

# The Light Side of Interval Temporal Logic: the Bernays-Schönfinkel fragment of CDT

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**Abstract** Decidability and complexity of the satisfiability problem for the logics of time intervals have been extensively studied in the recent years. Even though most interval logics turn out to be undecidable, meaningful exceptions exist, such as the logics of temporal neighborhood and (some of) the logics of the subinterval relation. In this paper, we explore a different path to decidability: instead of restricting the set of modalities or imposing severe semantic restrictions, we take the most expressive interval temporal logic studied so far, namely, Venema’s CDT, and we suitably limit the negation depth of modalities. The decidability of the satisfiability problem for the resulting fragment, called  $CDT_{BS}$ , over the class of all linear orders, is proved by embedding it into a well-known decidable quantifier prefix class of first-order logic, namely, Bernays-Schönfinkel class. In addition, we show that  $CDT_{BS}$  is in fact NP-complete (Bernays-Schönfinkel class is NEXPTIME-complete), and we prove its expressive completeness with respect to a suitable fragment of Bernays-Schönfinkel class. Finally, we show that any increase in the negation depth of  $CDT_{BS}$  modalities immediately yields undecidability.

**Keywords** Interval temporal logic · Tableau methods · Decidability · Complexity

## 1 Introduction

In the recent years, the study of temporal reasoning via interval-based (logical) approaches has been very intensive. Since the seminal work by Halpern and Shoham [18] and Ven-

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ema [33], a series of papers on interval temporal logics has been published, e.g., [5, 6, 9–11, 23, 24, 29]. As an effect, the problem of classifying all “natural”, genuinely interval-based (that is, all intervals over a linear order are considered, and no projection principle is applied [17]) logics with respect to their expressive and computational power has been extensively studied and almost completely solved.

Propositional interval temporal logics are modal logics, interpreted over linearly- or partially-ordered sets, whose proposition letters are evaluated over intervals instead of over points. They differ from each other in the number and type of basic relations between intervals that are captured by their modalities, by the linear order(s) over which they are interpreted, and by the inclusion or exclusion of point-intervals (intervals with coincident endpoints). In the hierarchy of existing interval temporal logics based on their expressive power, the top element is Venema’s CDT [33], whose language features three binary modalities, corresponding to the three possible ways to place a point with respect to the two endpoints of a given interval, and a modal constant, that identifies point-intervals. The second-highest logic in the hierarchy is Halpern and Shoham’s HS [19], which features one unary modality for each Allen’s relation between pairs of intervals [1]. Both in CDT and in HS, satisfiability turns out to be undecidable, no matters what class of linear orders is considered (all, discrete, dense, finite, the linear order of natural numbers, and so on) [19].

In the recent years, some fragments of HS with a better computational behavior have been identified. Meaningful examples include, but are not limited to,  $\overline{AA}$  (a.k.a. Propositional Neighborhood Logic, PNL), which features two modalities for Allen’s relations *meets* and *met by*, and is decidable over all meaningful classes of linear orders [8, 16]; its extension  $\overline{AA\overline{BB}}$  [28], that includes modalities for Allen’s relation’s *starts* and *started by*, and its mirror image  $\overline{AA\overline{EE}}$ , with additional modalities for Allen’s relations *finishes* and *finished by*, which are decidable over the class of finite linear orders and undecidable everywhere else; and  $\overline{BB\overline{DD}\overline{LL}}$  (and its mirror image  $\overline{EE\overline{DD}\overline{LL}}$ ), with modalities for Allen’s relations *starts*, *started by*, *during*, *contains*, *before*, and *after*, which is decidable over dense linear orders [27] and undecidable over finite and (weakly) discrete linear orders (as a matter of fact, one-modality logics  $\overline{D}$  and  $\overline{\overline{D}}$  are already undecidable over the classes of finite and discrete linear orders [23])<sup>1</sup>.

The situation with classical first-order logic is somehow similar. Since it has been shown that satisfiability for the full language is undecidable, a great effort has been made in order to identify more and more expressive decidable fragments. At least three different strategies have been pursued: (i) limiting the number of variables of the language, (ii) limiting the type of formulas allowed by relativizing quantification (*guarded fragments*), and (iii) limiting the structure and the shape of the quantifier prefix.

First-order logics with a restriction on the number of variables have been already studied in connection with interval temporal logics. Most notably,  $\overline{AA}$  has been proved to be expressively equivalent to the two-variable fragment of first-order logic over linear orders. Such a fragment of first-order logic has been shown to be NEXPTIME-complete over various classes of linear orders in [30]. Decidability of  $\overline{AA}$  over the same classes of orders immediately follows. Guarded fragments of first-order logic (see [2] for an introduction) have been shown to be quite useful to explain the good computational properties of modal logics, but, to the best of our knowledge, they have never been considered in the framework of interval temporal logics. As a matter of fact, mapping interval temporal logics into guarded fragments of first-order logic would require (i) the use of a relation in the guards

<sup>1</sup> In all these cases, including or excluding point-intervals makes no difference.

which is (or can be forced to behave as) a linear order, (ii) at least three distinct variables, (iii) uninterpreted predicates which are at least binary, and (iv) quantifications with Boolean combinations of atomic formulas as guards. Such requirements are not met by known decidable guarded fragments of first-order logic<sup>2</sup>.

In this paper, we explore an original path to decidability of interval temporal logics, which follows the third strategy: we look for meaningful interval temporal logics that can be embedded into decidable quantifier prefix classes of first-order logics. The decidability of the latter family of logics does not depend on the shape of the quantifier prefix only, but also on the number and the arity of predicate and function symbols that are allowed in the formulas, and on the presence/absence of equality. Seven different decidable classes have been identified in the literature (a survey on quantifier prefix classes of first-order logic can be found in [4]).

We focus our attention on the prefix vocabulary class identified by Bernays and Schönfinkel in 1928 (a.k.a. Bernays, Schönfinkel, and Ramsey class, as Ramsey proved that decidability is preserved even when equality is included) [4]. It consists of all and only formulas in prenex form whose quantifier prefix is of the form  $\exists x_1 \dots \exists x_n \forall y_1 \dots \forall y_m$  and whose matrix may include predicate symbols of any arity (but no function symbols) and, possibly, equality. It is well known that Bernays-Schönfinkel fragment of first-order logic is expressive enough to model a linear order devoid of specific properties such as discreteness or density. Moreover, it can express simple frame properties, commonly studied in the interval temporal logic literature, like, for instance, boundedness.

We identify a syntactic fragment of CDT [33], called  $\text{CDT}_{\text{BS}}$ , whose standard translation fits into Bernays-Schönfinkel class, by limiting the negation depth of the modalities to one, that is, by constraining temporal operators to occur in the scope of at most one negation. Decidability of  $\text{CDT}_{\text{BS}}$ , over the class of all linear orders, immediately follows. Then, a precise characterization of  $\text{CDT}_{\text{BS}}$  expressive power is given by showing that it is expressively complete with respect to a suitable fragment of Bernays and Schönfinkel class. A decision procedure for  $\text{CDT}_{\text{BS}}$  is then obtained by tailoring the non-terminating tableau-based deduction system for CDT developed in [15] to it. As a by-product, we prove that the satisfiability problem for  $\text{CDT}_{\text{BS}}$  is NP-complete, in sharp contrast with that of Bernays-Schönfinkel class, which is NEXPTIME-complete, when relation symbols of unbounded arity are allowed, and PSPACE, when relation symbols have bounded arity, e.g., only binary relations are allowed, as it is the case for interval logics. Finally, we show that any increase in the negation depth of  $\text{CDT}_{\text{BS}}$  modalities immediately yields undecidability.

The paper is structured as follows. In Section 2, we provide background knowledge about Bernays and Schönfinkel fragment of first-order logic. In Section 3, we define syntax and semantics of  $\text{CDT}_{\text{BS}}$ , and we define its standard translation. Decidability immediately follows from the inclusion of the resulting set of formulas in Bernays and Schönfinkel class. Next, in Section 4, we prove the expressive completeness of  $\text{CDT}_{\text{BS}}$  with respect to a suitable fragment of such a class. In Section 5, we devise a sound, complete, and terminating tableau method for  $\text{CDT}_{\text{BS}}$ . Finally, in Section 6, we show that fairly natural extensions of  $\text{CDT}_{\text{BS}}$  do not preserve decidability. An assessment of the work done and possible future research directions are given in Section 7.

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<sup>2</sup> Extended guarded fragments includes loosely guarded fragments, which allow guards to be more complex than simple atoms [3], and guarded fragments with transitive guards (in general, transitivity cannot be expressed as a guarded formula) [31].

## 2 Bernays-Schönfinkel class

Bernays-Schönfinkel prefix vocabulary class, denoted here by  $\text{FO}_{\text{BS}}$ , consists of all and only those first-order formulas, making use of any relational symbol of any arity, including equality, that can be put in prenex form by using a quantifier prefix of the form  $\exists \mathbf{x} \forall \mathbf{y}$ , where  $\mathbf{x} = x_1 \dots x_n$  and  $\mathbf{y} = y_1 \dots y_m$  are (possibly empty) vectors of first-order variables. It is well known that the satisfiability problem for  $\text{FO}_{\text{BS}}$  is NEXPTIME-complete [4]. Moreover,  $\text{FO}_{\text{BS}}$  is closed under conjunction and disjunction, since all its formulas can be thought of as sentences (free variables can be existentially quantified), but it is not closed under negation.

To simplify the proofs of the results given in the paper, we introduce an alternative definition of  $\text{FO}_{\text{BS}}$  via the following abstract grammar:

$$\alpha ::= \alpha_{\exists} \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \exists x. \alpha \mid \neg \alpha_{\exists} \text{ for } \alpha_{\exists} \text{ of the form } \exists x. \alpha_{\exists} \quad (1)$$

$$\alpha_{\exists} ::= A(\mathbf{x}) \mid \neg A(\mathbf{x}) \mid \alpha_{\exists} \wedge \alpha_{\exists} \mid \alpha_{\exists} \vee \alpha_{\exists} \mid \exists x. \alpha_{\exists} \quad (2)$$

$$A(\mathbf{x}) ::= \text{any relational symbol of arbitrary arity, including equality} \quad (3)$$

Grammar (1) generates a fragment of first-order logic consisting of all and only those formulas where existential quantifiers can occur in the scope of *at most one negation*. While any prenex formula of the form  $\exists \mathbf{x} \forall \mathbf{y} \beta$  can be generated by grammar (1), the converse is not true, since grammar (1) can generate also formulas which are not in prenex form. However, it is not difficult to show that any formula generated by grammar (1) can be transformed into an equivalent prenex formula of the correct form, as shown by the following proposition.

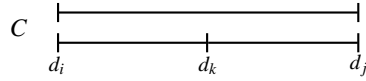
**Proposition 1** *Any formula generated by grammar (1) can be transformed into a prenex formula of the form  $\exists \mathbf{x} \forall \mathbf{y} \beta$ , with  $\beta$  quantifier-free.*

*Proof* Let  $\alpha$  be a formula generated by grammar (1). We show that there exists an equivalent formula  $\tau(\alpha)$  of the required form by structural induction. We start with the set of formulas generated by the sub-grammar for  $\alpha_{\exists}$ , and we show that each of them can be transformed into a formula of the form  $\exists \mathbf{x} \beta$ , with  $\beta$  quantifier-free. The case in which  $\alpha$  is a relation or the negation of a relation is trivial. Consider now the case of formulas  $\alpha = \alpha_{\exists} \wedge \alpha'_{\exists}$ . By inductive hypothesis,  $\tau(\alpha_{\exists}) = \exists \mathbf{z} \beta$  and  $\tau(\alpha'_{\exists}) = \exists \mathbf{w} \beta'$ , for some quantifier-free  $\beta$  and  $\beta'$ . Without loss of generality, we can assume  $\mathbf{z} \cap \mathbf{w} = \emptyset$  (if this is not the case, we can apply a suitable variable substitution), and thus  $\alpha$  is equivalent to  $\exists \mathbf{z} \mathbf{w} (\beta \wedge \beta')$ . The case of disjunction is similar, and thus omitted. Consider now the case of formulas  $\alpha = \exists x. \alpha_{\exists}$ . By inductive hypothesis,  $\tau(\alpha_{\exists}) = \exists \mathbf{w} \beta$ , for some quantifier-free  $\beta$ , with  $x \notin \mathbf{w}$ , and thus  $\alpha$  is equivalent to  $\exists x \exists \mathbf{w} \beta$ . Let us consider now an arbitrary formula generated by grammar (1). The only interesting case is the one for the negation of existential quantifiers. Let  $\alpha = \neg \exists x. \alpha_{\exists}$ . By inductive hypothesis,  $\tau(\exists x. \alpha_{\exists}) = \exists x \exists \mathbf{w} \beta$ , for some quantifier-free  $\beta$ , with  $x \notin \mathbf{w}$ . Hence,  $\alpha$  is equivalent to the formula (in prenex form)  $\forall x \forall \mathbf{w} \neg \beta$ .  $\square$

Thanks to the above result, from now on we will assume that any  $\text{FO}_{\text{BS}}$ -formula has been generated by grammar (1).

## 3 Decidability of the logic $\text{CDT}_{\text{BS}}$ over the class of all linear orders

Interval temporal logics are usually interpreted over a linearly ordered set  $\mathbb{D} = \langle D, < \rangle$ . In this setting, an *interval* on  $\mathbb{D}$  is an ordered pair  $[d_i, d_j]$  with  $d_i \leq d_j$  (we refer to such a case as



**Fig. 1** The ternary relation *chop*, splitting the interval  $[d_i, d_j]$  into the subintervals  $[d_i, d_k]$  and  $[d_k, d_j]$ .

the *non-strict* semantics, in contrast with the *strict* one, that excludes degenerate intervals of the form  $[d_i, d_i]$ . The set of all intervals on  $\mathbb{D}$  is denoted by  $\mathbb{I}(\mathbb{D})$ . The variety of all possible relations between any two intervals has been studied by Allen [1], who identified 12 distinct binary relations plus the equality relation. Halpern and Shoham modal logic of intervals, abbreviated HS, can be viewed as the modal logic of Allen's relations as it features one modality for each such relation. As we already mentioned, HS turns out to be undecidable over any meaningful class of linear orders [19]. In [33], the ternary relation *chop*, depicted in Figure 1, has been taken into consideration. The corresponding binary modality *C*, together with the two conjugated modalities *D* (*done*) and *T* (*to do*), and the modal constant  $\pi$  for point-intervals define the interval temporal logic CDT. It can be easily shown that CDT subsumes HS (in fact, it is strictly more expressive than HS), and thus it is undecidable whenever HS is. In [20], Hodkinson et al. systematically investigate the three fragments of CDT with only one binary modality each (*C*, *D*, or *T*), showing that each of them is undecidable.

