Metric Propositional Neighborhood Logics

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Please notice: these slides have been mostly produced by Dario Della Monica (University of Udine), who, in turn, borrowed many ideas from Davide Bresolin's slides (University of Verona).

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Outline

Outline

Time and logics

Studying time and its structure is of great importance in **computer science**:

- **Artificial Intelligence.**
  Planning, Natural Language Recognition, . . .
- **Databases.**
  Temporal Databases.
- **Formal methods.**
  Specification and Verification of Systems and Protocols, Model Checking, . . .

Points vs. intervals

Usually, time is formalized as a (usually linearly ordered) set of **points**.

In **point-based** temporal logics, formulas are interpreted directly over points. In **interval-based** ones, they are interpreted over **intervals**. In this case, intervals can also be given of a **duration**.

It is well-known that interval-based logics are much more difficult to deal with.

Definition

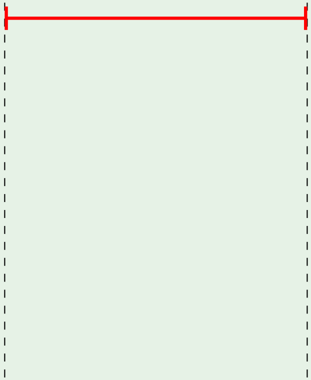
Given a linear order $\mathbb{D} = \langle D, < \rangle$:

- an interval in $\mathbb{D}$ is a pair $[d_0, d_1]$ such that $d_0 < d_1$ (or $d_0 \leq d_1$);
- $\mathbb{I}(\mathbb{D})$ is the set of all intervals on $\mathbb{D}$;
- $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ is an interval structure.

- We consider intervals as pairs of time points.
- A point $d \in D$ belongs to $[d_0, d_1]$ if $d_0 \leq d \leq d_1$.

Allen's binary relations

There are 13 different binary relations between intervals:



together with their inverses.

Between points we have only three binary relations!

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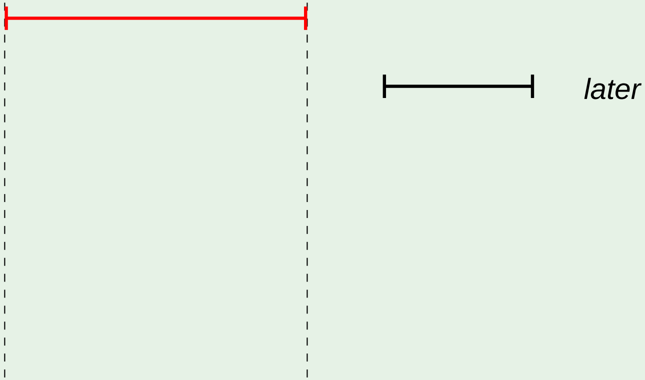

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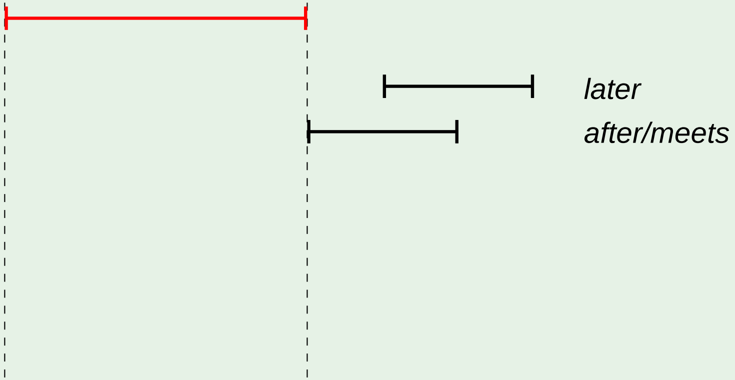

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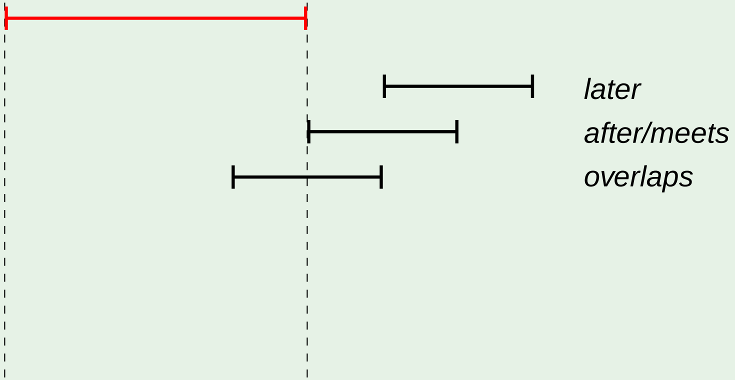

