Plane Sweep Approach

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Computational Geometry

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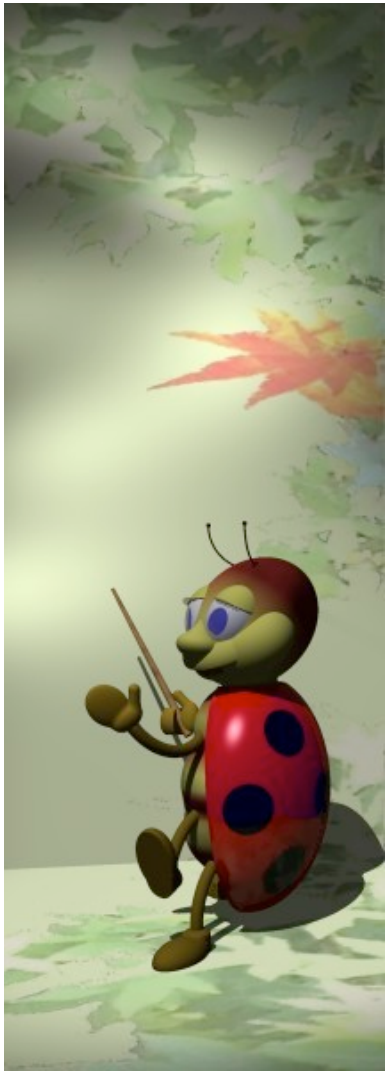
Outline



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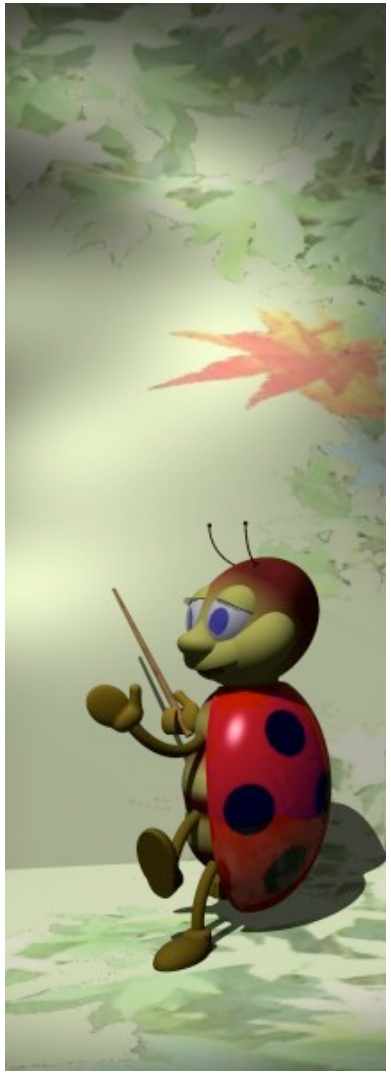

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- *Sweep line*:
  - sweeps the plane, e.g. moving left to right
  - keeps track of what it "sees" about the problem geometry
  - processing completed on the left
  - work in progress on the right
- *Events*:
  - sweep line positions where some critical change occurs...
  - ...in connection with what it sees
  - critical: important qualitative, e.g. topological, change

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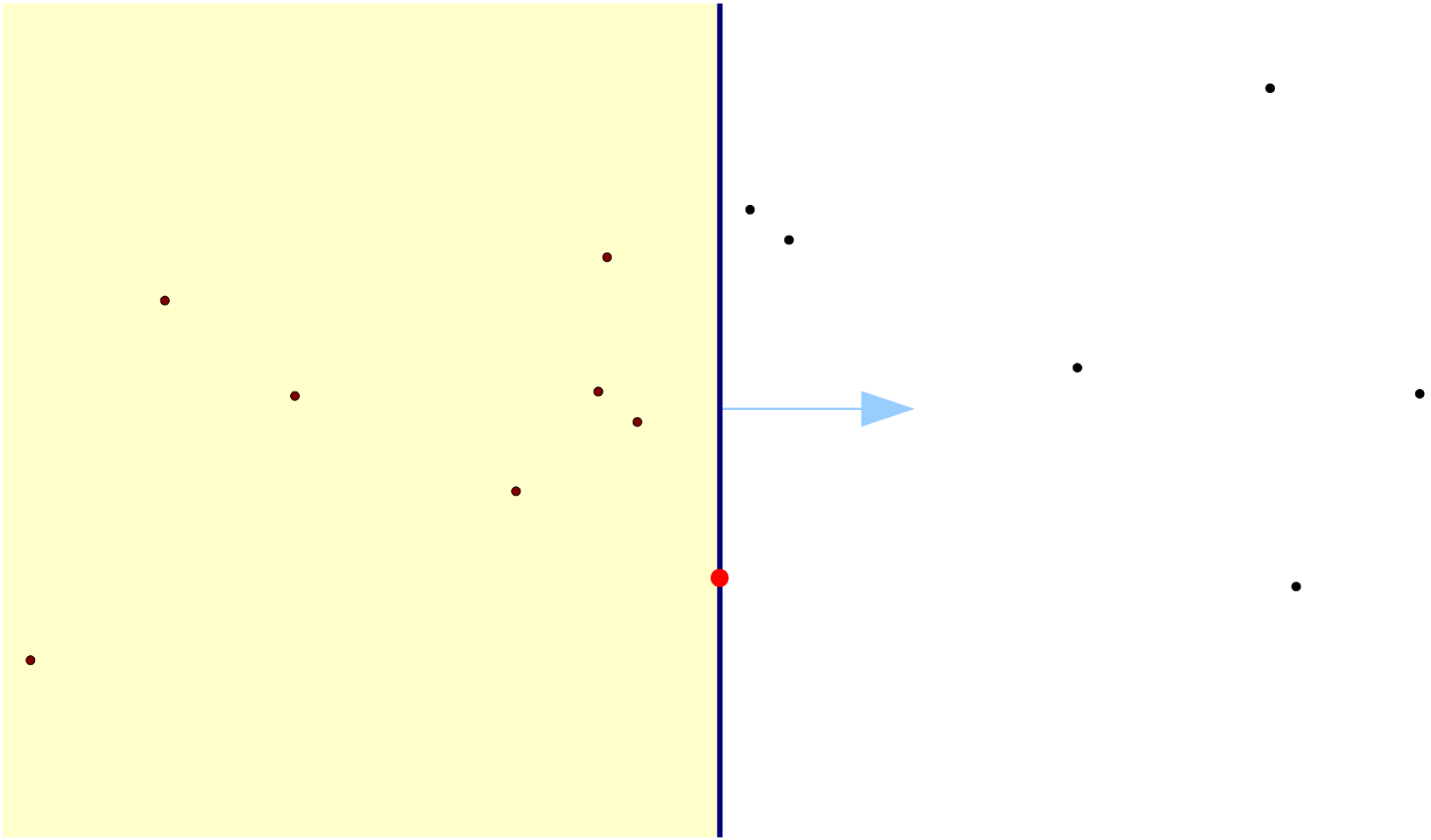