Formulas of CDT are built on a set of proposition letters  $\mathcal{A} \mathcal{P} = \{p, q, \dots\}$ , the Boolean connectives  $\neg$  and  $\vee$ , the three binary modalities *C*, *D*, and *T*, and the modal constant  $\pi$ , by the following abstract grammar [33]:

$$\varphi ::= p \mid \pi \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi C \psi \mid \varphi D \psi \mid \varphi T \psi.$$

The other Boolean connectives can be viewed as suitable short forms, as usual. Similarly, universal counterparts of the existential modalities *C*, *D*, and *T* can be defined by means of negation in the standard way; CDT has not any special notation for them.

The semantics of CDT-formulas can be given in terms of concrete *models* of the form  $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ , where  $V : \mathcal{A} \mathcal{P} \rightarrow 2^{\mathbb{I}(\mathbb{D})}$  is a *valuation function*, as follows:

- $M, [d_i, d_j] \Vdash p$  if and only if  $[d_i, d_j] \in V(p)$ ,
- $M, [d_i, d_j] \Vdash \pi$  if and only if  $d_i = d_j$ ,
- $M, [d_i, d_j] \Vdash \neg\varphi$  if and only if  $M, [d_i, d_j] \not\Vdash \varphi$ ,
- $M, [d_i, d_j] \Vdash \varphi \vee \psi$  if and only if  $M, [d_i, d_j] \Vdash \varphi$  or  $M, [d_i, d_j] \Vdash \psi$ ,
- $M, [d_i, d_j] \Vdash \varphi C \psi$  if and only if there exists  $d_k \leq d_j$  such that  $M, [d_i, d_k] \Vdash \varphi$  and that  $M, [d_k, d_j] \Vdash \psi$ ,
- $M, [d_i, d_j] \Vdash \varphi D \psi$  if and only if there exists  $d_k \leq d_i$  such that  $M, [d_k, d_i] \Vdash \varphi$  and that  $M, [d_k, d_j] \Vdash \psi$ ,
- $M, [d_i, d_j] \Vdash \varphi T \psi$  if and only if there exists  $d_k \geq d_j$  such that  $M, [d_j, d_k] \Vdash \varphi$  and that  $M, [d_i, d_k] \Vdash \psi$ .

The *standard translation* is the usual way to express the semantics of a modal or temporal formula in first-order logic. Let  $\varphi$  be a CDT-formula and, for every  $p \in \mathcal{A} \mathcal{P}$ , let us denote by the same symbol  $p$  the corresponding binary relation. The standard translation function  $ST(\varphi)[x, y]$  is defined as follows:

- $ST(\varphi)[x, y] = x \leq y \wedge ST'(\varphi)[x, y]$ ,

where  $x, y$  are two first-order variables and  $ST'(\varphi)[x, y]$  is inductively defined as follows:

- $ST'(p)[x, y] = p(x, y)$ ,

- $ST'(\pi)[x, y] = (x = y)$ ,
- $ST'(\neg\varphi)[x, y] = \neg ST'(\varphi)[x, y]$ ,
- $ST'(\varphi \vee \psi)[x, y] = ST'(\varphi)[x, y] \vee ST'(\psi)[x, y]$ ,
- $ST'(\varphi C \psi)[x, y] = \exists z(x \leq z \leq y \wedge ST'(\varphi)[x, z] \wedge ST'(\psi)[z, y])$ ,
- $ST'(\varphi D \psi)[x, y] = \exists z(z \leq x \wedge ST'(\varphi)[z, x] \wedge ST'(\psi)[z, y])$ ,
- $ST'(\varphi T \psi)[x, y] = \exists z(y \leq z \wedge ST'(\varphi)[y, z] \wedge ST'(\psi)[x, z])$ .

As a general rule, the standard translation makes it possible to reduce the satisfiability problem for a modal logic to a first-order satisfiability problem: a modal formula  $\varphi$  is satisfiable if and only if its standard translation, evaluated on a pair of points  $x, y$ , is (first-order) satisfiable. Now, we ask ourselves the following question: which CDT-formulas are such that their satisfiability problem can be reduced to a first-order satisfiability problem in Bernays-Schönfinkel class? To answer this question, we define an abstract grammar that generates only CDT-formulas suitably limited in the negation depth of modalities:

$$\varphi ::= \varphi_{\exists} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi C \varphi \mid \varphi D \varphi \mid \varphi T \varphi \mid \neg(\varphi_{\exists} C \varphi_{\exists}) \mid \neg(\varphi_{\exists} D \varphi_{\exists}) \mid \neg(\varphi_{\exists} T \varphi_{\exists}) \quad (4)$$

$$\varphi_{\exists} ::= \pi \mid \neg\pi \mid p \mid \neg p \mid \varphi_{\exists} \wedge \varphi_{\exists} \mid \varphi_{\exists} \vee \varphi_{\exists} \mid \varphi_{\exists} C \varphi_{\exists} \mid \varphi_{\exists} D \varphi_{\exists} \mid \varphi_{\exists} T \varphi_{\exists} \quad (5)$$

The above grammar generates a fragment of CDT, that we call  $CDT_{BS}$ , which consists of all and only those formulas where the modalities  $C$ ,  $D$ , and  $T$  can occur in the scope of *at most one negation*. The next lemma shows that the above-defined standard translation maps  $CDT_{BS}$ -formulas into Bernays-Schönfinkel class. It is easy to check that the syntactic limitations of  $CDT_{BS}$  do not prevent it from expressing all HS modalities (it only constrains the way in which they can be composed). As an example,  $\langle B \rangle \varphi$  is captured by  $\varphi C \neg\pi$ . Similar encodings can be given for the other HS modalities [33].

**Lemma 1** *For every  $CDT_{BS}$ -formula  $\varphi$ , its standard translation  $ST(\varphi)[x, y]$  is an  $FO_{BS}$ -formula, with free variables  $x$  and  $y$ .*

*Proof* The proof is by structural induction. We start with the set of formulas generated by the sub-grammar for  $\varphi_{\exists}$ , and we show that the standard translation of each of these formulas belongs to the sub-grammar for  $\alpha_{\exists}$  and it has  $x, y$  as its free variables.

As for the base case, let  $\varphi_{\exists} = p$ , for some proposition letter  $p$ . By definition,  $ST(p)[x, y] = x \leq y \wedge p(x, y)$ ; the thesis immediately follows. The cases  $\neg p, \pi$ , and  $\neg\pi$  are similar, and thus omitted. As for the case of conjunction, let  $\varphi_{\exists} = \varphi'_{\exists} \wedge \varphi''_{\exists}$ . By definition,  $ST(\varphi'_{\exists} \wedge \varphi''_{\exists})[x, y] = x \leq y \wedge ST'(\varphi'_{\exists})[x, y] \wedge ST'(\varphi''_{\exists})[x, y]$ . By inductive hypothesis, both  $ST'(\varphi'_{\exists})[x, y]$  and  $ST'(\varphi''_{\exists})[x, y]$ , and thus  $ST'(\varphi'_{\exists})[x, y]$  and  $ST'(\varphi''_{\exists})[x, y]$ , belong to the sub-grammar for  $\alpha_{\exists}$  and have  $x, y$  as their free variables. It immediately follows that  $ST(\varphi'_{\exists} \wedge \varphi''_{\exists})[x, y]$  has the required form. The case of disjunction is similar, and thus omitted.

Now, let  $\varphi_{\exists} = \varphi'_{\exists} C \varphi''_{\exists}$ . By definition,  $ST(\varphi'_{\exists} C \varphi''_{\exists})[x, y] = x \leq y \wedge \exists z(x \leq z \leq y \wedge ST'(\varphi'_{\exists})[x, z] \wedge ST'(\varphi''_{\exists})[z, y])$ . By inductive hypothesis,  $ST'(\varphi'_{\exists})[x, z]$  is an  $\alpha_{\exists}$ -formula with  $x, z$  as its free variables, and  $ST'(\varphi''_{\exists})[z, y]$  is an  $\alpha_{\exists}$ -formula with  $z, y$  as its free variables. Hence, the formula  $x \leq y \wedge \exists z(x \leq z \leq y \wedge ST'(\varphi'_{\exists})[x, z] \wedge ST'(\varphi''_{\exists})[z, y])$  is an  $\alpha_{\exists}$ -formula with  $x, y$  as its free variables. The other two cases for  $D$  and  $T$  can be dealt with in a similar way.

Let us consider now an arbitrary formula generated by the grammar. The only interesting cases are those for the negation of modalities. Let  $\varphi = \neg(\varphi'_{\exists} C \varphi''_{\exists})$ . By definition,  $ST(\neg(\varphi'_{\exists} C \varphi''_{\exists}))[x, y] = x \leq y \wedge \neg ST'(\varphi'_{\exists} C \varphi''_{\exists})[x, y]$ , and  $ST'(\varphi'_{\exists} C \varphi''_{\exists})[x, y] = \exists z(x \leq z \leq y \wedge ST'(\varphi'_{\exists})[x, z] \wedge ST'(\varphi''_{\exists})[z, y])$ . We have already shown that both  $ST'(\varphi'_{\exists})[x, z]$  and  $ST'(\varphi''_{\exists})[z, y]$  are  $\alpha_{\exists}$ -formulas with  $x, z$  and  $z, y$  as their free variables, respectively. Hence,

$\exists z(x \leq z \leq y \wedge ST'(\varphi'_\exists)[x, z] \wedge ST'(\varphi''_\exists)[z, y])$  is an  $\alpha_\exists$ -formula with  $x, y$  as its free variables. It immediately follows that  $\neg ST'(\varphi'_\exists C \varphi''_\exists)[x, y]$  is an  $\alpha$ -formula with  $x, y$  as its free variables, and thus the thesis, as the conjunction of two  $\alpha$ -formulas is an  $\alpha$ -formula. The other two cases can be dealt with in a similar way.  $\square$

In order to prove the main theorem, it suffices to observe that the linear order  $<$  is captured by the following axioms [4], whose conjunction  $\Phi$  belongs to  $\text{FO}_{\text{BS}}$ :

1.  $\forall x \neg(x < x)$ ;
2.  $\forall x, y(x < y \rightarrow \neg y < x)$ ;
3.  $\forall x, y, z(x < y \wedge y < z \rightarrow x < z)$ ;
4.  $\forall x, y(x = y \vee x < y \vee y < x)$ .

**Theorem 1** *The satisfiability problem for  $\text{CDT}_{\text{BS}}$  over the class of all linear orders is decidable.*

*Proof* By Lemma 1, if  $\varphi$  is a  $\text{CDT}_{\text{BS}}$ -formula, then  $\exists x, y ST(\varphi)[x, y]$  (the existential closure of  $ST(\varphi)[x, y]$ ) belongs to Bernays-Schönfinkel class. Satisfiability of  $\varphi$  can thus be reduced to satisfiability of the  $\text{FO}_{\text{BS}}$ -formula  $\Phi \wedge \exists x, y ST(\varphi)[x, y]$ . Since the satisfiability problem for  $\text{FO}_{\text{BS}}$  is decidable, decidability of  $\text{CDT}_{\text{BS}}$  immediately follows.  $\square$

The satisfiability problem for  $\text{FO}_{\text{BS}}$  has been shown to be NEXPTIME-complete. The proof relies on the observation that an  $\text{FO}_{\text{BS}}$ -formula is satisfiable if and only if it has a model with a number of elements bounded by the number of existential quantifiers [4, Proposition 6.2.17]. This immediately leads to a nondeterministic exponential-time procedure for satisfiability checking. However, when we restrict our attention to formulas where the arity of relational symbols is bounded (to two, in our case), the complexity of such a procedure becomes PSPACE, since in this case a candidate model for the formula can be represented using only a polynomial amount of memory. Hence, Theorem 1 gives us a PSPACE upper-bound to the complexity of  $\text{CDT}_{\text{BS}}$ . In Section 5, we will show that this bound is not tight, by providing an NP decision procedure for the satisfiability of  $\text{CDT}_{\text{BS}}$ .

#### 4 Expressive completeness of $\text{CDT}_{\text{BS}}$

In Section 3, we showed that  $\text{CDT}_{\text{BS}}$  formulas can be translated into Bernays-Schönfinkel class  $\text{FO}_{\text{BS}}$  of first-order logic with equality, thanks to the fact that the linear order  $<$  can be expressed in this fragment. Inspired by the observation that the translation uses only binary predicates, we now ask ourselves the following question: for every formula in Bernays-Schönfinkel class of first-order logic, interpreted over the linear order  $<$  and limited to binary predicates, is there an expressively equivalent  $\text{CDT}_{\text{BS}}$ -formula? Similar expressivity comparison issues have been already investigated for various point- and interval-based logics. A partial list includes basic results about the completeness of LTL with respect to the first-order fragment of monadic second-order logic over Dedekind-complete linear orders and generalizations (Kamp's Theorem and its extensions [12–14, 21, 22, 25]), the completeness of CDT with respect to the three-variable fragment of first-order logic over linear orders, where at most two variables are free [33], the completeness of  $\overline{\text{AA}}$  with respect to two-variable first-order logic over linear orders [8], and the completeness of its metric extension, called MPNL, with respect to a fragment of two-variable first-order logic extended with a successor function over  $\mathbb{N}$  [7].