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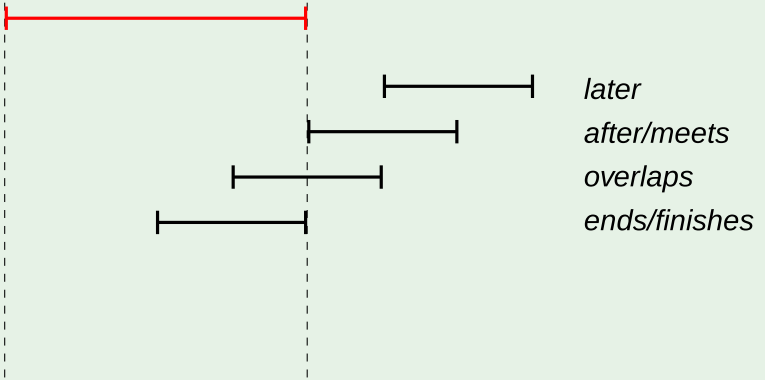

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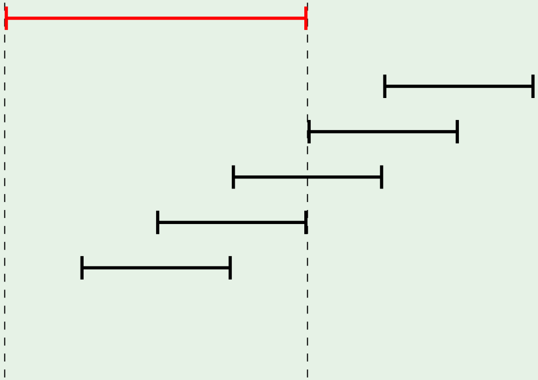

later

after/meets

overlaps

ends/finishes

during

together with their inverses.

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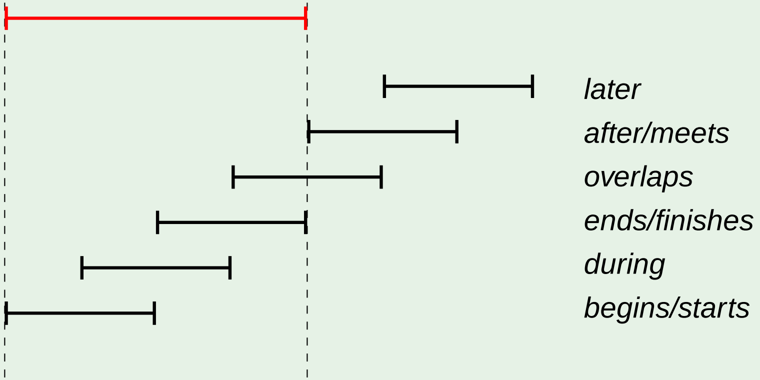

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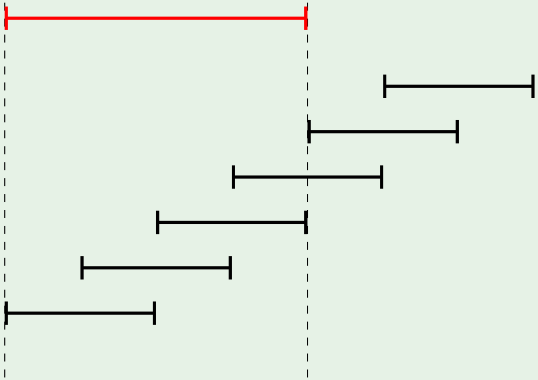

later	$\langle L \rangle$
after/meets	$\langle A \rangle$
overlaps	$\langle O \rangle$
ends/finishes	$\langle E \rangle$
during	$\langle D \rangle$
begins/starts	$\langle B \rangle$

together with their inverses.

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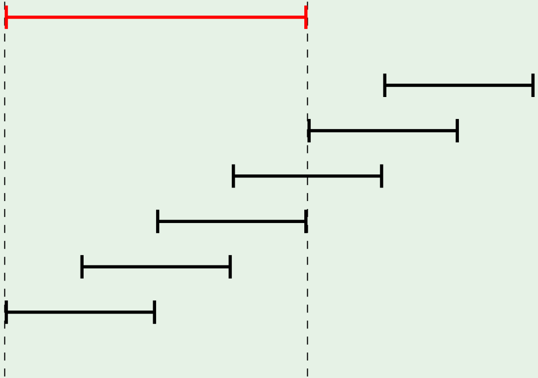

later	$\langle L\rangle$ $\langle\overline{L}\rangle$
after/meets	$\langle A\rangle$ $\langle\overline{A}\rangle$
overlaps	$\langle O\rangle$ $\langle\overline{O}\rangle$
ends/finishes	$\langle E\rangle$ $\langle\overline{E}\rangle$
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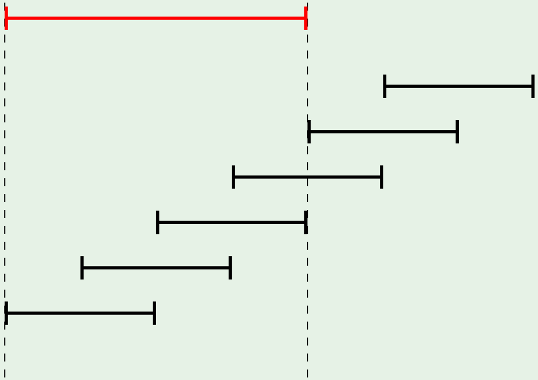

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together with their inverses.

Between points we have only three binary relations!