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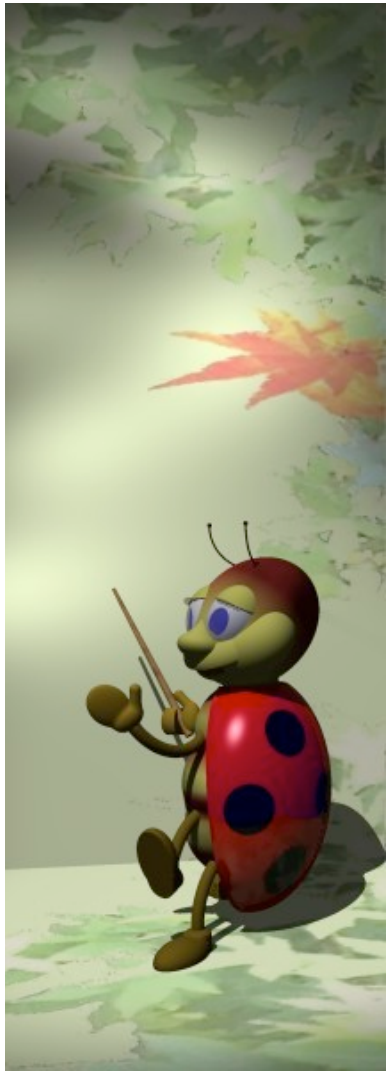

Line segment intersection problem

- Given a set S of n line segments in the plane
- Report all intersection points between segments in S
- Trivially $\Theta(n^2)$ for worst-case arrangements
- Interest in *output-sensitive* algorithms

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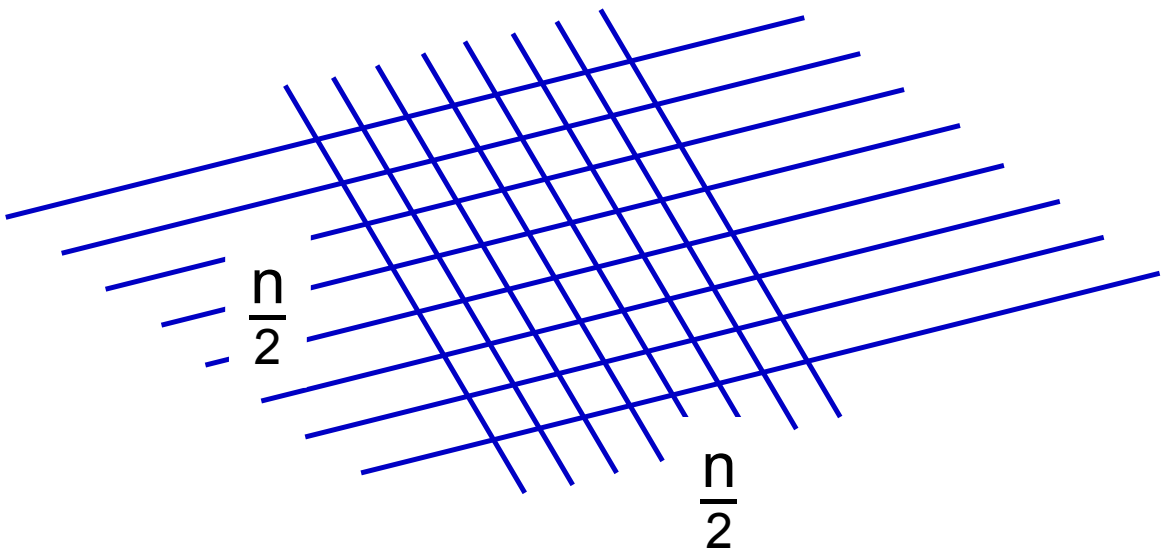

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First idea...

- Two line segments *may* intersect only if their projections onto (e.g.) the x-axis overlap
- The sweep line "sees" the line segments it crosses (whose projections do overlap)
- Critical changes occur at segment endpoints:
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Rudimentary algorithm

