Good-for-Game QPTL: An Alternating Hodges Semantics

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***Abstract*—An extension of QPTL is considered where *functional dependencies* among the quantified variables can be restricted in such a way that their current values are *independent of the future* values of the other variables. This restriction is tightly connected to the notion of *behavioral strategies* in game-theory and allows the resulting logic to naturally express game-theoretic concepts. The fragment where only restricted quantifications are considered, called *behavioral quantifications*, can be decided, for both *model checking* and *satisfiability*, in 2ExpTime and is *expressively equivalent* to QPTL, though significantly *less succinct*.**

I. Introduction

The tight connection between *logic* and *games* has been acknowledged since the sixties, when first Lorenzen [44] and later Lorenz [43] and Hintikka [27] proposed *game-theoretic semantics* for first-order logic [29], [32]. In this approach, the meaning of a sentence is given in terms of a zero-sum game played by two agents: the *verifier*, whose objective is to show the sentence true, and the *falsifier*, with the dual objective of showing the sentence false. Satisfiability of a sentence, then, becomes a game between these two players and the sentence is satisfiable (*resp.*, unsatisfiable) *iff* verifier (*resp.*, falsifier) has a strategy to win it. This tight connection can clearly be viewed in the other direction as well: logic can be used to reason about games, *i.e.*, we can encode the problem of solving a game into a *decision problem*, such as *satisfiability* or *model-checking*, of some logic. The idea is to describe the game and the *winning condition* with a formula of the logic and exploit the *game-theoretic interpretation* to reduce the solution of the game to a given specific decision problem for that logic. Essentially, the winning strategy for the game can be extracted from the winning strategy for the *decision game*.

Suppose we have a formula $\psi(x, y)$, expressing a required relation between the choice y made by a player, from now on called *Eloise*, and a choice x made by the adversary, namely *Abelard*, *i.e.*, $\psi(x, y)$ encodes the *objective* of a two-player game. We say that the game is won by Eloise if there exists a *strategy* for her such that, for each choice x made by Abelard, the corresponding response y of Eloise using that strategy guarantees that the resulting play satisfies the requirement $\psi(x, y)$. This condition can clearly be expressed via a sentence of the form $\forall x. \exists y. \psi(x, y)$. We could then solve the game by actually solving the satisfiability problem for this sentence. In other words, solving the game reduces to checking whether there exists a *Skolem function* f such that $\forall x. \psi(x, \mathsf{f}(x))$ is satisfied. This function basically dictates the response of Eloise to the choice of Abelard, thereby encoding her strategy.

The above approach works pretty well when we consider *single-round games*, *a.k.a.*, *normal-form games* [68], and can easily be extended to *finite-rounds games*, *a.k.a.*, *extensive-form games* [40], [41], [67], by extending the quantification prefix to a sequence of alternations of quantifiers, one for each round. Things, however, get much more complicated when *infinite-rounds games* come into play [17], [69]. For such a class of extensive-form games, indeed, plays are induced by infinite sequences of choices made by the players over time and a strategy dictates how a player at a given stage of a play responds to the choices made by the adversary up to that stage.

Extending the quantification prefix to match the rounds would immediately lead to *infinitary logics*, such as the one proposed in [38] and further studied in [23] (see also [30]). This technique has some interesting applications in logic [24], computer science [37], and even philosophy [16]. Besides its infinitary nature, however, this approach has also the drawback of heavily departing from the Tarskian viewpoint, as only non-compositional game-theoretic semantics has been provided.

A more viable route, instead, is to make the quantified variables x and y range over sequences of choices. For example, if the choices are simply *Boolean values*, *i.e.*, when *iterated Boolean games* are considered [21], [22], first-order extension of *temporal logics*, such as *Quantified Propositional Temporal Logic* (QPTL) [63], seem like a good place to start, as they predicate over infinite sequences of temporal points, the stages of the game. In this setting, however, the Skolem function f cannot be interpreted as a strategy in the game-theoretic sense, as its value at a given stage depends on the entire evaluation of its argument x, namely the entire sequence of choices made by the adversary, including all the future ones. By contrast, a strategy for a player can only dictate, step by step, what its responses should be, depending on the choices made so far by its opponent. What that means is that, in principle, the satisfiability and the game solution problems do not coincide anymore. A classic example of this problem already appears in [58]. Assume $\psi(x, y)$ is the LTL [56], [57] formula $\mathtt{G}(y \leftrightarrow \mathtt{X}x)$ (or just $y \leftrightarrow \mathtt{X}x$). Clearly, the sentence $\forall x. \exists y. \psi(x, y)$ is satisfiable. However, there is no *"feasible"* (*i.e.*, *implementable*) strategy that can enforce $\psi(x, y)$, without Eloise knowing in advance the future values that Abelard is going to choose for x in the rest of the play. The problem is that the standard interpretation of quantification treats the quantified objects as *atomic entities*, regardless of their *inner structure*, like their being sequences in the above example. This is by no means the only exemplification of the problem, which was already recognized in the theory of extensive-form games since its dawn [40], where the notion of feasible strategy, called *behavioral*, has been introduced (see also [39], [41], [53], [62]). Another important source of unfeasible strategic behaviors is hidden in the semantics of *Strategy Logic* (SL) [4], [5], [48], [50], an extension of LTL that allows for explicit *quantifications* over strategies and *binding* of strategies with players. In this logic, formulae can be written that can be satisfied only by allowing players to look at what other strategies dictate in the future or counterfactual situations [49], admitting infeasible behaviors. Once again, the problem lies in the intrinsic dependence among the variables quantified in the formula.

One way to reconcile quantifications and strategies in a temporal setting would be to extend the game-theoretic interpretation of the quantifiers (and of the logic in general) to account for the underlying temporal dynamics. This would imply allowing the players in the satisfiability game to play with partial information on the choices of the adversary, namely the players have no information about the future and can only choose based on the moves played so far in the game. Previous attempts to address the issue typically involve resorting to *ad hoc Skolem semantics* [33] for the specific logic. In the case of SL, for instance, the notion of *behavioral semantics* has been introduced [48], which prevents the players from looking at future choices when selecting their strategy, effectively limiting the player observation ability to the current history in the game. A more liberal semantics based on *timeline dependencies* has been also proposed [18], [19]. While these approaches do solve the problem in the specific case, they lead to non-compositional semantics [60], in that the interpretation of a formula is not defined in terms of the interpretation of its component subformulae. To obtain a compositional version of the game-theoretic semantics, a finer grained technical setting is required, compared to the classic Tarskian semantics, specifically, one that can accommodate some form of *partial independence* among the quantified variables.

Following Tarski approach, each choice for a quantified variable in a sentence is made with complete information about, hence it is (potentially) completely dependent on, the values of variables quantified before it in the sentence. This idiosyncrasy of the classic interpretation of quantifiers is well known and attempts have been made to overcome the *linear dependence* of quantifiers dictated by their relative syntactic position in a sentence [2], [25], [28], [61]. Most notably, Hintikka and Sandu [31] proposed *Independence-Friendly Logic* (IF), as a first-order logic where independence between quantified variables can be explicitly asserted in the formulae together with a game-theoretic, non-compositional, semantics [59] for the logic. A compositional semantics for IF was later proposed by Hodges [34], [35], whose idea was to replace the standard notion of *assignment* of the Tarskian semantics with that of a set of assignments (called *trump* [34] or *team* [65]), as the basic semantic element with respect to which the truth of a formula is evaluated. This multiplicity of assignments effectively allows one to express the notion of *dependence/independence* among variables, a distinction that makes very little sense when a single assignment is considered.

Taking inspiration from Hodges' work, the goal of this work is to devise a *compositional semantic framework* that can account for a game-theoretic interpretation of quantification over (possibly infinite) sequences of choices. The framework is specifically tailored to deal with quantifications in a linear-time settings and applied to the logic QPTL, which was introduced in [63] as a unifying *ω-regular language* allowing for both temporal operators and propositional quantifiers. Despite its expressiveness and theoretical interest, QPTL has not gained much traction in practical contexts, mainly due to the high complexity of its decision problems. Indeed, both the satisfiability and the model checking problems are *non-elementary* in the number of *alternations of the quantifiers* [64].

In this article we propose a novel semantics for QPTL, inspired by the body of work on (in)dependence logics [1], [45], [65]. Similarly to those works, the semantics provides a compositional formulation [60] for a game-theoretic interpretation of the quantifiers. In contrast to them, however, we require a *symmetric* treatment of the two quantifiers in order to preserve *closure under negation* and avoid *undetermined formulae* [34], [35]. The most significant feature of the new approach is the ability to encode various forms of *independence constraints* among the quantified variables and provide a powerful tool to fine-tune the semantics of the propositional quantifiers. In particular, we discuss a specific instantiation of the semantics that allows one to recover a game-theoretic interpretation of the quantifiers and reconcile the satisfiability and the game solution problems. This result is achieved by first generalizing classic *temporal assignments*, which give values to propositional variables at each time instant, to sets of sets of assignments, called *hyperassignments*. This also generalizes teams, defined as sets of assignments, used by Hodges. The second step is to introduce new *classes of functors* that maps temporal assignments to valuations of a given variable over time and, intuitively, correspond to the semantic counterparts of the Skolem functions. The dependence of functors on assignments allows us to impose various forms of independence constraints among the variables. In particular, we investigate two specific forms, called *behavioral* and *strongly-behavioral*, that require functors to choose the value of the variable at any given time instant based only upon the values dictated by the input assignment to the other variables up to that instant (possibly excluded). These are forms of independence constraints that make the choices of the value of a variable at a given time totally independent of the values that other variables assume in the future. The behavioral restrictions are precisely what allows us to recover the correspondence between Skolem functions and strategies and to reconcile the satisfiability and game solution problems, thus making the resulting version of QPTL, called *Good-for-Games* QPTL (GFG-QPTL), well suited to express game-theoretic concepts and a logical analogue of *Good-for-Games Automata* [3], [26].

On the technical side, the novel semantics under the behavioral interpretation of the quantifiers leads to 2EXPTIME decision procedures for both the satisfiability and model-checking problems. On the other hand, it does not give up expressiveness, as we show that the vanilla and behavioral semantics turn out to be expressively equivalent. These results also show that the high complexity of the decision problems for vanilla QPTL stems from the fact that unrestricted dependencies among the quantified variables are allowed. The properties expressible by exploiting such unrestricted dependencies can, however, still be expressed under the behavioral semantics, though with a non-elementary blowup.

II. Alternating Hodges Semantics

QPTL [63] extends LTL [56], [57] with quantifications over *atomic propositions* from a given set AP, with the intuition that the Boolean values of the same proposition in different time instants are independent of each other.

A. Quantified Propositional Temporal Logic

For convenience, we provide a syntax for QPTL where quantifications do not occur within temporal operators. This is equivalent to the original logic, thanks to the *prenex normal form* (*pnf*, for short) property enjoyed by QPTL [63], which allows to move quantifiers outside temporal operators.

Definition 1 (QPTL Syntax)**.** *The* Quantified Propositional Temporal Logic *is the set of formulae built accordingly to the following context-free grammar, where $\psi \in$ LTL and $p \in \mathrm{AP}$:*

$$\varphi := \psi \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists p.\, \varphi \mid \forall p.\, \varphi.$$

The classic semantics is given in terms of *temporal assignments* (simply *assignments*, from now on), which are functions associating each proposition with a *temporal valuation* mapping each time instant to a Boolean value, *i.e.*, infinite sequences of truth assignments. Let $\mathrm{Asg} \triangleq \mathrm{AP} \rightharpoonup (\mathbb{N} \to \mathbb{B})$ denote the set of assignments over arbitrary subsets of AP. For convenience, we also introduce the set of assignments defined exactly over the propositions in $\mathrm{P} \subseteq \mathrm{AP}$, *i.e.*, $\mathrm{Asg}(\mathrm{P}) \triangleq \{\chi \in \mathrm{Asg} \mid \mathsf{dom}(\chi) = \mathrm{P}\}$ and the set $\mathrm{Asg}_{\subseteq}(\mathrm{P}) \triangleq \{\chi \in \mathrm{Asg} \mid \mathrm{P} \subseteq \mathsf{dom}(\chi)\}$ of assignments defined at least over P.

The *satisfaction relation* $\models$ between an assignment χ and a QPTL formula φ is defined below, where $\models_{\mathsf{LTL}}$ is the standard LTL satisfiability and $\chi[p \mapsto \mathsf{f}]$ denotes the assignment that extends χ and maps proposition p to temporal valuation f. As usual, by $\mathsf{free}(\varphi)$ we denote the set of propositions free in φ.

Definition 2 (Tarski Semantics)**.** *The* Tarski-semantics relation *$\chi \models \varphi$ is inductively defined as follows, for all* QPTL *formulae φ and assignments $\chi \in \mathrm{Asg}_{\subseteq}(\mathsf{free}(\varphi))$.*

1) *$\chi \models \psi$, if $\chi \models_{\mathsf{LTL}} \psi$, whenever ψ is an* LTL *formula;*
2) *the semantics of Boolean connectives is defined as usual;*
3) *for all atomic propositions $p \in \mathrm{AP}$:*
   a) *$\chi \models \exists p.\, \phi$ if $\chi[p \mapsto \mathsf{f}] \models \phi$, for some $\mathsf{f} \in \mathbb{N} \to \mathbb{B}$;*
   b) *$\chi \models \forall p.\, \phi$ if $\chi[p \mapsto \mathsf{f}] \models \phi$, for all $\mathsf{f} \in \mathbb{N} \to \mathbb{B}$.*

B. A New Semantics for QPTL

We now introduce a novel compositional semantics for QPTL that, unlike Tarski's one, will allow us to specify, later on, independence constraints among quantified propositions. The new semantics follows an approach similar to [34], where a compositional semantics for IF was first proposed. Hodges' idea was to expand an assignment for the free variables to a set of assignments, a trump in his terminology, with the intuition of capturing all possible choices made by one of the two players for its own variables in the satisfiability game underlying the game-theoretic semantics of the logic [31]. Hodges' semantics, though able to correctly capture IF, is, however, not adequate for our purposes. Indeed, as design choice, it is intrinsically asymmetric, treating the two players differently. More specifically, a single set of assignments only provides complete information about the choices of one of the two players and only allows to restrict the choices of the adversary. This, in turn, limits the class of games expressible in the logic to asymmetric games where only the observation power of one player can be restricted. In order to capture symmetric games too, we need to get rid of this asymmetry, which requires a non-trivial generalization of Hodges' approach.

To give semantics to a QPTL formula φ, we proceed as follows. Similarly to Hodges, the idea is that the interpretations of the free atomic propositions correspond to the choices that the two players could make prior to the current stage of the game, *i.e.*, the stage where the formula φ has still to be evaluated. These possible choices can be organized on a two-level structure, *i.e.*, a set of sets of assignments, each level summarizing the information about the choices a player can make in its turns. In order to evaluate the formula φ, then, a player chooses a set of assignment, while its opponent chooses one assignment in that set where φ must hold. We shall use a flag $\alpha \in \{\exists\forall, \forall\exists\}$, called *alternation flag*, to keep track of which player is assigned to which level of choice. If $\alpha = \exists\forall$, Eloise chooses the set of assignments, while Abelard chooses one of those assignments; if $\alpha = \forall\exists$, the dual reasoning applies. Given a flag $\alpha \in \{\exists\forall, \forall\exists\}$, we denote by $\overline{\alpha}$ the dual flag, *i.e.*, $\overline{\alpha} \in \{\exists\forall, \forall\exists\}$ with $\overline{\alpha} \neq \alpha$. The idea above is captured by the following notion of *hyperassignment*, namely a non-empty set of non-trivial, *i.e.*, non-empty, sets of assignments defined over an arbitrary set $\mathrm{P} \subseteq \mathrm{AP}$:

$$\mathrm{HAsg} \triangleq \{\, \mathfrak{X} \subseteq 2^{\mathrm{Asg}(\mathrm{P})} \mid \emptyset \notin \mathfrak{X} \neq \emptyset \wedge \mathrm{P} \subseteq \mathrm{AP} \,\}.$$

Note that we require all the assignments contained in a hyperassignment to be defined on the same atomic propositions, though the domains of assignments in different hyperassignment may differ. By $\mathsf{ap}(\mathfrak{X}) \subseteq \mathrm{AP}$ we denote the set of atomic propositions over which the hyperassignment $\mathfrak{X}$ is defined. $\mathrm{HAsg}(\mathrm{P}) \triangleq \{\mathfrak{X} \in \mathrm{HAsg} \mid \mathfrak{X} \subseteq 2^{\mathrm{Asg}(\mathrm{P})}\}$ is the set of hyperassignments over the same set of atomic propositions P, while the set whose hyperassignments have domains that include P is $\mathrm{HAsg}_{\subseteq}(\mathrm{P}) \triangleq \{\mathfrak{X} \in \mathrm{HAsg} \mid \mathfrak{X} \subseteq 2^{\mathrm{Asg}_{\subseteq}(\mathrm{P})}\}$.

For any pair of hyperassignments $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathrm{HAsg}$, we write $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$ to express the fact that, for all sets of assignments

$X_1 \in \mathfrak{X}_1$, there is a set of assignments $X_2 \in \mathfrak{X}_2$ with $X_2 \subseteq X_1$. Obviously, $\mathfrak{X}_1 \subseteq \mathfrak{X}_2$ implies $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$, which in its turn implies $\mathsf{ap}(\mathfrak{X}_1) = \mathsf{ap}(\mathfrak{X}_2)$. Figure 1 reports a graphical representation of the relation $\sqsubseteq$. As usual, we write $\mathfrak{X}_1 \equiv \mathfrak{X}_2$ if both $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$ and $\mathfrak{X}_2 \sqsubseteq \mathfrak{X}_1$ hold true. It is clear that the relation $\sqsubseteq$ is both reflexive and transitive, hence it is a preorder. Consequently, $\equiv$ is an equivalence relation. In particular, we shall show (see Corollary 1) that $\equiv$ captures the intuitive notion of equivalence between hyperassignments, in the sense that two equivalent hyperassignments *w.r.t.* $\equiv$ do satisfy the same formulae.

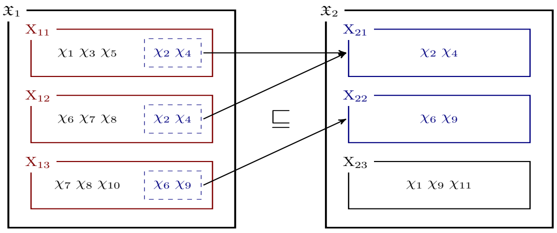

Figure 1. The preorder between hyperassignments: for every $X_{1i} \in \mathfrak{X}_1$, there is a $X_{2j} \in \mathfrak{X}_2$ with $X_{2j} \subseteq X_{1i}$. More than one set in $\mathfrak{X}_1$ could choose the same set in $\mathfrak{X}_2$; there may be sets in $\mathfrak{X}_2$ not used by any in $\mathfrak{X}_1$.

Our goal is to define a semantics for QPTL by providing a satisfaction relation between a hyperassignment $\mathfrak{X}$ and a QPTL formula φ, *w.r.t.* a given interpretation of the players of $\mathfrak{X}$, *i.e.*, *w.r.t.* an alternation flag $\alpha \in \{\exists\forall, \forall\exists\}$. Therefore, we shall have two satisfaction relations, namely $\models^{\exists\forall}$ and $\models^{\forall\exists}$, depending on how we interpret the levels of the hyperassignment. The idea is to capture the following intuition that relates, in a natural way, to the classic Tarskian semantics. When the alternation flag α is $\exists\forall$, then a set of assignments is chosen existentially by Eloise and all its assignments, chosen universally by Abelard, must satisfy φ. Conversely, when α is $\forall\exists$, then a set of assignments is chosen universally by Abelard and at least one assignment, chosen existentially by Eloise, must satisfy φ. Semantically, hyperassignments are a notion similar to *quasi-strategies* [38].

We break down the presentation of the semantics by introducing three operations: the *dualization* swaps the choices of the two players of a hyperassignment and allows to connect the two satisfaction relations; the *partitioning* deals with disjunction and conjunction; finally, the *extension* handles quantifications.

Let us consider the dualization operator first. The idea is that, given a hyperassignment $\mathfrak{X}$, the dual hyperassignment $\overline{\mathfrak{X}}$ exchanges the role of the two players *w.r.t.* $\mathfrak{X}$. This means that, if Eloise is the first to choose in $\mathfrak{X}$, then her choice will be postponed in $\overline{\mathfrak{X}}$ after that of Abelard. To ensure the game is still the same with only the order of the choices interchanged, we need to reshuffle the assignments in $\mathfrak{X}$ so as to simulate the original dependencies between the choices of the players. To this end, we introduce the set of choice functions for $\mathfrak{X}$ as follows, whose definition assumes the axiom of choice:

$$\mathrm{Chc}(\mathfrak{X}) \triangleq \{\, \Gamma : \mathfrak{X} \to \mathrm{Asg} \mid \forall X \in \mathfrak{X}.\, \Gamma(X) \in X \,\}.$$

$\mathrm{Chc}(\mathfrak{X})$ contains all the functions Γ that, for every set of assignments X in $\mathfrak{X}$, pick a specific assignment $\Gamma(X)$ in that set. Each such function simulates a possible choice of the second player of $\mathfrak{X}$ depending on the choice of (the set of assignments chosen by) its first player. The dual hyperassignment $\overline{\mathfrak{X}}$, then, simply collects the images of the choice functions in $\mathrm{Chc}(\mathfrak{X})$. We, thus, obtain a hyperassignment in which the order of the choices of the two players is inverted:

$$\overline{\mathfrak{X}} \triangleq \{\, \mathsf{img}(\Gamma) \mid \Gamma \in \mathrm{Chc}(\mathfrak{X}) \,\}.$$

Example 1. *Consider the hyperassignment* $\mathfrak{X} = \{X_1, X_2, X_3\}$, *where* $X_1 = \{\chi_{11}, \chi_{12}\}$, $X_2 = \{\chi_{21}, \chi_{22}\}$, *and* $X_3 = \{\chi_3\}$*:*

$$\mathfrak{X} = \left\{ \begin{array}{c} \{\chi_{11}, \chi_{12}\}, \\ \{\chi_{21}, \chi_{22}\}, \\ \{\chi_3\} \end{array} \right\}; \quad \overline{\mathfrak{X}} = \left\{ \begin{array}{l} \mathsf{img}(\Gamma_1) = \{\chi_{11}, \chi_{21}, \chi_3\}, \\ \mathsf{img}(\Gamma_2) = \{\chi_{11}, \chi_{22}, \chi_3\}, \\ \mathsf{img}(\Gamma_3) = \{\chi_{12}, \chi_{21}, \chi_3\}, \\ \mathsf{img}(\Gamma_4) = \{\chi_{12}, \chi_{22}, \chi_3\} \end{array} \right\}.$$

Every set of assignments in $\overline{\mathfrak{X}}$ *is obtained as the image of one of the four choice functions* $\Gamma_i \in \mathrm{Chc}(\mathfrak{X})$*, each choosing exactly one assignment from* X_1*, one from* X_2*, and one from* X_3*.*

The following property ensures that the dualization operator enjoys an *involution property*, similarly to the Boolean negation. Indeed, it states that, by applying the dualization operator twice, we obtain a hyperassignment equivalent to the original one.

Proposition 1. $\mathfrak{X} \subseteq \overline{\overline{\mathfrak{X}}}$ *and* $\mathfrak{X} \equiv \overline{\overline{\mathfrak{X}}}$*, for all* $\mathfrak{X} \in \mathrm{HAsg}$*.*

The following lemma, instead, formally states that the dualization operator swaps the role of the two players while still preserving their original choices.

Lemma 1 (Dualization)**.** *The following hold true, for all* QPTL *formulae* φ *and hyperassignments* $\mathfrak{X} \in \mathrm{HAsg}_{\subseteq}(\mathsf{free}(\varphi))$*.*

1) *Statements 1a and 1b are equivalent:*
   a) *there exists* $X \in \mathfrak{X}$ *such that* $\chi \models \varphi$*, for all* $\chi \in X$*;*
   b) *for all* $X \in \overline{\mathfrak{X}}$*, it holds that* $\chi \models \varphi$*, for some* $\chi \in X$*.*
2) *Statements 2a and 2b are equivalent:*
   a) *for all* $X \in \mathfrak{X}$*, it holds that* $\chi \models \varphi$*, for some* $\chi \in X$*;*
   b) *there exists* $X \in \overline{\mathfrak{X}}$ *such that* $\chi \models \varphi$*, for all* $\chi \in X$*.*

The *partition operator* decomposes hyperassignments and is instrumental in capturing the semantics of Boolean connectives. Given a hyperassignment $\mathfrak{X}$, the following set

$$\mathsf{par}(\mathfrak{X}) \triangleq \{\, (\mathfrak{X}_1, \mathfrak{X}_2) \in 2^{\mathfrak{X}} \times 2^{\mathfrak{X}} \mid \mathfrak{X}_1 \uplus \mathfrak{X}_2 = \mathfrak{X} \,\}$$

collects all the possible partitions of $\mathfrak{X}$ into two parts. Assume that the two players of $\mathfrak{X}$ are interpreted according to the alternation flag $\forall\exists$: Abelard chooses first and Eloise chooses second. The game-theoretic interpretation of the disjunction requires Eloise to choose one of two disjuncts to be proven true. In our setting, then, in order to satisfy $\varphi_1 \vee \varphi_2$, Eloise has to show that, for each set of assignments chosen by Abelard, she has a way to select one of the disjuncts φ_i in such a way that φ_i is satisfied by some assignment in that set. This selection is summarized by one of the pairs $(\mathfrak{X}_1, \mathfrak{X}_2)$ in $\mathsf{par}(\mathfrak{X})$, where $\mathfrak{X}_i$ collects the sets of assignments for which the i-th disjunct is selected, with $i \in \{1, 2\}$. A similar argument, with the role of the two players reversed and switching the quantifications throughout, leads to a dual interpretation for conjunction, where it is Abelard who chooses one of the two conjuncts to be proven false. This intuition is made precise by the following lemma.

Lemma 2 (Boolean Connectives)**.** *The following hold true, for all* QPTL *formulae* φ_1 *and* φ_2 *and hyperassignments* $\mathfrak{X} \in \mathrm{HAsg}_{\subseteq}(\mathrm{P})$*, with* $\mathrm{P} \triangleq \mathsf{free}(\varphi_1) \cup \mathsf{free}(\varphi_2)$.

1) *Statements 1a and 1b are equivalent:*
   a) *there exists* $\mathrm{X} \in \mathfrak{X}$ *such that* $\chi \models \varphi_1 \wedge \varphi_2$*, for all* $\chi \in \mathrm{X}$*;*
   b) *for each* $(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$*, there exist* $i \in \{1, 2\}$ *and* $\mathrm{X} \in \mathfrak{X}_i$ *such that* $\chi \models \varphi_i$*, for all* $\chi \in \mathrm{X}$.
2) *Statements 2a and 2b are equivalent:*
   a) *for all* $\mathrm{X} \in \mathfrak{X}$*, it holds that* $\chi \models \varphi_1 \vee \varphi_2$*, for some* $\chi \in \mathrm{X}$*;*
   b) *there exists* $(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$ *such that, for all* $i \in \{1, 2\}$ *and* $\mathrm{X} \in \mathfrak{X}_i$*, it holds that* $\chi \models \varphi_i$*, for some* $\chi \in \mathrm{X}$.

Quantifications are taken care of by the *extension operator*. Let $\mathrm{Fnc}(\mathrm{P}) \triangleq \mathrm{Asg}(\mathrm{P}) \to (\mathbb{N} \to \mathbb{B})$ be the set of *functors* that maps every assignment over P to a temporal valuation. Essentially, these objects play the role of Skolem functions in the non-compositional semantics. The *extension of an assignment* $\chi \in \mathrm{Asg}(\mathrm{P})$ *w.r.t.* a functor $\mathsf{F} \in \mathrm{Fnc}(\mathrm{P})$ for an atomic proposition $p \in \mathrm{AP}$ is defined as $\mathsf{ext}(\chi, \mathsf{F}, p) \triangleq \chi[p \mapsto \mathsf{F}(\chi)]$. Intuitively, it extends χ with p, by assigning to it the value $\mathsf{F}(\chi)$ prescribed by the functor F. The *extension operation* can then be lifted to sets of assignments $\mathrm{X} \subseteq \mathrm{Asg}(\mathrm{P})$ in the obvious way, *i.e.*, we set $\mathsf{ext}(\mathrm{X}, \mathsf{F}, p) \triangleq \{ \mathsf{ext}(\chi, \mathsf{F}, p) \mid \chi \in \mathrm{X} \}$. This operation embed into X the entire player strategy encoded by F. Finally, the *extension of a hyperassignment* $\mathfrak{X} \in \mathrm{HAsg}(\mathrm{P})$ with p is simply the set of extensions with p of all its sets of assignments *w.r.t.* all possible functors over the atomic propositions of $\mathfrak{X}$:

$$\mathsf{ext}(\mathfrak{X}, p) \triangleq \{ \mathsf{ext}(\mathrm{X}, \mathsf{F}, p) \mid \mathrm{X} \in \mathfrak{X}, \mathsf{F} \in \mathrm{Fnc}(\mathsf{ap}(\mathfrak{X})) \}.$$

Intuitively, this operation embeds into $\mathfrak{X}$ all possible strategies, encoded by the functors F, for choosing the value of p. The following lemma states that the extension operator provides an adequate semantics for quantifications, where statement 1 considers Eloise's choices, when the player interpretation of the hyperassignment is $\exists\forall$, and statement 2 takes care of Abelard's choices, when the player interpretation is $\forall\exists$.

Lemma 3 (Hyperassignment Extensions)**.** *The following hold true, for all* QPTL *formulae* φ*, atomic propositions* $p \in \mathrm{AP}$*, and hyperassignments* $\mathfrak{X} \in \mathrm{HAsg}_{\subseteq}(\mathsf{free}(\varphi) \setminus \{p\})$.

1) *Statements 1a and 1b are equivalent:*
   a) *there exists* $\mathrm{X} \in \mathfrak{X}$ *such that* $\chi \models \exists p. \varphi$*, for all* $\chi \in \mathrm{X}$*;*
   b) *there exists* $\mathrm{X} \in \mathsf{ext}(\mathfrak{X}, p)$ *such that* $\chi \models \varphi$*, for all* $\chi \in \mathrm{X}$.
2) *Statements 2a and 2b are equivalent:*
   a) *for all* $\mathrm{X} \in \mathfrak{X}$*, it holds that* $\chi \models \forall p. \varphi$*, for some* $\chi \in \mathrm{X}$*;*
   b) *for all* $\mathrm{X} \in \mathsf{ext}(\mathfrak{X}, p)$*, it holds that* $\chi \models \varphi$*, for some* $\chi \in \mathrm{X}$.

We can finally introduce the new semantics for QPTL based on the novel notion of hyperassignment.

Definition 3 (Alternating Hodges Semantics)**.** *The* alternating-Hodges-semantics relation $\mathfrak{X} \models^{\alpha} \varphi$ *is inductively defined as follows, for all* QPTL *formulae* φ*, hyperassignments* $\mathfrak{X} \in \mathrm{HAsg}_{\subseteq}(\mathsf{free}(\varphi))$*, and alternation flags* $\alpha \in \{\exists\forall, \forall\exists\}$.

1) *whenever* ψ *is an* LTL *formula:*
   a) $\mathfrak{X} \models^{\exists\forall} \psi$ *if there is a set of assignments* $\mathrm{X} \in \mathfrak{X}$ *such that, for each assignment* $\chi \in \mathrm{X}$*, it holds that* $\chi \models_{\mathsf{LTL}} \psi$*;*
   b) $\mathfrak{X} \models^{\forall\exists} \psi$ *if, for each set of assignments* $\mathrm{X} \in \mathfrak{X}$*, there is an assignment* $\chi \in \mathrm{X}$ *such that* $\chi \models_{\mathsf{LTL}} \psi$*;*
2) $\mathfrak{X} \models^{\alpha} \neg\phi$ *if* $\mathfrak{X} \not\models^{\overline{\alpha}} \phi$, i.e., *it is not the case that* $\mathfrak{X} \models^{\overline{\alpha}} \phi$*;*
3) a) $\mathfrak{X} \models^{\exists\forall} \phi_1 \wedge \phi_2$ *if, for each* $(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$*, it holds that* $\mathfrak{X}_1 \neq \emptyset$ *and* $\mathfrak{X}_1 \models^{\exists\forall} \phi_1$ *or* $\mathfrak{X}_2 \neq \emptyset$ *and* $\mathfrak{X}_2 \models^{\exists\forall} \phi_2$*;*
   b) $\mathfrak{X} \models^{\forall\exists} \phi_1 \wedge \phi_2$ *if* $\overline{\mathfrak{X}} \models^{\exists\forall} \phi_1 \wedge \phi_2$*;*
4) a) $\mathfrak{X} \models^{\exists\forall} \phi_1 \vee \phi_2$ *if* $\overline{\mathfrak{X}} \models^{\forall\exists} \phi_1 \vee \phi_2$*;*
   b) $\mathfrak{X} \models^{\forall\exists} \phi_1 \vee \phi_2$ *if there is* $(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$ *such that if* $\mathfrak{X}_1 \neq \emptyset$ *then* $\mathfrak{X}_1 \models^{\forall\exists} \phi_1$ *and if* $\mathfrak{X}_2 \neq \emptyset$ *then* $\mathfrak{X}_2 \models^{\forall\exists} \phi_2$*;*
5) *for all atomic propositions* $p \in \mathrm{AP}$*:*
   a) $\mathfrak{X} \models^{\exists\forall} \exists p. \phi$ *if* $\mathsf{ext}(\mathfrak{X}, p) \models^{\exists\forall} \phi$*;*
   b) $\mathfrak{X} \models^{\forall\exists} \exists p. \phi$ *if* $\overline{\mathfrak{X}} \models^{\exists\forall} \exists p. \phi$*;*
6) *for all atomic propositions* $p \in \mathrm{AP}$*:*
   a) $\mathfrak{X} \models^{\exists\forall} \forall p. \phi$ *if* $\overline{\mathfrak{X}} \models^{\forall\exists} \forall p. \phi$*;*
   b) $\mathfrak{X} \models^{\forall\exists} \forall p. \phi$ *if* $\mathsf{ext}(\mathfrak{X}, p) \models^{\forall\exists} \phi$.