- Interval temporal logics, such as HS [Halpern and Shoham, 1991] and CDT [Venema, 1991], are very expressive (compared to point-based temporal logics)
- Most interval temporal logics are (highly) undecidable

Problem

Find expressive, yet decidable, interval temporal logics.

A simple path to decidability

Interval logics make it possible to express properties of pairs of time points rather than of single time points.

How has decidability been achieved? By imposing suitable syntactic and/or semantic restrictions that allow one to reduce interval logics to point-based ones:

- **Constraining interval modalities**
  - $\langle B\rangle\langle\overline{B}\rangle$ and $\langle E\rangle\langle\overline{E}\rangle$ fragments of HS.
- **Constraining temporal structures**
  - Split Logics: any interval can be chopped in at most one way (Split Structures).
- **Constraining semantic interpretations**
  - Local QPITL: a propositional variable is true over an interval if and only if it is true over its starting point (Locality Principle).

An alternative path to decidability

A major challenge

Identify expressive enough, yet decidable, logics which are genuinely interval-based.

What is a genuinely interval-based logic?

A logic is genuinely interval-based if it cannot be directly translated into a point-based logic and does not invoke locality, or any other semantic restriction reducing the interval-based semantics to the point-based one.

Known decidability results

The picture of decidable/undecidable non-metric interval logics is almost complete

- Propositional Neighborhood Logic ($A\overline{A}$) is the first discovered decidable genuine interval logic (and maximal in most cases, including $\mathbb{N}$)
- the logic $AB\overline{BA}$ is maximal decidable over finite
- $D\overline{D}B\overline{B}L\overline{L}$ is maximal decidable over dense
- the vast majority of all other fragments is undecidable
- no previous known results for metric extension of any interval logic

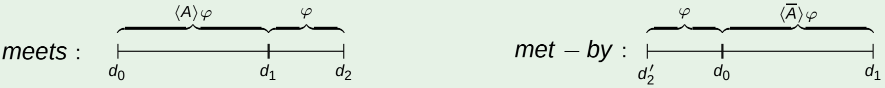
We will present a family of metric extensions of PNL over natural numbers:

- Decidability proof of the most expressive fragment (MPNL$_l$)
- Expressive completeness and undecidable extension ($\equiv FO^2_{[\mathbb{N},=,<,s]}$)
- Classification of all metric fragments w.r.t. expressive power

Outline

Semantics

PNL is based on the neighborhood operators *meets* and *met-by*:



Metric formulas can constrain the length of the current interval or the length of reachable intervals

The addition of a metric aspect

Metric extensions of PNL over the integers

1. Extensions of the modal operators $\langle A \rangle$ ($\equiv \Diamond_r$) and $\langle \overline{A} \rangle$ ($\equiv \Diamond_l$):
   $\Diamond_r^{=k}, \Diamond_r^{>k}, \Diamond_l^{[k,k']}, \Diamond_l^{(k,k')}, \ldots$
   - $\mathcal{S}$: set of all possible metric extensions of PNL modalities
2. Introduction of atomic length constraints: $\mathrm{len}_{>k}, \mathrm{len}_{\geq k}, \mathrm{len}_{=k}, \ldots$
   - $\mathcal{L}$: set of all atomic length constraints

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$$MPNL = \{MPNL_L^S \mid S \neq \emptyset, S \subseteq \mathcal{S}, L \subseteq \mathcal{L}\}$$

set of all metric extenstions of PNL

Propositional Neighborhood Logic with atomic length constraints

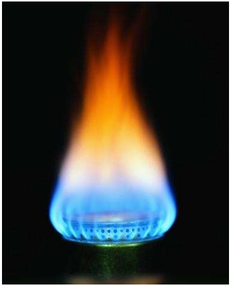
Syntax of MPNL$_l$

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle A \rangle \varphi \mid \langle \overline{A} \rangle \varphi \mid \mathrm{len}_{=k}$$

Proposition

MPNL$_l$ is the most powerful logic in MPNL

The leaking gas burner



- Every time the flame is ignited, a small amount of gas can leak from the burner.
- The propositional letter *Gas* is used to indicate the gas is flowing.
- The propositional letter *Flame* is true when the gas is burning.

Safety of the gas burner:

1. It is never the case that the gas is leaking for more than 2 seconds.
2. The gas burner will not leak for 30 seconds after the last leakage.

Safety of the gas burner in MPNL$_l$

Universal modality: φ holds everywhere in the future

$$[G]\varphi ::= \varphi \wedge [A]\varphi \wedge [A][A]\varphi$$

Leaking = gas flowing but not burning

$$[G](Leak \leftrightarrow Gas \wedge \neg Flame)$$

Safety properties:

1. $[G](Leak \rightarrow \mathrm{len}_{\leq 2})$
2. $[G](Leak \rightarrow \neg\langle A\rangle(\mathrm{len}_{<30} \wedge \langle A\rangle Leak))$

MPNL_l is simple but powerful

"Metric" Until

MPNL_l is expressive enough to encode a metric form of Until:

"p is true at a point in the future at distance k from the current interval and, until that point, q is true (pointwise)"

$$\langle A\rangle(\mathrm{len}_{=k} \wedge \langle A\rangle(\mathrm{len}_{=0} \wedge p)) \wedge [A](\mathrm{len}_{<k} \rightarrow \langle A\rangle(\mathrm{len}_{=0} \wedge q))$$

Unbounded until is not expressible in MPNL_l.