- At *ENTER* events, check for intersection of the "entering" segment against those crossed by the sweep line
- The list of crossed segments is updated at each event by either adding or removing a line segment
- Sorting endpoints: $O(\, n \log n \,)$
- However: $O(\, n^2 \,)$ checks in the worst case...
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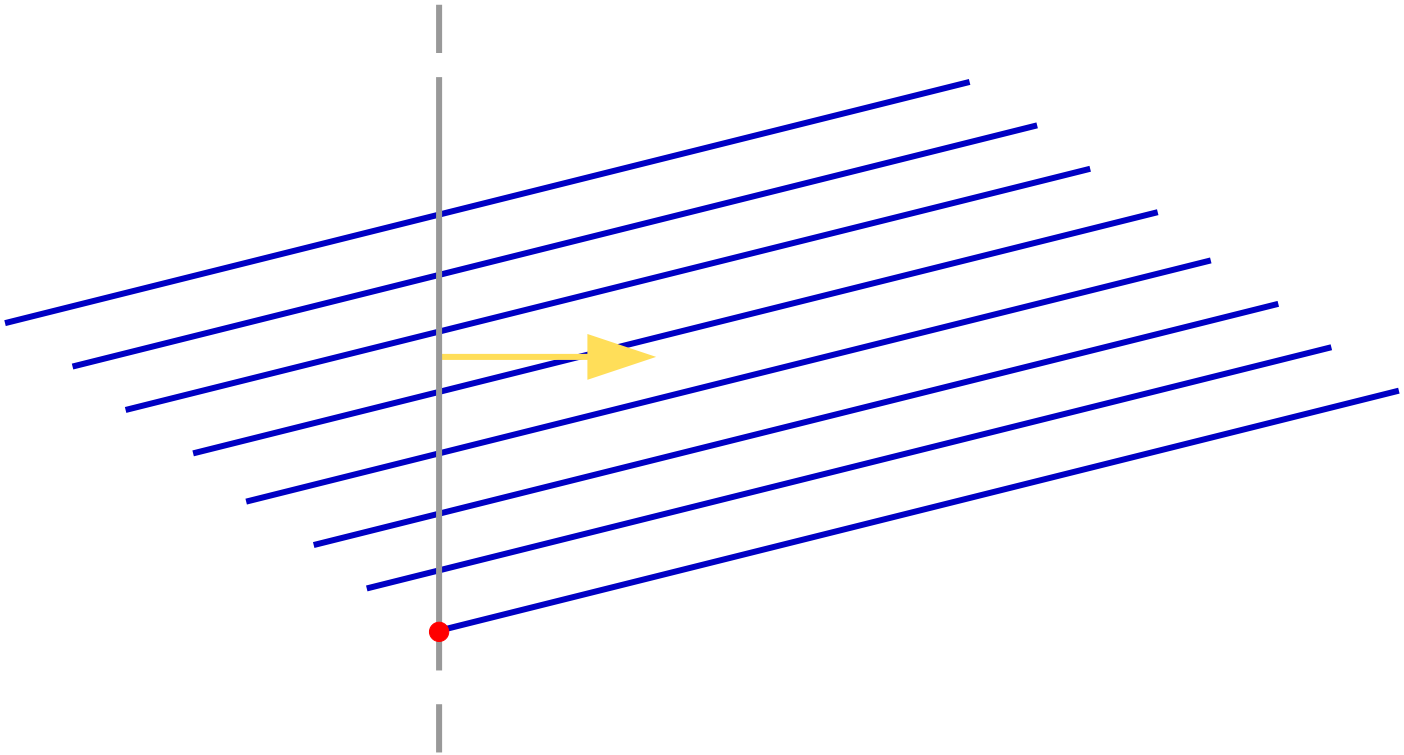
Rudimentary algorithm: Worst case arrangement



Second idea...

- At stage k, let L_k be the set of points where the sweep line crosses the segments of S
- In order for two line segments to intersect the corresponding points in L_k must be adjacent at some "previous" stage k
- The segments crossing the sweep line at stage k (S_k) can be organized according to the order of points in L_k (along the sweep line)

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Shamos & Hoey (1976)

- Sort endpoints of line segments in S: $O(\ n \log n\ )$
- Arrange S_k as a balanced search tree (initially empty)
- *ENTER* event — add $s \in S$ into S_k
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Intersection test

- When can new pairs of adjacent segments in S_k be found?
- *ENTER* event — test s for intersection with u and v if s is to be inserted into S_k between u and v: $O(\log n)$
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Operations on the tree structure

- $insert(s, T)$
- $find(s, T)$
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- All running in $O(\log |T|) = O(\log n)$

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Optimal intersection test algorithm

- Shamos & Hoey's $O(\, n \log n \,)$ algorithm for intersection test ...
- Do any two line segments in S intersect? (and if they do, report a "witness" pair)
- ... is optimal: $\Omega(\, n \log n \,)$ lower bound (worst case)
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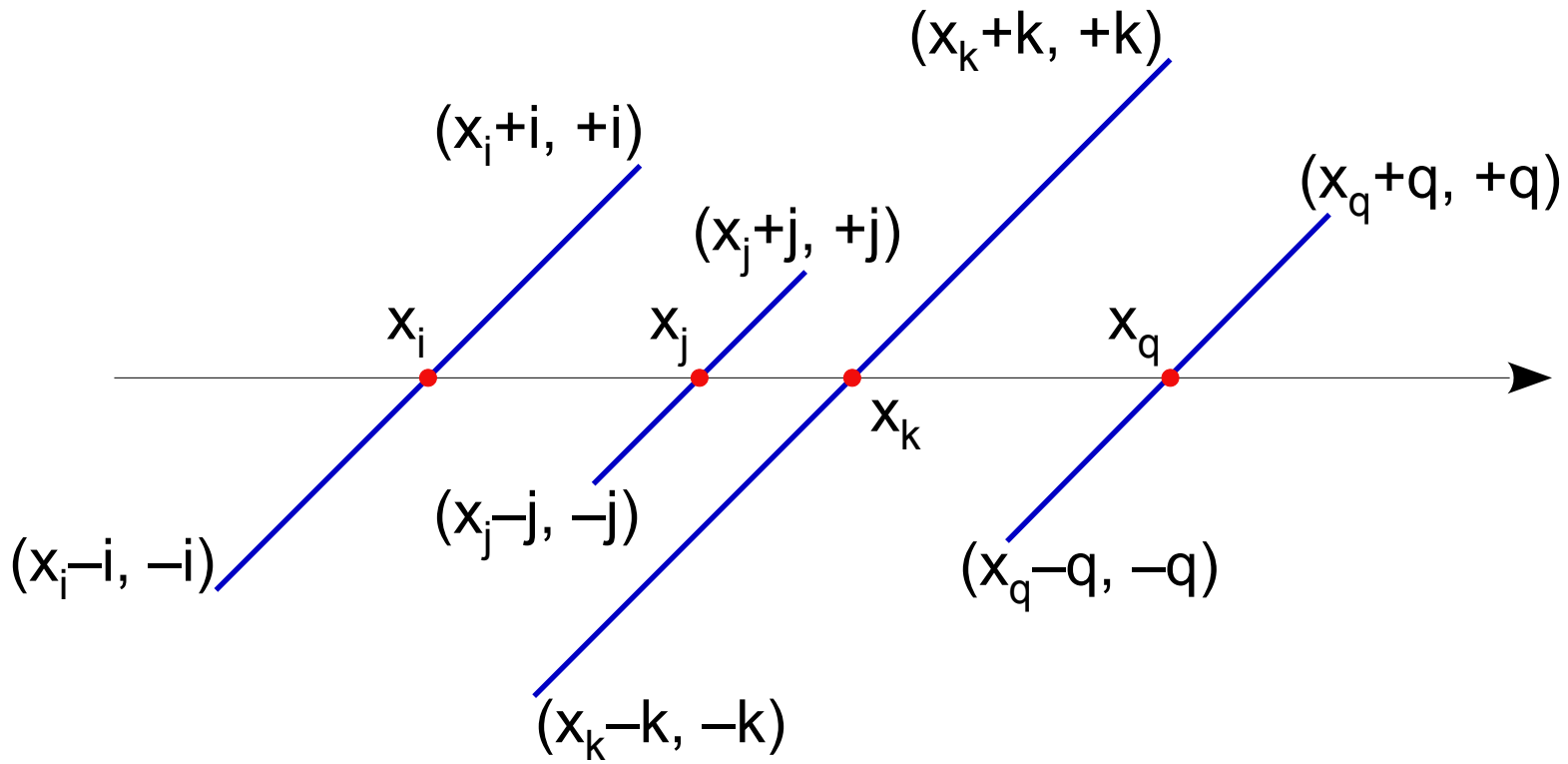
Problem reduction to "Uniqueness"