We focus our attention on first-order logic interpreted over the linear order  $<$  and limited to binary predicates, denoted by  $\text{FO}[<]$ . We will denote by  $\text{FO}^{n,m}[<]$  the  $n$ -variable fragment of  $\text{FO}[<]$ , where at most  $m$  variables are free, and by  $\text{FO}^{\omega,m}[<]$  the fragment of  $\text{FO}[<]$  with a denumerable set of variables, where at most  $m$  are free. Since interval logics are interpreted over intervals (represented as pairs of points), the standard translation of any interval logic formula is a formula with two free variables, and thus it belongs to  $\text{FO}^{\omega,2}[<]$ . By analogy with the case of other interval logics, e.g., [8, 33], to establish an expressive completeness result for  $\text{CDT}_{\text{BS}}$ , we will limit the number of variables of the corresponding first-order fragment. We denote by  $\text{FO}_{\text{BS}}^{n,m}[<]$  (resp.,  $\text{FO}_{\text{BS}}^{\omega,m}[<]$ ) the  $n$ -variable fragment (resp., the fragment with a denumerable set of variables) of the language defined by grammar (1), where at most  $m$  variables occur free.

In the following, we compare interval and first-order logics with respect to their ability of expressing properties of a given interval in a model. We distinguish three cases: (i) the comparison of two interval logics, (ii) the comparison of two fragments of first-order logic, and (iii) the comparison of an interval logic and a fragment of first-order logic.

Given two interval logics (resp., fragments of first-order logic)  $L$  and  $L'$ , we say that  $L'$  is *at least as expressive as*  $L$ , denoted by  $L \preceq L'$ , if there is an (effective) translation  $\tau$  from  $L$  to  $L'$  such that for every model  $M$ , interval  $[d_i, d_j]$  (resp., pair of points  $d_i, d_j$ ) in  $M$ , and formula  $\varphi$  of  $L$ ,  $M, [d_i, d_j] \models \varphi$  iff  $M, [d_i, d_j] \models \tau(\varphi)$  (resp.,  $M \models \varphi(d_i, d_j)$  iff  $M \models \tau(\varphi)(d_i, d_j)$ ). Furthermore, we say that  $L'$  is *as expressive as*  $L$ , denoted by  $L' \equiv L$ , if both  $L' \preceq L$  and  $L \preceq L'$ , and we say that  $L'$  is *strictly more expressive than*  $L$ , denoted by  $L \prec L'$ , if  $L \preceq L'$  and  $L' \not\preceq L$ .

To compare the expressive power of an interval logic and a fragment of first-order logic, we must cope with a technical problem: interval models constrain interval logic formulas to be evaluated on ordered pairs  $[d_i, d_j]$ , with  $d_i \leq d_j$ , only, while relational models do not impose such a constraint. To solve it, we map each binary relation  $p$  of the considered fragment of first-order logic into two distinct proposition letters  $p^{\leq}$  and  $p^{\geq}$  of the interval logic. From [8], we borrow the following definition.

**Definition 1** Let  $M = \langle \mathbb{I}(\mathbb{D}), V_M \rangle$  be an interval model. The corresponding relational model  $\eta(M)$  is the pair  $\langle \mathbb{D}, V_{\eta(M)} \rangle$ , where, for every proposition letter  $p$ ,  $V_{\eta(M)}(p) = \{(a, b) \in D \times D : [a, b] \in V_M(p)\}$ . Conversely, let  $M = \langle \mathbb{D}, V_M \rangle$  be a relational model. The corresponding interval model  $\zeta(M)$  is the pair  $\langle \mathbb{I}(\mathbb{D}), V_{\zeta(M)} \rangle$ , where, for every binary relation  $p$  and interval  $[d_i, d_j]$ ,  $[d_i, d_j] \in V_{\zeta(M)}(p^{\leq})$  iff  $(d_i, d_j) \in V_M(p)$  and  $[d_i, d_j] \in V_{\zeta(M)}(p^{\geq})$  iff  $(d_j, d_i) \in V_M(p)$ .

Given an interval logic  $L_I$  and a fragment of first-order logic  $L_{FO}$ , we say that  $L_{FO}$  is *at least as expressive as*  $L_I$ , denoted by  $L_I \preceq L_{FO}$ , if there exists an effective translation  $\tau$  from  $L_I$  to  $L_{FO}$  such that for any interval model  $M$ , interval  $[d_i, d_j]$ , and  $L_I$ -formula  $\varphi$ ,  $M, [d_i, d_j] \models \varphi$  iff  $\eta(M) \models \tau(\varphi)(d_i, d_j)$ . Conversely, we say that  $L_I$  is *at least as expressive as*  $L_{FO}$ , denoted by  $L_{FO} \preceq L_I$ , if there exists an effective translation  $\tau'$  from  $L_{FO}$  to  $L_I$  such that, for any relational model  $M$ , pair of points  $(d_i, d_j)$ , and  $L_{FO}$ -formula  $\varphi$ ,  $M \models \varphi(d_i, d_j)$  if and only if  $\zeta(M), [d_i, d_j] \models \tau'(\varphi)$ , if  $d_i \leq d_j$ , or  $\zeta(M), [d_j, d_i] \models \tau'(\varphi)$ , otherwise.  $L_I \equiv L_{FO}$ ,  $L_I \prec L_{FO}$ , and  $L_{FO} \prec L_I$  are defined as usual.

In [32], Venema shows that the hierarchy of fragments  $\text{FO}^{n,2}[<]$ , for  $n \geq 2$ , is strict.

**Theorem 2** For every  $n \geq 2$ ,  $\text{FO}^{n,2}[<] \prec \text{FO}^{n+1,2}[<]$  (over the class of all linear orders).

The expressive completeness of the interval logic of temporal neighborhood  $\overline{\text{AA}}$  with respect to  $\text{FO}^{2,2}[<]$  and of  $\text{CDT}$  with respect to  $\text{FO}^{3,2}[<]$  have been proved by Bresolin et al. in [8] and by Venema in [33], respectively.



$\tau^{i,j}(x_i = x_j) = \pi$ $\tau^{i,j}(x_j = x_i) = \pi$ $\tau^{i,j}(x_i = x_i) = \top$ $\tau^{i,j}(x_j = x_j) = \top$ $\tau^{i,j}(x_i < x_j) = \neg\pi$ $\tau^{i,j}(x_j < x_i) = \perp$ $\tau^{i,j}(x_i < x_i) = \perp$ $\tau^{i,j}(x_j < x_j) = \perp$ $\tau^{i,j}(p(x_i, x_j)) = p^{\leq}$ $\tau^{i,j}(p(x_j, x_i)) = p^{\geq}$ $\tau^{i,j}(p(x_i, x_i)) = (\pi \wedge p^{\leq}) C \top$ $\tau^{i,j}(p(x_j, x_j)) = \top C (\pi \wedge p^{\leq})$	$\tau^{i,j}(\neg\alpha(x_i, x_j)) = \neg\tau^{i,j}(\alpha(x_i, x_j))$ $\tau^{i,j}(\alpha(x_i, x_j) \wedge \beta(x_i, x_j)) = \tau^{i,j}(\alpha(x_i, x_j)) \wedge \tau^{i,j}(\beta(x_i, x_j))$ $\tau^{i,j}(\alpha(x_i, x_j) \vee \beta(x_i, x_j)) = \tau^{i,j}(\alpha(x_i, x_j)) \vee \tau^{i,j}(\beta(x_i, x_j))$ $\tau^{i,j}(\exists x_k (\alpha(x_i, x_k) \wedge \beta(x_k, x_j))) = \tau^{k,i}(\alpha(x_i, x_k)) D \tau^{k,j}(\beta(x_k, x_j)) \vee \tau^{i,k}(\alpha(x_i, x_k)) C \tau^{k,j}(\beta(x_k, x_j)) \vee \tau^{j,k}(\beta(x_k, x_j)) T \tau^{i,k}(\alpha(x_i, x_k))$
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**Table 1** The mapping of  $\text{FO}_{\text{sp}}^{3,2}[\prec]$  into  $\text{CDT}_{\text{BS}}$ : translation rules.

**Theorem 3**  $\text{A}\bar{\text{A}} \equiv \text{FO}^{2,2}[\prec]$ .

**Theorem 4**  $\text{CDT} \equiv \text{FO}^{3,2}[\prec]$ .

The proof of Theorem 2 shows that for any given  $n \geq 2$ , there exist two models  $M_1$  and  $M_2$  such that  $M_1$  and  $M_2$  satisfy the same set of  $\text{FO}^{n,2}[\prec]$ -formulas, and there exists an  $\text{FO}^{n+1,2}[\prec]$ -formula which is satisfied by  $M_1$  and not by  $M_2$ . Equivalence of  $M_1$  and  $M_2$  with respect to  $\text{FO}^{n,2}[\prec]$ -formulas is established by a game-theoretic argument, while the  $\text{FO}^{n+1,2}[\prec]$ -formula that differentiates the two models is the following one:

$$\exists x_1 \exists x_2 \dots \exists x_n \exists x_{n+1} \left( \bigwedge_{x_i \neq x_j} \neg p(x_i, x_j) \right). \quad (6)$$

Since such a formula belongs to Bernays-Schönfinkel fragment of first-order logic, the very same argument can be used to prove that  $\text{FO}_{\text{BS}}^{n,2}[\prec] \prec \text{FO}_{\text{BS}}^{n+1,2}[\prec]$ , for any  $n \geq 2$ . Moreover, by Theorem 4, it holds that  $\text{FO}_{\text{BS}}^{n,2}[\prec] \prec \text{FO}^{n,2}[\prec]$ , for every  $n \geq 3$ : on the one hand, it trivially holds that  $\text{FO}_{\text{BS}}^{n,2}[\prec] \preceq \text{FO}^{n,2}[\prec]$ ; on the other hand, decidability of  $\text{FO}_{\text{BS}}^{\omega,2}[\prec]$  and undecidability of  $\text{CDT}$  imply that  $\text{FO}^{n,2}[\prec] \not\preceq \text{FO}_{\text{BS}}^{n,2}[\prec]$ . Finally, we have that, for every  $n \geq 3$ ,  $\text{FO}^{n,2}[\prec]$  and  $\text{FO}_{\text{BS}}^{n+1,2}[\prec]$  are incomparable: on the one hand,  $\text{FO}_{\text{BS}}^{n+1,2}[\prec] \not\preceq \text{FO}^{n,2}[\prec]$ , as formula (6) belongs to  $\text{FO}_{\text{BS}}^{n+1,2}[\prec]$  and there is not an equivalent formula in  $\text{FO}^{n,2}[\prec]$ ; on the other hand,  $\text{FO}_{\text{BS}}^{n+1,2}[\prec]$  is decidable, while  $\text{FO}^{n,2}[\prec]$  is not, and thus  $\text{FO}^{n,2}[\prec] \not\preceq \text{FO}_{\text{BS}}^{n+1,2}[\prec]$ . Hence, the following theorem holds.

**Theorem 5** For every  $n \geq 3$ , it holds that:

1.  $\text{FO}_{\text{BS}}^{n-1,2}[\prec] \prec \text{FO}_{\text{BS}}^{n,2}[\prec]$ ;
2.  $\text{FO}_{\text{BS}}^{n,2}[\prec] \prec \text{FO}^{n,2}[\prec]$ ;
3.  $\text{FO}^{n,2}[\prec]$  and  $\text{FO}_{\text{BS}}^{n+1,2}[\prec]$  are incomparable

(over the class of all linear orders).

We conclude the section by showing that  $\text{CDT}_{\text{BS}}$  is expressively complete with respect to  $\text{FO}_{\text{BS}}^{3,2}[\prec]$ . One direction is straightforward: since the standard translation of  $\text{CDT}_{\text{BS}}$ -formulas given in Section 3 makes use of 3 variables only, it holds that  $\text{CDT}_{\text{BS}} \preceq \text{FO}_{\text{BS}}^{3,2}[\prec]$ .

We now show that the converse holds as well, that is,  $\text{FO}_{\text{BS}}^{3,2}[\prec] \preceq \text{CDT}_{\text{BS}}$ . By analogy to the case of the mapping from  $\text{FO}^{3,2}[\prec]$  to  $\text{CDT}$  defined by Venema [33], as a preliminary step, we provide a suitable characterization of  $\text{FO}_{\text{BS}}^{3,2}[\prec]$ -formulas.

**Definition 2** Let  $\{i, j, k\} \subseteq \{1, 2, 3\}$ . The language  $\text{FO}_{\text{sp}}^{3,2}[\prec]$  is defined by the following abstract grammar:

$$\begin{aligned} \beta(x_i, x_j) ::= & \beta_{\exists}(x_i, x_j) \mid \beta(x_i, x_j) \wedge \beta(x_i, x_j) \mid \beta(x_i, x_j) \vee \beta(x_i, x_j) \mid \\ & \exists x_k (\beta(x_i, x_k) \wedge \beta(x_k, x_j)) \mid \neg \beta_{\exists}(x_i, x_j) \text{ for } \beta_{\exists}(x_i, x_j) \\ & \text{of the form } \exists x_k (\beta_{\exists}(x_i, x_k) \wedge \beta_{\exists}(x_k, x_j)) \end{aligned} \quad (7)$$

$$\begin{aligned} \beta_{\exists}(x_i, x_j) ::= & A(x_i, x_j) \mid \neg A(x_i, x_j) \mid \beta_{\exists}(x_i, x_j) \wedge \beta_{\exists}(x_i, x_j) \mid \beta_{\exists}(x_i, x_j) \vee \beta_{\exists}(x_i, x_j) \mid \\ & \exists x_k (\beta_{\exists}(x_i, x_k) \wedge \beta_{\exists}(x_k, x_j)) \end{aligned} \quad (8)$$

$$\begin{aligned} A(x_i, x_j) ::= & x_i = x_j \mid x_j = x_i \mid x_i = x_i \mid x_j = x_j \mid x_i < x_j \mid x_j < x_i \mid x_i < x_i \mid x_j < x_j \mid \\ & p(x_i, x_j) \mid p(x_j, x_i) \mid p(x_i, x_i) \mid p(x_j, x_j) \end{aligned} \quad (9)$$

**Lemma 2** For every formula in  $\text{FO}_{\text{BS}}^{3,2}[\prec]$ , there is an equivalent formula in  $\text{FO}_{\text{sp}}^{3,2}[\prec]$ .

*Proof* We prove the following stronger claim on the 3-variable fragment  $\text{FO}_{\text{BS}}^{3,3}[\prec]$ , which includes formulas where all three variables occur free:

for every formula  $\alpha$  in  $\text{FO}_{\text{BS}}^{3,3}[\prec]$  there is equivalent formula  $\tau(\alpha)$ , which is a Boolean combination of  $\text{FO}_{\text{sp}}^{3,2}[\prec]$ -formulas, with the same free variables as  $\alpha$ .