MPNL$_l$ is expressive enough to encode some metric form of all (but one) Allen's relation:

p holds over intervals of length l, with $k \leq l \leq k'$

$$[G](p \rightarrow \text{len}_{\geq \text{k}} \wedge \text{len}_{\leq \text{k}'})$$

"Any p-interval begins a q-interval"

$$[G] \bigwedge_{i=k}^{k'} (p \wedge \text{len}_{=\text{i}} \rightarrow \Diamond_l \Diamond_r (\text{len}_{>\text{i}} \wedge q))$$

"Any p-interval contains a q-interval"

$$[G] \bigwedge_{i=k}^{k'} (p \wedge \text{len}_{=\text{i}} \rightarrow \bigvee_{j \neq 0, j+j' < i} (\Diamond_l \Diamond_r (\text{len}_{=\text{j}} \wedge \Diamond_r (\text{len}_{=\text{j}'} \wedge q))))$$

Outline

Basic ingredients

Definition

An atom is a maximal, locally consistent set of subformulae of φ.

A relation connecting atoms

Connect every pair of atoms that can be associated with neighbor intervals preserving the universal quantifiers:

$$A \, \mathrm{R}_{\varphi} \, B \quad \text{iff} \quad \begin{cases} ① & [A]\psi \in A \;\Rightarrow\; \psi \in B \\ ② & [\overline{A}]\psi \in B \;\Rightarrow\; \psi \in A \end{cases}$$

Labelled Interval Structures

Definition

A Labelled Interval Structure (LIS) is a pair $\langle \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ where:

- $\mathbb{I}(\mathbb{D})$ is the set of intervals over $\mathbb{D}$;
- the labelling function $\mathcal{L}$ assigns an atom to every interval $[d_i, d_j]$;
- atoms assigned to neighbor intervals are related by R_φ.

A LIS is fulfilling if:

- metric formulae in $\mathcal{L}([d_i, d_j])$ are consistent with respect to the interval length;
- for every $[d_i, d_j]$ and $\langle A \rangle \psi$ (resp., $\langle \overline{A} \rangle \psi$) $\in \mathcal{L}([d_i, d_j])$ there exists $d_k > d_j$ (resp., $d_k < d_i$) such that $\psi \in \mathcal{L}([d_j, d_k])$ (resp., $\mathcal{L}([d_k, d_i])$).

Theorem

A formula φ is satisfiable if and only if there exists a fulfilling LIS $\langle \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ and an interval $[d_i, d_j]$ such that $\varphi \in \mathcal{L}([d_i, d_j])$.

A small-model theorem for LIS

- We have reduced the satisfiability problem for MPNL$_l$ to the problem of finding a (fulfilling) LIS for φ.
- LIS can be of arbitrary size and even infinite!

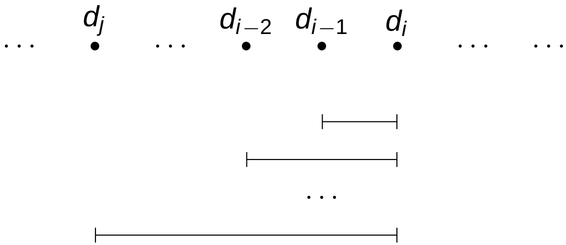
Problems

- How to bound the size of finite LIS?
- How to finitely represent infinite LIS?

Solution

Any large (resp., infinite) model can be turned into a bounded (resp., bounded periodic) one by progressively removing exceeding points

The set of requests of a point



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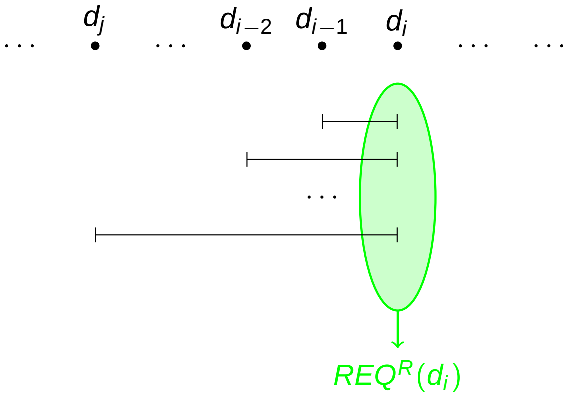

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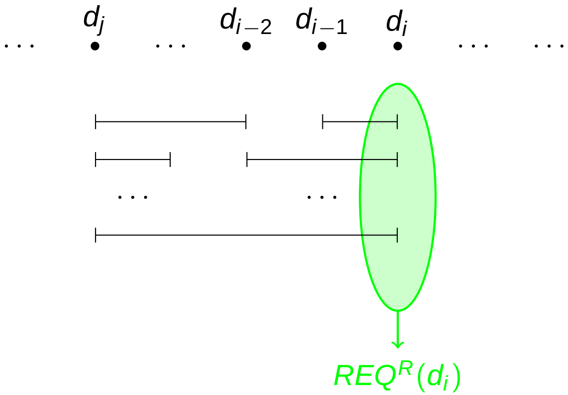