segment endpoints are all distinct, but segments overlap if x_i is the same

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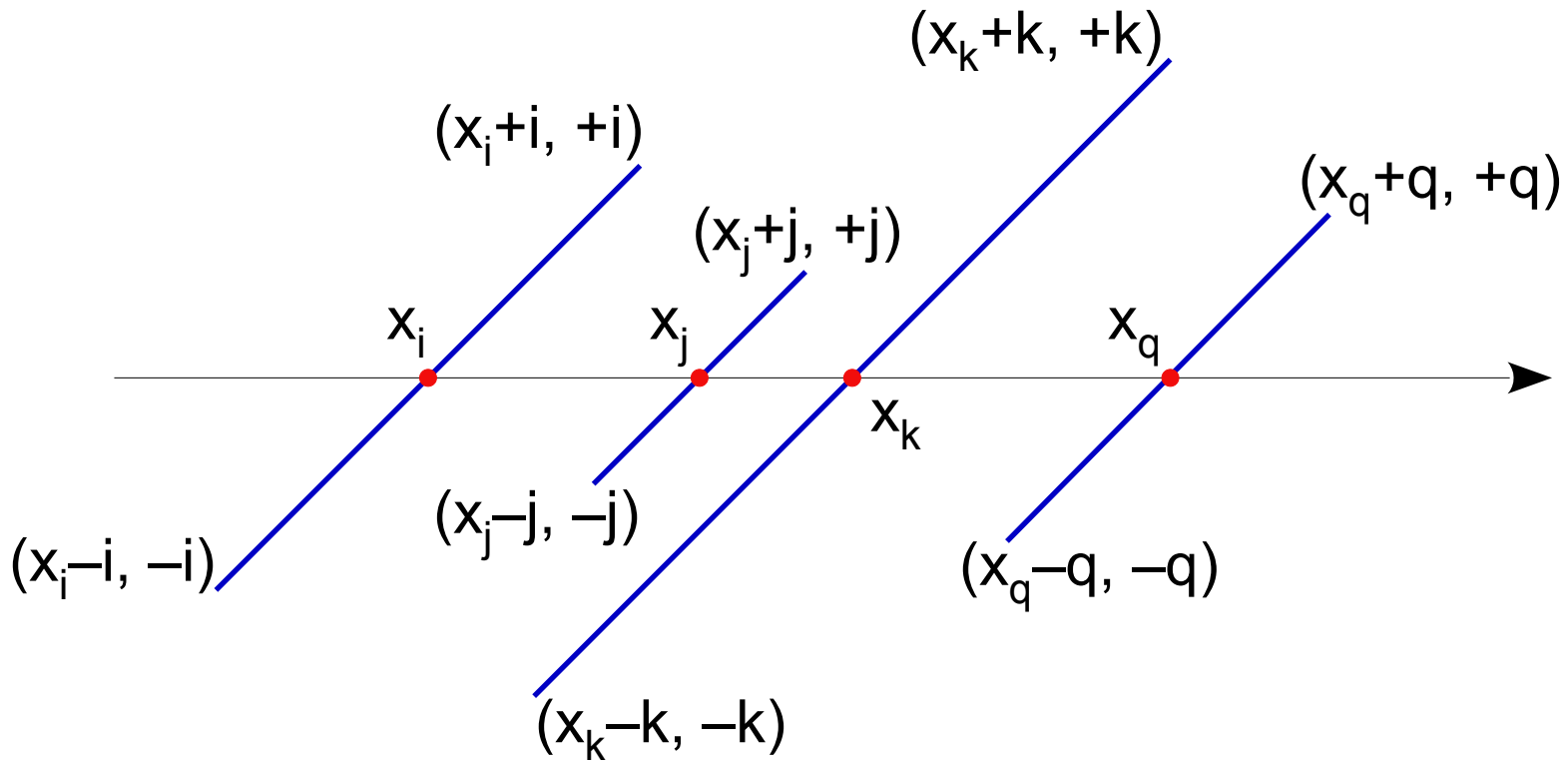