The proof is by structural induction.

The base cases ( $\alpha$  is an atomic formula or  $\alpha$  is the negation of an atomic formula) and the case of logical connectives ( $\alpha$  is a conjunction or a disjunction of formulas) are straightforward. In particular, as for the base case, it suffices to remind that we restricted our attention to fragments of first-order logic with binary predicates only.

Let  $\alpha$  be of the form  $\exists x_k \gamma(x_i, x_j, x_k)$ . By the inductive hypothesis,  $\gamma(x_i, x_j, x_k)$  is equivalent to a formula  $\tau(\gamma(x_i, x_j, x_k))$ , that we may assume, without loss of generality, to be a disjunction of conjunctions of formulas in  $\text{FO}_{\text{sp}}^{3,2}[\prec]$ . By distributing the existential quantifier  $\exists x_k$  over disjunctions, we obtain a formula of the form  $\bigvee_{h=1}^m \exists x_k \gamma_h(x_i, x_j, x_k)$ , where each  $\gamma_h(x_i, x_j, x_k)$  is a conjunction of formulas. Since only binary predicates are allowed, we can rewrite each  $\gamma_h(x_i, x_j, x_k)$  as  $\xi_h(x_i, x_j) \wedge \xi_h(x_i, x_k) \wedge \xi_h(x_j, x_k)$ . Since variable  $x_k$  does not occur free in  $\xi_h(x_i, x_j)$ , we can rewrite  $\exists x_k \gamma_h(x_i, x_j, x_k)$  as  $\xi_h(x_i, x_j) \wedge \exists x_k (\xi_h(x_i, x_k) \wedge \xi_h(x_j, x_k))$ . This latter formula is a conjunction of  $\text{FO}_{\text{sp}}^{3,2}[\prec]$ -formulas with the same free variables as  $\alpha$ .

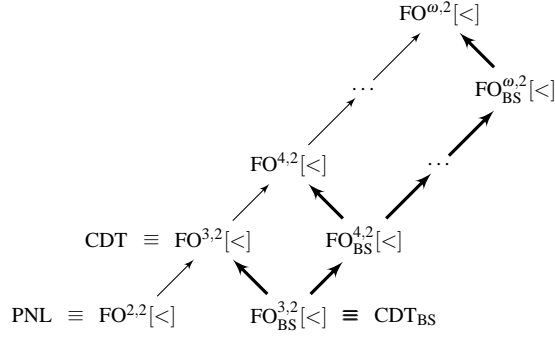
The case in which  $\alpha$  is of the form  $\neg \exists x_k \gamma(x_i, x_j, x_k)$  can be dealt with in a very similar way.  $\square$

We are now ready to define the translation  $\tau$  from  $\text{FO}_{\text{sp}}^{3,2}[\prec]$  to  $\text{CDT}_{\text{BS}}$ . For the sake of brevity, we write  $\tau^{i,j}$  for  $\tau[x_i, x_j]$ , with  $x_i \leq x_j$ . Translation rules for atomic and complex formulas are given in Table 1.

**Lemma 3** Let  $\alpha(x_i, x_j)$  be an  $\text{FO}_{\text{sp}}^{3,2}[\prec]$ -formula. Then, for every pair of points  $(d_i, d_j)$ ,  $M \models \alpha(d_i, d_j)$  if and only if  $d_i \leq d_j$  and  $\zeta(M), [d_i, d_j] \Vdash \tau^{i,j}(\alpha(x_i, x_j))$ , or  $d_j \leq d_i$  and  $\zeta(M), [d_j, d_i] \Vdash \tau^{j,i}(\alpha(x_i, x_j))$ .

*Proof* The proof is by induction on the structure of  $\alpha(x_i, x_j)$ . The cases of atomic formulas and Boolean connectives are straightforward.

Once more, the only interesting case is the one of existential quantifiers. Let  $\alpha(x_i, x_j)$  be the formula  $\exists x_k (\beta(x_i, x_k) \wedge \gamma(x_k, x_j))$  and  $d_i \leq d_j$ . By the semantic clauses for  $\text{FO}_{\text{sp}}^{3,2}[\prec]$ , it



**Fig. 2** A classification of the considered interval logics and fragments of first-order logic with respect to their expressive power.

holds that  $M \models \exists x_k (\beta(d_i, x_k) \wedge \gamma(x_k, d_j))$  if and only if there exists a point  $d_k$  such that  $M \models \beta(d_i, d_k)$  and  $M \models \gamma(d_k, d_j)$ . Since we are interpreting our formulas over a linear order, there are three possible ways to place  $d_k$  with respect to  $d_i$  and  $d_j$ : either  $d_k \leq d_i$ , or  $d_i \leq d_k \leq d_j$ , or  $d_j \leq d_k$ . By the inductive hypothesis, we have that  $M \models \alpha(d_i, d_j)$  if and only if:

$$\begin{aligned} & \left( \zeta(M), [d_k, d_i] \Vdash \tau^{k,i}(\beta(x_i, x_k)) \text{ and } \zeta(M), [d_k, d_j] \Vdash \tau^{k,j}(\gamma(x_k, x_j)) \right) \\ \text{or } & \left( \zeta(M), [d_i, d_k] \Vdash \tau^{i,k}(\beta(x_i, x_k)) \text{ and } \zeta(M), [d_k, d_j] \Vdash \tau^{k,j}(\gamma(x_k, x_j)) \right) \\ \text{or } & \left( \zeta(M), [d_j, d_k] \Vdash \tau^{j,k}(\gamma(x_k, x_j)) \text{ and } \zeta(M), [d_i, d_k] \Vdash \tau^{i,k}(\beta(x_i, x_k)) \right). \end{aligned}$$

By the semantics of the  $C$ ,  $D$ , and  $T$  operators, we can conclude that  $M \models \alpha(d_i, d_j)$  if and only if  $\zeta(M), [d_i, d_j] \Vdash \tau^{i,j}(\alpha(x_i, x_j))$ , as required.  $\square$

**Theorem 6**  $CDT_{BS}$  is as expressive as  $FO_{BS}^{3,2}[<]$ .

*Proof* By Lemma 2 and Lemma 3,  $FO_{BS}^{3,2}[<] \preceq FO_{sp}^{3,2}[<] \preceq CDT_{BS}$ . Moreover, by Lemma 1,  $CDT_{BS} \preceq FO_{BS}^{3,2}[<]$ . Hence,  $CDT_{BS} \equiv FO_{BS}^{3,2}[<]$ .  $\square$

Figure 2 gives a graphical account of the relationships among the considered logics (interval logics and fragments of first-order logic) in terms of their expressive power (the contributions of the present work are in boldface).

## 5 A tableau method for $CDT_{BS}$

In [15], Goranko et al. propose a tableau method for CDT interpreted over partial orders with the linear interval property, that is, partial orders in which every interval is linear ( $BCDT^+$  for short). The method provides a semi-decision procedure for  $BCDT^+$  (it is not guaranteed to terminate). This does not come as a surprise as  $BCDT^+$  is undecidable. In this section, we show how to turn the method into an NP decision procedure  $CDT_{BS}$ . In particular, we show how to exploit  $BCDT^+$  syntactic restrictions to guarantee termination.

Let us start with some basic terminology. A *finite tree* is a finite directed acyclic graph in which every node, apart from one (the *root*), has exactly one incoming edge. A *successor* of a node  $n$  is a node  $n'$  such that there is an edge from  $n$  to  $n'$ . A *leaf* is a node with no

successors. A path is a sequence of nodes  $n_0, \dots, n_k$  such that, for all  $i = 0 \dots k-1$ ,  $n_{i+1}$  is a successor of  $n_i$ ; a *branch* is a path from the root to a leaf. The *height* of a node  $n$  is the maximum length (number of edges) of a path from  $n$  to a leaf, while its *depth* is the length of the (unique) path from the root to it. If two nodes  $n$  and  $n'$  belong to the same branch and the height of  $n$  is less than (resp., less than or equal to) the height of  $n'$ , we write  $n \prec n'$  (resp.,  $n \preceq n'$ ).

**Definition 3** Let  $\mathbb{D}$  be a finite linear order. A *labeled formula over  $\mathbb{D}$*  is a pair  $(\psi, [d_i, d_j])$ , where  $\psi \in \text{CDT}_{\text{BS}}$  and  $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$ .

**Definition 4** Let  $\mathcal{T}$  be a (finite) tree and let  $n$  be a node of  $\mathcal{T}$ . The *decoration*  $v(n)$  of  $n$  is a tuple  $\langle \psi, [d_i, d_j], \mathbb{D}, p, u \rangle$ , where  $\mathbb{D}$  is a finite linear order,  $(\psi, [d_i, d_j])$  is a labeled formula over  $\mathbb{D}$ ,  $p \in \{0, 1\}$ , and  $u$  is a *local flag function* which associates the values 0 or 1 with every branch  $B$  containing  $n$ .

**Definition 5** A *decorated tree* is a finite tree  $\mathcal{T}$  enriched with a decoration  $v(n)$  for each node  $n$  of  $\mathcal{T}$ , apart from the root.

The tableau construction described below generates a decorated tree  $\mathcal{T}$ . Given a branch  $B$  and a node  $n$  belonging to it, with decoration  $v(n)$ ,  $u(B) = 1$  means that  $n$  can be expanded on  $B$ . Given a branch  $B$ ,  $B \cdot (n_1 \dots n_h)$  is the result of the expansion of  $B$  with the sequence of nodes  $n_1 \dots n_h$  (for  $h = 1$ , we simply write  $B \cdot n$ ), while  $B \cdot (n_{1,1} \dots n_{1,h}) \dots (n_{k,1} \dots n_{k,h})$  is the result of the expansion of  $B$  with  $k$  sequences of  $h$  nodes (for  $h = 1$ , we simply write  $B \cdot n_1 \dots n_k$ ). The auxiliary flag  $p$  has been added to simplify termination and complexity proofs. It records the nature of formula  $\psi$ : if  $\psi$  is a  $\varphi_{\exists}$ -formula, then  $p = 0$ ; otherwise,  $p = 1$ . Finally, if  $n$  is the leaf of a branch  $B$ , we denote by  $\mathbb{D}_B$  the finite linear order in  $v(n)$ .

Since in  $\text{CDT}_{\text{BS}}$  negation can occur only in front of proposition letters or modalities, we need to introduce the notion of *dual formula* of a formula  $\varphi$ , denoted by  $\overline{\varphi}$ . It is inductively defined as follows:

- $\overline{\neg p} = p$  and  $\overline{\neg \overline{p}} = \neg p$ , for every  $p \in \mathcal{A} \cup \{\pi\}$ ;
- $\overline{\varphi \vee \psi} = \overline{\varphi} \wedge \overline{\psi}$ ;
- $\overline{\varphi \wedge \psi} = \overline{\varphi} \vee \overline{\psi}$ ;
- $\overline{\varphi R \psi} = \neg(\varphi R \psi)$ , for  $R \in \{C, D, T\}$ ;
- $\overline{\neg(\varphi R \psi)} = \varphi R \psi$ , for  $R \in \{C, D, T\}$ .

Notice that the dual of a generic  $\text{CDT}_{\text{BS}}$ -formula does not necessarily belong to  $\text{CDT}_{\text{BS}}$ . This is the case, for instance, with the formula  $pC\neg(qCr)$ . However, the following lemma guarantees that dual formulas of  $\varphi_{\exists}$ -formulas are  $\text{CDT}_{\text{BS}}$ -formulas. Such a lemma will play a crucial role in the proof of correctness of the tableau method.

**Lemma 4** *Let  $\varphi$  be a  $\varphi_{\exists}$ -formula. Then,  $\overline{\varphi}$  is a  $\text{CDT}_{\text{BS}}$ -formula.*

*Proof* The cases of proposition letters and Boolean connectives can be proved by a straightforward structural induction. To prove that the thesis holds also for modalities, let us assume  $\varphi = \psi C \tau$  to be a  $\varphi_{\exists}$ -formula. By definition, the dual formula  $\overline{\varphi}$  is  $\neg(\psi C \tau)$ . Since  $\psi, \tau$  are  $\varphi_{\exists}$ -formulas, we can conclude that  $\overline{\varphi}$  is a  $\text{CDT}_{\text{BS}}$ -formula. The other cases can be dealt with in a similar way.  $\square$

The construction of a tableau for a  $\text{CDT}_{\text{BS}}$ -formula  $\varphi$  to be checked for satisfiability starts from a three-node tree (*initial tableau*) consisting of a root and two leaves with decorations  $\langle \varphi, [d_0, d_0], \{d_0\}, 1, 1 \rangle$  and  $\langle \varphi, [d_0, d_1], \{d_0 < d_1\}, 1, 1 \rangle$ , respectively. The procedure exploits

a set of expansion rules, adapted from those given in [15], to add new nodes to the tree. In particular, the original rules for modalities have been revised to restrict the search for possible models to linear orders only.

**Definition 6** Given a tree  $\mathcal{T}$ , a branch  $B$  in  $\mathcal{T}$ , and a node  $n \in B$  with decoration  $\langle \psi, [d_i, d_j], \mathbb{D}, p_n, u_n \rangle$  such that  $u_n(B) = 1$ , the *branch-expansion rule* for  $B$  and  $n$  is defined as follows (in all considered cases,  $u_{n'}(B') = 1$  for all new nodes  $n'$  and branches  $B'$ ).