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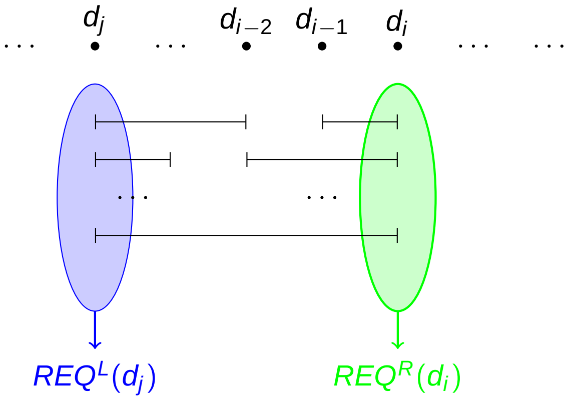

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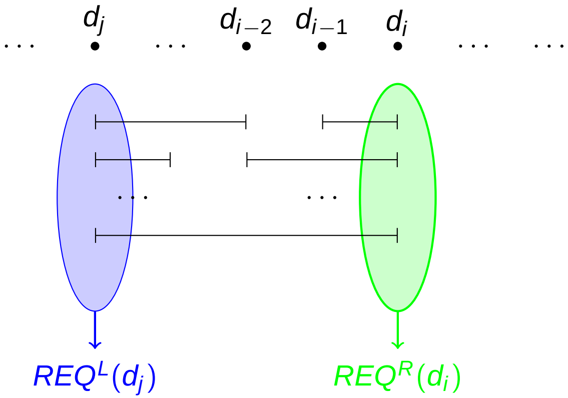


$$REQ(d_i) = REQ^R(d_i) \cup REQ^L(d_i)$$

Given a formula φ, let k be the greatest constant that appears in φ.

Definition

Given a LIS, a k-sequence is a sequence of k consecutive points. Given a sequence σ, its sequence of requests $REQ(\sigma)$ is defined as the sequence of temporal requests at the points in σ.

$$\cdots \quad \overbrace{\underbrace{\begin{array}{ccccc} d_1 & d_2 & d_3 & & d_k \\ \bullet & \bullet & \bullet & \cdots & \bullet \\ \mathrm{REQ}(d_1) & \mathrm{REQ}(d_2) & \mathrm{REQ}(d_3) & & \mathrm{REQ}(d_k) \end{array}}_{\mathrm{REQ}(\sigma)}}^{\sigma} \quad \cdots$$

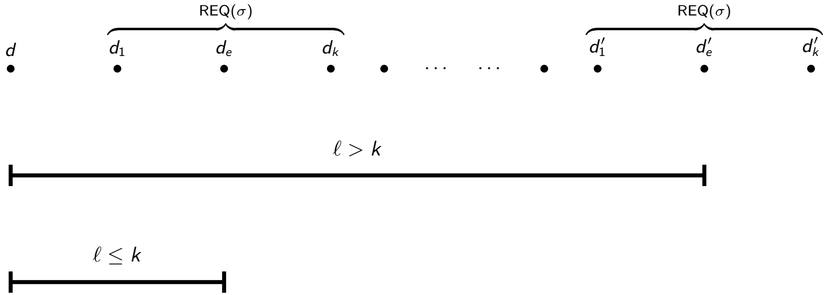
Removing k-sequences from a LIS

Lemma

- *Let m be the number of* $\langle \mathrm{A} \rangle$*-subformulae of φ and r the number of possible sets of requests* REQ*.*
- *Let $\langle \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ be a fulfilling LIS for φ and* $\mathrm{REQ}(\sigma)$ *be a k-sequence of request that occurs more than $2(m^2 + m)r + 1$ times.*

$\Rightarrow$ *We can remove one occurrence of* $\mathrm{REQ}(\sigma)$ *from the LIS in such a way that the resulting LIS is still fulfilling.*

The removal process: fixing the length of intervals



- Remove all points up to the next occurrence of REQ(σ)
- Some intervals became shorter, and do not respect metric formulas anymore
- Since REQ(d_e) = REQ(d'_e), we can relabel problematic intervals

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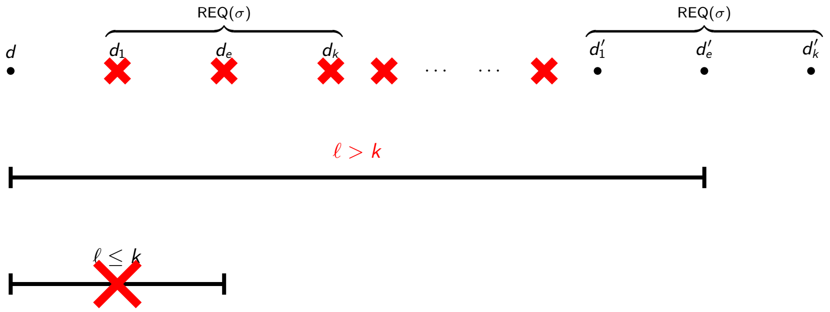