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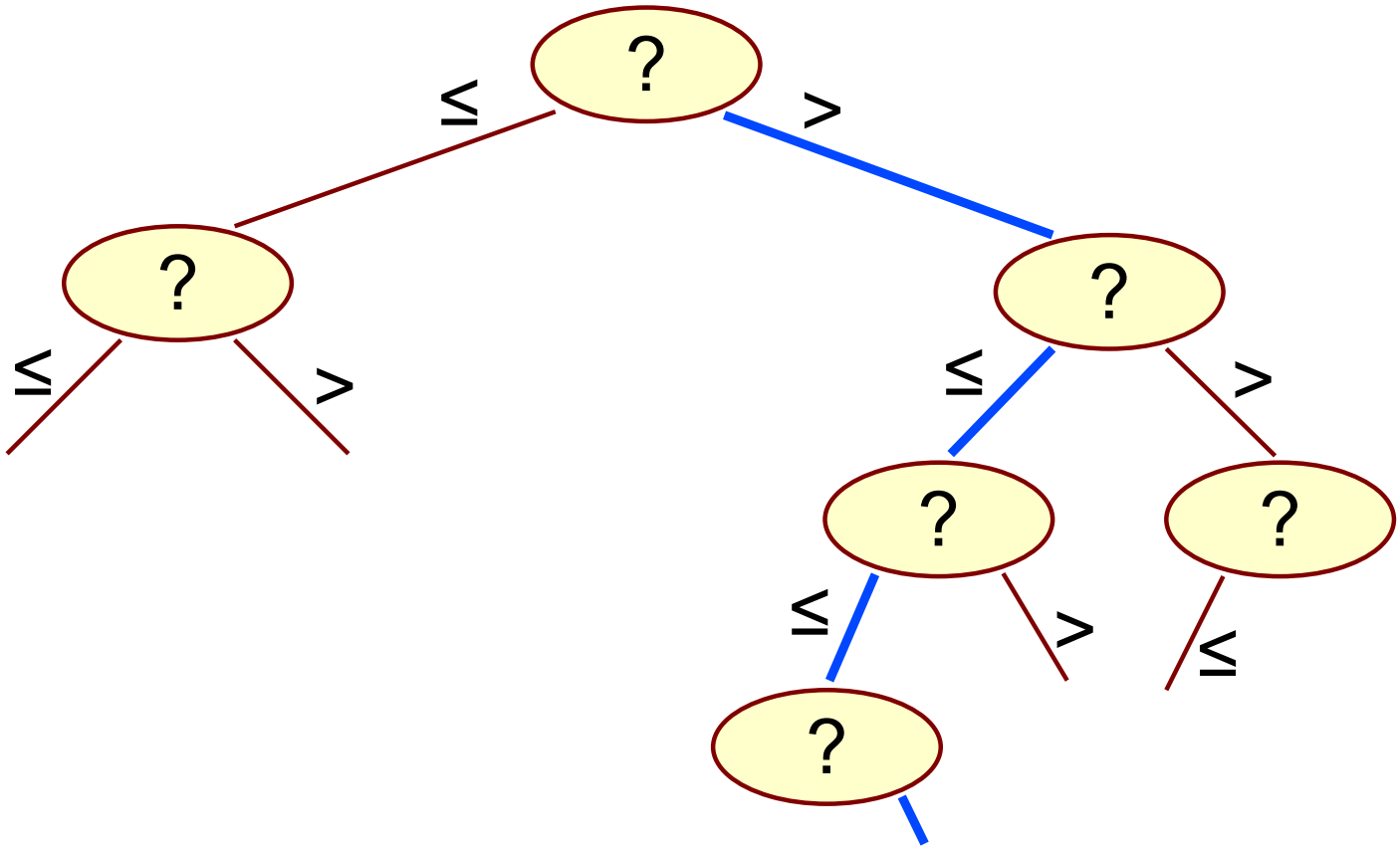
"Uniqueness": Algorithm decision tree



"Uniqueness": A bit of notation

- Collection of n numbers (linear order: <, =, >)
- $< x_1,\ x_2,\ \ldots\ x_n >$ and $< y_1,\ y_2,\ \ldots\ y_n >$ are two permutations of the same n-tuple
- $< z_1,\ z_2,\ \ldots\ z_n >$ such that $z_i = \alpha x_i + (1-\alpha) y_i$, where $\alpha \in [0,1]$
- i.e., $z_i = z_i(\alpha)$

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"Uniqueness": Property

If the decision paths for two distinct permutations of the same n-tuple

$< x_1, x_2, \ldots x_n >$, $< y_1, y_2, \ldots y_n >$

both reach node ν,

then the algorithm would follow the same path to ν starting with

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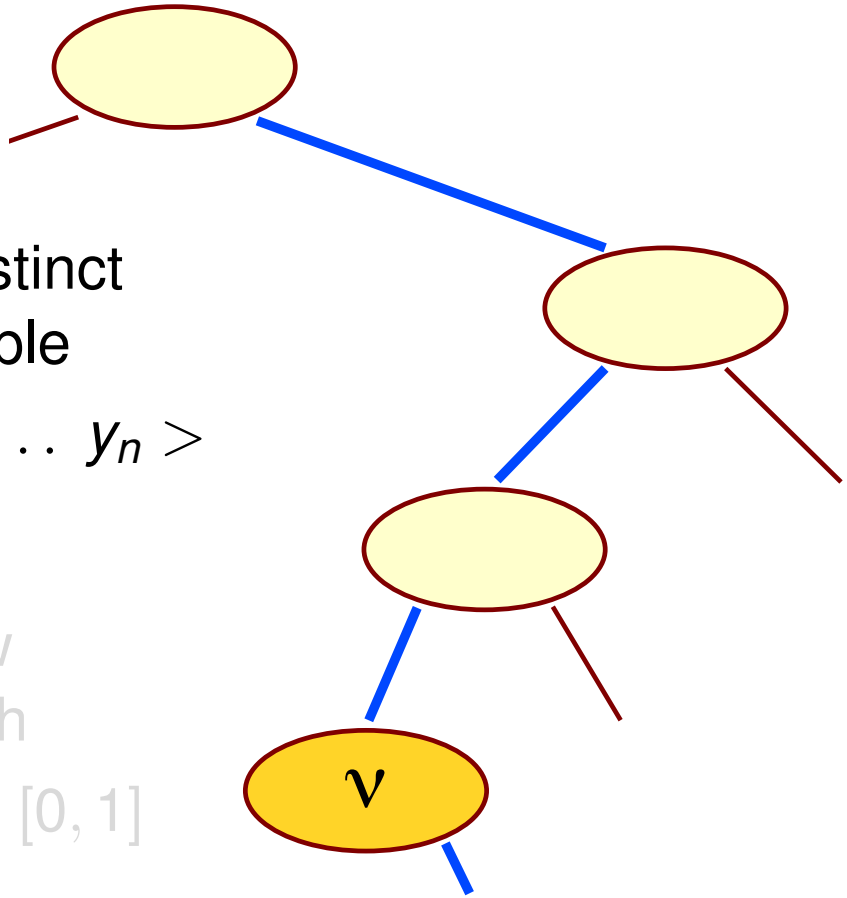