The base case (Item 1) for LTL formulae ψ simply formalizes the intuition about satisfaction relative to the alternation flag: if $\alpha = \exists\forall$, there exists a set of assignments whose elements satisfy ψ in the Tarski sense; the dual applies when $\alpha = \forall\exists$. Negation, in accordance with the classic game-theoretic interpretation, is dealt with by simply exchanging the player interpretation of the hyperassignment (Item 2). Observe that, from this semantic condition, it immediately follows that either $\mathfrak{X} \models^{\alpha} \varphi$ or $\mathfrak{X} \models^{\overline{\alpha}} \neg\varphi$. In other words, the semantics does not allow formulae with an undetermined truth value. The semantics of the remaining Boolean connectives (Items 3a and 4b) and quantifiers (Items 5a and 6b) is a direct application of Lemmata 2 and 3. Observe that swapping between $\models^{\exists\forall}$ and $\models^{\forall\exists}$ (Items 3b, 4a, 5b and 6a) is done according to Lemma 1 and it represents the main and fundamental point where our approach heavily departs from Hodges' semantics [34], [35]. The above three lemmata also imply the following theorem, which formalizes an *adequacy principle* that reduces the two satisfiability relations of the new semantics to the classic Tarskian satisfaction in a natural way.

Theorem 1 (Semantics Adequacy I)**.** *The following hold, for all* QPTL *formulae* φ *and hyperassignments* $\mathfrak{X} \in \mathrm{HAsg}_{\subseteq}(\mathsf{free}(\varphi))$*:*

1) $\mathfrak{X} \models^{\exists\forall} \varphi$ iff *there exists a set of assignments* $\mathrm{X} \in \mathfrak{X}$ *such that* $\chi \models \varphi$*, for all* $\chi \in \mathrm{X}$*;*

2) $\mathfrak{X} \models^{\forall\exists} \varphi$ iff, *for all sets of assignments* $\mathrm{X} \in \mathfrak{X}$, *it holds that* $\chi \models \varphi$, *for some* $\chi \in \mathrm{X}$.

III. GOOD-FOR-GAME QPTL

The semantic framework introduced in the previous section allows us to encode behavioral independence constraints among the quantified variables of QPTL. We thus obtain the logic GFG-QPTL, an extension of QPTL able to express the behavioralness of quantifications over temporal valuations.

A. Adding Behavioral Dependencies to QPTL

Given a set of assignments $\mathrm{Asg}(\mathrm{P})$ over some $\mathrm{P} \subseteq \mathrm{AP}$, a *behavioral quantification w.r.t.* a proposition $p \in \mathrm{P}$ should choose, for each assignment $\chi \in \mathrm{Asg}(\mathrm{P})$, a temporal valuation $\mathsf{f} \colon \mathbb{N} \to \mathbb{B}$ in such a way that, intuitively, at each instant of time $k \in \mathbb{N}$, the value $\mathsf{f}(k)$ of f at k only depends on the values $\chi(p)(t)$ of the temporal valuation $\chi(p)$ at the instants of time $t \leq k$; this means that $\mathsf{f}(k)$ is independent of the values $\chi(p)(t)$ at any future instant $t > k$. To be more precise, consider two assignments $\chi_1, \chi_2 \in \mathrm{Asg}(\mathrm{P})$ that may differ only on p strictly after k. Then, the functor $\mathsf{F} \in \mathrm{Fnc}(\mathrm{P})$ interpreting a quantification behavioral *w.r.t.* p must return the same value at k as a reply to both χ_1 and χ_2, *i.e.*, $\mathsf{F}(\chi_1)(k) = \mathsf{F}(\chi_2)(k)$; in other words, $\mathsf{F}(\chi)(k)$ cannot exploit the knowledge of the values $\chi(p)(t)$, with $t > k$. An analogous concept has been introduced in SL [48]. A stronger notion of behavioralness, similar to one reported in [18], requires the functor F to satisfy the above equality when χ_1 and χ_2 only (possibly) differ on p for $t \geq k$ and leads to the concept of *strongly behavioral quantification*. In game-theoretic terms, the interpretation of a *behavioral quantifier w.r.t.* p requires the corresponding player to choose the value of a proposition at each round only based on the choices for p made by the adversary up to that round. For a *strongly behavioral quantifier*, instead, the adversary keeps its choice for p at the current round hidden and the player can only access the choices made for p at previous rounds. Definitions 4 and 5 formalize these fundamental concepts.

Definition 4 (Assignment Distinguishability)**.** *Let* $\chi_1, \chi_2 \in \mathrm{Asg}(\mathrm{P})$ *be two assignments over some set* $\mathrm{P} \subseteq \mathrm{AP}$ *of propositions,* $p \in \mathrm{P}$ *one of these propositions, and* $k \in \mathbb{N}$ *a number. Then,* χ_1 *and* χ_2 *are* (p, k)-strict distinguishable *(*resp.*,* (p, k)-distinguishable*), in symbols* $\chi_1 \approx_p^{>k} \chi_2$ *(*resp.*,* $\chi_1 \approx_p^{\geq k} \chi_2$*), if the following holds:*

1) $\chi_1(q) = \chi_2(q)$*, for all* $q \in \mathrm{P}$ *with* $q \neq p$*;*
2) $\chi_1(p)(t) = \chi_2(p)(t)$*, for all* $t \leq k$ *(*resp.*,* $t < k$*).*

The notion of (p, k)-strict distinguishability (*resp.*, (p, k)-distinguishability) allows us to identify all the assignments that can only differ on the proposition p at some instant $t > k$ (*resp.*, $t \geq k$). Indeed, $\approx_p^{>k}$ (*resp.*, $\approx_p^{\geq k}$) is an equivalence relation on $\mathrm{Asg}(\mathrm{P})$, whose equivalence classes identify those assignments precisely. For a set of $\approx_p^{>k}$-equivalent (*resp.*, $\approx_p^{\geq k}$-equivalent) assignments, a *behavioral* (*resp.*, *strongly-behavioral*) functor must reply uniformly to them at time k.

Definition 5 (Behavioral Functor)**.** *Let* $\mathsf{F} \in \mathrm{Fnc}(\mathrm{P})$ *be a functor over some set* $\mathrm{P} \subseteq \mathrm{AP}$ *of propositions and* $p \in \mathrm{P}$ *one of these propositions. Then,* F *is* behavioral *(*resp.*,* strongly behavioral*)* w.r.t. p *if* $\mathsf{F}(\chi_1)(k) = \mathsf{F}(\chi_2)(k)$*, for all numbers* $k \in \mathbb{N}$ *and pairs of* $\approx_p^{>k}$*-equivalent (*resp.*,* $\approx_p^{\geq k}$*-equivalent) assignments* $\chi_1, \chi_2 \in \mathrm{Asg}(\mathrm{P})$.

	0	1	2	3	4	5	
$\chi_1 = \{\, p$:	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\cdots\}$
$\chi_2 = \{\, p$:	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\cdots\}$
$\mathsf{F}_{\mathtt{A}}(\chi_1) =$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\cdots$
$\mathsf{F}_{\mathtt{A}}(\chi_2) =$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\cdots$
$\mathsf{F}_{\mathtt{B}}(\chi_1) =$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\cdots$
$\mathsf{F}_{\mathtt{B}}(\chi_2) =$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\cdots$
$\mathsf{F}_{\mathtt{S}}(\chi_1) =$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\cdots$
$\mathsf{F}_{\mathtt{S}}(\chi_2) =$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\cdots$

Figure 2. Two $\approx_p^{>3}$ (*resp.*, $\approx_p^{\geq 4}$) -equivalent assignments and few functors.

Example 2. *Let* χ_1 *and* χ_2 *be two assignments over the singleton* $\{p\}$ *defined as in Figure 2. It is clear that* $\chi_1 \approx_p^{>3} \chi_2$*, but* $\chi_1 \not\approx_p^{\geq 4} \chi_2$*, and so* $\chi_1 \approx_p^{\geq 4} \chi_2$*, but* $\chi_1 \not\approx_p^{\geq 5} \chi_2$*. Also, consider the three functors* $\mathsf{F}_{\mathtt{A}}, \mathsf{F}_{\mathtt{B}}, \mathsf{F}_{\mathtt{S}} \in \mathrm{Fnc}(\{p\})$ *defined as follows, for all* $\mathfrak{X} \in \mathrm{Asg}(\{p\})$ *and* $t \in \mathbb{N}$*:* $\mathsf{F}_{\mathtt{A}}(\chi)(t) \triangleq \chi(p)(t+1)$*;* $\mathsf{F}_{\mathtt{B}}(\chi)(t) \triangleq \overline{\chi(p)(t)}$*;* $\mathsf{F}_{\mathtt{S}}(\chi)(t) \triangleq \top$*, if* $t = 0$*, and* $\mathsf{F}_{\mathtt{S}}(\chi)(t) \triangleq \chi(p)(t-1)$*, otherwise. It is immediate to see that* $\mathsf{F}_{\mathtt{B}}$ *is behavioral, while* $\mathsf{F}_{\mathtt{S}}$ *is strongly behavioral. However,* $\mathsf{F}_{\mathtt{A}}$ *does not enjoy any behavioral property, being defined as a future-dependent functor. Indeed,* $\mathsf{F}_{\mathtt{A}}(\chi_1)(3) \neq \mathsf{F}_{\mathtt{A}}(\chi_2)(3)$*, even though* $\chi_1 \approx_p^{>3} \chi_2$.

In order to capture the behavioral constraints on the functors within the logic, we extend the QPTL syntax with additional decorations for the quantifiers that express behavioral dependencies among the propositions involved. The result is a new logic, called *Good-for-Games* QPTL, able to express in a natural way game-theoretic concepts of Boolean games.

Definition 6 (GFG-QPTL Syntax)**.** Good-for-Games QPTL *(*GFG-QPTL*) is the set of formulae built accordingly to the following context-free grammar, where* $\psi \in \mathsf{LTL}$*,* $p \in \mathrm{AP}$*, and* $\mathrm{P}_{\mathtt{B}}, \mathrm{P}_{\mathtt{S}} \subseteq \mathrm{AP}$*:*

$$\varphi := \psi \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists p \colon \Theta.\, \varphi \mid \forall p \colon \Theta.\, \varphi;$$
$$\Theta := \left\langle \begin{matrix} \mathtt{B} : \mathrm{P}_{\mathtt{B}} \\ \mathtt{S} : \mathrm{P}_{\mathtt{S}} \end{matrix} \right\rangle.$$

Intuitively, a propositional quantifier of the form $\mathtt{Q}p \colon \left\langle \begin{smallmatrix} \mathtt{B}:\mathrm{P}_{\mathtt{B}} \\ \mathtt{S}:\mathrm{P}_{\mathtt{S}} \end{smallmatrix} \right\rangle$ explicitly expresses a $\mathtt{Q}$-quantification over p, *i.e.*, a choice of a functor to interpret p, that is behavioral *w.r.t.* all the propositions in $\mathrm{P}_{\mathtt{B}}$ and strongly-behavioral *w.r.t.* those in $\mathrm{P}_{\mathtt{S}}$.

To ease the notation, we may write $\mathtt{Q}^{\Theta} p.\, \varphi$ instead of $\mathtt{Q}p \colon \Theta.\, \varphi$, write $\langle \mathtt{B} : \mathrm{P}_{\mathtt{B}} \rangle$ and $\langle \mathtt{S} : \mathrm{P}_{\mathtt{S}} \rangle$ for $\left\langle \begin{smallmatrix} \mathtt{B}:\mathrm{P}_{\mathtt{B}} \\ \mathtt{S}:\emptyset \end{smallmatrix} \right\rangle$ and $\left\langle \begin{smallmatrix} \mathtt{B}:\emptyset \\ \mathtt{S}:\mathrm{P}_{\mathtt{S}} \end{smallmatrix} \right\rangle$, respectively, and $\mathtt{B}$ and $\mathtt{S}$ instead of $\langle \mathtt{B} : \mathrm{AP} \rangle$ and $\langle \mathtt{S} : \mathrm{AP} \rangle$. We also omit the quantifier specification $\left\langle \begin{smallmatrix} \mathtt{B}:\emptyset \\ \mathtt{S}:\emptyset \end{smallmatrix} \right\rangle$, using $\mathtt{Q}p.\, \varphi$ to denote $\mathtt{Q}p \colon \left\langle \begin{smallmatrix} \mathtt{B}:\emptyset \\ \mathtt{S}:\emptyset \end{smallmatrix} \right\rangle.\, \varphi$. Finally, we may drop the curly bracket for the sets $\mathrm{P}_{\mathtt{B}}$ and $\mathrm{P}_{\mathtt{S}}$ and write $\langle \mathtt{B} : p, q \rangle$ instead of $\langle \mathtt{B} : \{p, q\} \rangle$.

We say that a GFG-QPTL formula is *behavioral* (*resp.*, *strongly-behavioral*) if it is in prenex form and all its quantifier specifications are equal to $\mathtt{B}$ (*resp.*, $\mathtt{S}$). We denote by Qn (*resp.*, $\mathrm{Qn}_{\mathtt{B}}$) the set of (*resp.*, behavioral) quantifier prefixes and by Θ the set of quantifier specifications.

Given assignments $\chi_1, \chi_2 \in \mathrm{Asg}(\mathrm{P})$, we write $\chi_1 \sim^k_{\Theta} \chi_2$, for some $\Theta = \langle {}^{\mathtt{B}:\mathrm{P}_{\mathtt{B}}}_{\mathtt{S}:\mathrm{P}_{\mathtt{S}}} \rangle \in \Theta$ and $k \in \mathbb{N}$, if one of the following conditions holds: 1) $\chi_1 = \chi_2$; 2) $\chi_1 \approx^{>k}_{p} \chi_2$, for some $p \in \mathrm{P}_{\mathtt{B}}$; 3) $\chi_1 \approx^{\geq k}_{p} \chi_2$, for some $p \in \mathrm{P}_{\mathtt{S}}$. We denote by $\approx^k_{\Theta}$ the transitive closure of the reflexive and symmetric relation $\sim^k_{\Theta}$.

Proposition 2. *Let* $\mathrm{P} \subseteq \mathrm{AP}$, $\chi_1, \chi_2 \in \mathrm{Asg}(\mathrm{P})$, $\Theta \in \Theta$, *and* $k \in \mathbb{N}$. *Then,* $\chi_1 \approx^k_{\Theta} \chi_2$ iff *the following hold true:*

1) $\chi_1(q) = \chi_2(q)$, *for all* $q \in \mathrm{P} \setminus (\mathrm{P}_{\mathtt{B}} \cup \mathrm{P}_{\mathtt{S}})$*;*
2) $\chi_1(p)(t) = \chi_2(p)(t)$, *for all* $t \leq k$ *and* $p \in (\mathrm{P}_{\mathtt{B}} \cap \mathrm{P}) \setminus \mathrm{P}_{\mathtt{S}}$*;*
3) $\chi_1(p)(t) = \chi_2(p)(t)$, *for all* $t < k$ *and* $p \in \mathrm{P}_{\mathtt{S}} \cap \mathrm{P}$.

$$
\begin{array}{rcl}
 & & \begin{array}{cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & \end{array} \\
\chi_1 & = & \left\{ \begin{array}{cccccccc} p: & \top & \bot & \bot & \top & \bot & \top & \cdots \\ q: & \bot & \bot & \top & \top & \bot & \top & \cdots \end{array} \right\} \\
\chi_2 & = & \left\{ \begin{array}{cccccccc} p: & \top & \bot & \bot & \top & \top & \bot & \cdots \\ q: & \bot & \bot & \top & \top & \bot & \top & \cdots \end{array} \right\} \\
\chi_3 & = & \left\{ \begin{array}{cccccccc} p: & \top & \bot & \bot & \top & \top & \bot & \cdots \\ q: & \bot & \bot & \top & \bot & \top & \top & \cdots \end{array} \right\}
\end{array}
$$

Figure 3. Three $\approx^3_{\Theta}$-equivalent assignments, with $\Theta \triangleq \langle {}^{\mathtt{B}:p}_{\mathtt{S}:q} \rangle$.

Example 3. *Consider the three assignments* χ_1*,* χ_2*, and* χ_3 *over the doubleton* $\{p, q\}$ *depicted in Figure 3. It is easy to see that* $\chi_1 \approx^{>3}_{p} \chi_2$ *and* $\chi_2 \approx^{\geq 3}_{q} \chi_3$*. Therefore,* $\chi_1 \sim^3_{\Theta} \chi_2 \sim^3_{\Theta} \chi_3$*, where* $\Theta \triangleq \langle {}^{\mathtt{B}:p}_{\mathtt{S}:q} \rangle$*, which implies* $\chi_1 \approx^3_{\Theta} \chi_3$*.*

Given a set of propositions $\mathrm{P} \subseteq \mathrm{AP}$ and a quantifier specification $\Theta \triangleq \langle {}^{\mathtt{B}:\mathrm{P}_{\mathtt{B}}}_{\mathtt{S}:\mathrm{P}_{\mathtt{S}}} \rangle \in \Theta$, we introduce the set of *Θ-functors* $\mathrm{Fnc}_{\Theta}(\mathrm{P}) \subseteq \mathrm{Fnc}(\mathrm{P})$ containing exactly those $\mathsf{F} \in \mathrm{Fnc}(\mathrm{P})$ that are behavioral *w.r.t.* all the propositions in $\mathrm{P}_{\mathtt{B}} \cap \mathrm{P}$ and strongly behavioral *w.r.t.* those in $\mathrm{P}_{\mathtt{S}} \cap \mathrm{P}$.

Example 4. *Any* $\langle {}^{\mathtt{B}:p}_{\mathtt{S}:q} \rangle$*-functor* F *replies to all assignments of Figure 3 uniformly, for all time instants between* 0 *and* 3 *included. Indeed,* $\mathsf{F}(\chi_1)(3) = \mathsf{F}(\chi_2)(3)$*, since* $\chi_1 \approx^{>3}_{p} \chi_2$*, being* F *behavioral* w.r.t. p*. Similarly,* $\mathsf{F}(\chi_2)(3) = \mathsf{F}(\chi_3)(3)$*, since* $\chi_2 \approx^{\geq 3}_{q} \chi_3$*, being* F *strongly-behavioral* w.r.t. q*. Hence,* $\mathsf{F}(\chi_1)(3) = \mathsf{F}(\chi_3)(3)$*.*

The following proposition ensures that the above observation highlights a general phenomenon.

Proposition 3. *If* $\chi_1 \approx^k_{\Theta} \chi_2$ *then* $\mathsf{F}(\chi_1)(k) = \mathsf{F}(\chi_2)(k)$*, for all* $\chi_1, \chi_2 \in \mathrm{Asg}(\mathrm{P})$*,* $\Theta \in \Theta$*,* $k \in \mathbb{N}$*, and* $\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathrm{P})$*.*

We can now extend to GFG-QPTL the alternating Hodges semantics of QPTL reported in Definition 3. We simply need to parameterize the extension operation for hyperassignments with the corresponding specification of the behavioral dependencies:

$$\mathsf{ext}_{\Theta}(\mathfrak{X}, p) \triangleq \{ \mathsf{ext}(\mathrm{X}, \mathsf{F}, p) \mid \mathrm{X} \in \mathfrak{X}, \mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X})) \}.$$

Definition 7 (Alternating Hodges Semantics Revisited)**.** *The alternating-Hodges-semantics relation* $\mathfrak{X} \models^{\alpha} \varphi$ *is inductively defined as in Definition 3, for all but Items 5a and 6b that are modified, respectively, as follows, for all propositions* $p \in \mathrm{AP}$ *and quantifier specifications* $\Theta \in \Theta$*:*

5a') $\mathfrak{X} \models^{\exists\forall} \exists p\colon \Theta.\, \phi$ *if* $\mathsf{ext}_{\Theta}(\mathfrak{X}, p) \models^{\exists\forall} \phi$*;*
6b') $\mathfrak{X} \models^{\forall\exists} \forall p\colon \Theta.\, \phi$ *if* $\mathsf{ext}_{\Theta}(\mathfrak{X}, p) \models^{\forall\exists} \phi$*.*

Notice that one could easily extend both the syntax and semantics of the quantifier specification $\langle {}^{\mathtt{B}:\mathrm{P}_{\mathtt{B}}}_{\mathtt{S}:\mathrm{P}_{\mathtt{S}}} \rangle$ of GFG-QPTL in order to accommodate other types of (in)dependence constraints, like the ones already studied in first-order logic of incomplete information [20], [31], [34], [45], [65]. It would suffice to introduce suitable class of functors and corresponding construct, such as the *dependence atoms* of dependence logic, whose semantics can be easily defined via hyperassignments.

For every GFG-QPTL formula φ and alternation flag $\alpha \in \{\exists\forall, \forall\exists\}$, we say that φ is *α-satisfiable* if there exists a hyperassignment $\mathfrak{X} \in \mathrm{HAsg}(\mathsf{free}(\varphi))$ such that $\mathfrak{X} \models^{\alpha} \varphi$. Also, φ *α-implies* (*resp.*, is *α-equivalent* to) a GFG-QPTL formula ϕ, in symbols $\varphi \Rightarrow^{\alpha} \phi$ (*resp.*, $\varphi \equiv^{\alpha} \phi$), whenever $\mathsf{free}(\varphi) = \mathsf{free}(\phi)$ and if $\mathfrak{X} \models^{\alpha} \varphi$ then $\mathfrak{X} \models^{\alpha} \phi$ (*resp.*, $\mathfrak{X} \models^{\alpha} \varphi$ *iff* $\mathfrak{X} \models^{\alpha} \phi$), for all $\mathfrak{X} \in \mathrm{HAsg}_{\subseteq}(\mathsf{free}(\varphi))$. Finally, we say that φ is *satisfiable* if it is both $\exists\forall$- and $\forall\exists$-satisfiable, and write $\varphi \Rightarrow \phi$ (*resp.*, $\varphi \equiv \phi$) if both $\varphi \Rightarrow^{\exists\forall} \phi$ and $\varphi \Rightarrow^{\forall\exists} \phi$ (*resp.*, $\varphi \equiv^{\exists\forall} \phi$ and $\varphi \equiv^{\forall\exists} \phi$) hold.

At this point, let us consider some examples to provide some insight on the expressive power of the new logic.

Example 5. *Let us consider the example* QPTL *formula* $\varphi \triangleq \forall p \exists q.\, \psi$*, with* $\psi \triangleq \mathtt{G}(q \leftrightarrow \mathtt{X}p)$ *from the introduction, expressing a simple automatic synthesis problem. Intuitively, it asks for a temporal valuation of the proposition* q *that precisely mimics the values of proposition* p *one-instant in the future. The sentence is satisfiable, since there exists a functor, the* $\mathsf{F}_{\mathtt{A}}$ *of Example 2, encoding this mimicking technique. Indeed, according to the semantics,* φ *is satisfiable* iff $\mathsf{ext}(\{\mathrm{X}\}, q) \models^{\exists\forall} \psi$ *holds, with* $\mathrm{X} = \mathrm{Asg}(\{p\})$*, the latter being true since* $\mathsf{ext}(\{\mathrm{X}\}, \mathsf{F}_{\mathtt{A}}, q) \in \mathsf{ext}(\{\mathrm{X}\}, q)$*. Obviously,* φ *is not realizable, as* $\mathsf{F}_{\mathtt{A}}$ *is the unique functor satisfying the required property and it is non-behavioral, thus, not implementable by any real transducer. On the contrary, the behavioral* GFG-QPTL *sentence* $\forall^{\mathtt{B}} p \exists^{\mathtt{B}} q.\, \psi$ *is not satisfiable, since no behavioral functor for* q *can look at future values of* p*.*

Example 6. *Unlike the previous example, the* QPTL *sentence* $\exists q.\, \forall p.\, \psi$*, with* $\psi \triangleq p \leftrightarrow \mathtt{X}q$ *is unsatisfiable: Abelard can falsify* ψ *by looking at the value of* q *one instant in the future and choosing the opposite value as the present value for* p*. However, the two* GFG-QPTL *sentences* $\forall^{\mathtt{B}} p.\, \exists^{\mathtt{S}} q.\, \psi$ *and* $\exists^{\mathtt{B}} q.\, \forall^{\mathtt{B}} p.\, \psi$ *are both satisfiable. For the first one, it is enough to observe that the strongly-behavioral functor* $\mathsf{F}_{\mathtt{S}}$ *of Example 2 allows to mimic any temporal valuation assigned to the proposition* p *one-instant in the past, as required by the* LTL *property* ψ*. For the second one, we need to show that,* $\mathsf{ext}_{\mathtt{B}}(\{\mathrm{Y}\}, p) \models^{\forall\exists} \psi$*, with* $\mathrm{Y} = \mathrm{Asg}(\{q\})$*. Now, let* $\mathrm{X} \in \mathsf{ext}_{\mathtt{B}}(\{\mathrm{Y}\}, p)$ *be an arbitrary set of assignments obtained by extending those in* Y *as prescribed by the specification* $\mathtt{B}$*. Also, consider* $\chi_1, \chi_2 \in \mathrm{X}$ *as two of those assignments that differs on* q *at time* 1*, but are equal at time* 0*,* i.e.*,* $\chi_1(q)(0) = \chi_2(q)(0)$*, but* $\chi_1(q)(1) \neq \chi_2(q)(1)$*. Due to the* *required behavioralness* w.r.t. q *of the functors used in the extension of* Υ*, we necessarily have that* $\chi_1(p)(0) = \chi_2(p)(0)$*. As a consequence, either one between* χ_1 *and* χ_2 *satisfies* ψ*, as required by Item 1b of the semantics. In other words, Abelard is no longer allowed to look at the value of* q *in the future. Note that* $\exists^{\mathsf{B}} q. \forall^{\mathsf{B}} p. \psi$ *could not be expressed with an asymmetric Hodges-like semantics, as it cannot restrict the universal quantifiers.*

Example 7. *Information leaks via quantification of unused variables is a well-known phenomenon in* IF *[65]. The same occurs in* GFG-QPTL*, as the (in)equivalences below show:*

$$\forall p. \exists u. \exists^{\mathsf{B}} q. \phi \equiv \forall p. \exists q. \phi \not\equiv \forall p. \exists^{\mathsf{B}} q. \phi \equiv \forall p. \exists^{\mathsf{B}} u. \exists^{\mathsf{B}} q. \phi;$$

where $p, q \in \mathsf{free}(\phi)$*, but* $u \notin \mathsf{free}(\phi)$*. Indeed, an arbitrary functor* G_q *for* q *in* $\forall p. \exists q. \phi$ *can be simulated in* $\forall p. \exists u. \exists^{\mathsf{B}} q. \phi$ *by the functors* $\mathsf{F}_u = \mathsf{G}_q$*, for* u*, and* $\mathsf{F}_q(\chi) = \chi(u)$*, for* q*. Clearly,* F_q*, being the identity on* u*, is behavioral. Intuitively, the unused non-behaviorally-quantified proposition* u *leaks information about the future of* p *to* q *even if the latter is behaviorally quantified, as it can see the future of* p *through the present of* u*. This does not happen, however, if* u *is forced to be behavioral (*resp.*, strongly-behavioral). Indeed, the behavioral (*resp.*, strongly-behavioral) fragments of* GFG-QPTL *enjoys the classic property of elimination of unused propositions.*

The following example expands a bit more on the connection between GFG-QPTL and GFG-Automata briefly mentioned in the introduction and shows that GFG-QPTL can express the property of being good-for-game for an automaton.

Example 8. *It is well known that* QPTL *is able to express any* ω*-regular language [63]. This can be proved by encoding the existence of an accepting run of an arbitrary nondeterministic Büchi word automaton* $\mathcal{N}$ *via a formula* $\varphi \triangleq \exists s_1 \ldots \exists s_k. \psi$*, where* $\mathsf{free}(\varphi) = \{p_1, \ldots, p_n\}$ *is the set of propositions of the recognized language,* $s_1, \ldots, s_k$ *are mutually exclusive fresh propositions representing the* k *states of* $\mathcal{N}$*, and* ψ *is the* LTL *formula encoding the transition function and the acceptance condition. Via the behavioral* GFG-QPTL *formula* $\varphi_{\mathsf{B}} \triangleq \exists^{\mathsf{B}} s_1 \ldots \exists^{\mathsf{B}} s_k. \psi$ *we can identify precisely the sublanguages recognized by* $\mathcal{N}$ *when the nondeterminism is resolved in a* good-for-game *manner [26]. Therefore, the* GFG-QPTL *sentence* $\forall p_1 \ldots \forall p_n. (\varphi \leftrightarrow \varphi_{\mathsf{B}})$ *is satisfiable* iff $\mathcal{N}$ *is a good-for-game automaton.*

B. Model-Theoretic Analysis

Let us proceed with an elementary *model-theoretic analysis* of GFG-QPTL, showing that it enjoys several basic properties, like *De Morgan laws*, one would expect from a classic logic.

We start by observing the *monotonicity* of both the dualization and extension operators *w.r.t.* the preorder $\sqsubseteq$, a simple property that is a key tool in all subsequent statements.

Proposition 4. *Let* $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathrm{HAsg}$ *with* $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$*: 1)* $\overline{\mathfrak{X}_2} \sqsubseteq \overline{\mathfrak{X}_1}$*; 2)* $\mathsf{ext}_\Theta(\mathfrak{X}_1, p) \sqsubseteq \mathsf{ext}_\Theta(\mathfrak{X}_2, p)$*, for all* $p \in \mathrm{AP}$ *and* $\Theta \in \mathbb{O}$*.*

The preorder $\sqsubseteq$ between hyperassignments captures the intuitive notion of satisfaction strength *w.r.t.* GFG-QPTL formulae. Indeed, by analyzing Item 1 of Definition 3, it is not hard to see that if $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$, then the hyperassignment $\mathfrak{X}_1$ satisfies less (*resp.*, more) LTL formulae than the hyperassignment $\mathfrak{X}_2$ does *w.r.t.* the $\exists\forall$ (*resp.*, $\forall\exists$) semantics, *i.e.*, if $\mathfrak{X}_1$ (*resp.*, $\mathfrak{X}_2$) satisfies ψ, then $\mathfrak{X}_2$ (*resp.*, $\mathfrak{X}_1$) does the same. This property can easily be lifted to arbitrary GFG-QPTL formulae, by a standard structural induction using the monotonicity of the dualization and extension operators.

Theorem 2 (Hyperassignment Refinement)**.** *Let* φ *be a* GFG-QPTL *formula and* $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathrm{HAsg}_{\subseteq}(\mathsf{free}(\varphi))$ *with* $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$*. Then,* $\mathfrak{X}_1 \models^{\exists\forall} \varphi$ *implies* $\mathfrak{X}_2 \models^{\exists\forall} \varphi$ *and* $\mathfrak{X}_2 \models^{\forall\exists} \varphi$ *implies* $\mathfrak{X}_1 \models^{\forall\exists} \varphi$*.*

As an immediate consequence, we obtain the following result.

Corollary 1 (Hyperassignment Equivalence)**.** *Let* φ *be a* GFG-QPTL *formula and* $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathrm{HAsg}_{\subseteq}(\mathsf{free}(\varphi))$ *with* $\mathfrak{X}_1 \equiv \mathfrak{X}_2$*. Then,* $\mathfrak{X}_1 \models^\alpha \varphi$ iff $\mathfrak{X}_2 \models^\alpha \varphi$*.*

A fundamental feature of the proposed alternating semantics is the *duality* between swapping the players of a hyperassignment $\mathfrak{X}$, *i.e.*, swapping the alternation flag, and swapping the choices of the players, *i.e.*, dualizing $\mathfrak{X}$. Indeed, the following results states that dualizing both the alternation flag α and the hyperassignment preserves the truth of any formula. This also implies, as one might expect, that double dualization has no effect either. The latter fact is also a consequence of the previous corollary, since $\mathfrak{X} \equiv \overline{\overline{\mathfrak{X}}}$, due to Proposition 1.

Theorem 3 (Duality)**.** *Let* φ *be a* GFG-QPTL *formula and* $\mathfrak{X} \in \mathrm{HAsg}_{\subseteq}(\mathsf{free}(\varphi))$*. Then,* $\overline{\mathfrak{X}} \models^{\overline{\alpha}} \varphi$ iff $\mathfrak{X} \models^\alpha \varphi$ iff $\overline{\overline{\mathfrak{X}}} \models^\alpha \varphi$*.*

The duality property also grants that formulae satisfaction and equivalence do not depend on the specific interpretation α of hyperassignments: a positive answer for α implies the same for $\overline{\alpha}$. This *invariance* corresponds to the intuition that Eloise and Abelard both agree on the true and false formulae. Similarly, if φ is considered to be equivalent to, or to imply, some other property ϕ by Eloise, the same equivalence, or implication, holds for Abelard as well, and *vice versa*.