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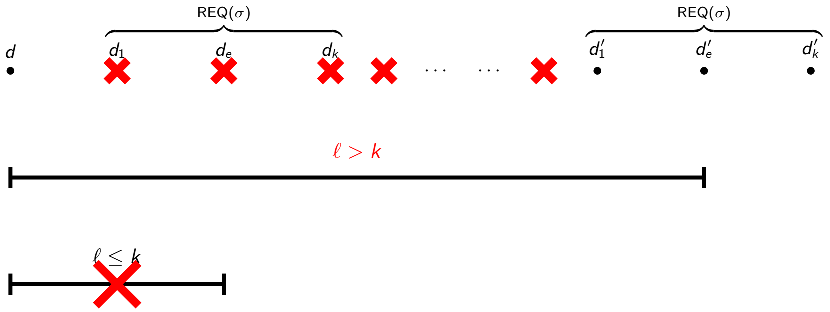

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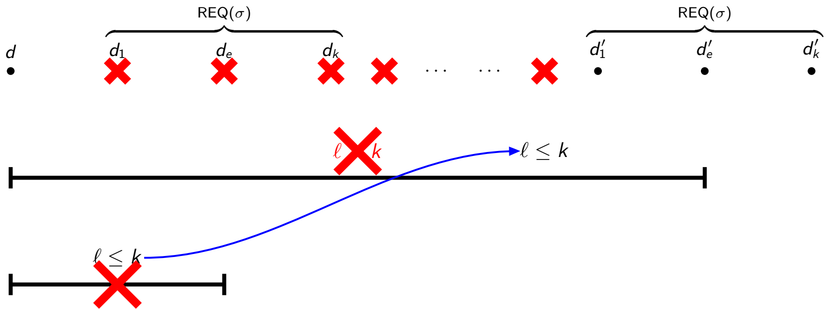

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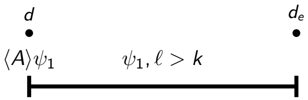
The removal process: fixing defects on $\langle A \rangle$-formulae



m points on the right of d_e
with the same set of requests of d_e

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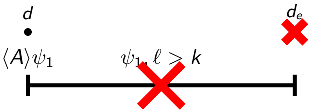

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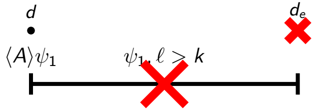

$\langle A \rangle \psi_2$

$\langle A \rangle \psi_3$

$\vdots$

$\langle A \rangle \psi_m$

m points on the right of d_e

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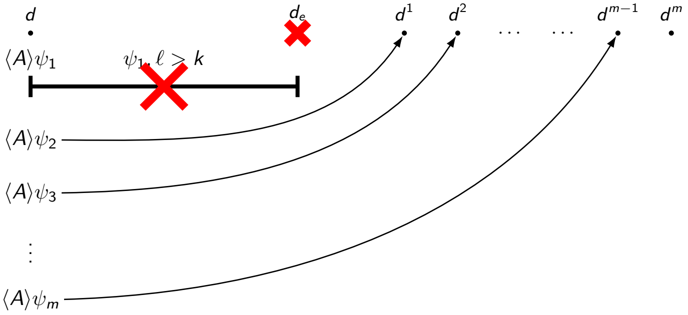
The removal process: fixing defects on $\langle A \rangle$-formulae



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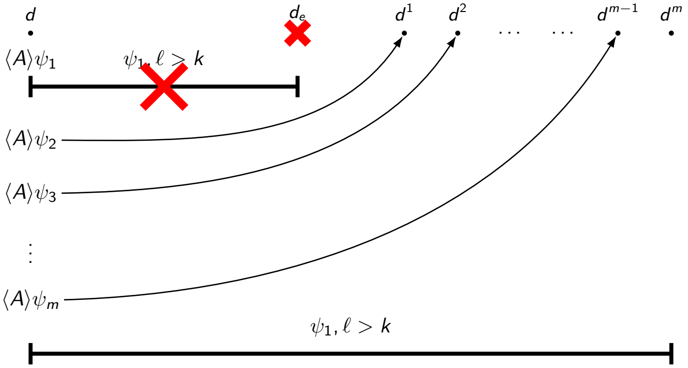

m points on the right of d_e
with the same set of requests of d_e

The small model theorem for MPNL$_l$

By taking advantage of such a removal process, we can prove the following theorem:

Theorem (Small model theorem)

A formula φ is satisfiable if and only if there exists a LIS $\langle \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ such that:

- *if $\mathbb{D}$ is finite, then every k-sequence of requests occurs at most $2(m^2 + m)r + 1$ times in $\mathbb{D}$*
- *if $\mathbb{D}$ is infinite, then the LIS is ultimately periodic with prefix and period bounded by $r^k(2(m^2 + m)r + 1)k + k - 1$*

Decidability and complexity

- "Plain" RPNL is known to be NEXPTIME-complete $\Rightarrow$ NEXPTIME-hardness
- A model for an MPNL_l formula φ can be obtained by a non-deterministic decision procedure that runs in time $O(2^{k \cdot n})$.