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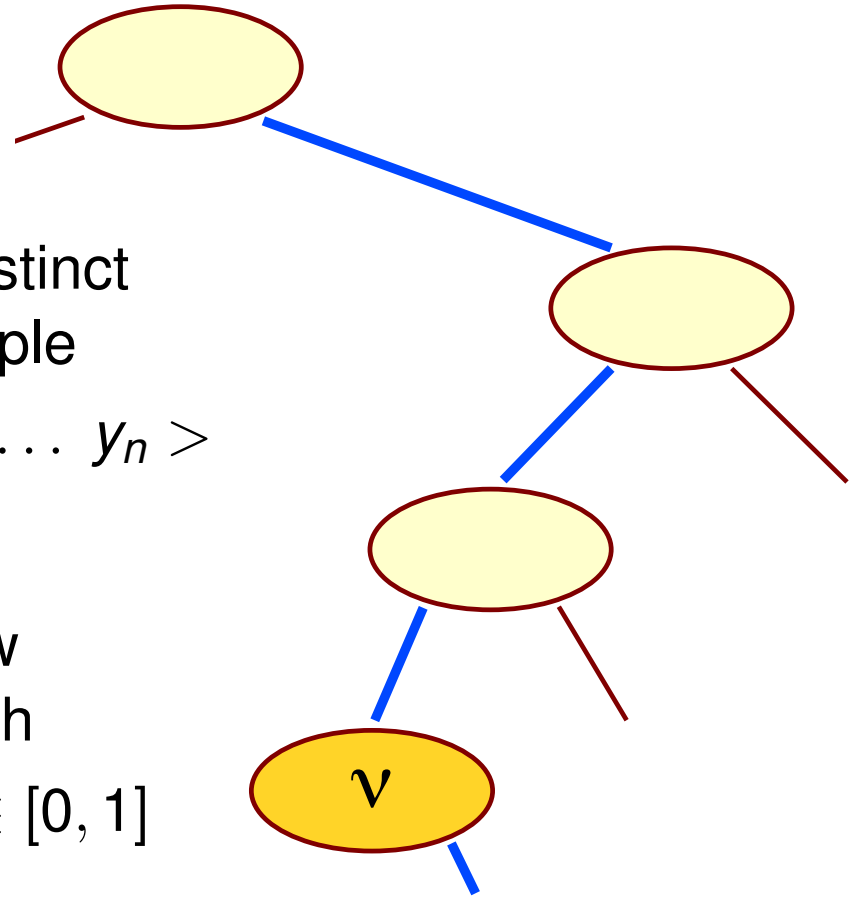

"Uniqueness": Contradiction

If not so, let μ be ν's parent node
where the path for $< z_1, z_2, \ldots z_n >$
is supposed to take a different route...

At node μ the algorithm compares
the i-th and j-th items of the input n-tuple,

but since $z_i = \alpha x_i + (1 - \alpha) y_i$
and $z_j = \alpha x_j + (1 - \alpha) y_j$,
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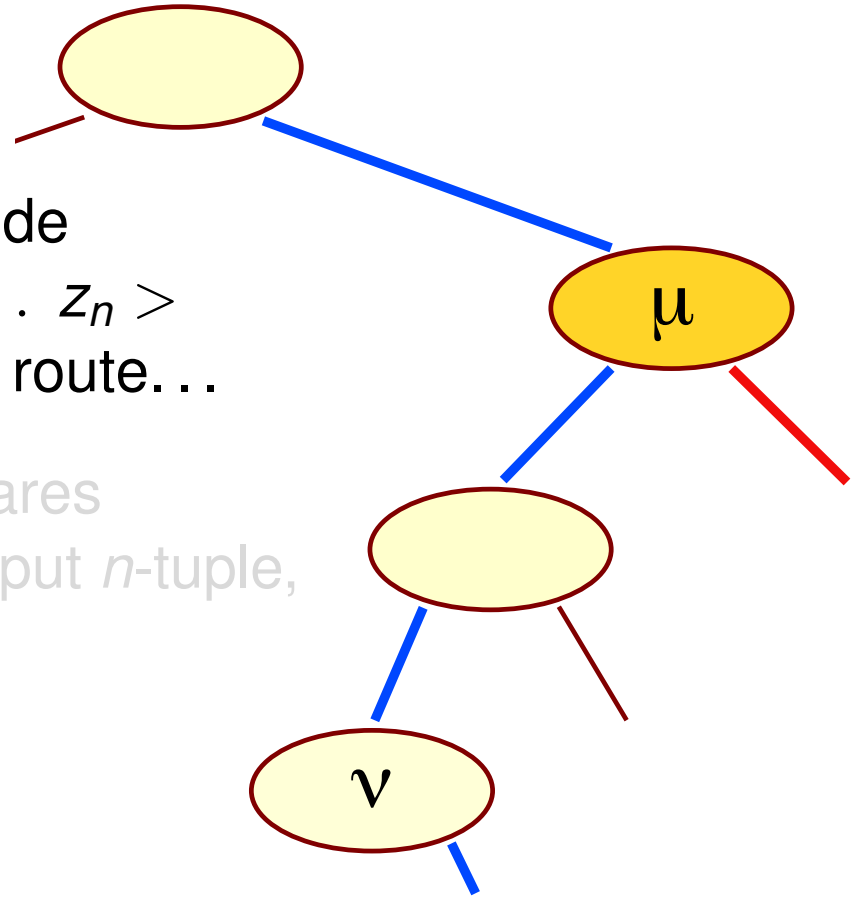