Corollary 2 (Interpretation Invariance)**.** *Let* φ *and* ϕ *be* GFG-QPTL *formulae.* φ *is* $\exists\forall$*-satisfiable* iff φ *is* $\forall\exists$*-satisfiable. Also,* $\varphi \Rightarrow^{\exists\forall} \phi$ iff $\varphi \Rightarrow^{\forall\exists} \phi$ *and* $\varphi \equiv^{\exists\forall} \phi$ iff $\varphi \equiv^{\forall\exists} \phi$*.*

Thanks to this invariance, the following Boolean laws hold.

Lemma 4 (Boolean Laws)**.** *Let* φ*,* φ_1*,* φ_2 *be* GFG-QPTL *formulae: 1)* $\varphi \equiv \neg\neg\varphi$*; 2)* $\varphi_1 \wedge \varphi_2 \Rightarrow \varphi_1$*; 3)* $\varphi_1 \Rightarrow \varphi_1 \vee \varphi_2$*; 4)* $\varphi_1 \wedge (\varphi \wedge \varphi_2) \equiv (\varphi_1 \wedge \varphi) \wedge \varphi_2$*; 5)* $\varphi_1 \vee (\varphi \vee \varphi_2) \equiv (\varphi_1 \vee \varphi) \vee \varphi_2$*; 6)* $\varphi_1 \wedge \varphi_2 \equiv \neg(\neg\varphi_1 \vee \neg\varphi_2)$*; 7)* $\varphi_1 \vee \varphi_2 \equiv \neg(\neg\varphi_1 \wedge \neg\varphi_2)$*; 8)* $\exists^\Theta p. \varphi \equiv \neg(\forall^\Theta p. \neg\varphi)$*; 9)* $\forall^\Theta p. \varphi \equiv \neg(\exists^\Theta p. \neg\varphi)$*.*

Unlike QPTL, GFG-QPTL in its full generality does not enjoy the *pnf* property. This is a consequence of the information-leak phenomenon reported in Example 7. Indeed,

$(\exists p.\, \phi) \wedge \varphi \not\equiv \exists p.\, (\phi \wedge \varphi)$ and $(\forall p.\, \phi) \vee \varphi \not\equiv \forall p.\, (\phi \vee \varphi)$, when $p \notin \mathsf{free}(\varphi)$. A similar problem arises in IF due to *signaling*, if one let quantifications depend on non-free variables [45]. Fortunately, for the purposes of this work, we can focus on *pnf* formulae, since, as we shall show, behavioral $\mathsf{GFG\text{-}QPTL}$ is powerful enough to express all ω-regular languages.

We now introduce an operator on quantifier prefixes, called *evolution*, that, given an arbitrary hyperassignment $\mathfrak{X}$ and one of its two interpretations α, computes the result $\mathsf{evl}_\alpha(\mathfrak{X}, \wp)$ of the application to $\mathfrak{X}$ of all quantifiers $\mathtt{Q}^\Theta p$ occurring in a prefix $\wp$ in that specific order. To this aim, we need to introduce the notion of *coherence* of a quantifier symbol $\mathtt{Q} \in \{\exists, \forall\}$ *w.r.t.* an alternation flag $\alpha \in \{\exists\forall, \forall\exists\}$ as follows: $\mathtt{Q}$ is α-*coherent* if either $\alpha = \exists\forall$ and $\mathtt{Q} = \exists$ or $\alpha = \forall\exists$ and $\mathtt{Q} = \forall$. Essentially, the evolution operator iteratively applies the semantics of quantifiers, as defined by Items 5a' and 6b' of Definition 7 and Items 5b and 6a of Definition 3, for all the quantifiers $\mathtt{Q}^\Theta p$ in the input prefix $\wp$. For a single quantifier, $\mathsf{evl}_\alpha(\mathfrak{X}, \mathtt{Q}^\Theta p)$ just corresponds to the Θ-extension of $\mathfrak{X}$ with p, when $\mathtt{Q}$ is α-coherent. On the contrary, when $\mathtt{Q}$ is $\overline{\alpha}$-coherent, we need to dualize the Θ-extension with p of the dual of $\mathfrak{X}$.

$$\mathsf{evl}_\alpha(\mathfrak{X}, \mathtt{Q}^\Theta p) \triangleq \begin{cases} \mathsf{ext}_\Theta(\mathfrak{X}, p), & \text{if } \mathtt{Q} \text{ is } \alpha\text{-coherent;} \\ \overline{\mathsf{ext}_\Theta(\overline{\mathfrak{X}}, p)}, & \text{otherwise.} \end{cases}$$

The operator lifts naturally to an arbitrary quantification prefix $\wp \in \mathrm{Qn}$ as follows: 1) $\mathsf{evl}_\alpha(\mathfrak{X}, \epsilon) \triangleq \mathfrak{X}$; 2) $\mathsf{evl}_\alpha(\mathfrak{X}, \mathtt{Q}^\Theta p.\, \wp) \triangleq \mathsf{evl}_\alpha(\mathsf{evl}_\alpha(\mathfrak{X}, \mathtt{Q}^\Theta p), \wp)$. We also set $\mathsf{evl}_\alpha(\wp) \triangleq \mathsf{evl}_\alpha(\{\{\varnothing\}\}, \wp)$.

It is easy to show that the evolution operator is monotone *w.r.t.* $\sqsubseteq$, by simply exploiting the monotonicity of the dualization and extension operators given in Proposition 4.

Proposition 5. *Let* $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathrm{HAsg}$ *with* $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$ *and* $\wp \in \mathrm{Qn}$*:* $\mathsf{evl}_\alpha(\mathfrak{X}_1, \wp) \sqsubseteq \mathsf{evl}_\alpha(\mathfrak{X}_2, \wp)$.

By simple structural induction on a quantifier prefix $\wp \in \mathrm{Qn}$, we can show that a hyperassignment $\mathfrak{X}$ α-satisfies a formula $\wp\phi$ *iff* its α-evolution *w.r.t.* $\wp$ α-satisfies ϕ.

Lemma 5 (Prefix Evolution)**.** *Let* $\wp\phi$ *be a* $\mathsf{GFG\text{-}QPTL}$ *formula with* $\wp \in \mathrm{Qn}$*. Then,* $\mathfrak{X} \models^\alpha \wp\phi$ iff $\mathsf{evl}_\alpha(\mathfrak{X}, \wp) \models^\alpha \phi$*, for all* $\mathfrak{X} \in \mathrm{HAsg}(\mathsf{free}(\wp\phi))$.

IV. Quantification Games

The solution of the satisfiability problem for the behavioral fragment of $\mathsf{GFG\text{-}QPTL}$ relies on the existence of a game, played by Eloise and Abelard, with the property that Eloise wins the game *iff* the corresponding formula is indeed satisfiable. We provide here a general result, showing that, for any quantifier prefix $\wp$ and Borelian property Ψ, there exists a game, called *quantification game*, such that Eloise wins the game *iff* the hyperassignment obtained by evaluating the prefix, namely $\mathsf{evl}_{\exists\forall}(\wp)$, contains a set of assignments completely included in Ψ. The correctness of this result depends, in turn, on the existence of canonical forms for the quantifier prefixes that allow one to reduce the alternations to at most one.

A. *Quantification Game for Sentences*

To define the quantification game, we first need a few notions. A two-player turn-based *arena* $\mathcal{A} = \langle \mathrm{Ps}_\mathtt{E}, \mathrm{Ps}_\mathtt{A}, v_I, Mv \rangle$ is a tuple where 1) $\mathrm{Ps}_\mathtt{E}$ and $\mathrm{Ps}_\mathtt{A}$ are the sets of *positions* of *Eloise* and *Abelard*, *a.k.a. Player* and *Opponent*, respectively, with $\mathrm{Ps}_\mathtt{E} \cap \mathrm{Ps}_\mathtt{A} = \emptyset$, 2) $v_I \in \mathrm{Ps} \triangleq \mathrm{Ps}_\mathtt{E} \cup \mathrm{Ps}_\mathtt{A}$ is the *initial position*, and 3) $Mv \subseteq \mathrm{Ps} \times \mathrm{Ps}$ is the binary relation describing all possible *moves* such that $\langle \mathrm{Ps}, Mv \rangle$ is a sinkless directed graph. A *game* $\mathcal{G} = \langle \mathcal{A}, \mathrm{Ob}, \mathrm{Wn} \rangle$ is a tuple where $\mathcal{A}$ is an arena, $\mathrm{Ob} \subseteq \mathrm{Ps}$ is the set of *observable positions*, and $\mathrm{Wn} \subseteq \mathrm{Ob}^\omega$ is the set of *observable sequences* that are *winning* for Eloise; the complement $\overline{\mathrm{Wn}} \triangleq \mathrm{Ob}^\omega \setminus \mathrm{Wn}$ is *winning* for Abelard. Eloise (*resp.*, Abelard) *wins* the game if she (*resp.*, he) has a strategy such that, for all adversary strategies, the corresponding play induces an observation sequence belonging (*resp.*, not belonging) to Wn. All notions necessary to formalize this intuition, as *history*, *strategy*, and *play*, are given in Appendix C.

Martin's determinacy theorem [46], [47] states that all games whose winning condition is a Borel set in the Cantor topological space of infinite words [54] are determined, *i.e.*, either one of the two players necessarily wins the game. To ensures that the quantification game we are about to define is indeed determined, we require a form of Borelian condition that can be applied to sets of assignments. This determinacy requirement is crucial here, since it is tightly connected with the fact that $\mathsf{GFG\text{-}QPTL}$ does not allow for undetermined formulae. To this end, let $\mathrm{Val} \triangleq \mathrm{AP} \rightharpoonup \mathbb{B}$ denote the set of Boolean valuations for sets of propositions and $\mathrm{Val}(\mathrm{P}) \triangleq \{\xi \in \mathrm{Val} \mid \mathsf{dom}(\xi) = \mathrm{P}\}$ the set of valuations for propositions in $\mathrm{P} \subseteq \mathrm{AP}$. Also, $\#(\xi) \triangleq |\mathsf{dom}(\xi)|$. We can now define a bijection between sets of assignments over P and languages of infinite words over the alphabet $\mathrm{Val}(\mathrm{P})$. Let $\mathsf{wrd}\colon \mathrm{Asg}(\mathrm{P}) \to \mathrm{Val}(\mathrm{P})^\omega$ be the *word function* mapping each assignment $\chi \in \mathrm{Asg}(\mathrm{P})$ to the word $w \triangleq \mathsf{wrd}(\chi) \in \mathrm{Val}(\mathrm{P})^\omega$ satisfying the equality $\chi(p)(t) = (w)_t(p)$, for all $p \in \mathrm{P}$ and $t \in \mathbb{N}$. Clearly wrd is a bijection. Now, every property $\Psi \subseteq \mathrm{Asg}(\mathrm{P})$, *i.e.*, every set of assignments, uniquely induces the language of infinite words $\mathsf{wrd}(\Psi) \triangleq \{\, \mathsf{wrd}(\chi) \mid \chi \in \Psi \,\} \subseteq \mathrm{Val}(\mathrm{P})^\omega$ over the alphabet $\mathrm{Val}(\mathrm{P})$. Thus, Ψ is said to be *Borelian* (*resp.*, regular) if the language $\mathsf{wrd}(\Psi)$ is a Borel (*resp.*, regular) set.

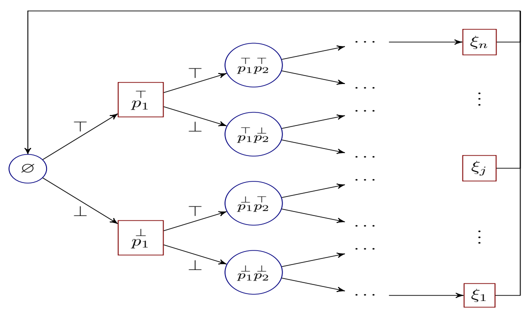

Figure 4. Quantification game for $\wp = \exists^\mathsf{B} p_1.\, \forall^\mathsf{B} p_2.\, \exists^\mathsf{B} p_3 \cdots$. Eloise owns the circled positions, while Abelard the squared ones. From the total-valuation positions $\xi_1, \ldots, \xi_n$, with $n = 2^{|\mathsf{ap}(\wp)|}$, Abelard moves to the initial position.

Given a behavioral sentence $\wp\psi$, let $\mathsf{L}(\psi) \subseteq \mathrm{Asg}(\mathsf{ap}(\wp))$ denote the set of assignments satisfying the LTL formula ψ. The quantification game $\mathcal{G}_{\wp}^{\psi} \triangleq \mathcal{G}_{\wp}^{\mathsf{L}(\psi)}$ is defined in Construction 1 and exemplified in Figure 4. *W.l.o.g.*, we assume that the prefix $\wp$ does not contain duplicates. The positions of the game are (partial) valuations of the propositions in $\wp$ and each position belongs to the player corresponding to the first quantifier in the prefix whose proposition is not defined at that position. The initial position of the game contains the empty valuation and in the example of Figure 4 belongs to Eloise, she being the first to play in $\wp$. Obviously, the game features an infinite number of rounds. Each round begins with the empty valuation and ends in a total valuation, after players have chosen (jointly) a value for all the propositions. A move in the round corresponds to a player choosing a value for the next proposition in the prefix. Take, for instance, position $\xi \triangleq \{p_1 \mapsto \bot\}$ in the figure, where the first proposition p_1 has been already assigned value $\bot$ by Eloise. From that position, Abelard first chooses a Boolean value, say $\top$, for the next proposition p_2 in the prefix. Then he moves to the position $\xi' \triangleq \{p_1 \mapsto \bot, p_2 \mapsto \top\}$, corresponding to the valuation $\xi[p_2 \mapsto \top]$, obtained by extending ξ with the value chosen for p_2. Position ξ' belongs to Eloise, since the next quantifier $\exists p_3$ in the prefix is existential. The last positions belong to Abelard and, from there, he can only move back to the starting position for the next turn. By sampling any infinite sequence of rounds of the games at the positions with total valuations, namely the observable positions, we obtain an infinite word w corresponding to some assignment $\chi \triangleq \mathsf{wrd}^{-1}(w)$. Then, w is winning for Eloise *iff* χ belongs to $\Psi \triangleq \mathsf{L}(\psi)$ (*i.e.*, $\chi \models \psi$), while it is winning for Abelard otherwise. This intuition is formalized by the following construction.

Construction 1 (Quantification Game I). *For every quantifier prefix $\wp \in \mathrm{Qn}_{\mathtt{B}}$ and property $\Psi \subseteq \mathrm{Asg}(\mathsf{ap}(\wp))$, the game $\mathcal{G}_{\wp}^{\Psi} \triangleq \langle \mathcal{A}_{\wp}, \mathrm{Ob}, \mathrm{Wn} \rangle$ with arena $\mathcal{A}_{\wp} \triangleq \langle \mathrm{Ps}_{\mathtt{E}}, \mathrm{Ps}_{\mathtt{A}}, v_I, Mv \rangle$ is defined as prescribed in the following:*

- *the set of positions $\mathrm{Ps} \subset \mathrm{Val}$ contains exactly those valuations $\xi \in \mathrm{Val}$ of the propositions in $\mathsf{ap}(\wp)$ that are quantified in the prefix $(\wp)_{<\#(\xi)}$ of $\wp$ having length $\#(\xi)$* i.e., $\mathsf{dom}(\xi) = \mathsf{ap}\big((\wp)_{<\#(\xi)}\big)$*;*
- *the set of Eloise's positions $\mathrm{Ps}_{\mathtt{E}} \subseteq \mathrm{Ps}$ only contains the valuations $\xi \in \mathrm{Ps}$ for which the proposition quantified in $\wp$ at index $\#(\xi)$ is existentially quantified,* i.e., $(\wp)_{\#(\xi)} = \exists^{\mathsf{B}} p$*, for some $p \in \mathsf{ap}(\wp)$;*
- *the initial position $v_I \triangleq \varnothing$ is just the empty valuation;*
- *the move relation $Mv \subseteq \mathrm{Ps} \times \mathrm{Ps}$ contains exactly those pairs of valuations $(\xi_1, \xi_2) \in \mathrm{Ps} \times \mathrm{Ps}$ such that:*
  - *$\xi_1 \subseteq \xi_2$ and $\#(\xi_2) = \#(\xi_1) + 1$, or*
  - *$\xi_1 \in \mathrm{Val}(\mathsf{ap}(\wp))$ and $\xi_2 = \varnothing$;*
- *the set of observable positions $\mathrm{Ob} \triangleq \mathrm{Val}(\mathsf{ap}(\wp))$ precisely contains the valuations of all the propositions in $\wp$;*
- *the winning condition induced by the property Ψ is the language of infinite words $\mathrm{Wn} \triangleq \mathsf{wrd}(\Psi)$ over $\mathrm{Val}(\mathsf{ap}(\wp))$.*

The game $\mathcal{G}_{\wp}^{\psi}$ above essentially provides a game-theoretic version of the semantics of behavioral quantifications. The correctness of the game is established by the following theorem.

Theorem 4 (Game-Theoretic Semantics I). *A behavioral* $\mathsf{GFG\text{-}QPTL}$ *sentence $\wp\psi$, with $\psi \in \mathsf{LTL}$, is satisfiable (*resp.*, unsatisfiable)* iff *Eloise (*resp.*, Abelard) wins $\mathcal{G}_{\wp}^{\psi}$.*

The proof of this result is split into the following three steps. First, for an arbitrary behavioral quantifier prefix $\wp$, we provide two transformations, $\mathtt{C}_{\exists\forall}(\wp)$ and $\mathtt{C}_{\forall\exists}(\wp)$, called *canonizations*, which allow one to reduce a behavioral $\mathsf{GFG\text{-}QPTL}$ sentence $\varphi = \wp\psi$ to the sentences $\mathtt{C}_{\exists\forall}(\wp)\psi$ and $\mathtt{C}_{\forall\exists}(\wp)\psi$ featuring at most one quantifier alternation. Second, in Theorem 5, we connect the winner of the game $\mathcal{G}_{\wp}^{\psi}$ with the satisfiability of one of the normal forms $\mathtt{C}_{\exists\forall}(\wp)\psi$ and $\mathtt{C}_{\forall\exists}(\wp)\psi$, showing also that $\mathtt{C}_{\exists\forall}(\wp)\psi$ implies $\mathtt{C}_{\forall\exists}(\wp)\psi$. Finally, in Theorem 6, we prove that the original sentence φ is equisatisfiable with the two normal forms.

Let us start with the definition of the two prefix canonizations based on the following syntactic quantifier-swap operations. Consider, *e.g.*, the formula $\forall^{\mathsf{B}} p.\, \exists^{\mathsf{B}} q.\, \phi$. A naïve quantifier-swap operator would simply swap the two quantifiers that, in game-theoretic terms, corresponds to a swap in the choices of the two players, which allows Abelard to see Eloise's move at the current round. To balance this additional power, we simply restrict the universal quantifier to be strictly behavioral, thus preventing Abelard from reading Eloise's choice. This leads to the formula $\exists^{\mathsf{B}} q.\, \forall^{\left\langle \substack{\mathtt{B}:\mathrm{AP} \\ \mathtt{S}:q} \right\rangle} p.\, \psi$. A symmetric swap operation would transform the formula $\exists^{\mathsf{B}} q.\, \forall^{\mathsf{B}} p.\, \phi$ into $\forall^{\mathsf{B}} p.\, \exists^{\left\langle \substack{\mathtt{B}:\mathrm{AP} \\ \mathtt{S}:p} \right\rangle} q.\, \phi$. Essentially, the swap operation exchange the positions of two adjacent dual behavioral quantifiers and restrict one of the two to be strongly behavioral *w.r.t.* the proposition of the other one. By iteratively swapping adjacent quantifiers and adjusting the quantifier specification accordingly, we can reduce the quantifier alternation to at most one.

For technical convenience we use a vector notation for the quantifier prefixes: $\mathtt{Q}^{\vec{\Theta}} \vec{p}.\, \phi \triangleq \mathtt{Q}^{(\vec{\Theta})_0} (\vec{p})_0 . \cdots \mathtt{Q}^{(\vec{\Theta})_k} (\vec{p})_k .\, \phi$, where $|\vec{p}| = |\vec{\Theta}| = k + 1$. We omit the vector symbol in $\vec{\Theta}$ if this is just a sequence of $\mathtt{B}$ or $\mathtt{S}$ specifications and consider $\vec{p}$ as sets of propositions when convenient. We also define in a natural way the union of two quantifier specifications as follows:

$$\left\langle \substack{\mathtt{B}:\mathrm{P}_{\mathtt{B}1} \\ \mathtt{S}:\mathrm{P}_{\mathtt{S}1}} \right\rangle \cup \left\langle \substack{\mathtt{B}:\mathrm{P}_{\mathtt{B}2} \\ \mathtt{S}:\mathrm{P}_{\mathtt{S}2}} \right\rangle \triangleq \left\langle \substack{\mathtt{B}:\mathrm{P}_{\mathtt{B}1} \cup \mathrm{P}_{\mathtt{B}2} \\ \mathtt{S}:\mathrm{P}_{\mathtt{S}1} \cup \mathrm{P}_{\mathtt{S}2}} \right\rangle.$$

We now introduce the two syntactic transformations $\mathtt{C}_{\exists\forall}(\cdot)$ and $\mathtt{C}_{\forall\exists}(\cdot)$ that, given a behavioral quantifier prefix $\wp \in \mathrm{Qn}_{\mathtt{B}}$, return the single-alternation ones $\mathtt{C}_{\exists\forall}(\wp)$ and $\mathtt{C}_{\forall\exists}(\wp)$, by applying all the quantifier swap operations at once. More specifically, the function $\mathtt{C}_{\exists\forall}(\cdot)$ provides an $\exists\forall$-*prefix*, *i.e.*, all existential quantifier precede the universal ones, while $\mathtt{C}_{\forall\exists}(\cdot)$ gives us the the dual $\forall\exists$-*prefix*.

For the definition of $\mathtt{C}_{\exists\forall}(\cdot)$, we observe that every behavioral quantifier prefix $\wp$ can be uniquely rewritten as $\exists^{\mathsf{B}} \vec{q}_0 .\, (\forall^{\mathsf{B}} \vec{p}_i .\, \exists^{\mathsf{B}} \vec{q}_i)_{i=1}^{k} .\, \forall^{\mathsf{B}} \vec{p}_{k+1}$, for some $k \in \mathbb{N}$ and vectors $\vec{q}_i$, with $i \in [0, k]$, and $\vec{p}_i$, with $i \in [1, k+1]$, where $|\vec{q}_i|, |\vec{p}_i| \geq 1$, for all $i \in [1, k]$. For a quantifier prefix $\wp$ in such a form, we then define

$$\mathtt{C}_{\exists\forall}(\wp) \triangleq (\exists^{\mathtt{B}} \vec{q}_i)_{i=0}^{k} . (\forall^{\vec{\Theta}_i} \vec{p}_i)_{i=1}^{k+1},$$

where $(\vec{\Theta}_i)_j \triangleq \mathtt{B} \cup \langle \mathtt{S} : \vec{q}_i \cdots \vec{q}_k \rangle$, for all $j \in [0, |\vec{p}_i|)$.

The definition of $\mathtt{C}_{\forall\exists}(\cdot)$ is analogous. First, we rewrite a behavioral quantifier prefix $\wp$ as $\forall^{\mathtt{B}} \vec{p}_0 . (\exists^{\mathtt{B}} \vec{q}_i . \forall^{\mathtt{B}} \vec{p}_i)_{i=1}^{k} . \exists^{\mathtt{B}} \vec{q}_{k+1}$, for some $k \in \mathbb{N}$ and vectors $\vec{p}_i$, with $i \in [0, k]$, and $\vec{q}_i$, with $i \in [1, k+1]$, where $|\vec{p}_i|, |\vec{q}_i| \geq 1$, for all $i \in [1, k]$. Then, we define

$$\mathtt{C}_{\forall\exists}(\wp) \triangleq (\forall^{\mathtt{B}} \vec{p}_i)_{i=0}^{k} . (\exists^{\vec{\Theta}_i} \vec{q}_i)_{i=1}^{k+1},$$

where $(\vec{\Theta}_i)_j \triangleq \mathtt{B} \cup \langle \mathtt{S} : \vec{p}_i \cdots \vec{p}_k \rangle$, for all $j \in [0, |\vec{q}_i|)$.

Example 9. *Consider the behavioral quantifier prefix* $\wp = \forall^{\mathtt{B}} p . \exists^{\mathtt{B}} q\, r . \forall^{\mathtt{B}} s . \exists^{\mathtt{B}} t$. *The corresponding* $\exists\forall$ *canonical-form is* $\mathtt{C}_{\exists\forall}(\wp) = \exists^{\mathtt{B}} q\, r\, t . \forall^{\Theta^p} p . \forall^{\Theta^s} s$, *where* $\Theta^p \triangleq \left\langle \begin{smallmatrix} \mathtt{B}:\mathrm{AP} \\ \mathtt{S}:q\,r\,t \end{smallmatrix} \right\rangle$ *and* $\Theta^s \triangleq \left\langle \begin{smallmatrix} \mathtt{B}:\mathrm{AP} \\ \mathtt{S}:t \end{smallmatrix} \right\rangle$. *The* $\forall\exists$ *canonical-form prefix is, instead,* $\mathtt{C}_{\forall\exists}(\wp) = \forall^{\mathtt{B}} p\, s . \exists^{\Theta} q\, r . \exists^{\mathtt{B}} t$, *where* $\Theta \triangleq \left\langle \begin{smallmatrix} \mathtt{B}:\mathrm{AP} \\ \mathtt{S}:s \end{smallmatrix} \right\rangle$.

For the second part of the proof of Theorem 4, we need to connect the winner of $\mathcal{G}_{\wp}^{\psi}$ with the satisfiability of (one among) $\mathtt{C}_{\exists\forall}(\wp)\psi$ and $\mathtt{C}_{\forall\exists}(\wp)\psi$. This also corresponds to showing that $\mathtt{C}_{\exists\forall}(\wp)\psi \Rightarrow \mathtt{C}_{\forall\exists}(\wp)\psi$. To this end, we exploit the ω-regularity of LTL languages, which ensures that the game is Borelian.

Theorem 5 (Quantification Game I). *For each behavioral quantification prefix* $\wp \in \mathrm{Qn}_{\mathtt{B}}$ *and Borelian property* $\Psi \subseteq \mathrm{Asg}(\mathsf{ap}(\wp))$, *the game* $\mathcal{G}_{\wp}^{\Psi}$ *satisfies the following two properties:*

1) *if Eloise wins then* $\mathrm{E} \subseteq \Psi$, *for some* $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathtt{C}_{\forall\exists}(\wp))$*;*
2) *if Abelard wins then* $\mathrm{E} \not\subseteq \Psi$, *for all* $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathtt{C}_{\exists\forall}(\wp))$.

The idea of the proof is to extract, from a winning strategy of Eloise (*resp.*, Abelard), a vector $\vec{\mathsf{F}}$ of functors, one for each proposition associated with that player, witnessing the existence (*resp.*, non-existence) of a set E of assignments satisfying the property Ψ. More precisely, assume Eloise has a strategy σ to win the game and let $\forall^{\mathtt{B}} \vec{p} . \exists^{\vec{\Theta}} \vec{q} = \mathtt{C}_{\forall\exists}(\wp)$ be the $\forall\exists$ canonical-form of $\wp$. Then, thanks to the bijection between plays π and assignments χ, we can operate as follows, for every round k and existential proposition q_i in $\vec{q}$: given Abelard's choices up to round k in π, we can extract, from Eloise's response for q_i in σ, the response to χ at time k of the functor F_i in $\vec{\mathsf{F}}$. As a consequence, for all $\chi \in \mathrm{Asg}(\vec{p})$ chosen by Abelard, Eloise's response corresponding to the extension of χ with $\vec{\mathsf{F}}$ on $\vec{q}$ satisfies, *i.e.*, belongs to, the property Ψ. The witness E is precisely the set of all those extensions. An analogous argument applies to Abelard for the $\exists\forall$ canonical-form. Notice that $\vec{\mathsf{F}}$ meets the specification $\vec{\Theta}$ thanks to the alternation of the players prescribed by $\wp$ in each round of $\mathcal{G}_{\wp}^{\Psi}$.

The final step establishes the equisatisfiability of a behavioral $\mathsf{GFG\text{-}QPTL}$ sentence with both its canonical forms.

Theorem 6 (Sentence Canonical Forms). *For every behavioral* $\mathsf{GFG\text{-}QPTL}$ *sentence* $\wp\psi$, *with* $\psi \in \mathsf{LTL}$, *it holds that* $\wp\psi$, $\mathtt{C}_{\exists\forall}(\wp)\psi$, *and* $\mathtt{C}_{\forall\exists}(\wp)\psi$ *are equisatisfiable.*

Towards the proof, we can derive the chain of implications $\mathtt{C}_{\forall\exists}(\wp)\psi \Rightarrow \wp\psi \Rightarrow \mathtt{C}_{\exists\forall}(\wp)\psi$ by exploiting the following property of the evolution function. Specifically, this asserts a total ordering *w.r.t.* the preorder $\sqsubseteq$ between a behavioral quantifier prefix $\wp$ and its two canonical forms $\mathtt{C}_{\exists\forall}(\wp)$ and $\mathtt{C}_{\forall\exists}(\wp)$ that can be proved by induction on the structure of $\wp$.

Proposition 6. $\mathsf{evl}_{\alpha}(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp)) \sqsubseteq \mathsf{evl}_{\alpha}(\mathfrak{X}, \wp) \sqsubseteq \mathsf{evl}_{\alpha}(\mathfrak{X}, \mathtt{C}_{\alpha}(\wp))$, *for all* $\mathfrak{X} \in \mathrm{HAsg}$ *and* $\wp \in \mathrm{Qn}_{\mathtt{B}}$, *with* $\mathsf{ap}(\wp) \cap \mathsf{ap}(\mathfrak{X}) = \emptyset$.

From this result, Lemma 5, and Theorem 2, the above implications immediately follow. To complete the proof, we need to show that $\mathtt{C}_{\exists\forall}(\wp)\psi \Rightarrow \mathtt{C}_{\forall\exists}(\wp)\psi$ holds. Thanks to Lemma 5, if $\{\{\varnothing\}\} \models^{\exists\forall} \mathtt{C}_{\exists\forall}(\wp)\psi$, then $\mathrm{E} \subseteq \Psi \triangleq \mathsf{L}(\psi)$, for some $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathtt{C}_{\exists\forall}(\wp))$. Thus, by Item 2 of Theorem 5, it follows that Abelard loses the game $\mathcal{G}_{\wp}^{\Psi}$, which means, by determinacy, that Eloise wins. As a consequence of Item 1 of the same theorem, there exists $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathtt{C}_{\forall\exists}(\wp))$ such that $\mathrm{E} \subseteq \Psi$. Hence, $\{\{\varnothing\}\} \models^{\exists\forall} \mathtt{C}_{\forall\exists}(\wp)\psi$, again by Lemma 5.

B. Quantification Game for Formulae

The game defined in the previous section can easily be adapted to deal with the satisfiability problem for behavioral $\mathsf{GFG\text{-}QPTL}$ as shown in the next section. Solving the model-checking problem requires, however, a generalization of Theorem 4, connecting a suitable game with satisfiability of arbitrary behavioral formulae *w.r.t.* a hyperassignment $\mathfrak{X}$. We can prove such a property under the assumption that $\mathfrak{X}$ is *well-behaved*, *i.e.*, if 1) $\mathfrak{X}$ is the evolution of a set of assignments X *w.r.t.* some behavioral prefix $\widetilde{\wp}$ and 2) $\mathfrak{X}$ is Borelian. The Borelian requirement is again connected to determinacy of the underlying game. The behavioral requirement, instead, allows for a simple proof that leverages the quantification game for sentences directly. At this stage, it is not clear whether the property actually holds for arbitrary Borelian hyperassignments.

To formalize the above assumption, we introduce the notion of *generator* for a hyperassignment $\mathfrak{X} \in \mathrm{HAsg}$ as a pair $\langle \widetilde{\wp}, \mathrm{X} \rangle$ of 1) a behavioral quantification prefix $\widetilde{\wp} \in \mathrm{Qn}$ and 2) a Borelian set of assignments $\emptyset \neq \mathrm{X} \subseteq \mathrm{Asg}(\mathsf{ap}(\mathfrak{X}) \setminus \mathsf{ap}(\widetilde{\wp}))$ such that $\mathfrak{X} = \mathsf{evl}_{\exists\forall}(\{\mathrm{X}\}, \widetilde{\wp})$. A hyperassignment $\mathfrak{X} \in \mathrm{HAsg}$ is *Borelian behavioral* if there is a generator for it. A *quantification-game schema* is a tuple $\mathfrak{Q} \triangleq \langle \mathfrak{X}, \wp, \Psi \rangle$ where 1) $\mathfrak{X} \in \mathrm{HAsg}$ is Borelian behavioral, 2) $\wp \in \mathrm{Qn}$ is behavioral, 3) $\Psi \subseteq \mathrm{Asg}(\mathsf{ap}(\wp) \cup \mathsf{ap}(\mathfrak{X}))$ is Borelian, and 4) $\mathsf{ap}(\wp) \cap \mathsf{ap}(\mathfrak{X}) = \emptyset$.