- R1** If  $\psi = \xi_0 \wedge \xi_1$ , then expand  $B$  to  $B \cdot n_0 \cdot n_1$ , where  $n_0$  is decorated with  $\langle \xi_0, [d_i, d_j], \mathbb{D}_B, p_n, u_{n_0} \rangle$  and  $n_1$  is decorated with  $\langle \xi_1, [d_i, d_j], \mathbb{D}_B, p_n, u_{n_1} \rangle$ .
- R2** If  $\psi = \xi_0 \vee \xi_1$ , then expand  $B$  to  $B \cdot n_0 | n_1$ , where  $n_0$  is decorated with  $\langle \xi_0, [d_i, d_j], \mathbb{D}_B, p_n, u_{n_0} \rangle$  and  $n_1$  is decorated with  $\langle \xi_1, [d_i, d_j], \mathbb{D}_B, p_n, u_{n_1} \rangle$ .
- R3** If  $\psi = \neg(\xi_0 C \xi_1)$  and  $d$  is a point in  $D_B$ , with  $d_i \leq d \leq d_j$ , which has not been used yet to expand  $n$  in  $B$ , then expand  $B$  to  $B \cdot n_0 | n_1$ , where  $n_0$  is decorated with  $\langle \xi_0, [d_i, d], \mathbb{D}_B, 0, u_{n_0} \rangle$  and  $n_1$  is decorated with  $\langle \xi_1, [d, d_j], \mathbb{D}_B, 0, u_{n_1} \rangle$ .
- R4** If  $\psi = \neg(\xi_0 D \xi_1)$ , and  $d$  is a point in  $D_B$ , with  $d \leq d_i$ , which has not been used yet to expand  $n$  in  $B$ , then expand  $B$  to  $B \cdot n_0 | n_1$ , where  $n_0$  is decorated with  $\langle \xi_0, [d, d_i], \mathbb{D}_B, 0, u_{n_0} \rangle$  and  $n_1$  is decorated with  $\langle \xi_1, [d, d_j], \mathbb{D}_B, 0, u_{n_1} \rangle$ .
- R5** If  $\psi = \neg(\xi_0 T \xi_1)$ , and  $d$  is a point in  $D_B$ , with  $d_j \leq d$ , which has not been used yet to expand  $n$  in  $B$ , then expand  $B$  to  $B \cdot n_0 | n_1$ , where  $n_0$  is decorated with  $\langle \xi_0, [d_j, d], \mathbb{D}_B, 0, u_{n_0} \rangle$  and  $n_1$  is decorated with  $\langle \xi_1, [d_i, d], \mathbb{D}_B, 0, u_{n_1} \rangle$ .
- R6** If  $\psi = \xi_0 C \xi_1$ , then expand  $B$  to  $B \cdot (n_i \cdot m_i) | \dots | (n_j \cdot m_j) | (n'_i \cdot m'_i) | \dots | (n'_{j-1} \cdot m'_{j-1})$ , where:
  - (a) for all  $i \leq k \leq j$ ,  $n_k$  is decorated with  $\langle \xi_0, [d_i, d_k], \mathbb{D}_B, p_n, u_{n_k} \rangle$  and  $m_k$  is decorated with  $\langle \xi_1, [d_k, d_j], \mathbb{D}_B, p_n, u_{m_k} \rangle$ ;
  - (b) for all  $i \leq k \leq j-1$ ,  $\mathbb{D}_k$  is the linear ordering obtained from  $\mathbb{D}_B$  by inserting a new point  $d$  between  $d_k$  and  $d_{k+1}$ ,  $n'_k$  is decorated with  $\langle \xi_0, [d_i, d], \mathbb{D}_k, p_n, u_{n'_k} \rangle$  and  $m'_k$  is decorated with  $\langle \xi_1, [d, d_j], \mathbb{D}_k, p_n, u_{m'_k} \rangle$ .
- R7** If  $\psi = \xi_0 D \xi_1$  and  $d_0$  is the least point of  $D_B$ , then expand  $B$  to  $B \cdot (n_0 \cdot m_0) | \dots | (n_i \cdot m_i) | (n'_0 \cdot m'_0) | \dots | (n'_i \cdot m'_i)$ , where:
  - (a) for all  $0 \leq k \leq i$ ,  $n_k$  is decorated with  $\langle \xi_0, [d_k, d_i], \mathbb{D}_B, p_n, u_{n_k} \rangle$  and  $m_k$  is decorated with  $\langle \xi_1, [d_k, d_j], \mathbb{D}_B, p_n, u_{m_k} \rangle$ ;
  - (b) for all  $0 \leq k \leq i$ ,  $\mathbb{D}_k$  is the linear ordering obtained from  $\mathbb{D}_B$  by inserting a new point  $d$  between  $d_{k-1}$  and  $d_k$  (for  $k=0$ ,  $d$  is placed immediately before  $d_0$ ),  $n'_k$  is decorated with  $\langle \xi_0, [d, d_i], \mathbb{D}_k, p_n, u_{n'_k} \rangle$  and  $m'_k$  is decorated with  $\langle \xi_1, [d, d_j], \mathbb{D}_k, p_n, u_{m'_k} \rangle$ .
- R8** If  $\psi = \xi_0 T \xi_1$  and  $d_N$  is the greatest point of  $D_B$ , then expand  $B$  to  $B \cdot (n_j \cdot m_j) | \dots | (n_N \cdot m_N) | (n'_j \cdot m'_j) | \dots | (n'_N \cdot m'_N)$ , where:
  - (a) for all  $j \leq k \leq N$ ,  $n_k$  is decorated with  $\langle \xi_0, [d_j, d_k], \mathbb{D}_B, p_n, u_{n_k} \rangle$  and  $m_k$  is decorated with  $\langle \xi_1, [d_i, d_k], \mathbb{D}_B, p_n, u_{m_k} \rangle$ ;
  - (b) for all  $j \leq k \leq N$ ,  $\mathbb{D}_k$  is the linear ordering obtained from  $\mathbb{D}_B$  by inserting a new point  $d$  between  $d_k$  and  $d_{k+1}$  (for  $k=N$ ,  $d$  is placed immediately after  $d_N$ ),  $n'_k$  is decorated with  $\langle \xi_0, [d_j, d], \mathbb{D}_k, p_n, u_{n'_k} \rangle$  and  $m'_k$  is decorated with  $\langle \xi_1, [d_i, d], \mathbb{D}_k, p_n, u_{m'_k} \rangle$ .

Finally, for each branch  $B'$  extending  $B$ , let  $u_m(B') = u_m(B)$ , for each node  $m \neq n$  in  $B$ , and let  $u_n(B') = 0$ , unless  $\psi = \neg(\xi_0 C \xi_1)$ ,  $\psi = \neg(\xi_0 D \xi_1)$ , or  $\psi = \neg(\xi_0 T \xi_1)$  (in such cases  $u_n(B') = 1$ ).

We briefly explain the behavior of the branch-expansion rule in cases **R6** ( $\xi_0 C \xi_1$ ) and **R3** ( $\neg(\xi_0 C \xi_1)$ ). The corresponding cases for modalities  $D$  and  $T$  are similar. **R6** deals with two possible scenarios: either there exists  $d_k \in D_B$  such that  $\xi_0$  holds over  $[d_i, d_k]$  and  $\xi_1$  holds over  $[d_k, d_j]$ , or such a point must be added to  $D_B$ . The successors  $(n_i \cdot m_i) | \dots | (n_j \cdot m_j)$  created by the rule cover the former case, while the successors  $(n'_i \cdot m'_i) | \dots | (n'_{j-1} \cdot m'_{j-1})$  cover the latter case. As for **R3**, the formula  $\neg(\xi_0 C \xi_1)$  states that, for all  $d_i \leq d \leq d_j$ , either  $\overline{\xi_0}$  holds over  $[d_i, d]$  or  $\overline{\xi_1}$  holds over  $[d, d_j]$ . **R3** imposes such a condition for a single point  $d \in D_B$  and keeps the flag equal to 1. In such a way, all points in  $D_B$  are eventually considered, including those points that will be added in subsequent steps of the tableau construction.

**Definition 7** A branch  $B$  is *closed* if one of the following conditions holds:

1. there are two nodes  $n, n'$  in  $B$  such that  $v(n) = \langle \psi, [d_i, d_j], \mathbb{D}, p, u \rangle$  and  $v(n') = \langle \overline{\psi}, [d_i, d_j], \mathbb{D}', p', u' \rangle$  for some formula  $\psi$  and  $d_i, d_j \in D$ ;
2. there is a node  $n$  such that  $v(n) = \langle \pi, [d_i, d_j], \mathbb{D}, p, u \rangle$  and  $d_i \neq d_j$ ;
3. there is a node  $n$  such that  $v(n) = \langle \neg\pi, [d_i, d_j], \mathbb{D}, p, u \rangle$  and  $d_i = d_j$ ;

If none of the above conditions hold, the branch is *open*.

**Definition 8** The branch-expansion *strategy* for a branch  $B$  in a decorated tree  $\mathcal{T}$  is defined as follows:

1. apply the branch-expansion rule to a branch  $B$  only if it is open;
2. if  $B$  is open, apply the branch-expansion rule to the closest to the root node  $n$  such that  $u_n(B) = 1$  and the application of the rule generates at least one node with a new decoration (if any).

**Definition 9** A tableau  $\mathcal{T}$  is any decorated tree obtained from the initial tableau by the application of the branch-expansion strategy.

We say that a tableau  $\mathcal{T}$  is *closed* if and only if all its branches are closed, otherwise it is *open*.

We conclude the section by giving a couple of examples of the application of the proposed method. As a first example, we consider the satisfiable formula  $\varphi = (\neg\pi D \neg\pi) C \neg\pi$ . A portion of a tableau for  $\varphi$  is given in Figure 3, where thick edges highlights an open branch representing a four-point model for the formula. As a second example, let  $\psi$  be the unsatisfiable formula  $p T \neg(\top C p)$ . A closed tableau for  $\psi$  is given in Figure 4. It is worth pointing out that there is an abuse of notation in the last component of the node decorations: while it is formally defined as a function from a set of branches to  $\{0, 1\}$ , in the pictures it is represented as a constant (either 0 or 1). The reason is that in the proposed examples the function is constant for each node, that is, for each  $n$  we have that the value of the function  $u_n(B)$  is the same for every branch  $B$  containing  $n$ .

In the following, we will show that to establish the satisfiability of a  $\text{CDT}_{\text{BS}}$ -formula  $\varphi$  it is sufficient to start with the initial tableau for  $\varphi$ , and keep expanding it for as long as it is possible: if the resulting tableau is open, then  $\varphi$  is satisfiable, otherwise it is not. Moreover, we will prove that this expansion procedure terminates and it can be executed by a nondeterministic machine that uses only a polynomial amount of time.

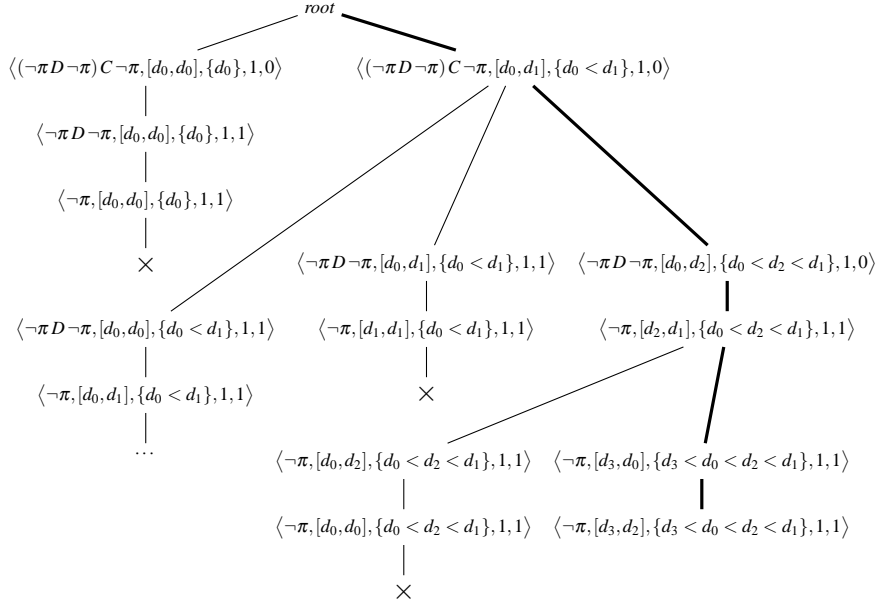


Fig. 3 A portion of an open tableau for the formula  $(\neg\pi D\neg\pi)C\neg\pi$ .

### 5.1 Soundness

In this subsection, we prove that the proposed tableau method is sound, that is, given a formula  $\varphi$  and a tableau  $\mathcal{T}$  for it, if  $\mathcal{T}$  is closed, then  $\varphi$  is not satisfiable. In the next subsection, we will show that the method is also complete.

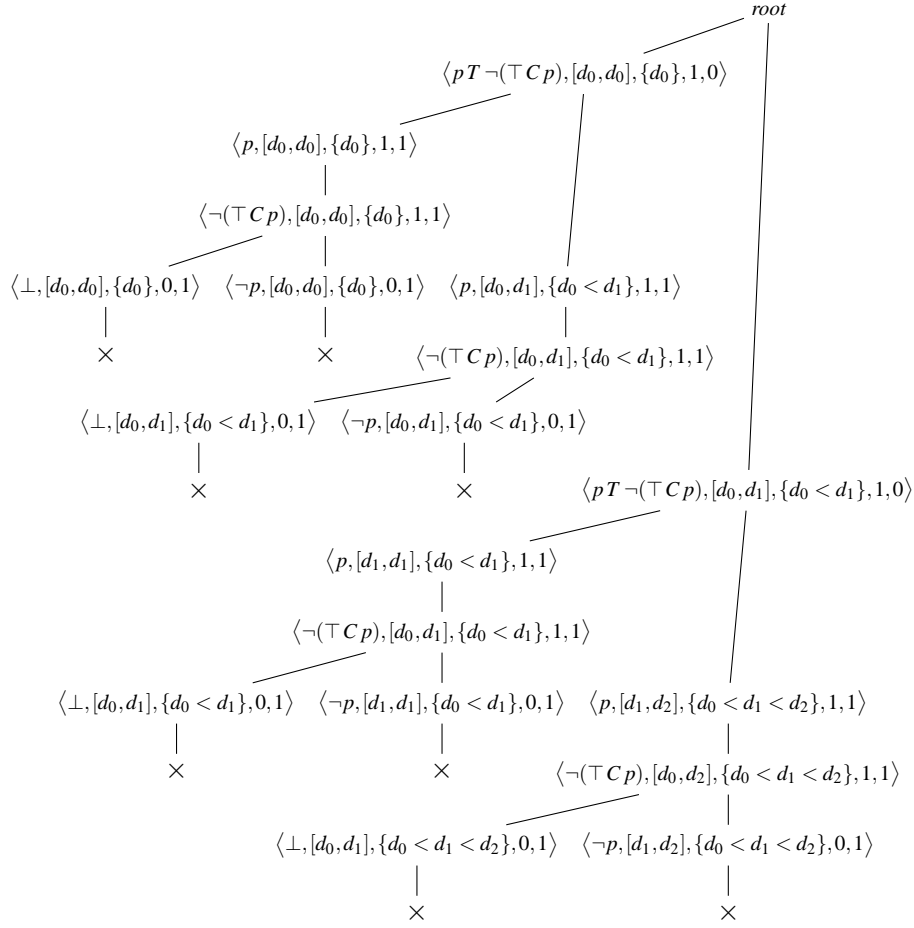
**Lemma 5 (Soundness)** *Let  $\varphi$  be a CDT<sub>BS</sub>-formula and  $\mathcal{T}$  be a tableau for it. If  $\mathcal{T}$  is closed, then  $\varphi$  is not satisfiable.*