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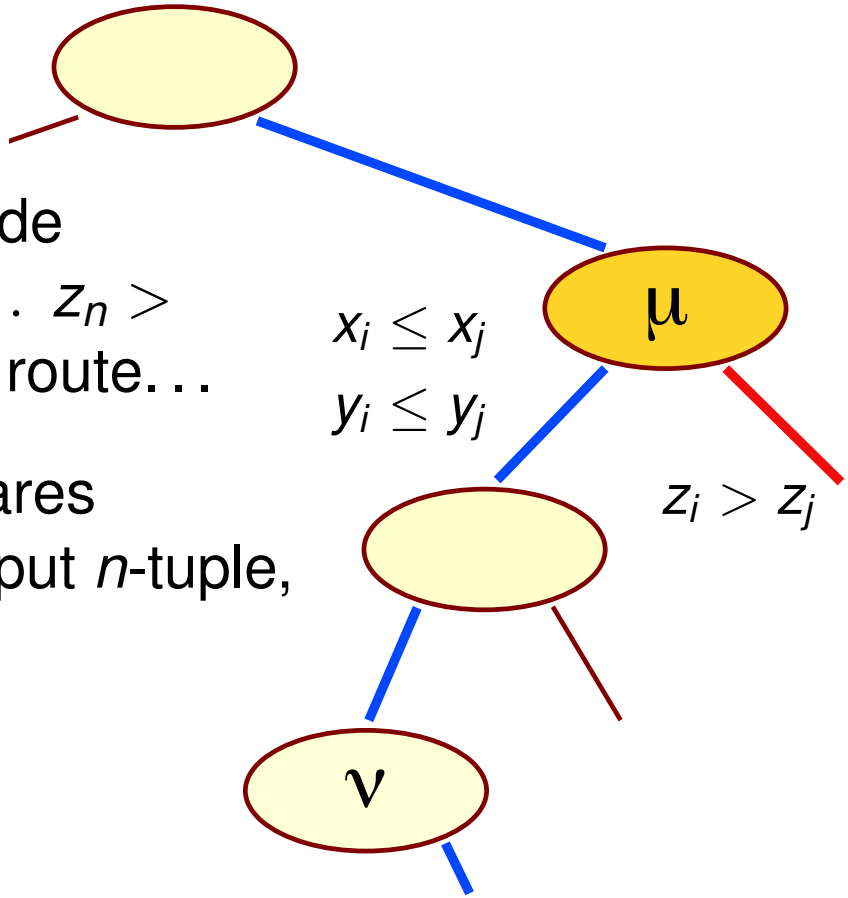

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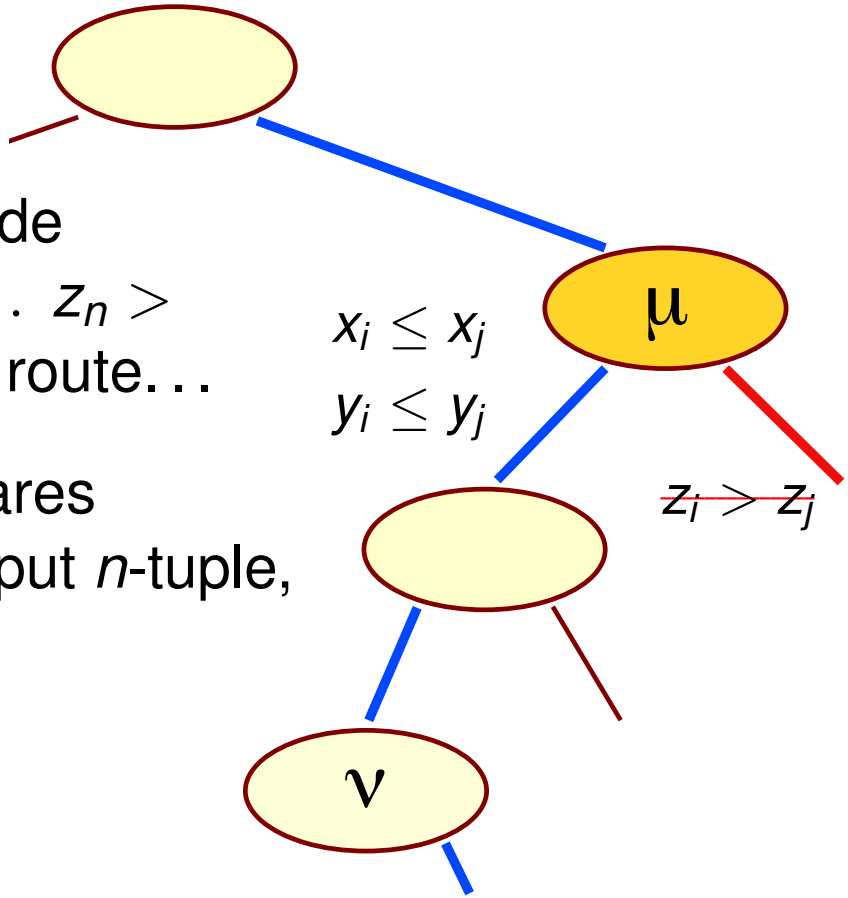