The idea behind the game-theoretic construction reported below is quite simple. Given a generator $\langle \widetilde{\wp}, \mathrm{X} \rangle$ for a behavioral hyperassignment $\mathfrak{X}$, we force the two players to simulate the given $\mathfrak{X}$ by playing according to the prefix $\widetilde{\wp}$, once Abelard has arbitrarily chosen the values of the atomic propositions $\vec{p}$ over which the set of assignments X is defined. Since $\mathsf{evl}_{\exists\forall}(\forall \vec{p}) = \{\mathrm{Asg}(\vec{p})\}$ and $\mathrm{X} \subseteq \mathrm{Asg}(\vec{p})$, it is clear that $\mathfrak{X} \sqsubseteq \mathsf{evl}_{\exists\forall}(\forall \vec{p} . \widetilde{\wp})$. Thus, if Eloise wins the game, she can ensure a given temporal property. Notice, however, that we gave Abelard the freedom to cheat and choose arbitrary values for $\vec{p}$. Thus, in principle, Eloise could be able to satisfy the property while loosing the game, since Abelard can choose assignments over $\vec{p}$ that do not belong to X. To remedy this, we add all those assignments to Eloise's winning set, thus deterring Abelard from cheating.

Construction 2 (Quantification Game II)**.** *For a quantification-game schema* $\mathfrak{Q} \triangleq \langle \mathfrak{X}, \wp, \Psi \rangle$*, we say that* $\mathcal{G}$ *is a* $\mathfrak{Q}$*-game if there is a generator* $\langle \widetilde{\wp}, \mathrm{X} \rangle$ *for* $\mathfrak{X}$ *such that* $\mathcal{G} \triangleq \mathcal{G}^{\widehat{\Psi}}_{\widehat{\wp}}$*, where*

- $\widehat{\wp} \triangleq \forall \vec{p}.\, \widetilde{\wp}.\, \wp$ *and*
- $\widehat{\Psi} \triangleq \Psi \cup \{ \chi \in \mathrm{Asg}(\mathrm{P}) \mid \chi \restriction \vec{p} \notin \mathrm{X} \}$,

with $\vec{p} \triangleq \mathsf{ap}(\mathfrak{X}) \setminus \mathsf{ap}(\widetilde{\wp})$ *and* $\mathrm{P} \triangleq \mathsf{ap}(\wp) \cup \mathsf{ap}(\mathfrak{X})$.

The quantification-game schema for a formula $\wp\psi$, with $\psi \in \mathsf{LTL}$, and a hyperassignment $\mathfrak{X}$ is the tuple $\mathfrak{Q}^{\mathfrak{X}}_{\wp\psi} \triangleq \langle \mathfrak{X}, \wp, \mathsf{L}(\psi) \rangle$.

Theorem 7 (Game-Theoretic Semantics II)**.** $\mathfrak{X} \models^{\exists\forall} \wp\psi$ iff *Eloise wins every* $\mathfrak{Q}^{\mathfrak{X}}_{\wp\psi}$*-game, for all behavioral* GFG-QPTL *formulae* $\wp\psi$ *and Borelian behavioral hyperassignments* $\mathfrak{X} \in \mathrm{HAsg}(\mathsf{free}(\wp\psi))$.

The proof is similar to the one of Theorem 4 and uses the following result, which generalizes Theorem 5.

Theorem 8 (Quantification Game II)**.** *Every* $\mathfrak{Q}$*-game* $\mathcal{G}$*, for some quantification-game schema* $\mathfrak{Q} \triangleq \langle \wp, \mathfrak{X}, \Psi \rangle$*, satisfies the following two properties:*

1) if Eloise wins then $\mathrm{E} \subseteq \Psi$*, for some* $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathfrak{X}, \mathsf{C}_{\forall\exists}(\wp))$*;*
2) if Abelard wins then $\mathrm{E} \not\subseteq \Psi$*, for all* $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathfrak{X}, \mathsf{C}_{\exists\forall}(\wp))$.

This result, together with Proposition 6, allows us to derive a generalization of Theorem 6 and, thus, to obtain Theorem7.

Theorem 9 (Formula Canonical Forms)**.** *For every behavioral* GFG-QPTL *formula* $\wp\psi$*, with* $\psi \in \mathsf{LTL}$*, it holds that* $\mathfrak{X} \models^{\alpha} \wp\psi$ iff $\mathfrak{X} \models^{\alpha} \mathsf{C}_{\exists\forall}(\wp)\psi$ iff $\mathfrak{X} \models^{\alpha} \mathsf{C}_{\forall\exists}(\wp)\psi$*, for all Borelian behavioral hyperassignments* $\mathfrak{X} \in \mathrm{HAsg}(\mathsf{free}(\wp\psi))$.

V. Decision Problems & Expressiveness

The results of the previous section can be exploited to solve optimally the decision problems for behavioral GFG-QPTL. More specifically, we can use the game of Constructions 1 for the satisfiability problem, and the game of Constructions 2 for the model-checking one. We also discuss the expressiveness relationship between QPTL and behavioral GFG-QPTL, showing, by means of a classic encoding of automata into logic, that they have the same expressive power, though QPTL is non-elementary more succinct then GFG-QPTL.

A. Decision Procedures

The first step in deciding the satisfiability problem is to derive from a behavioral sentence $\varphi = \wp\psi$ a *parity game* [9], [51] that is won by Eloise *iff* φ is satisfiable. To do that, we first construct a deterministic parity automaton $\mathcal{D}_{\psi}$ for the LTL formula ψ, by combining the Vardi-Wolper construction [66] with the Safra-like translation from Büchi to parity acceptance condition [55]. We then compute the synchronous product of the arena $\mathcal{A}_{\wp}$ of Construction 1 with $\mathcal{D}_{\psi}$, where the automaton component changes its state only when Abelard moves from observable positions containing a full valuation of the propositions. This valuation is, then, read by the transition function of $\mathcal{D}_{\psi}$ to determine its successor state. The resulting game simulates both the quantification game and the automaton, so that Eloise wins *iff* the play satisfies ψ.

Theorem 10 (Satisfiability Game)**.** *Every behavioral* GFG-QPTL *sentence* φ *has a parity game, with* $\mathrm{O}\left(2^{2^{|\varphi|}}\right)$ *positions and* $\mathrm{O}\left(2^{|\varphi|}\right)$ *priorities, in which Eloise wins* iff φ *is satisfiable.*

We can then obtain an upper bound on the complexity of the problem from the fact that parity games can be solved in time polynomial in the number of positions and exponential in that of the priorities [8], [10], [70]. For the lower bound, instead, we observe that the *reactive synthesis problem* [58] of an LTL formula ψ can be reduced to the satisfiability of a sentence of the form $\forall^{\mathsf{B}}\vec{p}.\, \exists^{\mathsf{B}}\vec{q}.\, \psi$, where $\vec{p}$ and $\vec{q}$ denote, respectively, the input and output signals of the desired system.

Theorem 11 (Satisfiability)**.** *The satisfiability problem for behavioral* GFG-QPTL *sentences is* 2ExpTime-complete.

For the universal (*resp.*, existential) model-checking problem, given a Kripke structure $\mathcal{K}$, we ask whether $\mathcal{K} \models \varphi$, in the sense that $\mathfrak{X}_{\mathcal{K}} \models^{\exists\forall} \varphi$ (*resp.*, $\mathfrak{X}_{\mathcal{K}} \models^{\forall\exists} \varphi$) holds, where $\mathfrak{X}_{\mathcal{K}} \triangleq \{\mathsf{wrd}^{-1}(\mathsf{L}(\mathcal{K}))\}$ is the hyperassignment obtained by collecting all those assignments $\chi \in \mathrm{Asg}(\mathsf{ap}(\mathcal{K}))$ over the propositions of $\mathcal{K}$ for which the infinite word $\mathsf{wrd}(\chi)$ belongs to the ω-regular language $\mathsf{L}(\mathcal{K})$ generated by $\mathcal{K}$. Obviously, $\mathfrak{X}_{\mathcal{K}}$ is a Borelian behavioral hyperassignment. As a consequence, Construction 2 applies. Thus, we can adopt the same synchronous product described above between the arena of the game and the union of the two automata $\mathcal{D}_{\psi}$ and $\mathcal{N}_{\overline{\mathcal{K}}}$, where $\mathcal{D}_{\psi}$ is obtained from the formula ψ, while $\mathcal{N}_{\overline{\mathcal{K}}}$ is a co-safety automaton of size linear in $|\mathcal{K}|$, recognizing the complement of the language $\mathsf{L}(\mathcal{K})$.

Theorem 12 (Model-Checking Game)**.** *Every Kripke structure* $\mathcal{K}$ *and behavioral* GFG-QPTL *formula* φ*, with* $\mathsf{free}(\varphi) \subseteq \mathsf{ap}(\mathcal{K})$*, have a parity game, with* $\mathrm{O}\left(2^{2^{|\varphi|}}|\mathcal{K}|\right)$ *positions and* $\mathrm{O}\left(2^{|\varphi|}\right)$ *priorities, in which Eloise wins* iff $\mathcal{K} \models \varphi$.

Upper bounds *w.r.t.* both formula and model complexity, and the lower bound *w.r.t.* formula complexity, are proved as in the case of the satisfiability problem. The lower bound *w.r.t.* model complexity is proved by reducing from reachability games [36].

Theorem 13 (Model-Checking)**.** *The model-checking problem for behavioral* GFG-QPTL *has* 2ExpTime-complete *formula complexity and* PTime-complete *model complexity.*

B. Expressive Power

We conclude the work by discussing the expressive power of the behavioral fragment of GFG-QPTL, showing that it precisely corresponds to the ω-regular languages. Similarly to Example 8, consider an arbitrary deterministic parity automaton $\mathcal{D}$ with k states over an alphabet 2^{P}, with $\mathrm{P} \subseteq \mathrm{AP}$. Via the standard technique of encoding the existence of an accepting run, we can construct an LTL formula ψ, over the set of propositions $\mathrm{P} \cup \{s_1, \ldots, s_k\}$, such that the existential projection on P of the language $\mathsf{L}(\psi)$ coincides with the language $\mathsf{L}(\mathcal{D})$ recognized by $\mathcal{D}$. Since $\mathcal{D}$ is deterministic, this projection is clearly behavioral. Hence, the behavioral $\mathsf{GFG\text{-}QPTL}$ formula $\exists^{\mathrm{B}} s_1, \ldots, \exists^{\mathrm{B}} s_k. \psi$ is satisfied by an hyperassignment $\{\{\chi\}\}$ *iff* $\mathsf{wrd}(\chi) \in \mathsf{L}(\mathcal{D})$. Since every QPTL formula can be translated into an equivalent nondeterministic Büchi automaton [64], which in turn can be determinized into a parity one [55], the obtain the following.

Theorem 14 (Vanilla to Behavioral)**.** *For every* QPTL *formula, there exists an equivalent behavioral* $\mathsf{GFG\text{-}QPTL}$ *one.*

It is not hard to see that the converse holds as well. Indeed, the satisfiability game $\mathcal{G}_\varphi$ can be transformed into a isomorphic alternating parity word automaton $\mathcal{A}_\varphi$, in the usual way, which can then be reduced to a nondeterministic parity automaton $\mathcal{N}_\varphi$ [52]. The emptiness of $\mathcal{N}_\varphi$ can then be encoded into a QPTL sentence. A similar reasoning applies also to formulae.

Theorem 15 (Behavioral to Vanilla)**.** *For every behavioral* $\mathsf{GFG\text{-}QPTL}$ *formula, there exists an equivalent* QPTL *one.*

Clearly, QPTL is also non-elementary more succinct than behavioral $\mathsf{GFG\text{-}QPTL}$. Indeed, the satisfiability problem for QPTL sentences with alternation of quantifiers k is $(k-1)$-EXPSPACE-COMPLETE [64], while behavioral $\mathsf{GFG\text{-}QPTL}$ is decidable in 2EXPTIME, so no elementary reduction exists.

Theorem 16 (Succinctness)**.** QPTL *is non-elementary more succinct than behavioral* $\mathsf{GFG\text{-}QPTL}$.

VI. DISCUSSION

We have introduced a novel semantics for QPTL extending Hodges' team semantics for Hintikka and Sandu's logic of imperfect information IF in a non-trivial way. On the one hand, the new semantic setting can express games with both symmetric and asymmetric restrictions on the players. On the other hand, it allows for encoding behavioral constraints on the quantified propositions, connecting the underlying logic with the game-theoretic notion of behavioral strategies. Based on this semantics, the extension of QPTL with constraints on the functional dependencies among propositions, called $\mathsf{GFG\text{-}QPTL}$, has surprisingly interesting properties. For one, its behavioral fragment enables reducing the solution of two-player zero-sum games to the decision problems for the logic. This fragment also enjoys good computational properties, being 2EXPTIME-COMPLETE for both satisfiability and model-checking. It is also very expressive, being equivalent to, though less succinct than, QPTL, hence able to describe all ω-regular properties. Second, the behavioral semantics also bears a connection to good-for-game automata, allowing to naturally express the property of being a GFG automata, the significance of which is probably worth investigating further.

To the best of our knowledge, this is the first attempt to provide a compositional account of behavioral constraints. We believe the generality and flexibility of the semantic settings opens up the possibility of a systematic investigation of the impact of this type of constraints in quantified temporal logics, such as QCTL [15], [42], HyperLTL/CTL^* [6], [7], [11], [12], [14], Coordination Logic [13], and Strategy Logic [5], [48].

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Appendix

A. Proofs for Section II

Proposition 1. *$\mathfrak{X} \subseteq \overline{\overline{\mathfrak{X}}}$ and $\mathfrak{X} \equiv \overline{\overline{\mathfrak{X}}}$, for all $\mathfrak{X} \in \mathrm{HAsg}$.*

Proof. To begin with, we show that $\mathfrak{X} \subseteq \overline{\overline{\mathfrak{X}}}$. By definition of $\overline{\mathfrak{X}}$, for every $\overline{\mathrm{X}} \in \overline{\mathfrak{X}}$ there is a function $\Gamma_{\overline{\mathrm{X}}} \in \mathrm{Chc}(\mathfrak{X})$ such that $\overline{\mathrm{X}} = \{ \Gamma_{\overline{\mathrm{X}}}(\mathrm{X}) \mid \mathrm{X} \in \mathfrak{X} \}$. Now, consider an arbitrary $\mathrm{X} \in \mathfrak{X}$ and define Γ as: $\Gamma(\overline{\mathrm{X}}) = \Gamma_{\overline{\mathrm{X}}}(\mathrm{X})$ for every $\overline{\mathrm{X}} \in \overline{\mathfrak{X}}$. Notice that $\Gamma(\overline{\mathrm{X}}) \in \overline{\mathrm{X}}$, for every $\overline{\mathrm{X}} \in \overline{\mathfrak{X}}$, and thus $\Gamma \in \mathrm{Chc}(\overline{\mathfrak{X}})$. Therefore, we have that $\{ \Gamma(\overline{\mathrm{X}}) \mid \overline{\mathrm{X}} \in \overline{\mathfrak{X}} \} \in \overline{\overline{\mathfrak{X}}}$. To conclude the proof, we are left to show that $\{ \Gamma(\overline{\mathrm{X}}) \mid \overline{\mathrm{X}} \in \overline{\mathfrak{X}} \} = \mathrm{X}$ holds as well. First, observe that $\Gamma(\overline{\mathrm{X}}) = \Gamma_{\overline{\mathrm{X}}}(\mathrm{X}) \in \mathrm{X}$ holds for every $\overline{\mathrm{X}} \in \overline{\mathfrak{X}}$, implying $\{ \Gamma(\overline{\mathrm{X}}) \mid \overline{\mathrm{X}} \in \overline{\mathfrak{X}} \} \subseteq \mathrm{X}$. In order to show the converse inclusion ($\{ \Gamma(\overline{\mathrm{X}}) \mid \overline{\mathrm{X}} \in \overline{\mathfrak{X}} \} \supseteq \mathrm{X}$), consider an arbitrary $\chi \in \mathrm{X}$ and a function $\Gamma_{\mathrm{X}_\chi} \in \mathrm{Chc}(\mathfrak{X})$ such that $\Gamma_{\mathrm{X}_\chi}(\mathrm{X}) = \chi$. Let $\mathrm{X}_\chi \triangleq \{ \Gamma_{\mathrm{X}_\chi}(\mathrm{X}) \mid \mathrm{X} \in \mathfrak{X} \}$. It holds $\mathrm{X}_\chi \in \overline{\mathfrak{X}}$. Since $\Gamma(\mathrm{X}_\chi) = \Gamma_{\mathrm{X}_\chi}(\mathrm{X}) = \chi$, we have that $\chi \in \{ \Gamma(\overline{\mathrm{X}}) \mid \overline{\mathrm{X}} \in \overline{\mathfrak{X}} \}$ and, since χ was chosen arbitrarily, we conclude $\{ \Gamma(\overline{\mathrm{X}}) \mid \overline{\mathrm{X}} \in \overline{\mathfrak{X}} \} \supseteq \mathrm{X}$.

Observe that, straightforwardly, $\mathfrak{X} \subseteq \overline{\overline{\mathfrak{X}}}$ implies $\mathfrak{X} \sqsubseteq \overline{\overline{\mathfrak{X}}}$.

Let us turn now to proving $\overline{\overline{\mathfrak{X}}} \sqsubseteq \mathfrak{X}$. Let $\overline{\overline{\mathrm{X}}} \in \overline{\overline{\mathfrak{X}}}$. By definition of $\overline{\overline{\mathfrak{X}}}$, there is a function $\Gamma_{\overline{\overline{\mathrm{X}}}} \in \mathrm{Chc}(\overline{\mathfrak{X}})$ such that $\overline{\overline{\mathrm{X}}} = \{ \Gamma_{\overline{\overline{\mathrm{X}}}}(\overline{\mathrm{X}}) \mid \overline{\mathrm{X}} \in \overline{\mathfrak{X}} \}$. Towards a contradiction, assume that for every $\mathrm{X} \in \mathfrak{X}$ there is $\chi_{\mathrm{X}} \in \mathrm{X} \setminus \overline{\overline{\mathrm{X}}}$. Let us define Γ as: $\Gamma(\mathrm{X}) = \chi_{\mathrm{X}}$ for every $\mathrm{X} \in \mathfrak{X}$. Notice that $\Gamma \in \mathrm{Chc}(\mathfrak{X})$ Thus, $\overline{\mathrm{X}} \triangleq \{ \chi_{\mathrm{X}} \mid \mathrm{X} \in \mathfrak{X} \} \in \overline{\mathfrak{X}}$ and $\overline{\mathrm{X}} \cap \overline{\overline{\mathrm{X}}} = \emptyset$. However, $\Gamma_{\overline{\overline{\mathrm{X}}}}(\overline{\mathrm{X}}) \in \overline{\overline{\mathrm{X}}} \cap \overline{\mathrm{X}}$, thus rising a contradiction. $\square$

Lemma 1 (Dualization)**.** *The following hold true, for all QPTL formulae φ and hyperassignments $\mathfrak{X} \in \mathrm{HAsg}_{\subseteq}(\mathsf{free}(\varphi))$.*

1) *Statements 1a and 1b are equivalent:*
   a) *there exists $\mathrm{X} \in \mathfrak{X}$ such that $\chi \models \varphi$, for all $\chi \in \mathrm{X}$;*
   b) *for all $\mathrm{X} \in \overline{\mathfrak{X}}$, it holds that $\chi \models \varphi$, for some $\chi \in \mathrm{X}$.*
2) *Statements 2a and 2b are equivalent:*
   a) *for all $\mathrm{X} \in \mathfrak{X}$, it holds that $\chi \models \varphi$, for some $\chi \in \mathrm{X}$;*
   b) *there exists $\mathrm{X} \in \overline{\mathfrak{X}}$ such that $\chi \models \varphi$, for all $\chi \in \mathrm{X}$.*

Proof. ($1a \Rightarrow 1b$) By $1a$, there is $\mathrm{X} \in \mathfrak{X}$ such that $\chi \models \varphi$ holds for every $\chi \in \mathrm{X}$. By definition of $\overline{\mathfrak{X}}$, for every $\overline{\mathrm{X}} \in \overline{\mathfrak{X}}$ there is $\Gamma_{\overline{\mathrm{X}}}$ such that $\Gamma_{\overline{\mathrm{X}}}(\mathrm{X}) \in \mathrm{X}$ and $\overline{\mathrm{X}} = \{\Gamma_{\overline{\mathrm{X}}}(\mathrm{X}) : \mathrm{X} \in \mathfrak{X}\}$; by $1a$, $\Gamma_{\overline{\mathrm{X}}}(\mathrm{X}) \models \varphi$; since, in addition, $\Gamma_{\overline{\mathrm{X}}}(\mathrm{X}) \in \overline{\mathrm{X}}$, the thesis holds.

($1b \Rightarrow 1a$) By $1b$, for every $\overline{\mathrm{X}} \in \overline{\mathfrak{X}}$ there is $\chi_{\overline{\mathrm{X}}} \in \overline{\mathrm{X}}$ such that $\chi_{\overline{\mathrm{X}}} \models \varphi$. Consider function $\Gamma \in \mathrm{Chc}(\overline{\mathfrak{X}})$ defined as: $\Gamma(\overline{\mathrm{X}}) = \chi_{\overline{\mathrm{X}}}$, for every $\overline{\mathrm{X}} \in \overline{\mathfrak{X}}$. By definition of $\overline{\overline{\mathfrak{X}}}$, we have that $\{\Gamma(\overline{\mathrm{X}}) : \overline{\mathrm{X}} \in \overline{\mathfrak{X}}\} \in \overline{\overline{\mathfrak{X}}}$. By Proposition 1, it holds $\overline{\overline{\mathfrak{X}}} \sqsubseteq \mathfrak{X}$, which means that there is $\mathrm{X} \in \mathfrak{X}$, with $\mathrm{X} \subseteq \{\Gamma(\overline{\mathrm{X}}) : \overline{\mathrm{X}} \in \overline{\mathfrak{X}}\}$. Since, by construction, $\Gamma(\overline{\mathrm{X}}) \models \varphi$ for every $\overline{\mathrm{X}} \in \overline{\mathfrak{X}}$, the thesis holds.

($2a \Leftrightarrow 2b$) By statement 1 of this lemma, we have that $1a$ is false if and only if $1b$ is false (*not* $1a \Leftrightarrow$ *not* $1b$, for short). By instantiating, in this last equivalence, φ with $\neg\varphi$, we have $1a' \Leftrightarrow 1b'$, where $1a'$ and $1b'$ are abbreviations for, respectively:

- for all sets of assignments $\mathrm{X} \in \mathfrak{X}$, there exists an assignment $\chi \in \mathrm{X}$ such that $\chi \not\models \neg\varphi$;
- there exists a set of assignments $\mathrm{X} \in \overline{\mathfrak{X}}$ such that, for all assignments $\chi \in \mathrm{X}$, it holds that $\chi \not\models \neg\varphi$.

By applying semantics of negation, it is straightforward to see that $1a'$ and $1b'$ correspond to $2a$ and $2b$, respectively, hence the thesis. $\square$

Lemma 2 (Boolean Connectives)**.** *The following hold true, for all QPTL formulae φ_1 and φ_2 and hyperassignments $\mathfrak{X} \in \mathrm{HAsg}_{\subseteq}(\mathrm{P})$, with $\mathrm{P} \triangleq \mathsf{free}(\varphi_1) \cup \mathsf{free}(\varphi_2)$.*

1) *Statements 1a and 1b are equivalent:*
   a) *there exists $\mathrm{X} \in \mathfrak{X}$ such that $\chi \models \varphi_1 \wedge \varphi_2$, for all $\chi \in \mathrm{X}$;*
   b) *for each $(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$, there exist $i \in \{1, 2\}$ and $\mathrm{X} \in \mathfrak{X}_i$ such that $\chi \models \varphi_i$, for all $\chi \in \mathrm{X}$.*
2) *Statements 2a and 2b are equivalent:*
   a) *for all $\mathrm{X} \in \mathfrak{X}$, it holds that $\chi \models \varphi_1 \vee \varphi_2$, for some $\chi \in \mathrm{X}$;*
   b) *there exists $(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$ such that, for all $i \in \{1, 2\}$ and $\mathrm{X} \in \mathfrak{X}_i$, it holds that $\chi \models \varphi_i$, for some $\chi \in \mathrm{X}$.*

Proof. ($1a \Rightarrow 1b$) Let $\mathrm{X} \in \mathfrak{X}$ be such that $\chi \models \varphi_1 \wedge \varphi_2$ holds for every $\chi \in \mathrm{X}$ and consider an arbitrary pair $(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$. Since $(\mathfrak{X}_1, \mathfrak{X}_2)$ is a partition of $\mathfrak{X}$, either $\mathrm{X} \in \mathfrak{X}_1$ or $\mathrm{X} \in \mathfrak{X}_2$: in the former case, let $i = 1$; in the latter, let $i = 2$. Since $\mathrm{X} \in \mathfrak{X}_i$ and $\chi \models \varphi_i$ holds for every $\chi \in \mathrm{X}$, the thesis holds.

($1b \Rightarrow 1a$) Consider the hyperassignment $\overline{\mathfrak{X}_1} = \{\mathrm{X} \in \mathfrak{X} : \forall \chi \in \mathrm{X} \,.\, \chi \models \varphi_1\}$ and the pair $(\mathfrak{X}_1 \triangleq \mathfrak{X} \setminus \overline{\mathfrak{X}_1}, \mathfrak{X}_2 \triangleq \overline{\mathfrak{X}_1}) \in \mathsf{par}(\mathfrak{X})$. Observe that, by definition of $\mathfrak{X}_1$, there is no $\mathrm{X} \in \mathfrak{X}_1$ such that $\chi \models \varphi_1$ holds for every $\chi \in \mathrm{X}$. Thus, by $1b$, there must exist $\mathrm{X} \in \mathfrak{X}_2$ such that $\chi \models \varphi_2$ holds for every $\chi \in \mathrm{X}$. By definition of $\mathfrak{X}_2$, it also holds that $\chi \models \varphi_1$ for every $\chi \in \mathrm{X}$, hence the thesis.

($2a \Leftrightarrow 2b$) By statement 1 of this lemma, we have that $1a$ is false if and only if $1b$ is false (*not* $1a \Leftrightarrow$ *not* $1b$, for short). By instantiating, in this last equivalence, φ_1 with $\neg\varphi_1$ and φ_2 with $\neg\varphi_2$, we have $1a' \Leftrightarrow 1b'$, where $1a'$ and $1b'$ are abbreviations for, respectively:

- for all sets of assignments $\mathrm{X} \in \mathfrak{X}$, there exists an assignment $\chi \in \mathrm{X}$ such that $\chi \not\models \neg\varphi_1 \wedge \neg\varphi_2$;
- there exists a pair of hyperassignments $(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$ such that, for all indexes $i \in \{1, 2\}$ and sets of assignments $\mathrm{X} \in \mathfrak{X}_i$, there exists an assignment $\chi \in \mathrm{X}$ for which it holds that $\chi \not\models \neg\varphi_i$.

By applying semantics of negation and De Morgan's laws, it is straightforward to see that $1a'$ and $1b'$ correspond to $2a$ and $2b$, respectively, hence the thesis. $\square$

Lemma 3 (Hyperassignment Extensions)**.** *The following hold true, for all QPTL formulae φ, atomic propositions $p \in \mathrm{AP}$, and hyperassignments $\mathfrak{X} \in \mathrm{HAsg}_{\subseteq}(\mathsf{free}(\varphi) \setminus \{p\})$.*

1) *Statements 1a and 1b are equivalent:*
   a) *there exists $\mathrm{X} \in \mathfrak{X}$ such that $\chi \models \exists p \,.\, \varphi$, for all $\chi \in \mathrm{X}$;*
   b) *there exists $\mathrm{X} \in \mathsf{ext}(\mathfrak{X}, p)$ such that $\chi \models \varphi$, for all $\chi \in \mathrm{X}$.*
2) *Statements 2a and 2b are equivalent:*

a) *for all* $\mathrm{X}\in\mathfrak{X}$, *it holds that* $\chi \models \forall p.\,\varphi$, *for some* $\chi \in \mathrm{X}$;
b) *for all* $\mathrm{X} \in \mathsf{ext}(\mathfrak{X}, p)$, *it holds that* $\chi \models \varphi$, *for some* $\chi\in\mathrm{X}$.

Proof. $(1a \Rightarrow 1b)$ Let $\mathrm{X} \in \mathfrak{X}$ be such that $\chi \models \exists p.\varphi$ holds for every $\chi \in \mathrm{X}$. By semantics (Def. 2, item 3a), for every $\chi \in \mathrm{X}$, there is a temporal function $\mathsf{f}_\chi \in \mathbb{N} \to \mathbb{B}$ such that $\chi[p \mapsto \mathsf{f}_\chi] \models \varphi$. Let $\mathsf{F} \in \mathrm{Fnc}(\mathsf{ap}(\mathfrak{X}))$ be such that $\mathsf{F}(\chi) = \mathsf{f}_\chi$ for every $\chi \in \mathrm{X}$ and let $\mathrm{X}_{\mathsf{F}} = \{\chi[p \mapsto \mathsf{F}(\chi)] : \chi \in \mathrm{X}\}$. Since $\mathrm{X}_{\mathsf{F}} \in \mathsf{ext}(\mathfrak{X}, p)$ and $\chi \models \varphi$ holds for every $\chi \in \mathrm{X}_{\mathsf{F}}$, the thesis holds.

$(1b \Rightarrow 1a)$ Let $\mathrm{X}_{\mathsf{F}} \in \mathsf{ext}(\mathfrak{X}, p)$ be such that $\chi \models \varphi$ holds for every $\chi \in \mathrm{X}_{\mathsf{F}}$. By definition of $\mathsf{ext}(\mathfrak{X}, p)$, there are $\mathrm{X} \in \mathfrak{X}$ and $\mathsf{F} \in \mathrm{Fnc}(\mathsf{ap}(\mathfrak{X}))$ such that $\mathrm{X}_{\mathsf{F}} = \{\chi[p \mapsto \mathsf{F}(\chi)] : \chi \in \mathrm{X}\}$. Clearly, by semantics (Def. 2, item 3a), $\chi \models \exists p.\varphi$ holds for every $\chi \in \mathrm{X}$, hence the thesis.

$(2a \Leftrightarrow 2b)$ By statement 1 of this lemma, we have that $1a$ is false if and only if $1b$ is false (*not* $1a \Leftrightarrow$ *not* $1b$, for short). By instantiating, in this last equivalence, φ with $\neg\varphi$, we have $1a' \Leftrightarrow 1b'$, where $1a'$ and $1b'$ are abbreviations for, respectively:

- for all sets of assignments $\mathrm{X} \in \mathfrak{X}$, there exists an assignment $\chi \in \mathrm{X}$ such that $\chi \not\models \exists p.\neg\varphi$;
- for all sets of assignments $\mathrm{X} \in \mathsf{ext}(\mathfrak{X}, p)$, there exists an assignment $\chi \in \mathrm{X}$ such that $\chi \not\models \neg\varphi$.

By applying semantics of negation and duality of $\exists$ and $\forall$, it is straightforward to see that $1a'$ and $1b'$ correspond to $2a$ and $2b$, respectively, hence the thesis. □

Theorem 1 (Semantics Adequacy I)**.** *The following hold, for all* QPTL *formulae* φ *and hyperassignments* $\mathfrak{X} \in \mathrm{HAsg}_{\subseteq}(\mathsf{free}(\varphi))$*:*

1) $\mathfrak{X} \models^{\exists\forall} \varphi$ iff *there exists a set of assignments* $\mathrm{X} \in \mathfrak{X}$ *such that* $\chi \models \varphi$, *for all* $\chi \in \mathrm{X}$;
2) $\mathfrak{X} \models^{\forall\exists} \varphi$ iff, *for all sets of assignments* $\mathrm{X} \in \mathfrak{X}$, *it holds that* $\chi \models \varphi$, *for some* $\chi \in \mathrm{X}$.

Proof. Both claims 1 and 2 are proved together, by induction on the structure of the formula.

(base case) If $\varphi \in \mathsf{LTL}$, then the claims immediately follows from the semantics (Definition 3, item 1).

(inductive step) If $\varphi = \neg\psi$, then we have, by semantics, $\mathfrak{X} \models^{\alpha} \varphi$ if and only if $\mathfrak{X} \not\models^{\overline{\alpha}} \psi$. If $\alpha = \exists\forall$, then, by inductive hypothesis, it is not the case that for every $\mathrm{X} \in \mathfrak{X}$ there is $\chi \in \mathrm{X}$ such that $\chi \models \psi$, which amounts to say that there is $\mathrm{X} \in \mathfrak{X}$ such that for every $\chi \in \mathrm{X}$ it holds $\chi \not\models \psi$, from which the thesis follows. If, instead, $\alpha = \forall\exists$, then, by inductive hypothesis, there is no $\mathrm{X} \in \mathfrak{X}$ such that for every $\chi \in \mathrm{X}$ it holds $\chi \models \psi$, which amounts to say that for every $\mathrm{X} \in \mathfrak{X}$ there is $\chi \in \mathrm{X}$ such that $\chi \not\models \psi$, from which the thesis follows.