*Proof* Let  $n$  be a node in the tableau  $\mathcal{T}$ , and let  $\mathbb{D}_n = \{d_0 < \dots < d_s\}$  be the linear ordering from  $v(n)$ . We will prove the following claim by induction on the height  $h$  of the node:

*if every branch including  $n$  is closed, then the set  $S(n)$  of all labeled formulas in the decorations of the nodes between  $n$  and the root is neither satisfiable in  $\mathbb{I}(\mathbb{D}_n)$  nor in any extension of it.*

If  $h = 0$ , then  $n$  is a leaf and the unique branch  $B$  containing  $n$  is closed. Then, either  $S(n)$  contains both the labeled formulas  $(\psi, [d_k, d_l])$  and  $(\neg\psi, [d_k, d_l])$ , for some CDT<sub>BS</sub>-formula  $\psi$  and  $d_k, d_l \in D_n$ , or the labeled formula  $(\pi, [d_k, d_l])$ , for some  $d_k \neq d_l$ , or the labeled formula  $(\neg\pi, [d_k, d_l])$ , for some  $d_k = d_l$ . Take any model  $M = \langle \mathbb{I}(\mathbb{D}'), V \rangle$ , where  $\mathbb{D}'$  extends  $\mathbb{D}_n$ . It holds that  $M, [d_k, d_l] \models \psi$  if and only if  $M, [d_k, d_l] \not\models \neg\psi$ , and, therefore,  $(\psi, [d_k, d_l])$  and  $(\neg\psi, [d_k, d_l])$  cannot be jointly satisfied. Similarly,  $M, [d_k, d_l] \models \pi$  (resp.,  $M, [d_k, d_l] \models \neg\pi$ ) if and only if  $d_k = d_l$  (resp.,  $d_k \neq d_l$ ), and therefore  $(\pi, [d_k, d_l])$  (resp.,  $(\neg\pi, [d_k, d_l])$ ) cannot be satisfied when  $d_k \neq d_l$  (resp.,  $d_k = d_l$ ).

Now, suppose that  $h > 0$ . Then, either  $n$  has been generated as one of the successors, but not the last one, when applying cases **R1**, **R6**, **R7**, or **R8** of the branch-expansion rule, or the branch-expansion rule has been applied to some labeled formula  $(\psi, [d_k, d_l]) \in S(n) \setminus \{\tau\}$ , where  $\tau$  is the labeled formula in the decoration  $v(n)$ , to extend the branch at  $n$ . We



**Fig. 4** A closed tableau for the formula  $pT\neg(\top Cp)$ .

detail the latter case; the former one can be dealt with in the same way, and thus its analysis is omitted. First, we observe that every branch passing through any successor of  $n$  must be closed. It immediately follows that the inductive hypothesis applies to all successors of  $n$ . We consider the possible cases for the application of the branch-expansion rule to extend the branch at  $n$ , restricting our attention to the conceptually different ones only (the other cases can be dealt with in a similar way):

- If  $\psi = \xi_0 \wedge \xi_1$ , **R1** has been applied. Then, there are two nodes  $n_0, n_1$  such that  $v(n_0) = \langle \xi_0, [d_k, d_l], \mathbb{D}, p_0, u_0 \rangle$ ,  $v(n_1) = \langle \xi_1, [d_k, d_l], \mathbb{D}, p_1, u_1 \rangle$ . Without loss of generality, we can assume  $n_0$  to be the successor of  $n$  and  $n_1$  to be the successor of  $n_0$ . Since each branch containing  $n$  is closed, then each branch containing  $n_1$  is closed as well. By the inductive hypothesis ( $n_1 \prec n$ ),  $S(n_1)$  is not satisfiable. Since every model satisfying  $S(n)$  must, in particular, satisfy  $(\xi_0 \wedge \xi_1, [d_k, d_l])$ , and hence  $(\xi_0, [d_k, d_l])$  and  $(\xi_1, [d_k, d_l])$ , it follows that  $S(n)$ ,  $S(n_0)$ , and  $S(n_1)$  are equi-satisfiable. Therefore,  $S(n)$  is not satisfiable.
- If  $\psi = \xi_1 \vee \xi_2$ , **R2** has been applied. Then, there exist two successor nodes  $n_0$  and  $n_1$  of  $n$  such that  $v(n_0) = \langle \xi_0, [d_k, d_l], \mathbb{D}, p_0, u_0 \rangle$ ,  $v(n_1) = \langle \xi_1, [d_k, d_l], \mathbb{D}, p_1, u_1 \rangle$ , and both



$n_0 \prec n$  and  $n_1 \prec n$ . Since each branch containing  $n$  is closed, then each branch containing  $n_0$  or  $n_1$  is closed as well. By the inductive hypothesis,  $S(n_0)$  and  $S(n_1)$  are not satisfiable. Since every model satisfying  $S(n)$  must also satisfy  $(\xi_0, [d_k, d_l])$  or  $(\xi_1, [d_k, d_l])$ , it follows that  $S(n)$  is not satisfiable.

- If  $\psi = \neg(\xi_0 C \xi_1)$ , **R3** has been applied. For the sake of contradiction, let us assume  $S(n)$  to be satisfiable. Then, since  $(\neg(\xi_0 C \xi_1), [d_k, d_l]) \in S(n)$ , there is a model  $M = \langle \mathbb{I}(\mathbb{D}'), V \rangle$  such that  $\mathbb{D}'$  extends  $\mathbb{D}_n$  and  $M, [d_k, d_l] \Vdash \neg(\xi_0 C \xi_1)$ . Hence, for each  $d_t$  such that  $d_k \leq d_t \leq d_l$ ,  $M, [d_k, d_t] \not\Vdash \xi_0$  or  $M, [d_t, d_l] \not\Vdash \xi_1$ . By construction, the two immediate successors of  $n$  are two nodes  $n_0$  and  $n_1$  and there exists a point  $d_t$ , with  $d_k \leq d_t \leq d_l$ , such that  $(\xi_0, [d_k, d_t])$  is in  $v(n_0)$  and  $(\xi_1, [d_t, d_l])$  is in  $v(n_1)$ . By the inductive hypothesis (both  $n_0 \prec n$  and  $n_1 \prec n$ ),  $S(n_0)$  and  $S(n_1)$  are not satisfiable. But, from the hypothesis of our reductio-ad-absurdum argument, there is a model  $M = \langle \mathbb{I}(\mathbb{D}'), V \rangle$ , where  $\mathbb{D}'$  is an extension of  $\mathbb{D}_n$ , such that  $M, [d_k, d_t] \Vdash \neg \xi_0$  or  $M, [d_t, d_l] \Vdash \neg \xi_1$ . Thus, either  $S(n_0)$  or  $S(n_1)$  is satisfiable (by model  $M$ ), leading to a contradiction.
- If  $\psi = \xi_0 C \xi_1$ , **R6** has been applied. For the sake of contradiction, let us assume  $S(n)$  to be satisfiable. Then, there is a model  $M = \langle \mathbb{I}(\mathbb{D}'), V \rangle$  such that  $\mathbb{D}'$  extends  $\mathbb{D}_n$  and  $M, [d_k, d_l] \Vdash \xi_0 C \xi_1$ . Hence,  $M, [d_k, d] \Vdash \xi_0$  and  $M, [d, d_l] \Vdash \xi_1$  for some  $d_k \leq d \leq d_l$ . Two cases are possible:
  1. If  $d \in D_n$ , then  $d = d_t$  for some  $d_k \leq d_t \leq d_l$ . By **R6**,  $n$  has a successor, say it  $n_t$ , which, in turn, has a successor, say it  $n'_t$ , with  $v(n_t) = \langle \xi_0, [d_k, d_t], \mathbb{D}_n, p_t, u_t \rangle$  and  $v(n'_t) = \langle \xi_1, [d_t, d_l], \mathbb{D}_n, p'_t, u'_t \rangle$ . By the inductive hypothesis ( $n_t \prec n$  and  $n'_t \prec n_t$ ),  $S(n'_t) = S(n) \cup \{(\xi_0, [d_k, d_t]), (\xi_1, [d_t, d_l])\}$  is not satisfiable. But, from the hypothesis of our reductio-ad-absurdum argument, there is a model  $M = \langle \mathbb{I}(\mathbb{D}'), V \rangle$ , where  $\mathbb{D}'$  is an extension of  $\mathbb{D}_n$ , such that  $M, [d_k, d_t] \Vdash \xi_0$  and  $M, [d_t, d_l] \Vdash \xi_1$ . Thus,  $S(n'_t)$  is satisfiable (by model  $M$ ), leading to a contradiction.
  2. If  $d \notin D_n$ , then there exists  $t$  such that  $k \leq t \leq l-1$  and  $d_t < d < d_{t+1}$ . By **R6**,  $n$  has a successor, say it  $n_t$ , which, in turn, has a successor, say it  $n'_t$ , with  $v(n_t) = \langle \xi_0, [d_k, d], \mathbb{D}_n \cup \{d\}, p_t, u_t \rangle$ ,  $v(n'_t) = \langle \xi_1, [d, d_l], \mathbb{D}_n \cup \{d\}, p'_t, u'_t \rangle$ . By the inductive hypothesis ( $n_t \prec n$  and  $n'_t \prec n_t$ ),  $S(n'_t) = S(n) \cup \{(\xi_0, [d_k, d]), (\xi_1, [d, d_l])\}$  is not satisfiable, which, as in the previous case, leads to a contradiction.  $\square$

## 5.2 Completeness

In this subsection, we prove that the proposed tableau method is complete, that is, whenever  $\varphi \in \text{CDT}_{\text{BS}}$  is valid, every tableau  $\mathcal{T}$  for  $\neg\varphi$  must be closed. To this end, we need to preliminary prove some partial results.

**Definition 10** Let  $\varphi$  be a  $\text{CDT}_{\text{BS}}$ -formula and  $\mathcal{T}_0$  be the initial tableau for it. The *limit tableau*  $\overline{\mathcal{T}}$  for  $\varphi$  is the decorated tree generated as follows. For all  $i \geq 0$ , let  $\mathcal{T}_{i+1}$  be the tableau generated by the simultaneous application of the branch-expansion strategy to each branch in  $\mathcal{T}_i$ . If we ignore all flags from the decorations of the nodes in every  $\mathcal{T}_i$ , we obtain a chain of decorated trees ordered by inclusion:  $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \dots \subseteq \mathcal{T}_k \subseteq \dots$ . The limit tableau  $\overline{\mathcal{T}}$  is equal to  $\bigcup_{i=0}^{\omega} \mathcal{T}_i$ .

Notice that the above definition does not preclude the limit tableau from being infinite. Later on, we will prove that it cannot be the case, that is, the limit tableau is always finite. Nevertheless, finiteness (of the limit tableau) is not necessary to prove that the tableau method is complete.

The definitions of open and closed branch and tableau directly apply to the limit tableau as well. In addition, we introduce the notion of saturated branch and tableau.

**Definition 11** A branch in a (limit) tableau is *saturated* if there are no nodes on that branch to which the branch-expansion rule is applicable on the branch. A (limit) tableau is *saturated* if every open branch in it is saturated.

We now show that the set of all labeled formulas on an open branch in a limit tableau has the saturation properties of a Hintikka set in first-order logic.

**Lemma 6** *Every limit tableau is saturated.*

*Proof* Let  $B$  be a branch  $B$  in the limit tableau  $\overline{\mathcal{T}}$  and  $n$  be a node in  $B$ . We prove that after every step of the expansion of that branch at which the branch-expansion rule becomes applicable to  $n$  (because  $n$  has just been introduced or a new point has been added) and the application of the rule generates at least a new node, then that rule is subsequently applied on  $B$  to that node. The proof is by induction on  $\text{depth}(n)$  (the depth of node  $n$ ).

Let us assume that  $\text{depth}(n) = l$  and the branch-expansion rule has become applicable to  $n$ . By the inductive hypothesis, the thesis holds for all nodes with  $\text{depth}(n) < l$ . If there are no nodes between the root (including the root) and  $n$  (excluding  $n$ ) to which the branch-expansion rule is applicable at that moment, the next application of the branch-expansion rule on  $B$  is necessarily to  $n$ . Otherwise, let  $n^*$  be the closest-to- $n$  node between the root and  $n$  to which the branch-expansion rule is applicable, or will become applicable, on  $B$  at least once thereafter. (Such a node exists because there are only finitely many nodes between  $n$  and the root.) Since  $\text{depth}(n^*) < \text{depth}(n)$ , by the inductive hypothesis, the branch-expansion rule has been subsequently applied to  $n^*$ . Then, the next application of the branch-expansion rule on  $B$  must have been to  $n$  and that completes the induction.