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The exact complexity class depends on how k is encoded:

- k is a constant: $k = O(1)$
  MPNL$_l$ is NEXPTIME-complete
- k is encoded in unary: $k = O(n)$
  MPNL$_l$ is NEXPTIME-complete
- k is encoded in binary: $k = O(2^n)$
  MPNL$_l$ is in 2NEXPTIME but ...
  ... is EXPSPACE-hard (since RPNL+INT is EXPSPACE-complete)
  The exact complexity class is an open problem!!!

Outline

PNL and Two-Variable Fragment of First Order Logic

Syntax of $FO^2[<,=]$:

$$\alpha ::= A_0 \mid A_1 \mid \neg\alpha \mid \alpha \vee \alpha \mid \exists x \alpha \mid \exists y \alpha$$
$$A_0 ::= x = x \mid x = y \mid y = x \mid y = y \mid x < y \mid y < x$$
$$A_1 ::= P(x,x) \mid P(x,y) \mid P(y,x) \mid P(y,y)$$

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$PNL^{\pi+} \equiv FO^2[<,=]$

$$\mathrm{PNL}^{\pi+} \xrightarrow{\equiv} \mathrm{FO}^2[\mathbb{N},=,<]$$

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$CDT \equiv FO^3_2[=,<]$

$$\begin{tikzcd}
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Theorem

The satisfiability problem for $FO^2[\mathbb{N}, <, =, s]$ is undecidable

Additional modalities $\Diamond_e^{+k}$, $\Diamond_b^{+k}$, $\Diamond_{be}^{+k}$

The logic MPNL$_l^+$: an extension of MPNL$_l$

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$\Diamond_e^{+k}\psi$

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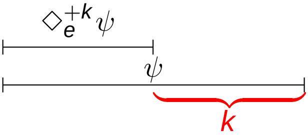

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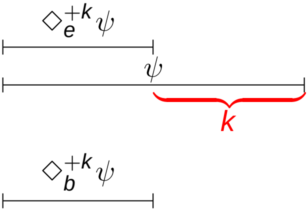

The logic MPNL_l^+: an extension of MPNL_l

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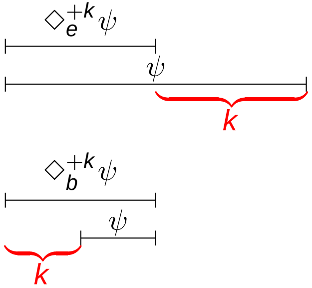

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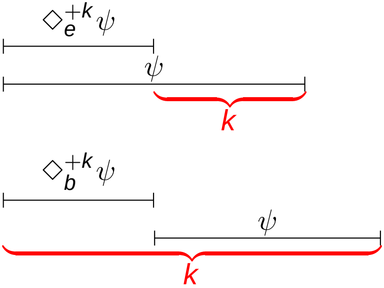

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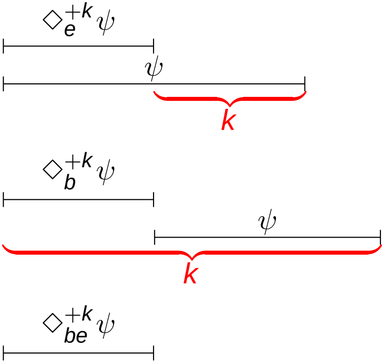

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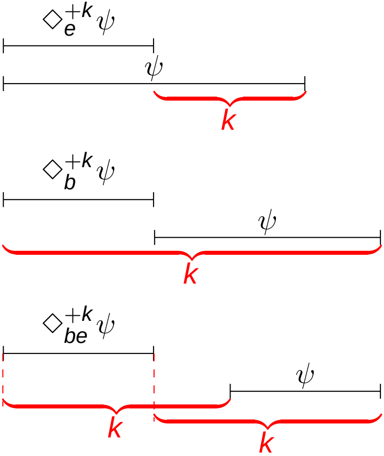

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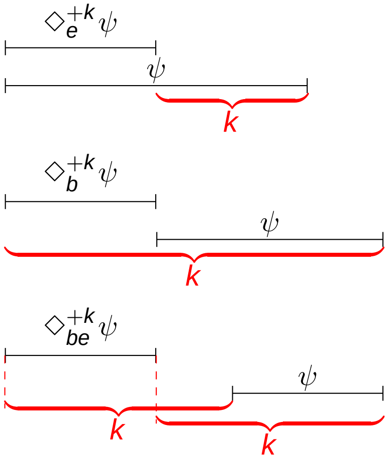

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$MPNL_l^+ \equiv FO^2[\mathbb{N}, <, =, s]$

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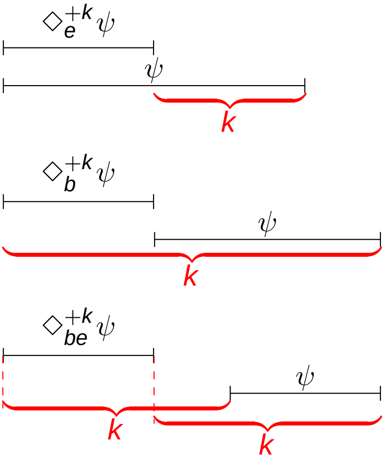