If $\varphi = \varphi_1 \wedge \varphi_2$ and $\alpha = \exists\forall$, then we have, by semantics, $\mathfrak{X} \models^{\alpha} \varphi$ if and only if for every $(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$ it holds true that $\mathfrak{X}_1 \neq \emptyset$ and $\mathfrak{X}_1 \models^{\alpha} \varphi_1$ or it holds true that $\mathfrak{X}_2 \neq \emptyset$ and $\mathfrak{X}_2 \models^{\alpha} \varphi_2$. By inductive hypothesis, this amounts to say that for every $(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$ there is $i \in \{1, 2\}$ and $\mathrm{X} \in \mathfrak{X}_i$ such that for every $\chi \in \mathrm{X}$ it holds $\chi \models \varphi_i$. The thesis follows from Lemma 2, item 1.

If $\varphi = \varphi_1 \wedge \varphi_2$ and $\alpha = \forall\exists$, then we have, by semantics, $\mathfrak{X} \models^{\alpha} \varphi$ if and only if $\overline{\mathfrak{X}} \models^{\overline{\alpha}} \varphi$. By proceeding as before, i.e., by applying semantics, inductive hypothesis, and Lemma 2, item 1, we have that there is $\overline{\mathrm{X}} \in \overline{\mathfrak{X}}$ such that for every $\overline{\chi} \in \overline{\mathrm{X}}$ it holds $\overline{\chi} \models \varphi$. The thesis follows from Lemma 1, item 2.

If $\varphi = \varphi_1 \vee \varphi_2$ and $\alpha = \forall\exists$, then we have, by semantics, $\mathfrak{X} \models^{\alpha} \varphi$ if and only if there is $(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$ such that $\mathfrak{X}_1 \neq \emptyset$ implies $\mathfrak{X}_1 \models^{\alpha} \varphi_1$ and $\mathfrak{X}_2 \neq \emptyset$ implies $\mathfrak{X}_2 \models^{\alpha} \varphi_2$. By inductive hypothesis, this amounts to say that there is $(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$ such that for every $i \in \{1, 2\}$ and $\mathrm{X} \in \mathfrak{X}_i$ there is $\chi \in \mathrm{X}$ for which it holds $\chi \models \varphi_i$. The thesis follows from Lemma 2, item 2.

If $\varphi = \varphi_1 \vee \varphi_2$ and $\alpha = \exists\forall$, then we have, by semantics, $\mathfrak{X} \models^{\alpha} \varphi$ if and only if $\overline{\mathfrak{X}} \models^{\overline{\alpha}} \varphi$. By proceeding as before, i.e., by applying semantics, inductive hypothesis, and Lemma 2, item 2, we have that for every $\overline{\mathrm{X}} \in \overline{\mathfrak{X}}$ there is $\overline{\chi} \in \overline{\mathrm{X}}$ such that $\overline{\chi} \models \varphi$. The thesis follows from Lemma 1, item 1.

If $\varphi = \exists p.\psi$ and $\alpha = \exists\forall$, then we have, by semantics, $\mathfrak{X} \models^{\alpha} \varphi$ if and only if $\mathsf{ext}(\mathfrak{X}, p) \models^{\alpha} \psi$. By inductive hypothesis, this amounts to say that there is $\mathrm{X} \in \mathsf{ext}(\mathfrak{X}, p)$ such that for every $\chi \in \mathrm{X}$ it holds $\chi \models \psi$. The thesis follows from Lemma 3, item 1.

If $\varphi = \exists p.\psi$ and $\alpha = \forall\exists$, then we have, by semantics, $\mathfrak{X} \models^{\alpha} \varphi$ if and only if $\overline{\mathfrak{X}} \models^{\overline{\alpha}} \varphi$. By proceeding as before, i.e., by applying semantics, inductive hypothesis, and Lemma 3, item 1, we have that there is $\overline{\mathrm{X}} \in \overline{\mathfrak{X}}$ such that for every $\overline{\chi} \in \overline{\mathrm{X}}$ it holds $\overline{\chi} \models \varphi$. The thesis follows from Lemma 1, item 2.

If $\varphi = \forall p.\psi$ and $\alpha = \forall\exists$, then we have, by semantics, $\mathfrak{X} \models^{\alpha} \varphi$ if and only if $\mathsf{ext}(\mathfrak{X}, p) \models^{\alpha} \psi$. By inductive hypothesis, this amounts to say that for every $\mathrm{X} \in \mathsf{ext}(\mathfrak{X}, p)$ there is $\chi \in \mathrm{X}$ such that $\chi \models \psi$. The thesis follows from Lemma 3, item 2.

If $\varphi = \forall p.\psi$ and $\alpha = \exists\forall$, then we have, by semantics, $\mathfrak{X} \models^{\alpha} \varphi$ if and only if $\overline{\mathfrak{X}} \models^{\overline{\alpha}} \varphi$. By proceeding as before, i.e., by applying semantics, inductive hypothesis, and Lemma 3, item 2, we have that for every $\overline{\mathrm{X}} \in \overline{\mathfrak{X}}$ there is $\overline{\chi} \in \overline{\mathrm{X}}$ such that $\overline{\chi} \models \varphi$. The thesis follows from Lemma 1, item 1. □

B. *Proofs for Section III*

Proposition 2. *Let* $\mathrm{P} \subseteq \mathrm{AP}$, $\chi_1, \chi_2 \in \mathrm{Asg}(\mathrm{P})$, $\Theta \in \Theta$, *and* $k \in \mathbb{N}$. *Then,* $\chi_1 \approx^{k}_{\Theta} \chi_2$ iff *the following hold true:*

1) $\chi_1(q) = \chi_2(q)$, *for all* $q \in \mathrm{P} \setminus (\mathrm{P}_{\mathsf{B}} \cup \mathrm{P}_{\mathsf{S}})$;
2) $\chi_1(p)(t) = \chi_2(p)(t)$, *for all* $t \leq k$ *and* $p \in (\mathrm{P}_{\mathsf{B}} \cap \mathrm{P}) \setminus \mathrm{P}_{\mathsf{S}}$;
3) $\chi_1(p)(t) = \chi_2(p)(t)$, *for all* $t < k$ *and* $p \in \mathrm{P}_{\mathsf{S}} \cap \mathrm{P}$.

Proof. Assume $\chi_1 \approx^{k}_{\Theta} \chi_2$, i.e., $\chi_1 = \chi^{(1)} \sim^{k}_{\Theta} \chi^{(2)} \sim^{k}_{\Theta} \ldots \sim^{k}_{\Theta} \chi^{(r)} = \chi_2$, for some $\chi^{(1)}, \ldots, \chi^{(r)}$, with $r \in \mathbb{N} \setminus \{0\}$ (observe that $\chi_1 = \chi_2$ if $r = 1$).

We prove, by induction on r, that items 1–3 hold. If $r = 1$, then the claim follows trivially. Let $r > 1$. Since $\chi^{(1)} \sim^{k}_{\Theta} \chi^{(2)}$, we have that 1–3 hold when instantiated with $\chi^{(1)}$ and $\chi^{(2)}$, by Definition 4. Moreover, by inductive hypothesis, 1–3 hold when instantiated with $\chi^{(2)}$ and $\chi^{(r)}$. The claim follows by transitivity of 1–3.

Now, in order to prove the converse direction, assume that items 1–3 hold. Let $\{ p_1, \ldots, p_r \}$ be an enumeration of $\mathrm{P_B} \cup \mathrm{P_S}$ and define $\chi^{(1)} \triangleq \chi_1$ and $\chi^{(i+1)} \triangleq \chi^{(i)}[p_i \mapsto \chi_2(p_i)]$ for $i \in [1, \ldots, r]$. It is not difficult to convince oneself that $\chi_1 = \chi^{(1)} \sim^k_\Theta \chi^{(2)} \sim^k_\Theta \ldots \sim^k_\Theta \chi^{(r+1)} = \chi_2$ holds, hence $\chi_1 \approx^k_\Theta \chi_2$. $\square$

Proposition 3. *If $\chi_1 \approx^k_\Theta \chi_2$ then $\mathsf{F}(\chi_1)(k) = \mathsf{F}(\chi_2)(k)$, for all $\chi_1, \chi_2 \in \mathrm{Asg}(\mathrm{P})$, $\Theta \in \mathbb{\Theta}$, $k \in \mathbb{N}$, and $\mathsf{F} \in \mathrm{Fnc}_\Theta(\mathrm{P})$.*

Proof. Assume $\chi_1 \approx^k_\Theta \chi_2$, i.e., $\chi_1 = \chi^{(1)} \sim^k_\Theta \chi^{(2)} \sim^k_\Theta \ldots \sim^k_\Theta \chi^{(r)} = \chi_2$, for some $\chi^{(1)}, \ldots, \chi^{(r)}$, with $r \in \mathbb{N} \setminus \{0\}$ (observe that $\chi_1 = \chi_2$ if $r = 1$).

We prove, by induction on r, that $\mathsf{F}(\chi_1)(k) = \mathsf{F}(\chi_2)(k)$. If $r = 1$, then the claim follows trivially. Let $r > 1$. Since $\chi^{(1)} \sim^k_\Theta \chi^{(2)}$ and $\mathsf{F} \in \mathrm{Fnc}_\Theta(\mathrm{P})$, we have that $\mathsf{F}(\chi^{(1)})(k) = \mathsf{F}(\chi^{(2)})(k)$. Moreover, by inductive hypothesis, $\mathsf{F}(\chi^{(2)})(k) = \mathsf{F}(\chi^{(r)})(k)$. The claim follows by transitivity. $\square$

Proposition 4. *Let $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathrm{HAsg}$ with $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$: 1) $\overline{\mathfrak{X}_2} \sqsubseteq \overline{\mathfrak{X}_1}$; 2) $\mathsf{ext}_\Theta(\mathfrak{X}_1, p) \sqsubseteq \mathsf{ext}_\Theta(\mathfrak{X}_2, p)$, for all $p \in \mathrm{AP}$ and $\Theta \in \mathbb{\Theta}$.*

Proof. Proof of point 1). Assume $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$ and let $\overline{\mathrm{X}_2} \in \overline{\mathfrak{X}_2}$. We have to show that there exists $\overline{\mathrm{X}_1} \in \overline{\mathfrak{X}_1}$ such that $\overline{\mathrm{X}_1} \subseteq \overline{\mathrm{X}_2}$. By $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$, there is a function $f : \mathfrak{X}_1 \to \mathfrak{X}_2$, such that $f(\mathrm{X}_1) \subseteq \mathrm{X}_1$. By definition of $\overline{\mathfrak{X}_2}$, we have that $\overline{\mathrm{X}_2} = \mathsf{img}(\Gamma_2)$ for some $\Gamma_2 \in \mathrm{Chc}(\mathfrak{X}_2)$.

Now, define Γ_1 as $\Gamma_1(\mathrm{X}_1) \triangleq \Gamma_2(f(\mathrm{X}_1))$ for every $\mathrm{X}_1 \in \mathfrak{X}_1$. Clearly, $\Gamma_1 \in \mathrm{Chc}(\mathfrak{X}_1)$, as $\Gamma_1(\mathrm{X}_1) = \Gamma_2(f(\mathrm{X}_1)) \in f(\mathrm{X}_1) \subseteq \mathrm{X}_1$, and thus $\mathsf{img}(\Gamma_1) \in \overline{\mathfrak{X}_1}$. The thesis follows from the fact that $\mathsf{img}(\Gamma_1) \subseteq \mathsf{img}(\Gamma_2) = \overline{\mathrm{X}_2}$.

Proof of point 2). Assume $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$ and let $\mathrm{X}_1' \in \mathsf{ext}_\Theta(\mathfrak{X}_1, p)$. We have to show that there exists $\mathrm{X}_2' \in \mathsf{ext}_\Theta(\mathfrak{X}_2, p)$ such that $\mathrm{X}_2' \subseteq \mathrm{X}_1'$. By definition of $\mathsf{ext}_\Theta(\mathfrak{X}_1, p)$, we have that $\mathrm{X}_1' = \mathsf{ext}(\mathrm{X}_1, \mathsf{F}, p)$ for some $\mathrm{X}_1 \in \mathfrak{X}_1$ and $\mathsf{F} \in \mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_1))$. By $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$ and $\mathrm{X}_1 \in \mathfrak{X}_1$, we have that there is $\mathrm{X}_2 \in \mathfrak{X}_2$ such that $\mathrm{X}_2 \subseteq \mathrm{X}_1$.

It clearly holds that $\mathsf{ext}(\mathrm{X}_2, \mathsf{F}, p) \subseteq \mathsf{ext}(\mathrm{X}_1, \mathsf{F}, p)$. The thesis follows, since $\mathsf{ext}(\mathrm{X}_2, \mathsf{F}, p) \in \mathsf{ext}_\Theta(\mathfrak{X}_2, p)$. $\square$

Theorem 2 (Hyperassignment Refinement)**.** *Let φ be a* $\mathsf{GFG\text{-}QPTL}$ *formula and $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathrm{HAsg}_\subseteq(\mathsf{free}(\varphi))$ with $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$. Then, $\mathfrak{X}_1 \models^{\exists\forall} \varphi$ implies $\mathfrak{X}_2 \models^{\exists\forall} \varphi$ and $\mathfrak{X}_2 \models^{\forall\exists} \varphi$ implies $\mathfrak{X}_1 \models^{\forall\exists} \varphi$.*

Proof. Assume $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$. Thus, there is a function $f : \mathfrak{X}_1 \to \mathfrak{X}_2$, such that $f(\mathrm{X}_1) \subseteq \mathrm{X}_1$. The claim is proved by induction on the structure of the formula and the alternation flags.

(base case) If $\varphi \in \mathsf{LTL}$, then the claims immediately follows from the semantics (Definition 3, item 1).

(inductive step)
- ($\varphi = \neg\psi$) We have, by semantics, $\mathfrak{X}_1 \models^{\exists\forall} \varphi$ if and only if $\mathfrak{X}_1 \not\models^{\forall\exists} \psi$. By inductive hypothesis, this implies $\mathfrak{X}_2 \not\models^{\forall\exists} \psi$, which amounts to $\mathfrak{X}_2 \models^{\exists\forall} \varphi$.

  On the other hand, we also have, by semantics, $\mathfrak{X}_2 \models^{\forall\exists} \varphi$ if and only if $\mathfrak{X}_2 \not\models^{\exists\forall} \psi$. By inductive hypothesis, this implies $\mathfrak{X}_1 \not\models^{\exists\forall} \psi$, which amounts to $\mathfrak{X}_1 \models^{\forall\exists} \varphi$.
- ($\varphi = \varphi_1 \wedge \varphi_2$) We have, by semantics, $\mathfrak{X}_1 \models^{\exists\forall} \varphi$ if and only if for every $(\mathfrak{X}_1', \mathfrak{X}_1'') \in \mathsf{par}(\mathfrak{X}_1)$ it holds true that $\mathfrak{X}_1' \neq \emptyset$ and $\mathfrak{X}_1' \models^{\exists\forall} \varphi_1$ or it holds true that $\mathfrak{X}_1'' \neq \emptyset$ and $\mathfrak{X}_1'' \models^{\exists\forall} \varphi_2$. Now, consider $(\mathfrak{X}_2', \mathfrak{X}_2'') \in \mathsf{par}(\mathfrak{X}_2)$ and $(\mathfrak{X}_1', \mathfrak{X}_1'') \in \mathsf{par}(\mathfrak{X}_1)$, where $\mathfrak{X}_1' \triangleq \{ \mathrm{X} \in \mathfrak{X}_1 \mid f(\mathrm{X}) \in \mathfrak{X}_2' \}$ and $\mathfrak{X}_1'' \triangleq \{ \mathrm{X} \in \mathfrak{X}_1 \mid f(\mathrm{X}) \in \mathfrak{X}_2'' \}$. We have that both $\mathfrak{X}_1' \sqsubseteq \mathfrak{X}_2'$ and $\mathfrak{X}_1'' \sqsubseteq \mathfrak{X}_2''$ hold. Thus, by inductive hypothesis, for every $(\mathfrak{X}_2', \mathfrak{X}_2'') \in \mathsf{par}(\mathfrak{X}_2)$ it holds true that $\mathfrak{X}_2' \neq \emptyset$ and $\mathfrak{X}_2' \models^{\exists\forall} \varphi_1$ or it holds true that $\mathfrak{X}_2'' \neq \emptyset$ and $\mathfrak{X}_2'' \models^{\exists\forall} \varphi_2$, which amounts to $\mathfrak{X}_2 \models^{\exists\forall} \varphi$.

  On the other hand, we also have, by semantics, $\mathfrak{X}_2 \models^{\forall\exists} \varphi$ if and only if $\overline{\mathfrak{X}_2} \models^{\exists\forall} \varphi$. By inductive hypothesis and Proposition 4, this implies $\overline{\mathfrak{X}_1} \models^{\exists\forall} \varphi$, which amounts to $\mathfrak{X}_1 \models^{\forall\exists} \varphi$.
- ($\varphi = \varphi_1 \vee \varphi_2$) We have, by semantics, $\mathfrak{X}_2 \models^{\forall\exists} \varphi$ if and only if there is $(\mathfrak{X}_2', \mathfrak{X}_2'') \in \mathsf{par}(\mathfrak{X}_2)$ such that $\mathfrak{X}_2' \neq \emptyset$ implies $\mathfrak{X}_2' \models^{\forall\exists} \varphi_1$ and $\mathfrak{X}_2'' \neq \emptyset$ implies $\mathfrak{X}_2'' \models^{\forall\exists} \varphi_2$. Now, let $(\mathfrak{X}_1', \mathfrak{X}_1'') \in \mathsf{par}(\mathfrak{X}_1)$, where $\mathfrak{X}_1' \triangleq \{ \mathrm{X} \in \mathfrak{X}_1 \mid f(\mathrm{X}) \in \mathfrak{X}_2' \}$ and $\mathfrak{X}_1'' \triangleq \{ \mathrm{X} \in \mathfrak{X}_1 \mid f(\mathrm{X}) \in \mathfrak{X}_2'' \}$. We have that both $\mathfrak{X}_1' \sqsubseteq \mathfrak{X}_2'$ and $\mathfrak{X}_1'' \sqsubseteq \mathfrak{X}_2''$ hold. Moreover, we have that $\mathfrak{X}_2' = \emptyset$ implies $\mathfrak{X}_1' = \emptyset$ and $\mathfrak{X}_2'' = \emptyset$ implies $\mathfrak{X}_1'' = \emptyset$. In addition, by inductive hypothesis, $\mathfrak{X}_2' \models^{\forall\exists} \varphi_1$ implies $\mathfrak{X}_1' \models^{\forall\exists} \varphi_1$ and $\mathfrak{X}_2'' \models^{\forall\exists} \varphi_2$ implies $\mathfrak{X}_1'' \models^{\forall\exists} \varphi_2$. Therefore, $\mathfrak{X}_1 \models^{\forall\exists} \varphi$ holds.

  On the other hand, we also have, by semantics, $\mathfrak{X}_1 \models^{\exists\forall} \varphi$ if and only if $\overline{\mathfrak{X}_1} \models^{\forall\exists} \varphi$. By inductive hypothesis and Proposition 4, this implies $\overline{\mathfrak{X}_2} \models^{\forall\exists} \varphi$, which amounts to $\mathfrak{X}_2 \models^{\exists\forall} \varphi$.
- ($\varphi = \exists p \colon \Theta. \psi$) We have, by semantics, $\mathfrak{X}_1 \models^{\exists\forall} \varphi$ if and only if $\mathsf{ext}_\Theta(\mathfrak{X}_1, p) \models^{\exists\forall} \psi$. By inductive hypothesis and Proposition 4, this implies $\mathsf{ext}_\Theta(\mathfrak{X}_2, p) \models^{\exists\forall} \psi$, which amounts to $\mathfrak{X}_2 \models^{\exists\forall} \varphi$.

  On the other hand, we also have, by semantics, $\mathfrak{X}_2 \models^{\forall\exists} \varphi$ if and only if $\overline{\mathfrak{X}_2} \models^{\exists\forall} \varphi$. By inductive hypothesis and Proposition 4, this implies $\overline{\mathfrak{X}_1} \models^{\exists\forall} \varphi$, which amounts to $\mathfrak{X}_1 \models^{\forall\exists} \varphi$.
- ($\varphi = \forall p \colon \Theta. \psi$) We have, by semantics, $\mathfrak{X}_2 \models^{\forall\exists} \varphi$ if and only if $\mathsf{ext}_\Theta(\mathfrak{X}_2, p) \models^{\forall\exists} \psi$. By inductive hypothesis and Proposition 4, this implies $\mathsf{ext}_\Theta(\mathfrak{X}_1, p) \models^{\forall\exists} \psi$, which amounts to $\mathfrak{X}_1 \models^{\forall\exists} \varphi$.

  On the other hand, we also have, by semantics, $\mathfrak{X}_1 \models^{\exists\forall} \varphi$ if and only if $\overline{\mathfrak{X}_1} \models^{\forall\exists} \varphi$. By inductive hypothesis and Proposition 4, this implies $\overline{\mathfrak{X}_2} \models^{\forall\exists} \varphi$, which amounts to $\mathfrak{X}_2 \models^{\exists\forall} \varphi$. $\square$

Theorem 3 (Duality)**.** *Let φ be a* $\mathsf{GFG\text{-}QPTL}$ *formula and $\mathfrak{X} \in \mathrm{HAsg}_\subseteq(\mathsf{free}(\varphi))$. Then, $\overline{\mathfrak{X}} \models^{\overline{\alpha}} \varphi$ iff $\mathfrak{X} \models^{\alpha} \varphi$ iff $\overline{\overline{\mathfrak{X}}} \models^{\alpha} \varphi$.*

Proof. The fact that $\mathfrak{X} \models^{\alpha} \varphi$ *iff* $\overline{\overline{\mathfrak{X}}} \models^{\alpha} \varphi$ immediately follows from $\mathfrak{X} \equiv \overline{\overline{\mathfrak{X}}}$ (Proposition 1) and Corollary 1.

We turn now on proving that $\mathfrak{X} \models^{\alpha} \varphi$ *iff* $\overline{\mathfrak{X}} \models^{\overline{\alpha}} \varphi$, for all $\mathfrak{X} \in \mathrm{HAsg}_{\subseteq}(\mathsf{free}(\varphi))$. The proof is by induction on the structure of the formula.

(base case) If $\varphi \in \mathsf{LTL}$, then the claim follows immediately from the semantics and Lemma 1.

(inductive step) If $\varphi = \neg\psi$, then we have: $\mathfrak{X} \models^{\alpha} \varphi \overset{\text{sem.}}{\Leftrightarrow} \mathfrak{X} \not\models^{\overline{\alpha}} \psi \overset{\text{ind.hp.}}{\Leftrightarrow} \overline{\mathfrak{X}} \not\models^{\alpha} \psi \overset{\text{sem.}}{\Leftrightarrow} \overline{\mathfrak{X}} \models^{\overline{\alpha}} \varphi$.

If $\varphi = \varphi_1 \wedge \varphi_2$, then we have:
- $\mathfrak{X} \models^{\exists\forall} \varphi \overset{\text{Thm. 3}}{\Leftrightarrow} \overline{\overline{\mathfrak{X}}} \models^{\exists\forall} \varphi \overset{\text{sem.}}{\Leftrightarrow} \overline{\mathfrak{X}} \models^{\forall\exists} \varphi$; and
- $\mathfrak{X} \models^{\forall\exists} \varphi \overset{\text{sem.}}{\Leftrightarrow} \overline{\mathfrak{X}} \models^{\exists\forall} \varphi$.

If $\varphi = \varphi_1 \vee \varphi_2$, then we have:
- $\mathfrak{X} \models^{\exists\forall} \varphi \overset{\text{sem.}}{\Leftrightarrow} \overline{\mathfrak{X}} \models^{\forall\exists} \varphi$; and
- $\mathfrak{X} \models^{\forall\exists} \varphi \overset{\text{Thm. 3}}{\Leftrightarrow} \overline{\overline{\mathfrak{X}}} \models^{\forall\exists} \varphi \overset{\text{sem.}}{\Leftrightarrow} \overline{\mathfrak{X}} \models^{\exists\forall} \varphi$.

If $\varphi = \exists p\colon \Theta.\, \psi$, then we have:
- $\mathfrak{X} \models^{\exists\forall} \varphi \overset{\text{Thm. 3}}{\Leftrightarrow} \overline{\overline{\mathfrak{X}}} \models^{\exists\forall} \varphi \overset{\text{sem.}}{\Leftrightarrow} \overline{\mathfrak{X}} \models^{\forall\exists} \varphi$; and
- $\mathfrak{X} \models^{\forall\exists} \varphi \overset{\text{sem.}}{\Leftrightarrow} \overline{\mathfrak{X}} \models^{\exists\forall} \varphi$.

If $\varphi = \forall p\colon \Theta.\, \psi$, then we have:
- $\mathfrak{X} \models^{\exists\forall} \varphi \overset{\text{sem.}}{\Leftrightarrow} \overline{\mathfrak{X}} \models^{\forall\exists} \varphi$;
- $\mathfrak{X} \models^{\forall\exists} \varphi \overset{\text{Thm. 3}}{\Leftrightarrow} \overline{\overline{\mathfrak{X}}} \models^{\forall\exists} \varphi \overset{\text{sem.}}{\Leftrightarrow} \overline{\mathfrak{X}} \models^{\exists\forall} \varphi$. □

Lemma 4 (Boolean Laws)**.** *Let φ, φ_1, φ_2 be* GFG-QPTL *formulae: 1) $\varphi \equiv \neg\neg\varphi$; 2) $\varphi_1 \wedge \varphi_2 \Rightarrow \varphi_1$; 3) $\varphi_1 \Rightarrow \varphi_1 \vee \varphi_2$; 4) $\varphi_1 \wedge (\varphi \wedge \varphi_2) \equiv (\varphi_1 \wedge \varphi) \wedge \varphi_2$; 5) $\varphi_1 \vee (\varphi \vee \varphi_2) \equiv (\varphi_1 \vee \varphi) \vee \varphi_2$; 6) $\varphi_1 \wedge \varphi_2 \equiv \neg(\neg\varphi_1 \vee \neg\varphi_2)$; 7) $\varphi_1 \vee \varphi_2 \equiv \neg(\neg\varphi_1 \wedge \neg\varphi_2)$; 8) $\exists^{\Theta} p.\, \varphi \equiv \neg(\forall^{\Theta} p.\, \neg\varphi)$; 9) $\forall^{\Theta} p.\, \varphi \equiv \neg(\exists^{\Theta} p.\, \neg\varphi)$.*

Proof. Thanks to Corollary 2, it suffices to prove the equivalence for $\equiv^{\alpha}$ for some $\alpha \in \{\exists\forall, \forall\exists\}$. Let φ be a QPTL formula and $\mathfrak{X} \in \mathrm{HAsg}(\mathsf{free}(\varphi))$ a hyperassignment.

1) $\mathfrak{X} \models^{\exists\forall} \neg\neg\varphi \overset{\text{sem.}}{\Leftrightarrow} \mathfrak{X} \not\models^{\forall\exists} \neg\varphi \overset{\text{sem.}}{\Leftrightarrow} \mathfrak{X} \models^{\exists\forall} \varphi$;

2-5) Trivial and omitted.

6) $\mathfrak{X} \models^{\exists\forall} \varphi_1 \wedge \varphi_2 \overset{\text{sem.}}{\Leftrightarrow} \forall(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$ it holds ($\mathfrak{X}_1 \neq \emptyset$ and $\mathfrak{X}_1 \models^{\exists\forall} \varphi_1$) or ($\mathfrak{X}_2 \neq \emptyset$ and $\mathfrak{X}_2 \models^{\exists\forall} \varphi_2$) $\overset{\text{1.,sem.}}{\Leftrightarrow}$ $\forall(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$ it holds ($\mathfrak{X}_1 \neq \emptyset$ and $\mathfrak{X}_1 \not\models^{\forall\exists} \neg\varphi_1$) or ($\mathfrak{X}_2 \neq \emptyset$ and $\mathfrak{X}_2 \not\models^{\forall\exists} \neg\varphi_2$) $\overset{\text{sem.}}{\Leftrightarrow} \mathfrak{X} \not\models^{\forall\exists} \neg\varphi_1 \vee \neg\varphi_2 \overset{\text{sem.}}{\Leftrightarrow} \mathfrak{X} \models^{\exists\forall} \neg(\neg\varphi_1 \vee \neg\varphi_2)$;

7) $\mathfrak{X} \models^{\forall\exists} \varphi_1 \vee \varphi_2 \overset{\text{sem.}}{\Leftrightarrow} \exists(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$ s.t. ($\mathfrak{X}_1 \neq \emptyset$ implies $\mathfrak{X}_1 \models^{\forall\exists} \varphi_1$) and ($\mathfrak{X}_2 \neq \emptyset$ implies $\mathfrak{X}_2 \models^{\forall\exists} \varphi_2$) $\overset{\text{1.,sem.}}{\Leftrightarrow}$ $\exists(\mathfrak{X}_1, \mathfrak{X}_2) \in \mathsf{par}(\mathfrak{X})$ s.t. ($\mathfrak{X}_1 \neq \emptyset$ implies $\mathfrak{X}_1 \not\models^{\exists\forall} \neg\varphi_1$) and ($\mathfrak{X}_2 \neq \emptyset$ implies $\mathfrak{X}_2 \not\models^{\exists\forall} \neg\varphi_2$) $\overset{\text{sem.}}{\Leftrightarrow} \mathfrak{X} \not\models^{\exists\forall} \neg\varphi_1 \wedge \neg\varphi_2 \overset{\text{sem.}}{\Leftrightarrow} \mathfrak{X} \models^{\forall\exists} \neg(\neg\varphi_1 \wedge \neg\varphi_2)$;

8) $\mathfrak{X} \models^{\exists\forall} \exists p\colon \Theta.\, \psi \overset{\text{sem.}}{\Leftrightarrow} \mathsf{ext}_{\Theta}(\mathfrak{X}, p) \models^{\exists\forall} \psi \overset{\text{1.,sem.}}{\Leftrightarrow} \mathsf{ext}_{\Theta}(\mathfrak{X}, p) \not\models^{\forall\exists} \neg\psi \overset{\text{sem.}}{\Leftrightarrow} \mathfrak{X} \not\models^{\forall\exists} \forall p\colon \Theta.\, \neg\psi \overset{\text{sem.}}{\Leftrightarrow} \mathfrak{X} \models^{\exists\forall} \neg\forall p\colon \Theta.\, \neg\psi$;

9) $\mathfrak{X} \models^{\forall\exists} \forall p\colon \Theta.\, \psi \overset{\text{sem.}}{\Leftrightarrow} \mathsf{ext}_{\Theta}(\mathfrak{X}, p) \models^{\forall\exists} \psi \overset{\text{1.,sem.}}{\Leftrightarrow} \mathsf{ext}_{\Theta}(\mathfrak{X}, p) \not\models^{\exists\forall} \neg\psi \overset{\text{sem.}}{\Leftrightarrow} \mathfrak{X} \not\models^{\exists\forall} \exists p\colon \Theta.\, \neg\psi \overset{\text{sem.}}{\Leftrightarrow} \mathfrak{X} \models^{\forall\exists} \neg\exists p\colon \Theta.\, \neg\psi$. □

Proposition 5. *Let $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathrm{HAsg}$ with $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$ and $\wp \in \mathrm{Qn}$:* $\mathsf{evl}_{\alpha}(\mathfrak{X}_1, \wp) \sqsubseteq \mathsf{evl}_{\alpha}(\mathfrak{X}_2, \wp)$.

Proof. The proof proceeds by induction on the length of the quantification prefix $\wp$.

If $\wp = \varepsilon$ (base case), then we have $\mathsf{evl}_{\alpha}(\mathfrak{X}_1, \wp) = \mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2 = \mathsf{evl}_{\alpha}(\mathfrak{X}_2, \wp)$.

If $\wp = \mathtt{Q}^{\Theta} p.\wp'$ (inductive step), then we distinguish two cases.

- If α and $\mathtt{Q}$ are coherent, then we have $\mathsf{evl}_{\alpha}(\mathfrak{X}_1, \wp) = \mathsf{evl}_{\alpha}(\mathsf{ext}_{\Theta}(\mathfrak{X}_1, p), \wp')$ and $\mathsf{evl}_{\alpha}(\mathfrak{X}_2, \wp) = \mathsf{evl}_{\alpha}(\mathsf{ext}_{\Theta}(\mathfrak{X}_2, p), \wp')$. By Proposition 4, $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$ implies $\mathsf{ext}_{\Theta}(\mathfrak{X}_1, p) \sqsubseteq \mathsf{ext}_{\Theta}(\mathfrak{X}_2, p)$ and, by inductive hypothesis, $\mathsf{evl}_{\alpha}(\mathsf{ext}_{\Theta}(\mathfrak{X}_1, p), \wp') \sqsubseteq \mathsf{evl}_{\alpha}(\mathsf{ext}_{\Theta}(\mathfrak{X}_2, p), \wp')$, hence the thesis.
- If α and $\mathtt{Q}$ are not coherent, then we have $\mathsf{evl}_{\alpha}(\mathfrak{X}_1, \wp) = \mathsf{evl}_{\alpha}(\overline{\mathsf{ext}_{\Theta}(\overline{\mathfrak{X}_1}, p)}, \wp')$ and $\mathsf{evl}_{\alpha}(\mathfrak{X}_2, \wp) = \mathsf{evl}_{\alpha}(\overline{\mathsf{ext}_{\Theta}(\overline{\mathfrak{X}_2}, p)}, \wp')$. By Proposition 4, $\mathfrak{X}_1 \sqsubseteq \mathfrak{X}_2$ implies $\overline{\mathsf{ext}_{\Theta}(\overline{\mathfrak{X}_1}, p)} \sqsubseteq \overline{\mathsf{ext}_{\Theta}(\overline{\mathfrak{X}_2}, p)}$, and, by inductive hypothesis, $\mathsf{evl}_{\alpha}(\overline{\mathsf{ext}_{\Theta}(\overline{\mathfrak{X}_1}, p)}, \wp') \sqsubseteq \mathsf{evl}_{\alpha}(\overline{\mathsf{ext}_{\Theta}(\overline{\mathfrak{X}_2}, p)}, \wp')$, hence the thesis. □

Lemma 5 (Prefix Evolution)**.** *Let $\wp\phi$ be a* GFG-QPTL *formula with $\wp \in \mathrm{Qn}$. Then, $\mathfrak{X} \models^{\alpha} \wp\phi$ iff $\mathsf{evl}_{\alpha}(\mathfrak{X}, \wp) \models^{\alpha} \phi$, for all $\mathfrak{X} \in \mathrm{HAsg}(\mathsf{free}(\wp\phi))$.*

Proof. (base case) When $\wp = \varepsilon$ the claim is trivial.