Suppose now that there exists a branch  $B$  in a limit tableau which is not saturated. Let  $n$  be the closest-to-the-root node on  $B$  to which the branch-expansion rule is applicable. If the case applicable to  $n$  is different from **R3**, **R4**, and **R5**, then the branch-expansion rule has become applicable to  $n$  at the step when  $n$  is introduced, and by the claim above, it has been subsequently applied. Hence, the node has become unavailable thereafter, which contradicts the assumption. Let us consider now the case of **R3**, that is, the formula in  $v(n)$  is  $\neg(\xi_0 C \xi_1)$  (cases **R4** and **R5** are similar, and thus they are omitted). An application of **R3** on  $B$  would create two immediate successors with labeled formulas  $(\overline{\xi}_0, [d_i, d])$  and  $(\overline{\xi}_1, [d, d_j])$ , at least one of them new on  $B$ . For **R3** to be applicable, points  $d_i, d_j$ , and  $d$  must have been already introduced at some step of the construction of  $B$ . Hence, at the moment when the three of them, and  $n$ , have appeared on the branch, the branch-expansion rule has become applicable to  $n$ . By the above claim, the rule has been subsequently applied on  $B$  and such an application must have introduced the labeled formulas  $(\xi_0, [d_i, d])$  and  $(\xi_1, [d, d_j])$  on  $B$ , which again contradicts the assumption.  $\square$

**Corollary 1** *Let  $\varphi$  be a  $\text{CDT}_{\text{BS}}$ -formula and  $\overline{\mathcal{T}}$  be the limit tableau for  $\varphi$ . For every open branch  $B$  in  $\overline{\mathcal{T}}$ , the following closure properties hold:*

- If there is a node  $n \in B$  such that  $v(n) = (\xi_0 \wedge \xi_1, [d_i, d_j], \mathbb{D}, p_n, u_n)$ , then there are a node  $n_0 \in B$  such that  $v(n_0) = (\xi_0, [d_i, d_j], \mathbb{D}, p_{n_0}, u_{n_0})$  and a node  $n_1 \in B$  such that  $v(n_1) = (\xi_1, [d_i, d_j], \mathbb{D}, p_{n_1}, u_{n_1})$ .
- If there is a node  $n \in B$  such that  $v(n) = (\xi_0 \vee \xi_1, [d_i, d_j], \mathbb{D}, p_n, u_n)$ , then there are a node  $n_0 \in B$  such that  $v(n_0) = (\xi_0, [d_i, d_j], \mathbb{D}, p_{n_0}, u_{n_0})$  or a node  $n_1 \in B$  such that  $v(n_1) = (\xi_1, [d_i, d_j], \mathbb{D}, p_{n_1}, u_{n_1})$ .

- If there is a node  $n \in B$  such that  $v(n) = (\xi_0 C \xi_1, [d_i, d_j], \mathbb{D}, p_n, u_n)$ , then there are two nodes  $n_0, n_1 \in B$  such that  $v(n_0) = (\xi_0, [d_i, d], \mathbb{D}', p_{n_0}, u_{n_0})$  and  $v(n_1) = (\xi_1, [d, d_j], \mathbb{D}', p_{n_0}, u_{n_0})$ , for some  $d \in D_B$ , with  $d_i \leq d \leq d_j$ .
- If there is a node  $n \in B$  such that  $v(n) = (\xi_0 D \xi_1, [d_i, d_j], \mathbb{D}, p_n, u_n)$ , then there are two nodes  $n_0, n_1 \in B$  such that  $v(n_0) = (\xi_0, [d, d_i], \mathbb{D}', p_{n_0}, u_{n_0})$  and  $v(n_1) = (\xi_1, [d, d_j], \mathbb{D}', p_{n_0}, u_{n_0})$ , for some  $d \in D_B$ , with  $d \leq d_i$ .
- If there is a node  $n \in B$  such that  $v(n) = (\xi_0 T \xi_1, [d_i, d_j], \mathbb{D}, p_n, u_n)$ , then there are two nodes  $n_0, n_1 \in B$  such that  $v(n_0) = (\xi_0, [d_i, d], \mathbb{D}', p_{n_0}, u_{n_0})$  and  $v(n_1) = (\xi_1, [d_j, d], \mathbb{D}', p_{n_0}, u_{n_0})$ , for some  $d \in D_B$ , with  $d \geq d_j$ .
- If there is a node  $n \in B$  such that  $v(n) = (\neg(\xi_0 C \xi_1), [d_i, d_j], \mathbb{D}, p_n, u_n)$ , then, for each  $d \in D_B$ , with  $d_i \leq d \leq d_j$ , there is a node  $n' \in B$  such that  $v(n') = (\bar{\xi}_0, [d_i, d], \mathbb{D}', p_{n'}, u_{n'})$  or a node  $n' \in B$  such that  $v(n') = (\bar{\xi}_1, [d, d_j], \mathbb{D}', p_{n'}, u_{n'})$ .
- If there is a node  $n \in B$  such that  $v(n) = (\neg(\xi_0 D \xi_1), [d_i, d_j], \mathbb{D}, p_n, u_n)$ , then for each  $d \in D_B$ , with  $d \leq d_i$ , there is a node  $n' \in B$  such that  $v(n') = (\bar{\xi}_0, [d, d_i], \mathbb{D}', p_{n'}, u_{n'})$  or a node  $n' \in B$  such that  $v(n') = (\bar{\xi}_1, [d, d_j], \mathbb{D}', p_{n'}, u_{n'})$ .
- If there is a node  $n \in B$  such that  $v(n) = (\neg(\xi_0 T \xi_1), [d_i, d_j], \mathbb{D}, p_n, u_n)$ , then, for each  $d \in D_B$ , with  $d \geq d_j$ , there is a node  $n' \in B$  such that  $v(n') = (\bar{\xi}_0, [d_i, d], \mathbb{D}', p_{n'}, u_{n'})$  or a node  $n' \in B$  such that  $v(n') = (\bar{\xi}_1, [d_j, d], \mathbb{D}', p_{n'}, u_{n'})$ .

The proof of Corollary 1 is straightforward, and thus it is omitted.

**Lemma 7 (Completeness)** *If the limit tableau for some formula  $\varphi \in \text{CDT}_{\text{BS}}$  is closed, then some finite tableau for  $\varphi$  is closed.*

*Proof* Let us assume the limit tableau for  $\varphi$  to be closed. Then, every branch closes at some finite step of the construction, and then it is not further expanded (it remains finite). Since the branch-expansion rule always produces finitely many successors, every finite tableau is finitely branching, and hence so is the limit tableau. Then, by König's lemma, the limit tableau, being a finitely branching tree with no infinite branches, must be finite. This allows us to conclude that its construction stabilizes at some finite stage. At that stage, a closed tableau for  $\varphi$  is constructed.  $\square$

### 5.3 Termination and Complexity

In this last subsection, we prove that the proposed tableau method is terminating, and we determine its computational complexity. The proof rests on a pair of basic lemmas.

As a preliminary step, we define a *counting function* Count on  $B$  as follows:

$$\text{Count}(B) = \sum_{n \in B} |\psi_n| \cdot p_n \cdot u_n(B),$$

where  $\psi_n$  and  $p_n$  are the formula and the  $p$ -flag in the decoration of  $n$ , respectively. The following lemma proves that Count is non-increasing with respect to branch expansions.

**Lemma 8** *Let  $\varphi$  be a  $\text{CDT}_{\text{BS}}$ -formula,  $B$  be a branch in a tableau for  $\varphi$ , and  $B'$  be an expansion of  $B$  generated by the application of the branch-expansion strategy of Definition 8. Then,  $\text{Count}(B') \leq \text{Count}(B)$ . Moreover, if  $B'$  is obtained from  $B$  by the application of **R1**, **R2**, **R6**, **R7**, or **R8** to a node  $n$  with  $p_n$  in  $v(n)$  equal to 1, then  $\text{Count}(B') < \text{Count}(B)$ .*

*Proof* Let  $\mathcal{T}$  be a tableau for  $\varphi$ ,  $B$  be a branch on it, and  $n$  be the closest-to-the-root node for which the branch-expansion rule is applicable. Moreover, let  $B'$  be a branch obtained by the application of the branch-expansion strategy on  $B$ . We consider the cases of the application of **R1**, **R3**, and **R6** to  $n$ . The missing cases are similar to the considered ones (**R2** is similar to **R1**, **R4** and **R5** to **R3**, **R7** and **R8** to **R6**), and thus they are omitted.

- **R1** is applied to  $n$ . Then,  $v(n) = \langle \xi_0 \wedge \xi_1, [d_i, d_j], \mathbb{D}, p_n, u_n \rangle$  and  $B' = B \cdot n' \cdot m'$ , with  $\xi_0$  belonging to  $v(n')$  and  $\xi_1$  belonging to  $v(m')$ . Since  $p_{n'} = p_{m'} = p_n$ ,  $u_{n'}(B') = u_{m'}(B') = 1$ ,  $u_n(B') = 0$ , and  $u_m(B') = u_m(B)$  for each  $m \notin \{n, n', m'\}$ ,  $\text{Count}(B') = \text{Count}(B) - |\xi_0 \wedge \xi_1| + |\xi_0| + |\xi_1| < \text{Count}(B)$ , when  $p_n = 1$ , and  $\text{Count}(B') = \text{Count}(B)$  when  $p_n = 0$ .
- **R3** is applied to  $B$ . Then,  $v(n) = \langle \neg(\xi_0 C \xi_1), [d_i, d_j], \mathbb{D}, p_n, u_n \rangle$  and  $B' = B \cdot n'$ , with  $\xi_0$  or  $\xi_1$  belonging to  $v(n')$ . In both cases,  $p_{n'}$  in  $v(n')$  is equal to 0, and thus  $\text{Count}(B') = \text{Count}(B)$ .
- **R6** is applied to  $B$ . Then,  $v(n) = \langle \xi_0 C \xi_1, [d_i, d_j], \mathbb{D}, p_n, u_n \rangle$  and  $B' = B \cdot n' \cdot m'$ , with  $\xi_0$  belonging to  $v(n')$  and  $\xi_1$  belonging to  $v(m')$ . Since  $p_{n'} = p_{m'} = p_n$ ,  $u_{n'}(B') = u_{m'}(B') = 1$ ,  $u_n(B') = 0$ , and  $u_m(B') = u_m(B)$  for each  $m \notin \{n, n', m'\}$ ,  $\text{Count}(B') = \text{Count}(B) - |\xi_0 C \xi_1| + |\xi_0| + |\xi_1| < \text{Count}(B)$  when  $p_n = 1$ , and  $\text{Count}(B') = \text{Count}(B)$  when  $p_n = 0$ .

Summing up, whatever **Ri** one applies,  $\text{Count}(B') \leq \text{Count}(B)$ . Moreover, when **R1**, **R2**, **R6**, **R7**, or **R8** are applied to a node  $n$  with  $p_n$  in  $v(n)$  equal to 1,  $\text{Count}(B') < \text{Count}(B)$ .  $\square$

**Lemma 9** *Let  $\varphi$  be a CDT<sub>BS</sub>-formula,  $\mathcal{T}$  be a tableau for  $\varphi$ , and  $n$  be a node in  $\mathcal{T}$  with decoration  $v(n) = \langle \psi, [d_i, d_j], \mathbb{D}, p_n, u_n \rangle$ . It holds that if  $p_n = 0$ , then  $\overline{\psi} \in \varphi_{\exists}$ .*

*Proof* Let  $n$  be a node on a branch  $B$  in  $\mathcal{T}$  with decoration  $v(n) = \langle \psi, [d_i, d_j], \mathbb{D}, p_n, u_n \rangle$ . We prove the claim by induction on  $\text{depth}(n)$ .

Base case. If  $\text{depth}(n) \leq 2$ , then  $n$  is either the root or one of the leaves of the initial tableau. In both cases, the claim follows trivially.

Inductive step. Let  $\text{depth}(n) > 2$ . By the inductive hypothesis, the claim holds for each ancestor of  $n$  in  $B$ . Let  $n'$  be the node to which the branch-expansion rule has been applied during the construction of  $\mathcal{T}$  to obtain node  $n$ . As in the proof of Lemma 8, we restrict our attention to **R1**, **R3**, and **R6**. The other cases can be dealt with in a similar way.

- Rule **R1** has been applied to  $n'$ . Then,  $v(n') = \langle \xi_0 \wedge \xi_1, [d_i, d_j], \mathbb{D}, p_{n'}, u_{n'} \rangle$  and either  $\xi_0$  or  $\xi_1$  belong to  $v(n)$ . Let us assume that  $\xi_0$  belongs to  $v(n)$  (the case in which  $\xi_1$  belongs to  $v(n)$  is analogous) and  $p_n = 0$ . By definition of **R1**,  $p_{n'} = p_n = 0$ . By the inductive hypothesis,  $\overline{\xi_0 \wedge \xi_1} = \overline{\xi_0} \vee \overline{\xi_1} \in \varphi_{\exists}$ . From the grammar rules for CDT<sub>BS</sub>, it follows that  $\overline{\xi_0} \in \varphi_{\exists}$ .
- Rule **R3** has been applied to  $B$ . Then,  $v(n') = \langle \neg(\xi_0 C \xi_1), [d_i, d_j], \mathbb{D}, p_{n'}, u_{n'} \rangle$  and either  $\overline{\xi_0}$  or  $\overline{\xi_1}$  belong to  $v(n)$ . Let us assume that  $\overline{\xi_0}$  belongs to  $v(n)$  (the case in which  $\overline{\xi_1}$  belongs to  $v(n)$  is analogous). By definition of **R3**,  $p_n = 0$ . By the grammar rules for CDT<sub>BS</sub>, it holds that  $\xi_0 \in \varphi_{\exists}$ . The thesis immediately follows from  $\overline{\overline{\xi_0}} = \xi_0$ .
- Rule **R6** has been applied to  $B$ . Then,  $v(n') = \langle \xi_0 C \xi_1, [d_i, d_j], \mathbb{D}, p_{n'}, u_{n'} \rangle$  and either  $\xi_0$  or  $\xi_1$  belong to  $v(n)$ . Let us assume  $p_{n'} = 0$ . By the inductive hypothesis, it follows that  $\overline{\xi_0 C \xi_1} = \neg(\xi_0 C \xi_1) \in \varphi_{\exists}$  (contradiction). Hence, it holds that  $p_{n'} = 1$ , and thus, by **R6**,  $p_n = 1$ .  $\square$

By exploiting Lemma 8 and Lemma 9, we now prove that the length of any branch  $B$  of any tableau for  $\varphi$  is polynomially bounded by the length of the formula.

**Lemma 10 (Termination)** *Let  $\varphi$  be a CDT<sub>BS</sub>-formula,  $\mathcal{T}$  be a tableau for  $\varphi$ , and  $B$  be a branch in  $\mathcal{T}$ . Then,  $|B| \leq 2 \cdot |\varphi|^3 + 8 \cdot |\varphi|^2 + 8 \cdot |\varphi|$ .*

*Proof* Let  $B$  be a branch in a tableau  $\mathcal{T}$  for  $\varphi$ . Given the branch-expansion rule and the branch-expansion strategy, there cannot be two nodes  $n, n'$  in  $B$  such that the same formula and the same interval belong to both  $v(n)$  and  $v(n')$ . Since for any node  $n$  in  $B$ , the formula in  $v(n)$  is either a subformula of  $\varphi$  or the dual of a subformula of  $\varphi$ , it holds that  $|B| \leq 2 \cdot |\varphi| \cdot |\mathbb{D}_B|^2$ .