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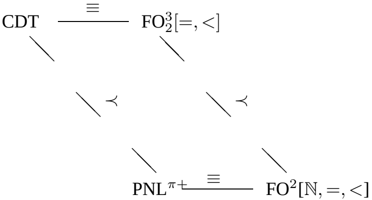

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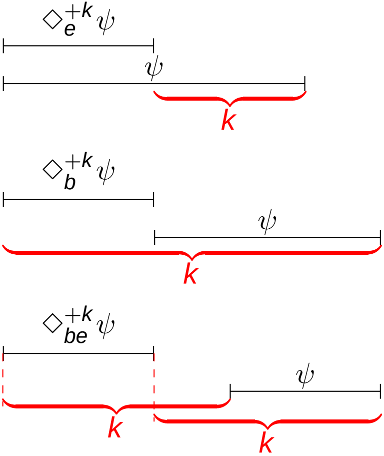

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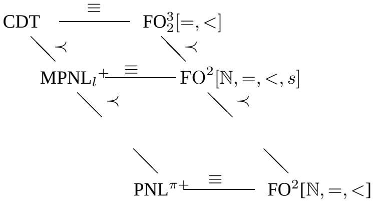

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Expressive completeness of MPNL_l

The fragment $\mathrm{FO}^2_r[\mathbb{N},<,=,s]$ of $\mathrm{FO}^2[\mathbb{N},<,=,s]$

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\end{tikzcd}
$$

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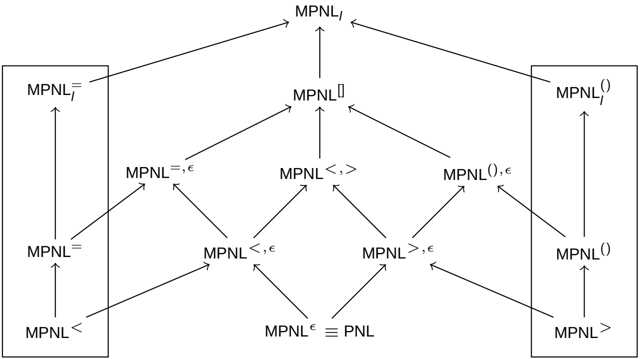
Relative expressive power of logics in MPNL

$$\mathrm{MPNL}^{<} \to \mathrm{MPNL}^{=} \to \mathrm{MPNL}_l^{=}$$

$$\mathrm{MPNL}^{>} \to \mathrm{MPNL}^{()} \to \mathrm{MPNL}_l^{()}$$

Relative expressive power of logics in MPNL



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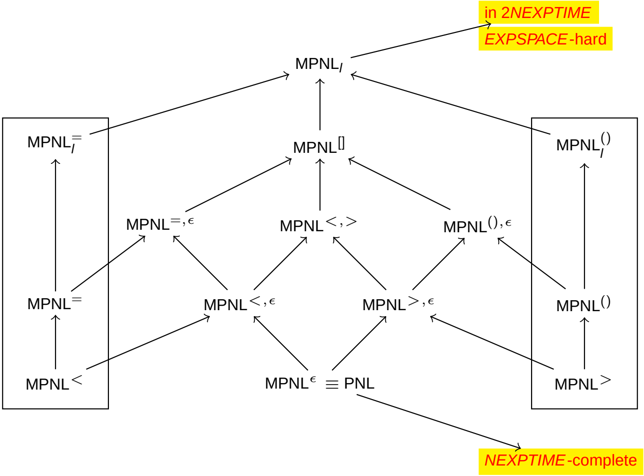

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Conclusions

- The class MPNL of metric logics based on PNL
- Decidability of the most expressive logic (MPNL_l)
- Undecidability of $\mathrm{FO}^2[\mathbb{N}, <, =, s]$
- Expressive completeness results:
  - $\mathrm{MPNL}_l^+ \equiv \mathrm{FO}^2[\mathbb{N}, <, =, s] \Rightarrow$ undecidability of MPNL_l^+
  - $\mathrm{MPNL}_l \equiv \mathrm{FO}_r^2[\mathbb{N}, <, =, s] \Rightarrow$ decidability of $\mathrm{FO}_r^2[\mathbb{N}, <, =, s]$
- Relative expressive power of logics in MPNL

To do

- From $\mathbb{N}$ to $\mathbb{Z}$ and all linear orderings
- From standard distance functions to other distance functions
- From constant constraint to "arithmetic" constraints
- Where is the complexity jump?
- To identify the precise complexity class of MPNL_l (2NEXPTIME or EXPSPACE?)

More research directions

- **Decidability/undecidability of other (Metric) Interval Logics:**
  - the sub-interval logic $\langle D \rangle$
  - other combinations of Allen's relations
- **Model Checking of (Metric) Interval logics:**
  - no known results;
- **Tableau method for Metric Interval Logics**
  - in particular, the extension of the tableau method for PNL to the metric case;
- **Metric PNL over dense orderings**
  - PNL is decidable even in the dense case ($\mathbb{Q}$); can we extend the language with metric features in this case too?