(inductive step)
- $\mathfrak{X} \models^{\exists\forall} \exists^{\Theta} p.\, \wp.\, \psi \Leftrightarrow \mathsf{ext}_{\Theta}(\mathfrak{X}, p) \models^{\exists\forall} \wp.\, \psi \Leftrightarrow \mathsf{evl}_{\exists\forall}(\mathsf{ext}_{\Theta}(\mathfrak{X}, p), \wp) \models^{\exists\forall} \psi \Leftrightarrow \mathsf{evl}_{\exists\forall}(\mathfrak{X}, \exists^{\Theta} p.\, \wp) \models^{\exists\forall} \psi$.
- $\mathfrak{X} \models^{\forall\exists} \exists^{\Theta} p.\, \wp.\, \psi \Leftrightarrow \overline{\mathfrak{X}} \models^{\exists\forall} \exists^{\Theta} p.\, \wp.\, \psi \Leftrightarrow \mathsf{ext}_{\Theta}(\overline{\mathfrak{X}}, p) \models^{\exists\forall} \wp.\, \psi \Leftrightarrow \overline{\mathsf{ext}_{\Theta}(\overline{\mathfrak{X}}, p)} \models^{\forall\exists} \wp.\, \psi \Leftrightarrow \mathsf{evl}_{\forall\exists}(\overline{\mathsf{ext}_{\Theta}(\overline{\mathfrak{X}}, p)}, \wp) \models^{\forall\exists} \psi \Leftrightarrow \mathsf{evl}_{\forall\exists}(\mathfrak{X}, \exists^{\Theta} p.\, \wp) \models^{\forall\exists} \psi$.
- $\mathfrak{X} \models^{\exists\forall} \forall^{\Theta} p.\, \wp.\, \psi \Leftrightarrow \overline{\mathfrak{X}} \models^{\forall\exists} \forall^{\Theta} p.\, \wp.\, \psi \Leftrightarrow \mathsf{ext}_{\Theta}(\overline{\mathfrak{X}}, p) \models^{\forall\exists} \wp.\, \psi \Leftrightarrow \overline{\mathsf{ext}_{\Theta}(\overline{\mathfrak{X}}, p)} \models^{\exists\forall} \wp.\, \psi \Leftrightarrow \mathsf{evl}_{\exists\forall}(\overline{\mathsf{ext}_{\Theta}(\overline{\mathfrak{X}}, p)}, \wp) \models^{\exists\forall} \psi \Leftrightarrow \mathsf{evl}_{\exists\forall}(\mathfrak{X}, \forall^{\Theta} p.\, \wp) \models^{\exists\forall} \psi$.
- $\mathfrak{X} \models^{\forall\exists} \forall^{\Theta} p.\, \wp.\, \psi \Leftrightarrow \mathsf{ext}_{\Theta}(\mathfrak{X}, p) \models^{\forall\exists} \wp.\, \psi \Leftrightarrow \mathsf{evl}_{\forall\exists}(\mathsf{ext}_{\Theta}(\mathfrak{X}, p), \wp) \models^{\forall\exists} \psi \Leftrightarrow \mathsf{evl}_{\forall\exists}(\mathfrak{X}, \forall^{\Theta} p.\, \wp) \models^{\forall\exists} \psi$. □

We now introduce an alternative to evl that we use to prove theorems presented in this paper. To use this alternative, we prove propositions that have not been expressed in the main article.

Proposition 7. *Let $\mathfrak{X} \in \mathrm{HAsg}(\mathrm{P})$ with $\mathrm{P} \subseteq \mathrm{AP}$, $\Theta \in \mathbb{O}$, $p \in \mathrm{AP}\backslash\mathrm{P}$, and $\Psi \subseteq \mathrm{Asg}(\mathrm{P} \cup \{p\})$. There exists $\mathrm{W} \in \mathsf{evl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta} p)$ such that $\mathrm{W} \subseteq \Psi$* iff *the following conditions hold true:*

1) *there exist $\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathrm{P})$ and $\mathrm{X} \in \mathfrak{X}$ such that $\mathsf{ext}(\mathrm{X}, \mathsf{F}, p) \subseteq \Psi$, whenever α and $\mathtt{Q}$ are coherent;*
2) *for all $\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathrm{P})$, there is $\mathrm{X} \in \mathfrak{X}$ such that $\mathsf{ext}(\mathrm{X}, \mathsf{F}, p) \subseteq \Psi$, whenever α and $\mathtt{Q}$ are not coherent.*

Proof. If α and $\mathtt{Q}$ are coherent, then $\mathsf{evl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta} p) = \mathsf{ext}_{\Theta}(\mathfrak{X}, p) = \{\, \mathsf{ext}(\mathrm{X}, \mathsf{F}, p) \mid \mathrm{X} \in \mathfrak{X}, \mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X})) \,\}$, and we have: $\exists \mathrm{W} \in \mathsf{evl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta} p)\ .\ \mathrm{W} \subseteq \Psi \Leftrightarrow \exists \mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathrm{P})\ \exists \mathrm{X} \in \mathfrak{X}\ .\ \mathsf{ext}(\mathrm{X}, \mathsf{F}, p) \subseteq \Psi$.

If α and $\mathtt{Q}$ are not coherent, then we proceed as follows: $\exists \mathrm{W} \in \mathsf{evl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta} p)\ .\ \mathrm{W} \subseteq \Psi \Leftrightarrow$

$\exists \mathrm{W} \in \overline{\mathsf{ext}_{\Theta}(\overline{\mathfrak{X}}, p)} \,.\, \mathrm{W} \subseteq \Psi \Leftrightarrow \exists \Gamma \in \mathrm{Chc}(\mathsf{ext}_{\Theta}(\overline{\mathfrak{X}}, p)) \,.\, \mathsf{img}(\Gamma) = \{ \Gamma(\mathrm{Z}) \mid \mathrm{Z} \in \mathsf{ext}_{\Theta}(\overline{\mathfrak{X}}, p) \} \subseteq \Psi \Leftrightarrow \exists \Gamma \in \mathrm{Chc}(\mathsf{ext}_{\Theta}(\overline{\mathfrak{X}}, p)) \,.\, \forall \mathrm{Z} \in \mathsf{ext}_{\Theta}(\overline{\mathfrak{X}}, p) \,.\, \Gamma(\mathrm{Z}) \in \Psi \Leftrightarrow \forall \mathrm{Z} \in \mathsf{ext}_{\Theta}(\overline{\mathfrak{X}}, p) \,.\, \exists \chi_{\mathrm{Z}} \in \mathrm{Z} \,.\, \chi_{\mathrm{Z}} \in \Psi \Leftrightarrow \forall \mathsf{G} \in \mathrm{Fnc}_{\Theta}(\mathrm{P}) \,.\, \forall \mathrm{Y} \in \overline{\mathfrak{X}} \,.\, \exists \chi_{\mathsf{G},\mathrm{Y}} \in \mathsf{ext}(\mathrm{Y}, \mathsf{G}, p) \,.\, \chi_{\mathsf{G},\mathrm{Y}} \in \Psi \Leftrightarrow \forall \mathsf{G} \in \mathrm{Fnc}_{\Theta}(\mathrm{P}) \,.\, \forall \Lambda \in \mathrm{Chc}(\mathfrak{X}) \,.\, \exists \chi_{\mathsf{G},\Lambda} \in \mathsf{ext}(\mathsf{img}(\Lambda), \mathsf{G}, p) \,.\, \chi_{\mathsf{G},\Lambda} \in \Psi \Leftrightarrow \forall \mathsf{G} \in \mathrm{Fnc}_{\Theta}(\mathrm{P}) \,.\, \forall \Lambda \in \mathrm{Chc}(\mathfrak{X}) \,.\, \exists \chi_{\mathsf{G},\Lambda} \in \mathsf{ext}(\{ \Lambda(\mathrm{X}) \mid \mathrm{X} \in \mathfrak{X} \}, \mathsf{G}, p) \,.\, \chi_{\mathsf{G},\Lambda} \in \Psi \Leftrightarrow \forall \mathsf{G} \in \mathrm{Fnc}_{\Theta}(\mathrm{P}) \,.\, \forall \Lambda \in \mathrm{Chc}(\mathfrak{X}) \,.\, \exists \mathrm{X} \in \mathfrak{X} \,.\, \mathsf{ext}(\Lambda(\mathrm{X}), \mathsf{G}, p) \in \Psi \Leftrightarrow \forall \mathsf{G} \in \mathrm{Fnc}_{\Theta}(\mathrm{P}) \,.\, \exists \mathrm{X} \in \mathfrak{X} \,.\, \forall \chi \in \mathrm{X} \,.\, \mathsf{ext}(\chi, \mathsf{G}, p) \in \Psi \Leftrightarrow \forall \mathsf{G} \in \mathrm{Fnc}_{\Theta}(\mathrm{P}) \,.\, \exists \mathrm{X} \in \mathfrak{X} \,.\, \mathsf{ext}(\mathrm{X}, \mathsf{G}, p) \subseteq \Psi$. □

We now introduce nevl. First, we define it only on $\mathtt{Q}^{\Theta}p$. $\mathsf{nevl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta}p) \triangleq \mathsf{ext}_{\Theta}(\mathfrak{X}, p)$, if α and $\mathtt{Q}$ are coherent, and $\mathsf{nevl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta}p) \triangleq \{ \mathsf{ext}(\eth, p) \mid \eth \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X})) \to \mathfrak{X} \}$, otherwise, where $\mathsf{ext}(\eth, p) \triangleq \bigcup_{\mathsf{F} \in \mathsf{dom}(\eth)} \mathsf{ext}(\eth(\mathsf{F}), \mathsf{F}, p)$.

Proposition 8. *If* $\mathfrak{X}_1 \equiv \mathfrak{X}_2$ *then* $\mathsf{nevl}_{\alpha}(\mathfrak{X}_1, \mathtt{Q}^{\Theta}p) \equiv \mathsf{evl}_{\alpha}(\mathfrak{X}_2, \mathtt{Q}^{\Theta}p)$*, for all* $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathrm{HAsg}$*,* $\mathtt{Q} \in \{\exists, \forall\}$*,* $\Theta \in \mathbb{O}$*, and* $p \in \mathrm{AP} \setminus \mathsf{ap}(\mathfrak{X})$*.*

Proof. Assume α and $\mathtt{Q}$ to be coherent. Then, $\mathsf{nevl}_{\alpha}(\mathfrak{X}_1, \mathtt{Q}^{\Theta}p) = \mathsf{ext}_{\Theta}(\mathfrak{X}_1, p) \equiv \mathsf{ext}_{\Theta}(\mathfrak{X}_2, p) = \mathsf{evl}_{\alpha}(\mathfrak{X}_2, \mathtt{Q}^{\Theta}p)$.

Assume, now, α and $\mathtt{Q}$ not to be coherent. Then, $\mathsf{nevl}_{\alpha}(\mathfrak{X}_1, \mathtt{Q}^{\Theta}p) = \{ \bigcup_{\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}_1))} \mathsf{ext}(\eth(\mathsf{F}), \mathsf{F}, p) \mid \eth \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}_1)) \to \mathfrak{X}_1 \}$.

In order to prove that $\mathsf{nevl}_{\alpha}(\mathfrak{X}_1, \mathtt{Q}^{\Theta}p) \sqsubseteq \mathsf{evl}_{\alpha}(\mathfrak{X}_2, \mathtt{Q}^{\Theta}p)$, we let $\Psi \in \mathsf{nevl}_{\alpha}(\mathfrak{X}_1, \mathtt{Q}^{\Theta}p)$ and we show that there is $W \in \mathsf{evl}_{\alpha}(\mathfrak{X}_2, \mathtt{Q}^{\Theta}p)$ with $W \subseteq \Psi$. By the definition of $\mathsf{nevl}_{\alpha}(\mathfrak{X}_1, \mathtt{Q}^{\Theta}p)$, we have that $\Psi = \bigcup_{\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}_1))} \mathsf{ext}(\eth(\mathsf{F}), \mathsf{F}, p)$ for some $\eth \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}_1)) \to \mathfrak{X}_1$. This means that for every $\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}_1))$ there is $\mathrm{X}_{\mathsf{F}} \triangleq \eth(\mathsf{F}) \in \mathfrak{X}_1$ such that $\mathsf{ext}(\mathrm{X}_{\mathsf{F}}, \mathsf{F}, p) \subseteq \Psi$. By Proposition 7, there exists $\mathrm{W}' \in \mathsf{evl}_{\alpha}(\mathfrak{X}_1, \mathtt{Q}^{\Theta}p)$ such that $\mathrm{W}' \subseteq \Psi$. Since, due to Proposition 5, $\mathfrak{X}_1 \equiv \mathfrak{X}_2$ implies $\mathsf{evl}_{\alpha}(\mathfrak{X}_1, \mathtt{Q}^{\Theta}p) \equiv \mathsf{evl}_{\alpha}(\mathfrak{X}_2, \mathtt{Q}^{\Theta}p)$, we have that there is $\mathrm{W} \in \mathsf{evl}_{\alpha}(\mathfrak{X}_2, \mathtt{Q}^{\Theta}p)$ such that $\mathrm{W} \subseteq \mathrm{W}' \subseteq \Psi$.

In order to prove the converse, *i.e.*, $\mathsf{evl}_{\alpha}(\mathfrak{X}_2, \mathtt{Q}^{\Theta}p) \sqsubseteq \mathsf{nevl}_{\alpha}(\mathfrak{X}_1, \mathtt{Q}^{\Theta}p)$, let $\Psi \in \mathsf{evl}_{\alpha}(\mathfrak{X}_2, \mathtt{Q}^{\Theta}p)$. By instantiating W with Ψ in Proposition 7, we have that for all $\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}_2))$, there is $\mathrm{X}'_{\mathsf{F}} \in \mathfrak{X}_2$ such that $\mathsf{ext}(\mathrm{X}'_{\mathsf{F}}, \mathsf{F}, p) \subseteq \Psi$. By $\mathfrak{X}_1 \equiv \mathfrak{X}_2$, we have that there is $\mathrm{X}_{\mathsf{F}} \in \mathfrak{X}_1$ such that $\mathrm{X}_{\mathsf{F}} \subseteq \mathrm{X}'_{\mathsf{F}}$, which, in turn, implies $\mathsf{ext}(\mathrm{X}_{\mathsf{F}}, \mathsf{F}, p) \subseteq \mathsf{ext}(\mathrm{X}'_{\mathsf{F}}, \mathsf{F}, p) \subseteq \Psi$. Now, define $\eth$ as $\eth(\mathsf{F}) \triangleq \mathrm{X}_{\mathsf{F}} \in \mathfrak{X}_1$ for every $\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}_1)) = \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}_2))$. Clearly, both of the following hold: $\bigcup_{\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}_1))} \mathsf{ext}(\eth(\mathsf{F}), \mathsf{F}, p) \in \mathsf{nevl}_{\alpha}(\mathfrak{X}_1, \mathtt{Q}^{\Theta}p)$ and $\bigcup_{\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}_1))} \mathsf{ext}(\eth(\mathsf{F}), \mathsf{F}, p) \subseteq \Psi$; hence the thesis. □

We extend the definition of nevl to any quantifier prefix.

- $\mathsf{nevl}_{\alpha}(\mathfrak{X}, \epsilon) \triangleq \mathfrak{X}$;
- $\mathsf{nevl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta}p.\,\wp) \triangleq \mathsf{nevl}_{\alpha}(\mathsf{nevl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta}p), \wp)$.

$$\mathsf{nevl}_{\alpha}(\wp) \triangleq \mathsf{nevl}_{\alpha}(\{\{\varnothing\}\}, \wp)$$

Functions evl and nevl preserve equivalence between hyperassignments, as stated in the next proposition. Thus, in what follows, we will use the two functions interchangeably, as they are only involved in functions and relations that are invariant with respect to equivalence between hyperassignments.

Proposition 9. *If* $\mathfrak{X}_1 \equiv \mathfrak{X}_2$ *then* $\mathsf{nevl}_{\alpha}(\mathfrak{X}_1, \wp) \equiv \mathsf{evl}_{\alpha}(\mathfrak{X}_2, \wp)$*, for all* $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathrm{HAsg}$ *and* $\wp \in \mathrm{Qn}$*, with* $\mathsf{ap}(\mathfrak{X}) \cap \mathsf{ap}(\wp) = \emptyset$*.*

Proof. The proof proceeds by induction on the length of the quantification prefix $\wp$.

(base case) If $\wp = \varepsilon$, then we have $\mathsf{nevl}_{\alpha}(\mathfrak{X}_1, \wp) = \mathfrak{X}_1 \equiv \mathfrak{X}_2 = \mathsf{evl}_{\alpha}(\mathfrak{X}_2, \wp)$.

(inductive step) If $\wp = \mathtt{Q}^{\Theta}p.\wp'$, then $\mathsf{nevl}_{\alpha}(\mathfrak{X}_1, \wp) = \mathsf{nevl}_{\alpha}(\mathsf{nevl}_{\alpha}(\mathfrak{X}_1, \mathtt{Q}^{\Theta}p), \wp')$. The thesis follows by a straightforward application of Proposition 8 and the inductive hypothesis. □

Let $\mathrm{P}, \mathrm{P}' \subseteq \mathrm{AP}$, $\chi \in \mathrm{Asg}(\mathrm{P})$, and $\mathrm{X} \subseteq \mathrm{Asg}(\mathrm{P})$. We define $(\chi \setminus \mathrm{P}') \triangleq \chi \upharpoonright (\mathrm{P} \setminus \mathrm{P}')$ and $\mathrm{X} \setminus \mathrm{P}' \triangleq \{ \chi \setminus \mathrm{P}' \mid \chi \in \mathrm{X} \}$. Notice that $\mathrm{X} \subseteq \mathrm{Y}$ implies $\mathrm{X} \setminus \{p\} \subseteq \mathrm{Y} \setminus \{p\}$, for all $\mathrm{X}, \mathrm{Y} \subseteq \mathrm{Asg}(\mathrm{P})$, $\mathrm{P} \subseteq \mathrm{AP}$, and $p \in \mathrm{P}$, and $(\bigcup_{\mathrm{X} \in \mathfrak{X}} \mathrm{X}) \setminus \{p\} = \bigcup_{\mathrm{X} \in \mathfrak{X}} (\mathrm{X} \setminus \{p\})$ for all $\mathfrak{X} \in \mathrm{HAsg}(\mathrm{P})$, $\mathrm{P} \subseteq \mathrm{AP}$, and $p \in \mathrm{P}$.

Proposition 10. $\mathrm{Y} \setminus \mathsf{ap}(\wp) \subseteq \bigcup \mathfrak{X}$*, for all* $\mathrm{Y} \in \mathsf{nevl}_{\alpha}(\mathfrak{X}, \wp)$*,* $\mathfrak{X} \in \mathrm{HAsg}(\mathrm{P})$*, and* $\wp \in \mathrm{Qn}$*, with* $\mathrm{P} \subseteq \mathrm{AP}$ *and* $\mathsf{ap}(\wp) \cap \mathrm{P} = \emptyset$*.*

Proof. The proof proceeds by induction on the length of the quantification prefix $\wp$.

(base case) If $\wp = \varepsilon$, then we have $\mathsf{nevl}_{\alpha}(\mathfrak{X}, \wp) = \mathfrak{X}$, and the claim follows trivially.

If $\wp = \mathtt{Q}^{\Theta}p$, let $\mathrm{Y} \in \mathsf{nevl}_{\alpha}(\mathfrak{X}, \wp)$. If α and $\mathtt{Q}$ are coherent, then $\mathsf{nevl}_{\alpha}(\mathfrak{X}, \wp) = \mathsf{ext}_{\Theta}(\mathfrak{X}, p)$. Thus, $\mathrm{Y} = \mathsf{ext}(\mathrm{X}, \mathsf{F}, p)$ for some $\mathrm{X} \in \mathfrak{X}$ and $\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}))$. Trivially, $\mathrm{Y} \setminus \{p\} = \mathrm{X}$, and we are done. If, on the other hand, α and $\mathtt{Q}$ are not coherent, then $\mathsf{nevl}_{\alpha}(\mathfrak{X}, \wp) = \{ \bigcup_{\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}))} \mathsf{ext}(\eth(\mathsf{F}), \mathsf{F}, p) \mid \eth \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X})) \to \mathfrak{X} \}$. Thus, $\mathrm{Y} = \bigcup_{\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}))} \mathsf{ext}(\eth(\mathsf{F}), \mathsf{F}, p)$ for some $\eth \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X})) \to \mathfrak{X}$. Clearly, $\mathrm{Y} \setminus \{p\} = \bigcup_{\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}))} \eth(\mathsf{F})$. Since $\eth(\mathsf{F}) \in \mathfrak{X}$ for every $\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}))$, it holds that $\bigcup_{\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}))} \eth(\mathsf{F}) \subseteq \bigcup \mathfrak{X}$, hence the thesis.

(inductive step) If $\wp = \mathtt{Q}^{\Theta}p.\wp'$ (with $\wp' \neq \varepsilon$), then $\mathsf{nevl}_{\alpha}(\mathfrak{X}, \wp) = \mathsf{nevl}_{\alpha}(\mathsf{nevl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta}p), \wp')$. Let $\mathrm{Y} \in \mathsf{nevl}_{\alpha}(\mathfrak{X}, \wp)$. By inductive hypothesis, $\mathrm{Y} \setminus \mathsf{ap}(\wp') \subseteq \bigcup_{\mathrm{X} \in \mathsf{nevl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta}p)} \mathrm{X}$. Since by inductive hypothesis it also holds that $\mathrm{X} \setminus \{p\} \subseteq \bigcup \mathfrak{X}$ for every $\mathrm{X} \in \mathsf{nevl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta}p)$, we have that $\bigcup_{\mathrm{X} \in \mathsf{nevl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta}p)} (\mathrm{X} \setminus \{p\}) \subseteq \bigcup \mathfrak{X}$. Therefore, it holds $\mathrm{Y} \setminus \mathsf{ap}(\wp) = (\mathrm{Y} \setminus \mathsf{ap}(\wp')) \setminus \{p\} \subseteq (\bigcup_{\mathrm{X} \in \mathsf{nevl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta}p)} \mathrm{X}) \setminus \{p\} = \bigcup_{\mathrm{X} \in \mathsf{nevl}_{\alpha}(\mathfrak{X}, \mathtt{Q}^{\Theta}p)} (\mathrm{X} \setminus \{p\}) \subseteq \bigcup \mathfrak{X}$. □

C. *Proofs for Section IV*

A *path* $\pi \in \mathrm{Pth} \subseteq \mathrm{Ps}^{\infty}$ is a finite or infinite sequence of positions compatible with the move relation, *i.e.*, $((\pi)_i, (\pi)_{i+1}) \in Mv$, for all $i \in [0, |\pi| - 1)$; it is *initial* if $|\pi| > 0$ and $\mathsf{fst}(\pi) = v_I$. A *history* for player $\alpha \in \{\mathtt{E}, \mathtt{A}\}$ is a finite initial path $\rho \in \mathrm{Hst}_{\alpha} \subseteq \mathrm{Pth} \cap (\mathrm{Ps}^* \cdot \mathrm{Ps}_{\alpha})$ terminating in an α-position. A *play* $\pi \in \mathrm{Play} \subseteq \mathrm{Pth} \cap \mathrm{Ps}^{\omega}$ is just an infinite initial path.

A *strategy* for player $\alpha \in \{\mathtt{E}, \mathtt{A}\}$ is a function $\sigma_\alpha \in \mathrm{Str}_\alpha \subseteq \mathrm{Hst}_\alpha \to \mathrm{Ps}$ mapping each α-history $\rho \in \mathrm{Hst}_\alpha$ to a position $\sigma_\alpha(\rho) \in \mathrm{Ps}$ compatible with the move relation, *i.e.*, $(\mathsf{lst}(\rho), \sigma_\alpha(\rho)) \in Mv$. A path $\pi \in \mathrm{Pth}$ is *compatible* with a pair of strategies $(\sigma_\mathtt{E}, \sigma_\mathtt{A}) \in \mathrm{Str}_\mathtt{E} \times \mathrm{Str}_\mathtt{A}$ if, for all $i \in [0, |\pi| - 1)$, it holds that $(\pi)_{i+1} = \sigma_\mathtt{E}((\pi)_{\leq i})$, if $(\pi)_i \in \mathrm{Ps}_\mathtt{E}$, and $(\pi)_{i+1} = \sigma_\mathtt{A}((\pi)_{\leq i})$, otherwise. The *play function* $\mathsf{play}\colon \mathrm{Str}_\mathtt{E} \times \mathrm{Str}_\mathtt{A} \to \mathrm{Play}$ returns, for each pair of strategies $(\sigma_\mathtt{E}, \sigma_\mathtt{A}) \in \mathrm{Str}_\mathtt{E} \times \mathrm{Str}_\mathtt{A}$, the unique play $\mathsf{play}(\sigma_\mathtt{E}, \sigma_\mathtt{A}) \in \mathrm{Play}$ compatible with them.

The *observation function* $\mathsf{obs}\colon \mathrm{Pth} \to \mathrm{Ob}^\infty$ associates with each path $\pi \in \mathrm{Pth}$ the ordered sequence $w \triangleq \mathsf{obs}(\pi) \in \mathrm{Ob}^\infty$ of all observable positions occurring in it; formally, there exists a monotone bijection $\mathsf{f}\colon [0, |w|) \to \{j \in [0, |\pi|) \mid (\pi)_j \in \mathrm{Ob}\}$ satisfying the equality $(w)_i = (\pi)_{\mathsf{f}(i)}$, for all $i \in [0, |w|)$.

Eloise *wins* the game if there exists a strategy $\sigma_\mathtt{E} \in \mathrm{Str}_\mathtt{E}$ such that $\mathsf{obs}(\mathsf{play}(\sigma_\mathtt{E}, \sigma_\mathtt{A})) \in \mathrm{Wn}$, for all $\sigma_\mathtt{A} \in \mathrm{Str}_\mathtt{A}$. Similarly, Abelard *wins* the game if there exists a strategy $\sigma_\mathtt{A} \in \mathrm{Str}_\mathtt{A}$ such that $\mathsf{obs}(\mathsf{play}(\sigma_\mathtt{E}, \sigma_\mathtt{A})) \in \overline{\mathrm{Wn}}$, for all $\sigma_\mathtt{E} \in \mathrm{Str}_\mathtt{E}$.

Theorem 5 (Quantification Game I)**.** *For each behavioral quantification prefix $\wp \in \mathrm{Qn}_\mathsf{B}$ and Borelian property $\Psi \subseteq \mathrm{Asg}(\mathsf{ap}(\wp))$, the game $\mathcal{G}_\wp^\Psi$ satisfies the following two properties:*

1) if Eloise wins then $\mathrm{E} \subseteq \Psi$, for some $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathsf{C}_{\forall\exists}(\wp))$;
2) if Abelard wins then $\mathrm{E} \not\subseteq \Psi$, for all $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathsf{C}_{\exists\forall}(\wp))$.

Proof. Let $\mathcal{G}_\wp^\Psi$ be the game defined as prescribed in Construction 1. Obviously, this is a Borelian game, due to the hypothesis on the property Ψ.

Before continuing, observe that, due to the specific structure of the game, every history $\rho \cdot v \in \mathrm{Hst}_\alpha$ is bijectively correlated with the sequence of positions $\mathsf{obs}(\rho) \cdot v \in \mathrm{Ob}^* \cdot \mathrm{Ps}_\alpha$, for any player $\alpha \in \{\mathtt{E}, \mathtt{A}\}$. In other words, the functions $\jmath_\alpha\colon \mathrm{Hst}_\alpha \to \mathrm{Ob}^* \cdot \mathrm{Ps}_\alpha$ defined as $\jmath_\alpha(\rho \cdot v) \triangleq \mathsf{obs}(\rho) \cdot v$ are bijective.

Thanks to this observation, it is thus immediate to show that, for each strategy $\sigma_\mathtt{E} \in \mathrm{Str}_\mathtt{E}$, there is a unique function $\widehat{\sigma_\mathtt{E}}\colon \mathrm{Ob}^* \cdot \mathrm{Ps}_\mathtt{E} \to \mathrm{Ps}$ and, *vice versa*, for each function $\widehat{\sigma_\mathtt{E}}\colon \mathrm{Ob}^* \cdot \mathrm{Ps}_\mathtt{E} \to \mathrm{Ps}$, there is a unique strategy $\sigma_\mathtt{E} \in \mathrm{Str}_\mathtt{E}$ such that

$$\widehat{\sigma_\mathtt{E}}(\jmath_\mathtt{E}(\rho)) = \sigma_\mathtt{E}(\rho), \text{ for all histories } \rho \in \mathrm{Hst}_\mathtt{E}.$$

Similarly, for each strategy $\sigma_\mathtt{A} \in \mathrm{Str}_\mathtt{A}$, there is a unique function $\widehat{\sigma_\mathtt{A}}\colon \mathrm{Ob}^* \cdot \mathrm{Ps}_\mathtt{A} \to \mathrm{Ps}$ and, *vice versa*, for each function $\widehat{\sigma_\mathtt{A}}\colon \mathrm{Ob}^* \cdot \mathrm{Ps}_\mathtt{A} \to \mathrm{Ps}$, there is a unique strategy $\sigma_\mathtt{A} \in \mathrm{Str}_\mathtt{A}$ satisfying the equality

$$\widehat{\sigma_\mathtt{A}}(\jmath_\mathtt{A}(\rho)) = \sigma_\mathtt{A}(\rho), \text{ for all histories } \rho \in \mathrm{Hst}_\mathtt{A}.$$

We can now proceed with the proof of the two properties.