To give a bound on the number of points in  $D_B$ , it suffices to observe that:

1. only the application of **R6**, **R7**, and **R8** add new points to  $D_B$ ;
2. by Lemma 9, they can be applied only to nodes where the flag  $p$  is equal to 1;
3. by Lemma 8, every application of them strictly decreases the value of  $\text{Count}(B)$ .

Now, let  $B_0$  be the two-node prefix of  $B$  consisting of the root and one of its successors labeled with  $\varphi$ . Since  $|\mathbb{D}_{B_0}| \leq 2$  and  $\text{Count}(B_0) = |\varphi|$ ,  $|\mathbb{D}_B| \leq |\varphi| + 2$ , and thus  $|B| \leq 2 \cdot |\varphi| \cdot (|\varphi| + 2)^2 \leq 2 \cdot |\varphi|^3 + 8 \cdot |\varphi|^2 + 8 \cdot |\varphi|$ .  $\square$

**Theorem 7** *The proposed tableau method for  $\text{CDT}_{\text{BS}}$  is sound and complete, and the satisfiability problem for  $\text{CDT}_{\text{BS}}$  is NP-complete.*

*Proof* By Lemma 5 (soundness) and Lemma 7 (completeness), it holds that satisfiability of a formula  $\varphi$  can be reduced to the search for an open limit tableau for it. A direct consequence of Lemma 10 is that this search can be performed by a nondeterministic algorithm that guesses an open and saturated branch of the limit tableau, using only a polynomial amount of time. NP-hardness immediately follows from that of propositional logic.  $\square$

## 6 Undecidable extensions of $\text{CDT}_{\text{BS}}$

In the previous section, we have proved that the satisfiability problem for  $\text{CDT}_{\text{BS}}$  is NP-complete. Since the full logic CDT is undecidable, one may wonder whether  $\text{CDT}_{\text{BS}}$  can be extended preserving decidability. In this section, we show that the most natural extension of  $\text{CDT}_{\text{BS}}$  is already undecidable.

In  $\text{CDT}_{\text{BS}}$ -formulas, modalities can occur in the scope of at most one negation. We slightly extend  $\text{CDT}_{\text{BS}}$  by allowing one more nesting of negations and modalities. The resulting logic includes formulas like  $\neg(\neg(pCq)Cq)$  or  $\neg(pC\neg(qCr))$ . In [20], Hodkinson et al. have shown that CDT is undecidable over the class of all linearly-ordered domains even if we restrict ourselves to formulas where only one modality occurs. Undecidability has been proved by reducing the problem of finding a solution to the octant tiling problem to the satisfiability problem for the logic. The undecidability proof below is based on the observation that the entire construction given in [20] exploits formulas where modalities occur in the scope of at most *two* negations.

Given a set of tiles  $\mathcal{T} = \{\tau_1, \dots, \tau_k\}$ , the *octant tiling problem* is the problem of establishing whether  $\mathcal{T}$  can tile an octant of the Cartesian plane over the integers. Let us consider the second octant  $\mathcal{O} = \{(p, q) \mid p, q \in \mathbb{N}, p \leq q\}$ . Each tile  $\tau_i$  has four colors, namely, *right*( $\tau_i$ ), *left*( $\tau_i$ ), *up*( $\tau_i$ ), and *down*( $\tau_i$ ). Neighboring tiles must have matching colors. Formally, we say that a set  $\mathcal{T}$  can tile  $\mathcal{O}$  if there exists a function  $f : \mathcal{O} \mapsto \mathcal{T}$  such that *right*( $f(p, q)$ ) = *left*( $f(p + 1, q)$ ) and *up*( $f(p, q)$ ) = *down*( $f(p, q + 1)$ ), where  $f(p, q)$  represents the tile to be placed in the position  $(p, q)$ , provided that all relevant coordinates  $((p, q), (p + 1, q), \text{etc.})$  lie in  $\mathcal{O}$ . Using König's lemma, one can prove that a tiling system tiles the second octant if and only if it tiles arbitrarily large squares if and only if it tiles  $\mathbb{N} \times \mathbb{N}$  if and only if it tiles  $\mathbb{Z} \times \mathbb{Z}$ . Undecidability of the first problem immediately follows from that of the last one [4].

Let  $\mathcal{T} = \{\tau_1, \dots, \tau_k\}$  be an instance of the octant tiling problem. We will assume that  $\mathcal{A}(\mathcal{P})$  contains at least the propositional letters  $u, t_1, \dots, t_k$ .  $\text{CDT}_{\text{BS}}$  makes it possible to define the “somewhere in the future” operator  $F$  (we assume future to be non-strict) as follows:

$$F\varphi ::= \top T(\varphi T \top). \quad (10)$$

The *universal* operator  $G$  can be defined as the dual of  $F$ , that is:

$$G\varphi ::= \neg F \neg \varphi. \quad (11)$$

Making use of  $G$ , we set our framework by forcing the existence of *unit-intervals* (or *u-intervals*) working like atomic elements. Such intervals will be denoted by the proposition letter  $u$ . We force  $u$ -intervals to be disposed in an unbounded unique (uninterrupted) sequence by means of the following formula:

$$uT\top \wedge G(u \rightarrow uT\neg u). \quad (12)$$

**Lemma 11** *Let  $M$  be a model such that  $M, [d, d'] \Vdash (12)$ . Then, there exists an infinite sequence of points  $d_0 < d_1 < \dots$ , such that*

1.  $d' = d_0$ ;
2. for every  $l \in \mathbb{N}$ ,  $M, [d_l, d_{l+1}] \Vdash u$ .

The following formulas associate a unique tile with every  $u$ -interval; moreover, they guarantee that tiles are placed in such a way that they respect conditions on colors (a graphical account of the encoding is given in Figure 5):

$$G(u \rightarrow \bigvee_{i=1}^{|\mathcal{T}|} t_i), \quad (13)$$

$$G \bigwedge_{i,j=1, i \neq j}^{|\mathcal{T}|} \neg(t_i \wedge t_j), \quad (14)$$

$$G \bigwedge_{i=1}^{|\mathcal{T}|} (t_i \rightarrow \neg(uT\neg \bigvee_{j=1, \text{up}(\tau_i)=\text{down}(\tau_j)}^{|\mathcal{T}|} t_j)), \quad (15)$$

$$G(u \rightarrow \bigwedge_{i,j=1, \text{right}(\tau_j) \neq \text{left}(\tau_i)}^{|\mathcal{T}|} \neg(t_i T t_j)). \quad (16)$$

It is easy to check that, in (12), (13), and (14), modalities occur in the scope of at most two negations. Moreover, formulas (15) and (16) can be easily rewritten in such a way that modalities occur in the scope of at most two negations as well. Now, let  $\varphi_{\mathcal{T}}$  be the following formula:

$$(12) \wedge (13) \wedge (14) \wedge (15) \wedge (16). \quad (17)$$

We prove that the encoding is sound and complete.

**Lemma 12** *Let  $\mathcal{T} = \{\tau_1, \dots, \tau_k\}$  be a set of tiles. It holds that  $\varphi_{\mathcal{T}}$  is satisfiable if and only if  $\mathcal{T}$  tiles the second octant  $\mathcal{O}$ .*

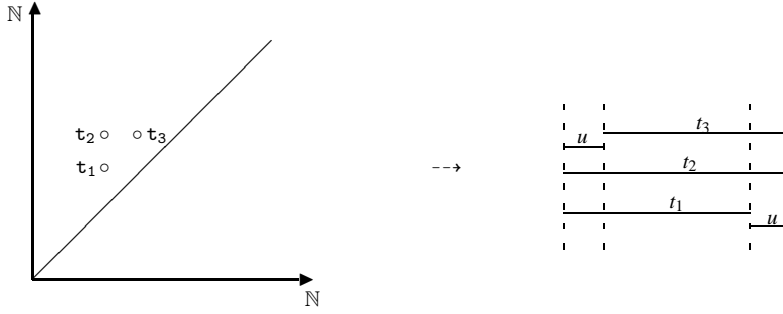


Fig. 5 A pictorial representation of the encoding.

*Proof (Soundness)* Let  $M, [d, d'] \models \varphi_{\mathcal{F}}$ . We show that there exists a tiling function  $f : \mathcal{O} \mapsto \mathcal{T}$ . By Lemma 11, we know that there exists an infinite sequence of points  $d_0 < d_1 < \dots$  such that  $d' = d_0$  and, for every  $i \in \mathbb{N}$ ,  $M, [y_i, y_{i+1}] \models u$ . Now, for each  $l, m \in \mathbb{N}$ , with  $l \leq m$ , we put:

$$f(l, m) = \tau \text{ whenever } M, [d_l, d_{m+1}] \models \tau.$$

First, we have to show that  $f$  is well-defined, that is, that each  $f(l, m)$  is a tile. We proceed by induction on  $(m - l)$ . If  $(m - l) = 0$ , then, by Lemma 11, we are on a  $u$ -interval and thus, by (13), there exists a tile associated with it. Since by (14) such a tile is unique,  $f$  is well-defined. Suppose now that  $f(l, m)$  is a tile whenever  $m - l \leq p$ , and consider  $m - l = p + 1$ . Since  $(m - 1) - l \leq p$ , by the inductive hypothesis  $f(l, m - 1)$  is a tile, say  $\tau_i$ , which means that  $M, [d_l, d_m] \models \tau_i$ . By (15),  $M, [d_l, d_m] \models \neg(uT \neg \bigvee_{j=1, \text{up}(\tau_i)=\text{down}(\tau_j)}^{|\mathcal{F}|} t_j)$ . Hence, for every  $\bar{d} \geq d_m$ , if  $M, [d_m, \bar{d}] \models u$ , then it must be the case that  $M, [d_l, \bar{d}] \models \bigvee_{j=1, \text{up}(\tau_i)=\text{down}(\tau_j)}^{|\mathcal{F}|} t_j$ . Since  $M, [d_m, d_{m+1}] \models u$ , this applies to the particular case  $\bar{d} = d_{m+1}$ . Thus, we have that  $M, [d_l, d_{m+1}] \models t_j$ , that is,  $f(l, m) = \tau_j$ , for some  $j$  such that  $\text{down}(\tau_j) = \text{up}(\tau_i)$  (again, since by (14) such a tile is unique,  $f$  is well-defined). This not only guarantees us that  $f$  is well-defined, but also that it respects the ‘vertical’ condition of a tiling function. To conclude the proof, we need to show that the ‘horizontal’ condition is respected as well. To this end, let us consider  $f(l, m)$  and  $f(l + 1, m)$ . By definition, the corresponding tiles are those associated with  $[d_l, d_{m+1}]$  and  $[d_{l+1}, d_{m+1}]$ . Since, by definition, the interval  $[d_l, d_{l+1}]$  is a  $u$ -interval, by (16) it cannot be the case that  $\text{left}(f(l + 1, m)) \neq \text{right}(f(l, m))$ , which implies that  $\text{left}(f(l + 1, m)) = \text{right}(f(l, m))$ .  $\square$

*(Completeness)* For simplicity, let us assume the linearly ordered set to be  $(\mathbb{N}, <)$ . One can force the truth of  $\varphi_{\mathcal{F}}$  over  $[0, 0]$  by letting  $u$  be true over all intervals of length 1 and each  $t_i$  be true over all intervals of the form  $[l, m + 1]$ , where  $f(l, m) = \tau_i$ .  $\square$

**Theorem 8** *The satisfiability problem for any syntactic extension of  $\text{CDT}_{\text{BS}}$  where modal operators occur in the scope of two negations is undecidable.*

*Proof* The thesis directly follows from Lemma 12.

It is worth noticing that only modality  $T$  occurs in  $\varphi_{\mathcal{F}}$ . An alternative proof of Lemma 12 can be given by making use of modality  $C$  or of modality  $D$  only. This shows that any fragment of CDT containing at least one modality among  $C, D$ , and  $T$ , where modalities are allowed to occur in the scope of two negations, is undecidable.

## 7 Conclusions and future work

In this paper, we studied a syntactic fragment of Venema’s CDT logic, that we called  $\text{CDT}_{\text{BS}}$ , whose standard translation to first-order logic fits into Bernays-Schönfinkel class of quantifier prefix formulas. Decidability of  $\text{CDT}_{\text{BS}}$  directly follows from that of Bernays-Schönfinkel class.

We first focused our attention on expressiveness issues. We considered the following question: “can every formula in Bernays-Schönfinkel class of first-order logic over the linear order  $<$ , limited to binary predicates, be turned into a  $\text{CDT}_{\text{BS}}$ -formula?”. We proved that this is not the case. In [33], Venema showed that CDT is expressively complete with respect to  $\text{FO}^{3,2}[<]$ . In this paper, we showed that  $\text{CDT}_{\text{BS}}$  is expressively complete with respect to the corresponding fragment of Bernays-Schönfinkel class  $\text{FO}_{\text{BS}}^{3,2}[<]$ . Next, we developed a tableau-based decision procedure for  $\text{CDT}_{\text{BS}}$ , and we proved that the satisfiability problem for  $\text{CDT}_{\text{BS}}$  is NP-complete. Finally, we showed that any natural relaxation of the syntactic restrictions we imposed on  $\text{CDT}_{\text{BS}}$  yields undecidability, as it makes the resulting logic expressive enough to encode the (undecidable) octant tiling problem.

The present work can be developed in a number of future research directions. From a theoretical point of view, one can think of the possibility of identifying the interval temporal logic counterparts of other decidable classes of first-order formulas. Moreover, the relationships between interval temporal logics and (extended) guarded fragments are still unexplored. For instance, It would be interesting to give an account of the good computational properties of decidable fragments of CDT and HS (including  $\text{CDT}_{\text{BS}}$ ) in terms of suitable guarded fragments. From a more practical point of view, we expect  $\text{CDT}_{\text{BS}}$  to be applicable in a variety of areas such as, for instance, planning and synthesis of plan controllers, temporal description logics, and sequencing problems in computational genetics.

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