target}$ itself. To encode a generic computation $(q_1, \bar{z}_1) \ldots (q_n, \bar{z}_n)$ of $\mathcal{A}$, we first introduce $|Q| + k$ proposition letters that label intervals of the form $[x, x + 1]$); the first $|Q|$ proposition letters will identify the control states of $\mathcal{A}$, while the last $k$ proposition letters, denoted here by $c_1, .., c_k$, will identify the $k$ counters. We then divide the underlying domain $\mathbb{D} = \{0, \ldots, N\}$ into exactly $n + 2$ intervals $[0, x_1], [x_1, x_2], \ldots, [x_n, x_{n+1}], [x_{n+1}, N]$. Such intervals, except for the first and the last one, will be used to encode the configurations $(q_1, \bar{z}_1) \ldots (q_n, \bar{z}_n)$, while the other two intervals will be used to correctly move between the various configurations via the modal operators $\langle \overline{A} \rangle$ and $\langle B \rangle$. Finally, the unit intervals which are subinterval of a generic $[x_t, x_{t+1}]$ will be labeled by proposition letters in $Q \cup \{c_1, \ldots, c_k\}$ as follows: the subinterval $[x_t, x_t + 1]$ is labeled by the control state $q_t$, and, for every $1 \leq i \leq k$, the number of $c_i$-labeled intervals of the form $[x, x + 1]$, with $x_t < x < x_{t+1}$, coincides with the value $\bar{z}_t(i)$ of the counter $i$. Notice that there may exist different encodings of the same computation of $\mathcal{A}$.

**Lemma 10** *There exists a reduction from the satisfiability problem for* $\overline{A}B$ *to the reachability problem for lossy counter machines.*

**Proof.** According to the above-sketched schema, we now provide the formulas which are needed to encode the reachability problem for a given lossy machine $\mathcal{A} = (Q, k, \Delta)$ whose initial and final configurations have all counters set to 0. First, we introduce the following shorthands:

$$\psi_{\exists q} = \bigvee_{q \in Q} q \wedge \bigwedge_{q \neq q'} \neg(q \wedge q'), \; \psi_{\exists \overline{q}} = \bigvee_{q \in Q} \overline{q} \wedge \bigwedge_{\overline{q} \neq \overline{q}'} \neg(\overline{q} \wedge \overline{q}'),$$

$$\psi_{\exists c} = \bigvee_{c \in C} c \wedge \bigwedge_{c \neq c'} \neg(c \wedge q').$$

We denote by the proposition letter *new* (resp., *del*) a counter which has been incremented (resp., decremented) by one, while the proposition letter *conf* uniquely identifies the interval corresponding to a

configuration. Moreover, for each proposition letter $p$ that appears in a given configuration, the proposition letter $\bar{p}$ is used to transfer this piece of information to the intervals that start with the beginning point of the model (this is done for technical reasons). Finally, proposition letter $t$ links the value of a given counter in a given configuration with the value of the same counter in the next configuration. The following formulas set up this schema and guarantee that we actually start from the initial configuration:

$$
\begin{aligned}
\psi_{prop} = & \left( \bigwedge_{q \in Q} q \rightarrow [\overline{A}]\overline{q} \right) \wedge \left( \bigwedge_{c \in C} c \rightarrow [\overline{A}]\overline{c} \right) \wedge \\
& (new \rightarrow [\overline{A}]\overline{new}) \wedge (del \rightarrow [\overline{A}]\overline{del}) \wedge \\
& (conf \rightarrow [\overline{A}]\overline{conf}), \\
\psi_{transfer} = & \bigvee_{c \in C} c \wedge \neg new \rightarrow \langle \overline{A} \rangle t, \\
\psi_0 = & \langle \overline{A} \rangle \langle A \rangle ([\overline{A}]\bot \wedge q_0 \wedge \psi_{\exists \overline{q}}), \ \psi_f = q_f.
\end{aligned}
$$

The following formulas behave as follows: $\psi_{conf}$ guarantees the existence of a sequence of configurations; $\psi_{new}$ ensures that at most one counter is incremented at each step; $\psi_t$ sets the correspondences between counters of successive configurations; finally, $\psi_{\Delta}$ implements the transition function:

$$
\begin{aligned}
\psi_{conf} = & conf \leftrightarrow (\psi_{\exists q} \vee \langle B \rangle \psi_{\exists q}) \wedge \psi_{\exists \overline{q}} \wedge [B] \neg \psi_{\exists \overline{q}}, \\
\psi_{new} = & conf \rightarrow \langle B \rangle (\overline{new} \rightarrow [B] \neg \overline{new}), \\
\psi_t = & t \rightarrow \neg \overline{new} \wedge [B](\neg t \wedge \neg del) \wedge \left( \bigvee_{c \in C} (\overline{c} \wedge \langle B \rangle c) \right) \wedge \\
& new \wedge \langle B \rangle \overline{conf} \wedge [B](\overline{conf} \rightarrow [B] \neg \overline{conf}), \\
\psi_{\Delta} = & \bigwedge_{(q_i, c+1, q_j) \in \Delta} (conf \wedge \overline{q}_i \rightarrow \langle B \rangle q_i) \wedge \\
& (conf \wedge \langle B \rangle q_j \rightarrow \langle B \rangle (\overline{c} \wedge \overline{new})) \wedge \\
& \bigwedge_{(q_i, c?0, q_j, q_h) \in \Delta} (conf \wedge \langle B \rangle q_i \wedge \overline{q}_j \rightarrow [B](\neg \overline{c} \wedge \neg \overline{new})) \wedge \\
& (conf \wedge \langle B \rangle q_i \wedge \overline{q}_h \rightarrow \langle B \rangle (\overline{c} \wedge \overline{del}) \wedge [B] \neg \overline{new}).
\end{aligned}
$$

It is not difficult to check that, for a given lossy machine $\mathcal{A}$ that starts with the configuration $(q_0, \bar{0})$, $\mathcal{A}$ reaches $(q_f, \bar{0})$ if and only if the following formula is satisfiable:

$$
\psi_0 \wedge \psi_f \wedge [\overline{U}](\psi_{prop} \wedge \psi_{transfer} \wedge \psi_{conf} \wedge \psi_{\Delta} \wedge \psi_t),
$$

where $[\overline{U}]$ is the transposed universal operator, which is defined as shown in the previous section with $\langle A \rangle$ replaced by $\langle \overline{A} \rangle$. ∎

**Theorem 11** *The finite satisfiability problem for* $A\overline{A}B\overline{B}$ *and all its fragments that contain* $\langle \overline{A} \rangle$ *and at least one between* $\langle B \rangle$ *and* $\langle \overline{B} \rangle$ *is non-primitive recursive.*

## 8 Conclusions

In this paper, we focused our attention on the satisfiability problem for interval temporal logics over finite linear orders, which is of interest for various application domains. We provided a complete classification of HS (decidable) fragments with respect to their expressive power and complexity. Such a classification cannot be automatically transferred to any other class of linear orders. As an example, the fragment D, which is undecidable over finite linear orders, turns out to be PSPACE-complete over dense ones [4].

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