- **[1]** Since Eloise wins the game, she has a winning strategy, *i.e.*, there is $\sigma_\mathtt{E} \in \mathrm{Str}_\mathtt{E}$ such that $\mathsf{obs}(\mathsf{play}(\sigma_\mathtt{E}, \sigma_\mathtt{A})) \in \mathrm{Wn}$, for all $\sigma_\mathtt{A} \in \mathrm{Str}_\mathtt{A}$. We want to prove that there exists $\mathrm{E} \in \mathsf{nevl}_{\exists\forall}(\mathsf{C}_{\forall\exists}(\wp))$ such that $\mathrm{E} \subseteq \Psi$.
  First, recall that $\mathsf{C}_{\forall\exists}(\wp) = \forall^\mathsf{B}\vec{p}.\, \exists^{\vec{\Theta}}\vec{q}$, for some vectors of atomic propositions $\vec{p}, \vec{q} \in \mathrm{AP}^*$ and quantifier specifications $\vec{\Theta} \in \Theta^{|\vec{q}|}$. Moreover, thanks to Propositions 7 and 9, the following claim can be proved.
  Claim 1. $\mathrm{E} \subseteq \Psi$*, for some* $\mathrm{E} \in \mathsf{nevl}_{\exists\forall}(\mathsf{C}_{\forall\exists}(\wp))$, iff *there exists* $\vec{\mathsf{F}} \in \mathrm{Fnc}_{\vec{\Theta}}(\vec{p})$ *such that* $\mathsf{ext}\Big(\chi, \vec{\mathsf{F}}, \vec{q}\Big) \in \Psi$*, for all* $\chi \in \mathrm{Asg}(\vec{p})$.
  Due to the above characterization of the existence of a set $\mathrm{E} \in \mathsf{nevl}_{\exists\forall}(\mathsf{C}_{\forall\exists}(\wp))$ such that $\mathrm{E} \subseteq \Psi$, the thesis can be proved by defining a suitable vector of functors $\vec{\mathsf{F}} \in \mathrm{Fnc}_{\vec{\Theta}}(\vec{p})$.
  Consider an arbitrary assignment $\chi \in \mathrm{Asg}(\vec{p})$ and define the function $\widehat{\sigma_\mathtt{A}}^\chi\colon \mathrm{Ob}^* \cdot \mathrm{Ps}_\mathtt{A} \to \mathrm{Ps}$ as follows, for all finite sequences of observable positions $w \in \mathrm{Ob}^*$ and Abelard's positions $\xi \in \mathrm{Ps}_\mathtt{A}$:
  $$\widehat{\sigma_\mathtt{A}}^\chi(w \cdot \xi) \triangleq \begin{cases} \varnothing, & \text{if } \xi \in \mathrm{Ob}; \\ \xi[x \mapsto \chi(x)(|w|)], & \text{otherwise}; \end{cases}$$
  where $x \in \vec{p}$ is the atomic proposition at position $\#(\xi)$ in the prefix $\wp$, *i.e.*, $(\wp)_{\#(\xi)} = \forall^\mathsf{B} x$. Due to the bijective correspondence previously described, there is a unique strategy $\sigma_\mathtt{A}{}^\chi \in \mathrm{Str}_\mathtt{A}$ such that $\sigma_\mathtt{A}{}^\chi(\rho) = \widehat{\sigma_\mathtt{A}}^\chi(\jmath_\mathtt{A}(\rho))$, for all histories $\rho \in \mathrm{Hst}_\mathtt{A}$. Obviously, the induced play $\pi^\chi \triangleq \mathsf{play}(\sigma_\mathtt{E}, \sigma_\mathtt{A}{}^\chi)$ is won by Eloise, *i.e.*, $w^\chi \triangleq \mathsf{obs}(\pi^\chi) \in \mathrm{Wn}$.
  Thanks to all the infinite sequences w^χ, one for each assignment $\chi \in \mathrm{Asg}(\vec{p})$, we can define every component $(\vec{\mathsf{F}})_i$ of the vector of functors $\vec{\mathsf{F}} \in (\mathrm{Fnc}(\vec{p}))^{|\vec{q}|}$ as follows, for all instants of time $t \in \mathbb{N}$, where $i \in [0, |\vec{q}|)$:
  $$(\vec{\mathsf{F}})_i(\chi)(t) \triangleq (w^\chi)_t((\vec{q})_i).$$
  It is not too hard to show that this functor complies with the vector of quantifier specifications $\vec{\Theta}$.
  Claim 2. $\vec{\mathsf{F}} \in \mathrm{Fnc}_{\vec{\Theta}}(\vec{p})$.
  At this point, for all assignments $\chi \in \mathrm{Asg}(\vec{p})$, let $\chi_{\vec{\mathsf{F}}} \triangleq \mathsf{ext}\Big(\chi, \vec{\mathsf{F}}, \vec{q}\Big)$. We can easily argue that $\chi_{\vec{\mathsf{F}}} \in \Psi$. Indeed, by construction of the strategy $\sigma_\mathtt{A}{}^\chi$ and the vector of functors $\vec{\mathsf{F}}$, it holds that $\chi_{\vec{\mathsf{F}}}(x)(t) = (w^\chi)_t(x)$, for all instants of time $t \in \mathbb{N}$ and atomic propositions $x \in \vec{p} \cdot \vec{q}$. Hence, $\mathsf{wrd}(\chi_{\vec{\mathsf{F}}}) = w^\chi$, which implies $\chi_{\vec{\mathsf{F}}} \in \Psi$, since $w^\chi \in \mathrm{Wn}$.
- **[2]** Since Abelard wins the game, he has a winning strategy, *i.e.*, there is $\sigma_\mathtt{A} \in \mathrm{Str}_\mathtt{A}$ such that $\mathsf{obs}(\mathsf{play}(\sigma_\mathtt{E}, \sigma_\mathtt{A})) \notin \mathrm{Wn}$, for all $\sigma_\mathtt{E} \in \mathrm{Str}_\mathtt{E}$. We want to prove that, for all $\mathrm{E} \in \mathsf{nevl}_{\exists\forall}(\mathsf{C}_{\exists\forall}(\wp))$, it holds that $\mathrm{E} \not\subseteq \Psi$.
  First, recall that $\mathsf{C}_{\exists\forall}(\wp) = \exists^\mathsf{B}\vec{q}.\, \forall^{\vec{\Theta}}\vec{p}$, for some vectors of atomic propositions $\vec{p}, \vec{q} \in \mathrm{AP}^*$ and quantifier specifications $\vec{\Theta} \in \Theta^{|\vec{p}|}$. Moreover, thanks to Propositions 7 and 9, the following claim can be proved.
  Claim 3. $\mathrm{E} \not\subseteq \Psi$*, for all* $\mathrm{E} \in \mathsf{nevl}_{\exists\forall}(\mathsf{C}_{\exists\forall}(\wp))$, iff *there exists* $\vec{\mathsf{G}} \in \mathrm{Fnc}_{\vec{\Theta}}(\vec{q})$ *such that* $\mathsf{ext}\Big(\chi, \vec{\mathsf{G}}, \vec{p}\Big) \notin \Psi$*, for all* $\chi \in \mathrm{Asg}(\vec{q})$.
  Due to the above characterization of non-existence of a set $\mathrm{E} \in \mathsf{nevl}_{\exists\forall}(\mathsf{C}_{\exists\forall}(\wp))$ such that $\mathrm{E} \subseteq \Psi$, the thesis can be proved by defining a suitable vector of functors $\vec{\mathsf{G}} \in \mathrm{Fnc}_{\vec{\Theta}}(\vec{q})$.
  Consider an arbitrary assignment $\chi \in \mathrm{Asg}(\vec{q})$ and define the function $\widehat{\sigma_\mathtt{E}}^\chi\colon \mathrm{Ob}^* \cdot \mathrm{Ps}_\mathtt{E} \to \mathrm{Ps}$ as follows, for all finite

sequences of observable positions $w \in \mathrm{Ob}^*$ and Eloise's positions $\xi \in \mathrm{Ps}_{\mathtt{E}}$:

$$\widehat{\sigma_{\mathtt{E}}}^{\chi}(w \cdot \xi) \triangleq \xi[x \mapsto \chi(x)(|w|)],$$

where $x \in \vec{q}$ is the atomic proposition at position $\#(\xi)$ in the prefix $\wp$, *i.e.*, $(\wp)_{\#(\xi)} = \exists^{\mathtt{B}} x$. Due to the bijective correspondence previously described, there is a unique strategy $\sigma_{\mathtt{E}}{}^{\chi} \in \mathrm{Str}_{\mathtt{E}}$ such that $\sigma_{\mathtt{E}}{}^{\chi}(\rho) = \widehat{\sigma_{\mathtt{E}}}^{\chi}(\jmath_{\mathtt{E}}(\rho))$, for all histories $\rho \in \mathrm{Hst}_{\mathtt{E}}$. Obviously, the induced play $\pi^{\chi} \triangleq \mathsf{play}(\sigma_{\mathtt{E}}{}^{\chi}, \sigma_{\mathtt{A}})$ is won by Abelard, *i.e.*, $w^{\chi} \triangleq \mathsf{obs}(\pi^{\chi}) \notin \mathrm{Wn}$.

Thanks to all the infinite sequences w^{χ}, one for each assignment $\chi \in \mathrm{Asg}(\vec{q})$, we can define every component $(\vec{\mathsf{G}})_i$ of the vector of functors $\vec{\mathsf{G}} \in (\mathrm{Fnc}(\vec{q}))^{|\vec{p}|}$ as follows, for all instants of time $t \in \mathbb{N}$, where $i \in [0, |\vec{p}|)$:

$$(\vec{\mathsf{G}})_i(\chi)(t) \triangleq (w^{\chi})_t((\vec{p})_i).$$

It is not too hard to show that this functor complies with the vector of quantifier specifications $\vec{\Theta}$.

Claim 4. $\vec{\mathsf{G}} \in \mathrm{Fnc}_{\vec{\Theta}}(\vec{q})$.

At this point, for all assignments $\chi \in \mathrm{Asg}(\vec{q})$, let $\chi_{\vec{\mathsf{G}}} \triangleq \mathsf{ext}\Big(\chi, \vec{\mathsf{G}}, \vec{p}\Big)$. We can easily argue that $\chi_{\vec{\mathsf{G}}} \notin \Psi$. Indeed, by construction of the strategy $\sigma_{\mathtt{E}}{}^{\chi}$ and the vector of functors $\vec{\mathsf{G}}$, it holds that $\chi_{\vec{\mathsf{G}}}(x)(t) = (w^{\chi})_t(x)$, for all instants of time $t \in \mathbb{N}$ and atomic propositions $x \in \vec{q} \cdot \vec{p}$. Hence, $\mathsf{wrd}(\chi_{\vec{\mathsf{G}}}) = w^{\chi}$, which implies $\chi_{\vec{\mathsf{G}}} \notin \Psi$, since $w^{\chi} \notin \mathrm{Wn}$. □

$\Theta_{\mathtt{B}}$ is the set of behavioral quantifier specifications, *i.e.*, quantifier specifications of the form $\mathtt{B} \cup \langle \mathtt{S} : \mathrm{P}_{\mathtt{S}} \rangle$ for some $\mathrm{P}_{\mathtt{S}} \subseteq \mathrm{AP}$.

Proposition 11. $\mathsf{evl}_{\exists\forall}(\mathfrak{X}, \forall^{\mathtt{B}} p.\exists^{\Theta \cup \langle \mathtt{S}:p \rangle} q) \sqsubseteq \mathsf{evl}_{\exists\forall}(\mathfrak{X}, \exists^{\Theta} q.\forall^{\mathtt{B}} p)$ *and* $\mathsf{evl}_{\forall\exists}(\mathfrak{X}, \exists^{\mathtt{B}} p.\forall^{\Theta \cup \langle \mathtt{S}:p \rangle} q) \sqsubseteq \mathsf{evl}_{\forall\exists}(\mathfrak{X}, \forall^{\Theta} q.\exists^{\mathtt{B}} p)$, *for all* $\mathfrak{X} \in \mathrm{HAsg}(\mathrm{P})$ *with* $\mathrm{P} \subseteq \mathrm{AP}$, $p, q \in \mathrm{AP} \setminus \mathrm{P}$, *and* $\Theta \in \Theta_{\mathtt{B}}$.

Proof. Due to the specific definition of the normal evaluation function $\mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \wp)$, and by exploiting Propositions 7 and 9, the following claim can be proved.

Claim 5. $\mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \forall^{\mathtt{B}} p.\exists^{\Theta \cup \langle \mathtt{S}:p \rangle} q) \sqsubseteq \mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \exists^{\Theta} q.\forall^{\mathtt{B}} p)$ iff, *for all* $\mathsf{J} \in \mathrm{Fnc}_{\Theta \cup \langle \mathtt{S}:p \rangle}(\mathsf{ap}(\mathfrak{X}) \cup \{p\})$, $\eth \in \mathrm{Fnc}_{\mathtt{B}}(\mathsf{ap}(\mathfrak{X})) \to \mathfrak{X}$, *and* $\mathsf{G} \in \mathrm{Fnc}_{\mathtt{B}}(\mathsf{ap}(\mathfrak{X}) \cup \{q\})$, *there exists* $\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}))$ *and* $\mathrm{X} \in \mathfrak{X}$ *such that* $\mathsf{ext}(\mathsf{ext}(\mathrm{X}, \mathsf{F}, q), \mathsf{G}, p) \subseteq \mathsf{ext}(\mathsf{ext}(\eth, p), \mathsf{J}, q)$.

An analogous claim can be proved stating that the same characterization also holds for $\mathsf{nevl}_{\forall\exists}(\mathfrak{X}, \exists^{\mathtt{B}} p.\forall^{\Theta \cup \langle \mathtt{S}:p \rangle} q) \sqsubseteq \mathsf{nevl}_{\forall\exists}(\mathfrak{X}, \forall^{\Theta} q.\exists^{\mathtt{B}} p)$. Thanks to such characterizations, the thesis can be shown by choosing a suitable functor F and set of assignments X in dependence of the functors J and G and the selection map $\eth$.

In order to define the functor F, let us inductively construct, for every given assignment $\chi \in \mathrm{Asg}(\mathsf{ap}(\mathfrak{X}))$, the following infinite families of assignments $\{a_t^{\chi} \in \mathrm{Asg}(\mathsf{ap}(\mathfrak{X}) \cup \{p\})\}_{t \in \mathbb{N}}$, Boolean values $\{v_t^{\chi} \in \mathbb{B}\}_{t \in \mathbb{N}}$, and assignments $\{b_t^{\chi} \in \mathrm{Asg}(\mathsf{ap}(\mathfrak{X}) \cup \{q\})\}_{t \in \mathbb{N}}$:

- as base step $t = 0$, we choose $a_0^{\chi} \in \mathrm{Asg}(\mathsf{ap}(\mathfrak{X}) \cup \{p\})$ as an arbitrary assignment for which the equality $a_0^{\chi} \restriction \mathsf{ap}(\mathfrak{X}) = \chi$ holds true, the Boolean value $v_0^{\chi} \in \mathbb{B}$ as $\mathsf{J}(a_0^{\chi})(0)$, *i.e.*, $v_0^{\chi} \triangleq \mathsf{J}(a_0^{\chi})(0)$, and $b_0^{\chi} \in \mathrm{Asg}(\mathsf{ap}(\mathfrak{X}) \cup \{q\})$ as an arbitrary assignment with $b_0^{\chi} \restriction \mathsf{ap}(\mathfrak{X}) = \chi$ such that, at time 0 on the variable q, assumes v_0^{χ} as value, *i.e.*, $b_0^{\chi}(q)(0) = v_0^{\chi}$;
- as inductive step $t > 0$, we derive the assignment $a_t^{\chi} \in \mathrm{Asg}(\mathsf{ap}(\mathfrak{X}) \cup \{p\})$ from $\mathsf{G}(b_{t-1}^{\chi})$, *i.e.*, $a_t^{\chi} \triangleq \chi[p \mapsto \mathsf{G}(b_{t-1}^{\chi})]$, and the Boolean value $v_t^{\chi} \in \mathbb{B}$ from $\mathsf{J}(a_t^{\chi})(t)$, *i.e.*, $v_t^{\chi} \triangleq \mathsf{J}(a_t^{\chi})(t)$; moreover, we choose $b_t^{\chi} \in \mathrm{Asg}(\mathsf{ap}(\mathfrak{X}) \cup \{q\})$ as an arbitrary assignment with $b_t^{\chi} \restriction \mathsf{ap}(\mathfrak{X}) = \chi$ such that, on the variable q, is equal to b_{t-1}^{χ} up to time t excluded and assumes v_t^{χ} as value at time t, *i.e.*, $b_t^{\chi}(q)(h) = b_{t-1}^{\chi}(q)(h)$, for all $h \in [0, t)$, and $b_t^{\chi}(q)(t) = v_t^{\chi}$.

Thanks to the infinite family of Boolean values $\{v_t^{\chi} \in \mathbb{B}\}_{t \in \mathbb{N}}$, one for each assignment $\chi \in \mathrm{Asg}(\mathsf{ap}(\mathfrak{X}))$, we can define the functor $\mathsf{F} \in \mathrm{Fnc}(\mathsf{ap}(\mathfrak{X}))$ as follows, for every instant of time $t \in \mathbb{N}$:

$$\mathsf{F}(\chi)(t) \triangleq v_t^{\chi}.$$

It is easy to show that this functor complies with the quantifier specification Θ, since the functor J, from which F is derived, is compliant with the quantifier specification $\Theta \cup \langle \mathtt{S} : p \rangle$.

Claim 6. $\mathsf{F} \in \mathrm{Fnc}_{\Theta}(\mathsf{ap}(\mathfrak{X}))$.

Before continuing, let us first introduce the functor $\mathsf{H} \in \mathrm{Fnc}(\mathsf{ap}(\mathfrak{X}))$ as follows, for every assignment $\chi \in \mathrm{Asg}(\mathsf{ap}(\mathfrak{X}))$:

$$\mathsf{H}(\chi) \triangleq \mathsf{ext}(\mathsf{ext}(\chi, \mathsf{F}, q), \mathsf{G}, p)(p).$$

It is quite immediate to verify that such a functor is behavioral.

Claim 7. $\mathsf{H} \in \mathrm{Fnc}_{\mathtt{B}}(\mathsf{ap}(\mathfrak{X}))$.

At this point, consider the set of assignments $\mathrm{X} \triangleq \eth(\mathsf{H})$. Thanks to the specific definitions of the two functors F and H, the following claim can be proved.

Claim 8. $\mathsf{ext}(\mathsf{ext}(\mathrm{X}, \mathsf{F}, q), \mathsf{G}, p) \subseteq \mathsf{ext}(\mathsf{ext}(\mathrm{X}, \mathsf{H}, p), \mathsf{J}, q)$.

Now, it is obvious that $\mathsf{ext}(\mathrm{X}, \mathsf{H}, p) \subseteq \mathsf{ext}(\eth, p)$, due to the definition of the latter and the choice of the set X, which immediately implies $\mathsf{ext}(\mathsf{ext}(\mathrm{X}, \mathsf{H}, p), \mathsf{J}, q) \subseteq \mathsf{ext}(\mathsf{ext}(\eth, p), \mathsf{J}, q)$. Therefore, $\mathsf{ext}(\mathsf{ext}(\mathrm{X}, \mathsf{F}, q), \mathsf{G}, p) \subseteq \mathsf{ext}(\mathsf{ext}(\eth, p), \mathsf{J}, q)$, which concludes the proof. □

Proposition 6. $\mathsf{evl}_{\alpha}(\mathfrak{X}, \mathsf{C}_{\overline{\alpha}}(\wp)) \sqsubseteq \mathsf{evl}_{\alpha}(\mathfrak{X}, \wp) \sqsubseteq \mathsf{evl}_{\alpha}(\mathfrak{X}, \mathsf{C}_{\alpha}(\wp))$, *for all* $\mathfrak{X} \in \mathrm{HAsg}$ *and* $\wp \in \mathrm{Qn}_{\mathtt{B}}$, *with* $\mathsf{ap}(\wp) \cap \mathsf{ap}(\mathfrak{X}) = \emptyset$.

Proof. First, we notice that, by repeatedly applying Proposition 11, we are able to derive the following claim.

Claim 9. $\mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \forall^{\mathtt{B}} \vec{p}.\exists^{\mathtt{B} \cup \langle \mathtt{S}:\vec{p} \rangle} \vec{q}) \sqsubseteq \mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \exists^{\mathtt{B}} \vec{q}.\forall^{\mathtt{B}} \vec{p})$ *and* $\mathsf{nevl}_{\forall\exists}(\mathfrak{X}, \exists^{\mathtt{B}} \vec{p}.\forall^{\mathtt{B} \cup \langle \mathtt{S}:\vec{p} \rangle} \vec{q}) \sqsubseteq \mathsf{nevl}_{\forall\exists}(\mathfrak{X}, \forall^{\mathtt{B}} \vec{q}.\exists^{\mathtt{B}} \vec{p})$, *for all* $\mathfrak{X} \in \mathrm{HAsg}(\mathrm{P})$ *with* $\mathrm{P} \subseteq \mathrm{AP}$ *and* $\vec{p}, \vec{q} \subseteq \mathrm{AP} \setminus \mathrm{P}$.

Let us first focus on proving $\mathsf{nevl}_{\alpha}(\mathfrak{X}, \mathsf{C}_{\overline{\alpha}}(\wp)) \sqsubseteq \mathsf{nevl}_{\alpha}(\mathfrak{X}, \wp)$ and consider the case $\alpha = \exists\forall$. First, let us rewrite $\wp$ as

$\forall^{\mathsf{B}}\vec{p}_0.\,(\exists^{\mathsf{B}}\vec{q}_i.\,\forall^{\mathsf{B}}\vec{p}_i)_{i=1}^{k}.\,\exists^{\mathsf{B}}\vec{q}_{k+1}$, where $|\vec{p}_0|, |\vec{q}_{k+1}| \geq 0$, $k \geq 0$, and $|\vec{q}_i|, |\vec{p}_i| \geq 1$ for all $i \in \{1, \ldots, k\}$. The proof proceeds by induction on the $\exists\forall$-*alternation degree* of $\wp$, that is, k.

If $k = 0$ (base case), then $\mathtt{C}_{\forall\exists}(\wp) = \wp$, and we are done. If $k > 0$ (inductive step), then $\mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \wp) = \mathsf{nevl}_{\exists\forall}(\mathsf{nevl}_{\exists\forall}(\mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \forall^{\mathsf{B}}\vec{p}_0.\,(\exists^{\mathsf{B}}\vec{q}_i.\,\forall^{\mathsf{B}}\vec{p}_i)_{i=1}^{k-1}), \exists^{\mathsf{B}}\vec{q}_k.\,\forall^{\mathsf{B}}\vec{p}_k), \exists^{\mathsf{B}}\vec{q}_{k+1})$. Thanks to the above claim and to the monotonicity of nevl, we have $\mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \wp) \sqsupseteq \mathsf{nevl}_{\exists\forall}(\mathsf{nevl}_{\exists\forall}(\mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \forall^{\mathsf{B}}\vec{p}_0.\,(\exists^{\mathsf{B}}\vec{q}_i.\,\forall^{\mathsf{B}}\vec{p}_i)_{i=1}^{k-1}), \forall^{\mathsf{B}}\vec{p}_k.\,\exists^{\mathsf{B}\cup\langle \mathsf{S}:\vec{p}_k\rangle}\vec{q}_k), \exists^{\mathsf{B}}\vec{q}_{k+1}) = \mathsf{nevl}_{\exists\forall}(\mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \wp'), \exists^{\mathsf{B}\cup\langle \mathsf{S}:\vec{p}_k\rangle}\vec{q}_k.\,\exists^{\mathsf{B}}\vec{q}_{k+1})$, where $\wp' = \forall^{\mathsf{B}}\vec{p}_0.\,(\exists^{\mathsf{B}}\vec{q}_i.\,\forall^{\mathsf{B}}\vec{p}_i)_{i=1}^{k-2}.\,\exists^{\mathsf{B}}\vec{q}_{k-1}.\,\forall^{\mathsf{B}}\vec{p}_{k-1}.\,\forall^{\mathsf{B}}\vec{p}_k$ has $\exists\forall$-alternation degree $k-1$. Thus, by inductive hypothesis, it holds that $\mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \mathtt{C}_{\forall\exists}(\wp')) \sqsubseteq \mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \wp')$. The thesis follows by observing that $\mathtt{C}_{\forall\exists}(\wp').\,\exists^{\mathsf{B}\cup\langle \mathsf{S}:\vec{p}_k\rangle}\vec{q}_k.\,\exists^{\mathsf{B}}\vec{q}_{k+1} = \mathtt{C}_{\forall\exists}(\wp)$.

In order to prove that $\mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp)) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \wp)$ holds when $\alpha = \forall\exists$, it suffices to replace quantifier $\forall$ with $\exists$, and vice versa, inside the quantifier prefixes, and to switch $\exists\forall$ and $\forall\exists$ throughout the previous proof. We omit the details.

Finally, in order to prove that $\mathsf{nevl}_\alpha(\mathfrak{X}, \wp) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp))$ for every $\alpha \in \{\forall\exists, \forall\exists\}$, we first state two auxiliary results.

Claim 10. $\mathsf{nevl}_\alpha(\overline{\mathfrak{X}}, \overline{\wp}) \equiv \overline{\mathsf{nevl}_\alpha(\mathfrak{X}, \wp)}$*, for all* $\mathfrak{X} \in \mathrm{HAsg}$ *and* $\wp \in \mathrm{Qn}$.

Proof. The proof is by induction on the length of $\wp$. If $\wp = \varepsilon$, then the claim follows immediately. Let $\wp = \mathtt{Q}^{\Theta}p.\,\wp'$, then we distinguish two cases.

If α and $\mathtt{Q}$ are coherent, then $\mathsf{nevl}_\alpha(\overline{\mathfrak{X}}, \overline{\wp}) \equiv \mathsf{nevl}_\alpha(\overline{\mathsf{ext}_\Theta\big(\overline{\overline{\mathfrak{X}}}, p\big)}, \overline{\wp'})$ (recall that $\mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{Q}^{\Theta}p) \equiv \mathsf{evl}_\alpha(\mathfrak{X}, \mathtt{Q}^{\Theta}p)$ due to Proposition 8). By Propositions 1, 4, 5, and 9, we have that $\mathsf{nevl}_\alpha(\overline{\mathsf{ext}_\Theta\big(\overline{\overline{\mathfrak{X}}}, p\big)}, \overline{\wp'}) \equiv \mathsf{nevl}_\alpha(\overline{\mathsf{ext}_\Theta(\mathfrak{X}, p)}, \overline{\wp'})$. Moreover, we have that $\overline{\mathsf{nevl}_\alpha(\mathfrak{X}, \wp)} = \overline{\mathsf{nevl}_\alpha(\mathsf{ext}_\Theta(\mathfrak{X}, p), \wp')}$. By inductive hypothesis, it holds that $\mathsf{nevl}_\alpha(\overline{\mathsf{ext}_\Theta(\mathfrak{X}, p)}, \overline{\wp'}) \equiv \overline{\mathsf{nevl}_\alpha(\mathsf{ext}_\Theta(\mathfrak{X}, p), \wp')}$, hence the thesis.

If α and $\mathtt{Q}$ are not coherent, then $\mathsf{nevl}_\alpha(\overline{\mathfrak{X}}, \overline{\wp}) = \mathsf{nevl}_\alpha(\mathsf{ext}_\Theta\big(\overline{\mathfrak{X}}, p\big), \overline{\wp'})$. Moreover, $\overline{\mathsf{nevl}_\alpha(\mathfrak{X}, \wp)} \equiv \overline{\mathsf{nevl}_\alpha(\overline{\mathsf{ext}_\Theta\big(\overline{\mathfrak{X}}, p\big)}, \wp')}$. By inductive hypothesis, $\overline{\mathsf{nevl}_\alpha(\overline{\mathsf{ext}_\Theta\big(\overline{\mathfrak{X}}, p\big)}, \wp')} \equiv \mathsf{nevl}_\alpha(\overline{\overline{\mathsf{ext}_\Theta\big(\overline{\mathfrak{X}}, p\big)}}, \overline{\wp'}) \equiv \mathsf{nevl}_\alpha(\mathsf{ext}_\Theta\big(\overline{\mathfrak{X}}, p\big), \overline{\wp'}) = \mathsf{nevl}_\alpha(\overline{\mathfrak{X}}, \overline{\wp})$ ☐

It is not difficult to convince oneself that the following claim holds.

Claim 11. $\mathtt{C}_{\overline{\alpha}}(\overline{\wp}) = \overline{\mathtt{C}_\alpha(\wp)}$ *for all* $\wp \in \mathrm{Qn}$.

Now, by instantiating $\mathfrak{X}$ and $\wp$ with $\overline{\mathfrak{X}}$ and $\overline{\wp}$, respectively, in the first "squared inclusion" of this proposition, proved in the first part of this proof, we know that $\mathsf{nevl}_\alpha(\overline{\mathfrak{X}}, \mathtt{C}_{\overline{\alpha}}(\overline{\wp})) \sqsubseteq \mathsf{nevl}_\alpha(\overline{\mathfrak{X}}, \overline{\wp})$. By Claims 10 and 11, we have $\overline{\mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp))} \sqsubseteq \overline{\mathsf{nevl}_\alpha(\mathfrak{X}, \wp)}$. The thesis follows from Proposition 4. ☐

Proposition 12. $\mathsf{evl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp_1.\,\wp_2)) \sqsubseteq \mathscr{Y} \sqsubseteq \mathsf{evl}_\alpha(\mathfrak{X}, \wp_1.\,\wp_2)$, *for all* $\mathscr{Y} \in \{\mathsf{evl}_\alpha(\mathfrak{X}, \wp_1.\,\mathtt{C}_{\overline{\alpha}}(\wp_2)), \mathsf{evl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp_1).\,\wp_2)\}$, $\mathfrak{X} \in \mathrm{HAsg}$ *and* $\wp_1, \wp_2 \in \mathrm{Qn}_{\mathsf{B}}$, *with* $\mathsf{ap}(\wp_1.\,\wp_2) \cap \mathsf{ap}(\mathfrak{X}) = \emptyset$.

Proof. We will make use of the following result.

Claim 12. $\mathtt{C}_\alpha(\mathtt{C}_\alpha(\wp_1).\,\wp_2) = \mathtt{C}_\alpha(\wp_1.\,\wp_2) = \mathtt{C}_\alpha(\wp_1.\,\mathtt{C}_\alpha(\wp_2))$ *for all* $\wp_1, \wp_2 \in \mathrm{Qn}$.

Remark 1. *Notice that due to the fact that* $\mathtt{C}_\alpha(\wp_1) \notin \mathrm{Qn}_{\mathsf{B}}$ *and* $\mathtt{C}_\alpha(\wp_2) \notin \mathrm{Qn}_{\mathsf{B}}$*, the two expressions* $\mathtt{C}_\alpha(\mathtt{C}_\alpha(\wp_1).\,\wp_2)$ *and* $\mathtt{C}_\alpha(\wp_1.\,\mathtt{C}_\alpha(\wp_2))$ *are not well defined. Thus, in order for the above claim to make sense, we need generalized definitions for functions* $\mathtt{C}_{\exists\forall}$ *and* $\mathtt{C}_{\forall\exists}$ *to cope with quantifier prefixes featuring arbitrary quantifier specifications.*

1) In order to show that $\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\mathtt{C}_{\overline{\alpha}}(\wp_2)) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\wp_2)$, we proceed as follows: $\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\wp_2) = \mathsf{nevl}_\alpha(\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1), \wp_2) \sqsupseteq \mathsf{nevl}_\alpha(\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1), \mathtt{C}_{\overline{\alpha}}(\wp_2)) = \mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\mathtt{C}_{\overline{\alpha}}(\wp_2))$.
2) In order to show that $\mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp_1).\,\wp_2) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\wp_2)$, we proceed as follows: $\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\wp_2) = \mathsf{nevl}_\alpha(\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1), \wp_2) \sqsupseteq \mathsf{nevl}_\alpha(\mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp_1)), \wp_2) = \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp_1).\,\wp_2)$.
3) In order to show that $\mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp_1.\,\wp_2)) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\mathtt{C}_{\overline{\alpha}}(\wp_2))$, we proceed as follows: $\mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp_1.\,\wp_2)) = \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp_1.\,\mathtt{C}_{\overline{\alpha}}(\wp_2))) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\mathtt{C}_{\overline{\alpha}}(\wp_2))$.
4) In order to show that $\mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp_1.\,\wp_2)) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp_1).\,\wp_2)$, we proceed as follows: $\mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp_1.\,\wp_2)) = \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\mathtt{C}_{\overline{\alpha}}(\wp_1).\,\wp_2)) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_{\overline{\alpha}}(\wp_1).\,\wp_2)$. ☐

Proposition 13. $\mathsf{evl}_\alpha(\mathfrak{X}, \wp_1.\,\wp_2) \sqsubseteq \mathscr{Y} \sqsubseteq \mathsf{evl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp_1.\,\wp_2))$, *for all* $\mathscr{Y} \in \{\mathsf{evl}_\alpha(\mathfrak{X}, \wp_1.\,\mathtt{C}_\alpha(\wp_2)), \mathsf{evl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp_1).\,\wp_2)\}$, $\mathfrak{X} \in \mathrm{HAsg}$ *and* $\wp_1, \wp_2 \in \mathrm{Qn}_{\mathsf{B}}$, *with* $\mathsf{ap}(\wp_1.\,\wp_2) \cap \mathsf{ap}(\mathfrak{X}) = \emptyset$.

Proof.
1) In order to show that $\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\wp_2) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\mathtt{C}_\alpha(\wp_2))$, we proceed as follows: $\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\wp_2) = \mathsf{nevl}_\alpha(\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1), \wp_2) \sqsubseteq \mathsf{nevl}_\alpha(\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1), \mathtt{C}_\alpha(\wp_2)) = \mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\mathtt{C}_\alpha(\wp_2))$.
2) In order to show that $\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\wp_2) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp_1).\,\wp_2)$, we proceed as follows: $\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\wp_2) = \mathsf{nevl}_\alpha(\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1), \wp_2) \sqsubseteq \mathsf{nevl}_\alpha(\mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp_1)), \wp_2) = \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp_1).\,\wp_2)$.
3) In order to show that $\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\mathtt{C}_\alpha(\wp_2)) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp_1.\,\wp_2))$, we proceed as follows: $\mathsf{nevl}_\alpha(\mathfrak{X}, \wp_1.\,\mathtt{C}_\alpha(\wp_2)) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp_1.\,\mathtt{C}_\alpha(\wp_2))) = \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp_1.\,\wp_2))$.
4) In order to show that $\mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp_1).\,\wp_2) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp_1.\,\wp_2))$, we proceed as follows: $\mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp_1).\,\wp_2) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\mathtt{C}_\alpha(\wp_1).\,\wp_2)) = \mathsf{nevl}_\alpha(\mathfrak{X}, \mathtt{C}_\alpha(\wp_1.\,\wp_2))$. ☐

Theorem 8 (Quantification Game II)**.** *Every* $\mathfrak{Q}$*-game* $\mathcal{G}$*, for some quantification-game schema* $\mathfrak{Q} \triangleq \langle \wp, \mathfrak{X}, \Psi \rangle$*, satisfies the following two properties:*

1) if Eloise wins then $\mathrm{E} \subseteq \Psi$*, for some* $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathfrak{X}, \mathtt{C}_{\forall\exists}(\wp))$*;*
2) if Abelard wins then $\mathrm{E} \not\subseteq \Psi$*, for all* $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathfrak{X}, \mathtt{C}_{\exists\forall}(\wp))$*.*

Proof. First of all, recall that the game $\mathcal{G}_{\mathfrak{Q}}$ of Construction 2 is obtained from the game $\mathcal{G}_{\widehat{\wp}}^{\widehat{\Psi}}$ given in Construction 1, where

- $\widehat{\wp} \triangleq \forall\vec{p}.\,\wp^{\bullet}.\,\wp$ and

- $\widehat{\Psi} \triangleq \Psi \cup \{ \chi \in \mathrm{Asg}(\mathrm{P}) \mid \chi \upharpoonright \vec{p} \notin \mathrm{X} \}$,

with $\vec{p} \triangleq \mathsf{ap}(\mathfrak{X}) \setminus \mathsf{ap}(\wp^\bullet)$ and $\mathrm{P} \triangleq \mathsf{ap}(\wp) \cup \mathsf{ap}(\mathfrak{X})$ and being $\langle \wp^\bullet, \mathrm{X} \rangle$ a generator for $\mathfrak{X}$.

We can now proceed with the proof of the two properties.

- **[1]** If Eloise wins the game, by Theorem 5, there exists a set of assignments $\widehat{\mathrm{E}} \in \mathsf{nevl}_{\exists\forall}(\mathsf{C}_{\forall\exists}(\widehat{\wp}))$ such that $\widehat{\mathrm{E}} \subseteq \widehat{\Psi}$. Thanks to Propositions 5 and 12, we can easily prove the following inclusion between normal evaluations.

  Claim 13. $\mathsf{nevl}_{\exists\forall}(\mathsf{C}_{\forall\exists}(\widehat{\wp})) \sqsubseteq \mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \mathsf{C}_{\forall\exists}(\wp))$.

  Due to the specific definition of the ordering $\sqsubseteq$ between hyperassignments, it follows that the above inclusion necessarily implies the existence of a set of assignments $\mathrm{E} \in \mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \mathsf{C}_{\forall\exists}(\wp))$ such that $\mathrm{E} \subseteq \widehat{\mathrm{E}}$. Therefore, $\mathrm{E} \subseteq \widehat{\Psi}$.

  At this point, we can immediately prove that $\mathrm{E} \subseteq \Psi$, being $\widehat{\Psi} = \Psi \cup \{ \chi \in \mathrm{Asg}(\mathrm{P}) \mid \chi \upharpoonright \vec{p} \notin \mathrm{X} \}$, thanks to the following claim, which can be derived from Proposition 10.

  Claim 14. *$\mathrm{E} \cap \{ \chi \in \mathrm{Asg}(\mathrm{P}) \mid \chi \upharpoonright \vec{p} \notin \mathrm{X} \} = \emptyset$, for all $\mathrm{E} \in \mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \mathsf{C}_{\forall\exists}(\wp))$.*

- **[2]** If Abelard wins the game, by Theorem 5, for all sets of assignments $\widehat{\mathrm{E}} \in \mathsf{nevl}_{\exists\forall}(\mathsf{C}_{\exists\forall}(\widehat{\wp}))$ it holds that $\widehat{\mathrm{E}} \not\subseteq \widehat{\Psi}$. First, given $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathrm{HAsg}$, with $\mathsf{ap}(\mathfrak{X}_1) = \mathsf{ap}(\mathfrak{X}_2)$, and $\mathrm{X} \subseteq \mathrm{Asg}(\mathrm{P})$ for some $\mathrm{P} \subseteq \mathrm{AP}$, we define $\mathfrak{X}_1 \sqsubseteq_{\mathrm{X}} \mathfrak{X}_2$ if and only if for every $\mathrm{X}_1 \in \mathfrak{X}_1$ there is $\mathrm{X}_2 \in \mathfrak{X}_2$ such that $\mathrm{X}_2 \setminus \{ \chi \in \mathrm{Asg} \mid \chi \upharpoonright \mathrm{P} \in \mathrm{X} \} \subseteq \mathrm{X}_1$. The following claim holds.

  Claim 15. *Let $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathrm{HAsg}$ and $\mathrm{X} \subseteq \mathrm{Asg}(\mathrm{P})$ for some $\mathrm{P} \subseteq \mathrm{AP}$. Then, $\mathfrak{X}_1 \sqsubseteq_{\mathrm{X}} \mathfrak{X}_2$ implies $\mathsf{evl}_\alpha(\mathfrak{X}_1, \wp) \sqsubseteq_{\mathrm{X}} \mathsf{evl}_\alpha(\mathfrak{X}_2, \wp)$, for all $\wp \in \mathrm{Qn}$ with $\mathsf{ap}(\mathfrak{X}_1) \cap \mathsf{ap}(\wp) = \emptyset$.*

  Proof. The proof is by induction on the length of $\wp$.

  If $\wp = \varepsilon$ (base case), then the claim follows trivially.

  Let $\wp = \mathtt{Q}^{\Theta} p.\, \wp'$. If α and $\mathtt{Q}$ are coherent, then $\mathsf{evl}_\alpha(\mathfrak{X}_i, \wp) = \mathsf{evl}_\alpha(\mathsf{ext}_\Theta(\mathfrak{X}_i, p), \wp')$, for every $i \in \{1, 2\}$. We show that $\mathsf{ext}_\Theta(\mathfrak{X}_1, p) \sqsubseteq_{\mathrm{X}} \mathsf{ext}_\Theta(\mathfrak{X}_2, p)$ holds, and the thesis follows by applying the inductive hypothesis. Since $\mathsf{ext}_\Theta(\mathfrak{X}_i, p) = \{ \mathsf{ext}(\mathrm{X}_i, \mathsf{F}_i, p) \mid \mathrm{X}_i \in \mathfrak{X}_1, \mathsf{F}_i \in \mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_i)) \}$ $(i \in \{1, 2\})$, we have to show that for every $\mathrm{X}_1 \in \mathfrak{X}_1$ and $\mathsf{F}_1 \in \mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_1))$ $(= \mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_2)))$ there are $\mathrm{X}_2 \in \mathfrak{X}_2$ and $\mathsf{F}_2 \in \mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_1))$ such that $\mathsf{ext}(\mathrm{X}_2, \mathsf{F}_2, p) \setminus \{ \chi \in \mathrm{Asg} \mid \chi \upharpoonright \mathrm{P} \in \mathrm{X} \} \subseteq \mathsf{ext}(\mathrm{X}_1, \mathsf{F}_1, p)$. It is easy to see that for every $\mathrm{X}_1 \in \mathfrak{X}_1$ and $\mathsf{F}_1 \in \mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_1))$, it holds that $\mathsf{ext}(f(\mathrm{X}_1), \mathsf{F}_1, p) \setminus \{ \chi \in \mathrm{Asg} \mid \chi \upharpoonright \mathrm{P} \in \mathrm{X} \} \subseteq \mathsf{ext}(\mathrm{X}_1, \mathsf{F}_1, p)$, where $f : \mathfrak{X}_1 \to \mathfrak{X}_2$ is a witness for $\mathfrak{X}_1 \sqsubseteq_{\mathrm{X}} \mathfrak{X}_2$.

  Instead, if α and $\mathtt{Q}$ are not coherent, then $\mathsf{evl}_\alpha(\mathfrak{X}_i, \wp) \equiv \mathsf{evl}_\alpha(\mathsf{nevl}_\alpha(\mathfrak{X}_i, \mathtt{Q}^{\Theta} p), \wp')$, for every $i \in \{1, 2\}$. We show that $\mathsf{nevl}_\alpha(\mathfrak{X}_1, \mathtt{Q}^{\Theta} p) \sqsubseteq_{\mathrm{X}} \mathsf{nevl}_\alpha(\mathfrak{X}_2, \mathtt{Q}^{\Theta} p)$ holds, and the thesis follows by applying the inductive hypothesis. Since $\mathsf{nevl}_\alpha(\mathfrak{X}_i, \mathtt{Q}^{\Theta} p) = \{ \mathsf{ext}(\mathfrak{F}_i, p) \mid \mathfrak{F}_i \in \mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_i)) \to \mathfrak{X}_i \}$, where $\mathsf{ext}(\mathfrak{F}_i, p) = \bigcup_{\mathsf{F} \in \mathsf{dom}(\mathfrak{F}_i)} \mathsf{ext}(\mathfrak{F}_i(\mathsf{F}), \mathsf{F}, p)$ $(i \in \{1, 2\})$, we have to show that for every $\mathfrak{F}_1 \in \mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_1)) \to \mathfrak{X}_1$ there is $\mathfrak{F}_2 \in \mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_1)) \to \mathfrak{X}_2$ (recall that $\mathsf{ap}(\mathfrak{X}_1) = \mathsf{ap}(\mathfrak{X}_2)$) such that $\mathsf{ext}(\mathfrak{F}_2, p) \setminus \{ \chi \in \mathrm{Asg} \mid \chi \upharpoonright \mathrm{P} \in \mathrm{X} \} \subseteq \mathsf{ext}(\mathfrak{F}_1, p)$. To this end, we define a function $g : (\mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_1)) \to \mathfrak{X}_1) \to (\mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_1)) \to \mathfrak{X}_2)$ as follows. For every $\mathfrak{F}_1 \in \mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_1)) \to \mathfrak{X}_1$ and $\mathsf{F} \in \mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_1))$, we define $g(\mathfrak{F}_1)(\mathsf{F}) = f(\mathfrak{F}_1(\mathsf{F}))$. Clearly, since f is a witness for $\mathfrak{X}_1 \sqsubseteq_{\mathrm{X}} \mathfrak{X}_2$, it holds that $g(\mathfrak{F}_1)(\mathsf{F}) \setminus \{ \chi \in \mathrm{Asg} \mid \chi \upharpoonright \mathrm{P} \in \mathrm{X} \} \subseteq \mathfrak{F}_1(\mathsf{F})$, for every $\mathsf{F} \in \mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_1))$. Thus, for every $\mathfrak{F}_1 \in \mathrm{Fnc}_\Theta(\mathsf{ap}(\mathfrak{X}_1)) \to \mathfrak{X}_1$, the following holds: $\mathsf{ext}(g(\mathfrak{F}_1), p) \setminus \{ \chi \in \mathrm{Asg} \mid \chi \upharpoonright \mathrm{P} \in \mathrm{X} \} = \left(\bigcup_{\mathsf{F} \in \mathsf{dom}(g(\mathfrak{F}_1))} \mathsf{ext}(g(\mathfrak{F}_1)(\mathsf{F}), \mathsf{F}, p) \right) \setminus \{ \chi \in \mathrm{Asg} \mid \chi \upharpoonright \mathrm{P} \in \mathrm{X} \} = \bigcup_{\mathsf{F} \in \mathsf{dom}(g(\mathfrak{F}_1))} (\mathsf{ext}(g(\mathfrak{F}_1)(\mathsf{F}), \mathsf{F}, p) \setminus \{ \chi \in \mathrm{Asg} \mid \chi \upharpoonright \mathrm{P} \in \mathrm{X} \}) \subseteq \bigcup_{\mathsf{F} \in \mathsf{dom}(g(\mathfrak{F}_1))} \mathsf{ext}(g(\mathfrak{F}_1)(\mathsf{F}) \setminus \{ \chi \in \mathrm{Asg} \mid \chi \upharpoonright \mathrm{P} \in \mathrm{X} \}, \mathsf{F}, p) \subseteq \bigcup_{\mathsf{F} \in \mathsf{dom}(\mathfrak{F}_1)} \mathsf{ext}(\mathfrak{F}_1(\mathsf{F}), \mathsf{F}, p) = \mathsf{ext}(\mathfrak{F}_1, p)$ (notice that $\mathsf{dom}(g(\mathfrak{F}_1)) = \mathsf{dom}(\mathfrak{F}_1)$). ☐

  Now, notice that, by definition of generator, $\mathrm{X} \subseteq \mathrm{Asg}(\vec{p})$. Let $\overline{\mathrm{X}} \triangleq \mathrm{Asg}(\vec{p}) \setminus \mathrm{X}$. Thanks to Propositions 5, it is possible to prove the following claim, by observing that, clearly, $\{\mathrm{X}\} \sqsubseteq_{\overline{\mathrm{X}}} \{\mathrm{Asg}(\vec{p})\}$ holds, as $\mathrm{Asg}(\vec{p}) \setminus \{ \chi \in \mathrm{Asg} \mid \chi \upharpoonright \vec{p} \in \overline{\mathrm{X}} \} = \mathrm{X}$.

  Claim 16. $\mathsf{evl}_{\exists\forall}(\mathfrak{X}, \mathsf{C}_{\exists\forall}(\wp)) \sqsubseteq_{\overline{\mathrm{X}}} \mathsf{evl}_{\exists\forall}(\mathsf{C}_{\exists\forall}(\widehat{\wp}))$.

  Due to the specific definition of the ordering $\sqsubseteq$ between hyperassignments, it follows that the above inclusion necessarily implies the non existence of a set of assignments $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathfrak{X}, \mathsf{C}_{\exists\forall}(\wp))$ such that $\mathrm{E} \subseteq \widehat{\Psi}$. Indeed, assume, towards a contradiction that there is $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathfrak{X}, \mathsf{C}_{\exists\forall}(\wp))$ such that $\mathrm{E} \subseteq \widehat{\Psi}$. By the above claim, there is $\widehat{\mathrm{E}} \in \mathsf{evl}_{\exists\forall}(\mathsf{C}_{\exists\forall}(\widehat{\wp}))$ such that $\widehat{\mathrm{E}} \setminus \{ \chi \in \mathrm{Asg} \mid \chi \upharpoonright \vec{p} \in \overline{\mathrm{X}} \} \subseteq \mathrm{E} \subseteq \widehat{\Psi}$. Since $\widehat{\mathrm{E}} \cap \{ \chi \in \mathrm{Asg} \mid \chi \upharpoonright \vec{p} \in \overline{\mathrm{X}} \} \subseteq \{ \chi \in \mathrm{Asg}(\mathrm{P}) \mid \chi \upharpoonright \vec{p} \notin \mathrm{X} \} \subseteq \widehat{\Psi}$, we have that $\widehat{\mathrm{E}} \subseteq \widehat{\Psi}$, which is in contradiction with Abelard winning the game. Hence, $\mathrm{E} \not\subseteq \widehat{\Psi}$ holds for all $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathfrak{X}, \mathsf{C}_{\exists\forall}(\wp))$, which implies $\mathrm{E} \not\subseteq \Psi$, for all $\mathrm{E} \in \mathsf{evl}_{\exists\forall}(\mathfrak{X}, \mathsf{C}_{\exists\forall}(\wp))$, being $\Psi \subseteq \widehat{\Psi}$. ☐

Theorem 9 (Formula Canonical Forms). *For every behavioral* GFG-QPTL *formula $\wp\psi$, with $\psi \in$ LTL, it holds that $\mathfrak{X} \models^\alpha \wp\psi$ iff $\mathfrak{X} \models^\alpha \mathsf{C}_{\exists\forall}(\wp)\psi$ iff $\mathfrak{X} \models^\alpha \mathsf{C}_{\forall\exists}(\wp)\psi$, for all Borelian behavioral hyperassignments $\mathfrak{X} \in \mathrm{HAsg}(\mathsf{free}(\wp\psi))$.*

Proof. By Proposition 9 and Lemma 5, we have that $\mathfrak{X} \models^\alpha \wp\psi$ *iff* $\mathsf{nevl}_\alpha(\mathfrak{X}, \wp) \models^\alpha \psi$, $\mathfrak{X} \models^\alpha \mathsf{C}_{\forall\exists}(\wp)\psi$ *iff* $\mathsf{nevl}_\alpha(\mathfrak{X}, \mathsf{C}_{\forall\exists}(\wp)) \models^\alpha \psi$, and $\mathfrak{X} \models^\alpha \mathsf{C}_{\exists\forall}(\wp)\psi$ *iff* $\mathsf{nevl}_\alpha(\mathfrak{X}, \mathsf{C}_{\exists\forall}(\wp)) \models^\alpha \psi$.

If $\alpha = \exists\forall$, then, by Proposition 6, it holds that $\mathsf{nevl}_\alpha(\mathfrak{X}, \mathsf{C}_{\forall\exists}(\wp)) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \wp) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \mathsf{C}_{\exists\forall}(\wp))$. Therefore, by Theorem 2, we have that $\mathfrak{X} \models^\alpha \mathsf{C}_{\forall\exists}(\wp)\psi$ implies $\mathfrak{X} \models^\alpha \wp\psi$, which, in turn, implies $\mathfrak{X} \models^\alpha \mathsf{C}_{\exists\forall}(\wp)\psi$.

If $\alpha = \forall\exists$, then, by Proposition 6, it holds that $\mathsf{nevl}_\alpha(\mathfrak{X}, \mathsf{C}_{\exists\forall}(\wp)) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \wp) \sqsubseteq \mathsf{nevl}_\alpha(\mathfrak{X}, \mathsf{C}_{\forall\exists}(\wp))$. Therefore, by Theorem 2, we have that $\mathfrak{X} \models^\alpha \mathsf{C}_{\exists\forall}(\wp)\psi$ implies $\mathfrak{X} \models^\alpha \wp\psi$, which, in turn, implies $\mathfrak{X} \models^\alpha \mathsf{C}_{\forall\exists}(\wp)\psi$.

We are now left to show that $\mathfrak{X} \models^{\exists\forall} \mathsf{C}_{\exists\forall}(\wp)\psi$ implies $\mathfrak{X} \models^{\exists\forall} \mathsf{C}_{\forall\exists}(\wp)\psi$ and that $\mathfrak{X} \models^{\forall\exists} \mathsf{C}_{\forall\exists}(\wp)\psi$ implies $\mathfrak{X} \models^{\forall\exists} \mathsf{C}_{\exists\forall}(\wp)\psi$.

In order to prove the former, we proceed as follows. Let $\Psi \triangleq \{ \chi \in \mathrm{Asg}(\mathsf{free}(\psi)) \mid \chi \models_{\mathsf{LTL}} \psi \}$ and notice that Ψ is Borelian [54]. Observe that $\mathfrak{Q} = \langle \mathfrak{X}, \wp, \Psi \rangle$ is a quantification-game schema. Since $\mathfrak{X}$ is Borelian behavioral, then there is a generator $\langle \tilde{\wp}, \mathrm{X} \rangle$ for it. Therefore, the game $\mathcal{G}^{\widehat{\Psi}}_{\widehat{\wp}}$ is a $\mathfrak{Q}$-game, where $\vec{p} \triangleq \mathsf{ap}(\mathfrak{X}) \setminus \mathsf{ap}(\tilde{\wp})$, $\widehat{\wp} \triangleq \forall \vec{p}.\, \tilde{\wp}.\, \wp$, and $\widehat{\Psi} \triangleq \Psi \cup \{ \chi \in \mathrm{Asg}(\mathsf{free}(\psi)) \mid \chi \restriction \vec{p} \notin \mathrm{X} \}$. By Proposition 9 and Item 1 of Theorem 1, $\mathfrak{X} \models^{\exists\forall} \mathsf{C}_{\exists\forall}(\wp)\psi$ implies that there exists $\mathrm{X} \in \mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \mathsf{C}_{\exists\forall}(\wp))$ such that $\mathrm{X} \subseteq \Psi$. By Item 2 of Theorem 8, it holds that Abelard does not win the game $\mathcal{G}^{\widehat{\Psi}}_{\widehat{\wp}}$. Now, notice that $\mathcal{G}^{\widehat{\Psi}}_{\widehat{\wp}}$ is Borelian; thus, as a consequence of Martin's determinacy theorem [46], [47], $\mathcal{G}^{\widehat{\Psi}}_{\widehat{\wp}}$ is won by Eloise. Hence, from Item 1 of Theorem 8, it follows that there exists $\mathrm{X} \in \mathsf{nevl}_{\exists\forall}(\mathfrak{X}, \mathsf{C}_{\forall\exists}(\wp))$ such that $\chi \models_{\mathsf{LTL}} \psi$ for all $\chi \in \mathrm{X}$, which, by Proposition 9 and Item 1 of Theorem 1, implies $\mathfrak{X} \models^{\exists\forall} \mathsf{C}_{\forall\exists}(\wp)\psi$.

Finally, we turn to showing that $\mathfrak{X} \models^{\forall\exists} \mathsf{C}_{\forall\exists}(\wp)\psi$ implies $\mathfrak{X} \models^{\forall\exists} \mathsf{C}_{\exists\forall}(\wp)\psi$. By Proposition 6, it holds that $\mathsf{nevl}_{\forall\exists}(\mathfrak{X}, \mathsf{C}_{\exists\forall}(\wp)) \sqsubseteq \mathsf{nevl}_{\forall\exists}(\mathfrak{X}, \mathsf{C}_{\forall\exists}(\wp))$. By Theorem 2, we have that $\mathsf{nevl}_{\forall\exists}(\mathfrak{X}, \mathsf{C}_{\forall\exists}(\wp)) \models^{\forall\exists} \psi$ implies $\mathsf{nevl}_{\forall\exists}(\mathfrak{X}, \mathsf{C}_{\exists\forall}(\wp)) \models^{\forall\exists} \psi$. The thesis follows from Lemma 5 and Proposition 9. $\square$

D. Proofs for Section V

Theorem 10 (Satisfiability Game)**.** *Every behavioral* GFG-QPTL *sentence φ has a parity game, with $\mathrm{O}\left(2^{2^{|\varphi|}}\right)$ positions and $\mathrm{O}\left(2^{|\varphi|}\right)$ priorities, in which Eloise wins* iff *φ is satisfiable.*

Proof. Let $\varphi = \wp\psi$ be a behavioral QPTL prenex sentence with $\wp$ a quantification prefix and ψ an LTL formula. The idea of the proof is to construct a parity game from the automaton recognizing models of ψ and the quantification prefix that will be equivalent to the game $\mathcal{G}^{\psi}_{\wp}$ defined in Construction 1.

From ψ, we construct a non-deterministic Büchi automaton A_ψ recognizing models of ψ using the Vardi-Wolper construction [66]. We will consider a complete determinized parity Automaton $\mathcal{D}_\psi = \langle Q, q_0, \Sigma, \delta, \mathrm{Acc} \rangle$ equivalent to A_ψ (that can be obtained via a Safra-like determinization procedure [55]) with

- Q is the set of states of $\mathcal{D}_\psi$,
- $q_0 \in Q$ is the initial state,
- $\Sigma = \mathrm{Val}$ is the alphabet of $\mathcal{D}_\psi$,
- $\delta : Q \times \Sigma \to Q$ is the transition function,
- Acc is the parity condition.

The parity game associated is defined as $\mathcal{G}_\varphi \triangleq \langle \mathcal{A}, \mathrm{Ob}, \mathrm{Wn} \rangle$ with arena $\mathcal{A} \triangleq \langle \mathrm{Ps}_{\mathtt{E}}, \mathrm{Ps}_{\mathtt{A}}, v_I, Mv \rangle$ and is constructed as follows:

- the set of positions $\mathrm{Ps}(\mathcal{G}_\varphi) \subseteq Q \times \mathrm{Ps}(\mathcal{G}^{\psi}_{\wp})$ contains exactly the pairs of one state of the automaton $\mathcal{D}_\psi$ and a valuation $\xi \in \mathrm{Val}$ that is a position of $\mathcal{G}^{\psi}_{\wp}$;
- the set of Eloise's positions $\mathrm{Ps}_{\mathtt{E}} \subseteq \mathrm{Ps}(\mathcal{G}_\varphi)$ only contains the positions $(q, \xi) \in \mathrm{Ps}(\mathcal{G}_\varphi)$ for ξ is an Eloise's position in $\mathcal{G}^{\psi}_{\wp}$;
- the initial position $v_I \triangleq (q_0, \varnothing)$ is just the initial state of $\mathcal{D}_\psi$ paired with the initial state of $\mathcal{G}^{\psi}_{\wp}$;
- the move relation $Mv \subseteq \mathrm{Ps}(\mathcal{G}_\varphi) \times \mathrm{Ps}(\mathcal{G}_\varphi)$ contains exactly those pairs of positions $((q_1, \xi_1), (q_2, \xi_2)) \in \mathrm{Ps}(\mathcal{G}_\varphi) \times \mathrm{Ps}(\mathcal{G}_\varphi)$ such that:
  - (ξ_1, ξ_2) is a move in $\mathcal{G}^{\psi}_{\wp}$;
  - if $\xi_2 = \varnothing$ then $q_2 = \delta(q_1, \xi_1)$, otherwise, $q_1 = q_2$;
- the set of observable positions $\mathrm{Ob} \triangleq Q \times \{ \varnothing \}$;
- the winning condition is deduced from the accepting condition of the automaton $\mathcal{D}_\psi$. More precisely, the priority of a position $(q, \varnothing)$ is defined as the priority of q.

We first show that the arena of the game $\mathcal{G}_\varphi$ is very similar to the arena of $\mathcal{G}^{\psi}_{\wp}$ trough the definition of a bijection between initial paths on $\mathcal{G}_\varphi$ (denoted $\mathrm{Pth}_{\mathsf{init}}(\mathcal{G}^{\psi}_{\wp})$) and initial paths on $\mathcal{G}^{\psi}_{\wp}$ (denoted $\mathrm{Pth}_{\mathsf{init}}(\mathcal{G}_\varphi)$) $f : \mathrm{Pth}_{\mathsf{init}}(\mathcal{G}^{\psi}_{\wp}) \to \mathrm{Pth}_{\mathsf{init}}(\mathcal{G}_\varphi)$.

We consider the morphism $\mathtt{f} \colon \mathrm{Ps}(\mathcal{G}_\varphi) \to \mathrm{Ps}(\mathcal{G}^{\psi}_{\wp})$ defined on (q, ξ) by the following.

$$\mathtt{f}((q, \xi)) \triangleq \xi$$

We now define f on $\pi \in \mathrm{Pth}_{\mathsf{init}}(\mathcal{G}_\varphi)$ by the following.

$$f(\pi) \triangleq \mathtt{f}(\pi)$$

Claim 17. *The function f between $\mathrm{Pth}_{init}(\mathcal{G}^{\psi}_{\wp})$ and $\mathrm{Pth}_{init}(\mathcal{G}_\varphi)$ is a bijection.*

Proof. First, we show that $\mathtt{f}$ is indeed a morphism:

- there is a move $(q, \xi_1)(q, \xi_2)$ for some $q \in Q$ in $\mathcal{G}_\varphi$ *iff* there is a move $\xi_1\xi_2$ in $\mathcal{G}^{\psi}_{\wp}$,
- there is a move $(q, \xi)(q', \varnothing)$ for some $\xi \in \mathrm{Val}(\mathsf{ap}(\wp))$ in $\mathcal{G}_\varphi$ *iff* there is a move $\xi\varnothing$ in $\mathcal{G}^{\psi}_{\wp}$.

We conclude that if $(q_1, \xi_1)(q_2, \xi_2)$ is a move in $\mathcal{G}_\varphi$, then $\mathtt{f}((q_1, \xi_1)(q_2, \xi_2))$ is a move in $\mathcal{G}^{\psi}_{\wp}$. Then f is well defined.

The inverse function is also well defined because we consider only initial paths and the automaton $\mathcal{D}_\psi$ is deterministic and complete. Thus, from an history ρ of $\mathcal{G}^{\psi}_{\wp}$, there is only one state q reached by $\mathcal{D}_\psi$ by reading $\mathsf{obs}(\rho)$. Formally f^{-1} can be defined inductively on a α-history $\rho = \rho'\xi$ as follows:

- $f^{-1}(\varnothing) = (q_0, \varnothing')$
- if $\mathsf{lst}\big(f^{-1}(\rho')\big) = (q, \xi)$ with $\mathsf{dom}(\xi) = \mathsf{ap}\big((\wp)_{<\#(\xi)}\big)$, then $f^{-1}(\rho) = f^{-1}(\rho')(\delta(q, \xi), \varnothing)$,
- otherwise, $\mathsf{lst}\big(f^{-1}(\rho')\big) = (q, \xi')$ for some $q \in Q$ thus $f^{-1}(\rho) = f^{-1}(\rho')(q, \xi)$.

with $\mathsf{lst}(w)$ being the last character of the finite word w.

We just showed that f is a bijection between initial paths $\mathrm{Pth}_{\mathsf{init}}(\mathcal{G}_\varphi)$ and initial paths $\mathrm{Pth}_{\mathsf{init}}(\mathcal{G}^{\psi}_{\wp})$. $\square$

We want to state that a play in $\mathcal{G}_\varphi$ is won by Eloise *iff* its image by f is also won by Eloise. To do so, we first characterize the plays in $\mathcal{G}_\varphi$ that Eloise wins by linking them with an execution of $\mathcal{D}_\psi$. To precise this link, we associate an assignment χ to a play π of $\mathcal{G}_\varphi$.

We define a function $g : \mathrm{Play}(\mathcal{G}_\varphi) \to \mathrm{Val}(\mathsf{ap}(\wp))^\omega$ as follow where $\pi = (\pi)_{i\in\mathbb{N}}$ is a play on $\mathcal{G}_\varphi$.

$$g(\pi) \in \mathrm{Val}(\mathsf{ap}(\wp))^\omega$$
$$g(\pi)_t \triangleq \mathrm{snd}(\pi_{(t+1)\times(|\mathsf{ap}(\wp)|+1)-1})$$

Claim 18. *The function g is a bijection and a play π is won by Eloise* iff *$g(\pi)$ is recognized by the automaton $\mathcal{D}^{\psi}_{\psi}$.*

Proof. We first prove that g is a bijection by proving it is well defined and it is a surjection and an injection. By definition of the game $\mathcal{G}_\varphi$, for every $t \in \mathbb{N}$, it holds that $\pi_{(t+1)\times(|\mathsf{ap}(\wp)|+1)}$ has the form $(q, \varnothing)$. Thus, the previous positions necessarily has the form (q', ξ) with $\xi \in \mathrm{Val}(\mathsf{ap}(\wp))$. Then g is well defined.

To show that g is a bijection, we show that every word w is the image of exactly one play. Consider a word $w \in \mathrm{Val}(\mathsf{ap}(\wp))^\omega$. For every natural t and every position $(q, \varnothing)$ there exist a unique finite path of $|\mathsf{ap}(\wp)|$ moves reaching the position (q, w_t), by definition of the game $\mathcal{G}_\varphi$. This path π is defined as $\pi_0 = (q, \varnothing)$ and $\pi_{i+1} = (q, \xi_{i+1})$ with $\xi_{i+1} = (w_t)_{<i+1}$ for every $i < |\mathsf{ap}(\wp)|$. Thanks to this construction, we can build step by step the full play, starting at $(q_0, \varnothing)$ with $t = 0$.

1) From a state of the form $(q, \varnothing)$, goes to (q, w_t) in $|\mathsf{ap}(\wp)|$ moves.
2) The next move in necessarily from (q, w_t) to $(\delta(q, w_t), \varnothing)$.
3) Go to 1) with $t := t + 1$

Steps 1) and 2) use $|\mathsf{ap}(\wp)| + 1$ moves. Positions of the form (q, w_t) are reached at the end of the first step. Thus we have $\pi_{(t+1)\times(|\mathsf{ap}(\wp)|+1)-1} = (q, w_t)$.

Because of the parity condition of the game $\mathcal{G}_\varphi$, it is clear that a play π is won by Eloise if and only if $g(\pi)$ is accepted by the automaton $\mathcal{D}_\psi$. $\square$

We now explicit the link between f and g: the word associated with a play in $\mathcal{G}_\varphi$ is the sequence of observable position in the associated play in $\mathcal{G}^{\psi}_{\wp}$.

Claim 19. *Given a play π in the game $\mathcal{G}_\varphi$, we have for every t natural number, $g(\pi)_t = \mathsf{obs}(f(\pi))_t$.*

Proof. This claim derive directly from definitions of f, g and $\mathcal{G}^{\psi}_{\wp}$. $\square$

We now show that Eloise wins a play in $\mathcal{G}_\varphi$ *iff* she wins the corresponding play in $\mathcal{G}^{\psi}_{\wp}$.

Claim 20. *The bijection f preserves the winner: π is won by Eloise in $\mathcal{G}_\varphi$* iff *$f(\pi)$ is won by Eloise in $\mathcal{G}^{\psi}_{\wp}$.*

Proof. Consider a play π in the game $\mathcal{G}_\varphi$. Thanks to Claim 18, we know that π is won by Eloise *iff* $g(\pi)$ is accepted by $\mathcal{D}_\psi$ which means that $\mathsf{wrd}^{-1}(g(\pi)) \models \psi$. Claim 19 assure that $g(\pi)(n) = \mathsf{obs}(f(\pi))_n$. We can deduce that $\mathsf{wrd}^{-1}(\mathsf{obs}(f(\pi)))$ define an assignment that satisfy ψ. Then $f(\pi)$ is won by Eloise.

The reciprocity is ensured by the bijective property of f (Claim 17.) $\square$

Because f is a bijection, we can derive a bijection between strategies on $\mathcal{G}_\varphi$ and strategies on $\mathcal{G}^{\psi}_{\wp}$. It is not hard to see that a strategy for Eloise is winning *iff* its associated strategy is also winning. Thus Eloise wins $\mathcal{G}_\varphi$ *iff* she wins $\mathcal{G}^{\psi}_{\wp}$.

The automaton A_ψ has a size exponential in the size of ψ. The procedure to determinize in a parity automaton adds one exponential; thus $|\mathcal{D}_\psi| = \mathrm{O}\Big(2^{2^{|\psi|}}\Big)$. The quantification game have a size in $\mathrm{O}\big(2^{|\wp|}\big)$. We conclude that the game constructed has a size in $\mathrm{O}\Big(2^{|\wp|}2^{2^{|\psi|}}\Big) = \mathrm{O}\Big(2^{2^{|\varphi|}}\Big)$. The game has the same number of priorities as the automaton $\mathcal{D}_\psi$ which is in $\mathrm{O}\big(2^{|\psi|}\big)$. $\square$