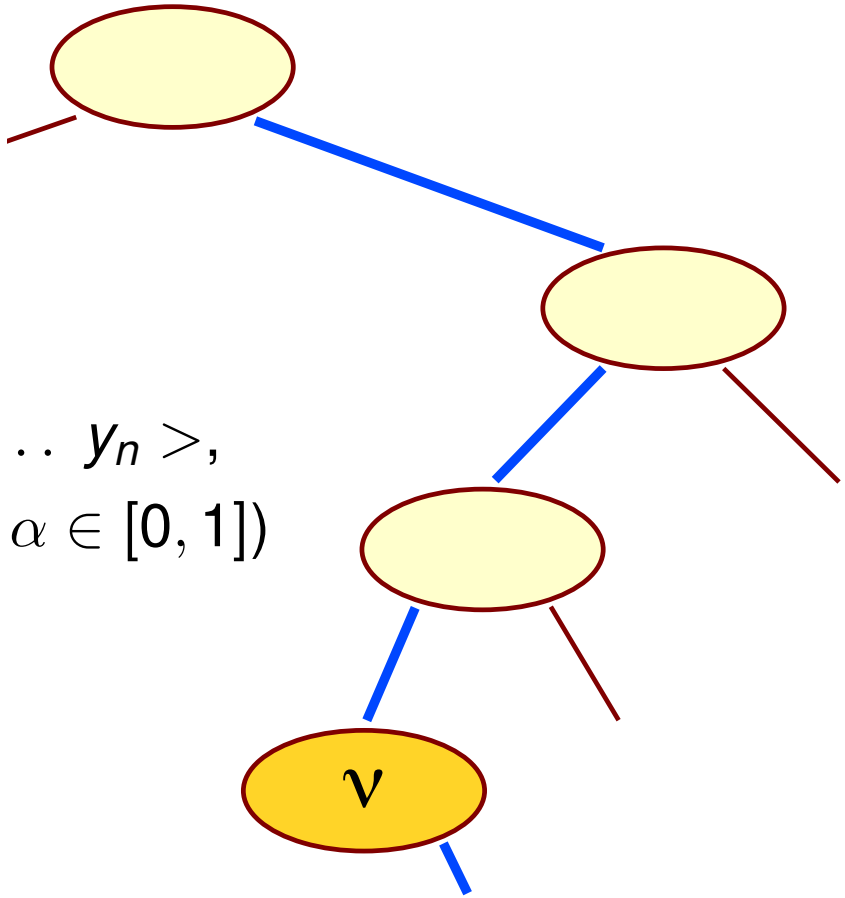
"Uniqueness": Implication

Thus, the input n-tuples

$< x_1, x_2, \ldots x_n >$, $< y_1, y_2, \ldots y_n >$,

or $< z_1, z_2, \ldots z_n >$ (for any $\alpha \in [0, 1]$)

all lead to node ν ...



"Uniqueness": At node ν ...

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  are permutations of a same n-tuple of *distinct* items
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  (notice that the denominator is strictly positive)

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- Shamos & Hoey's algorithm reports a "witness" intersection
- How to proceed in order to find the other intersections?
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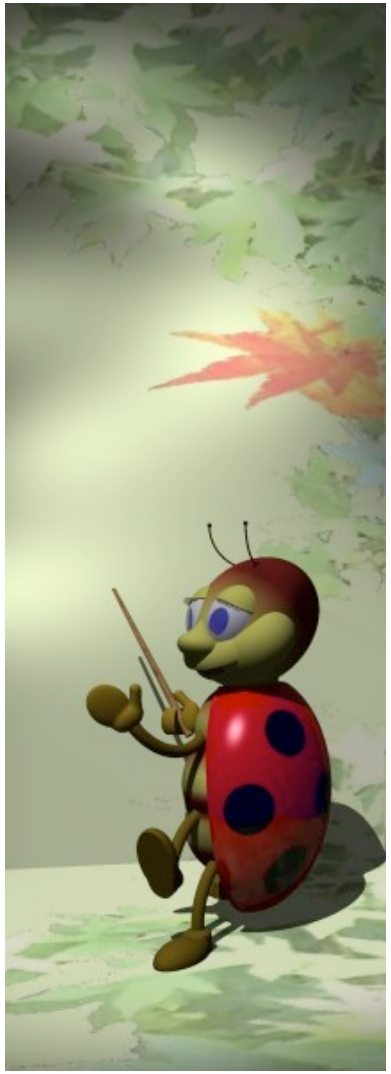

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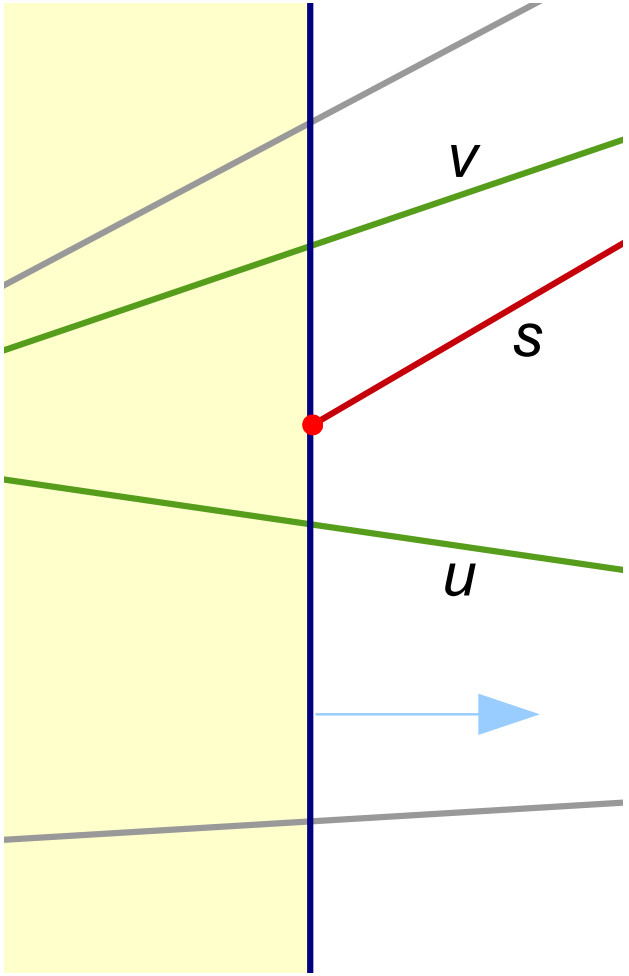
ENTER event:

- leftmost endpoint of segment s
- falling between u and v
- new adjacencies: u–s, s–v
- both are to be checked for intersection



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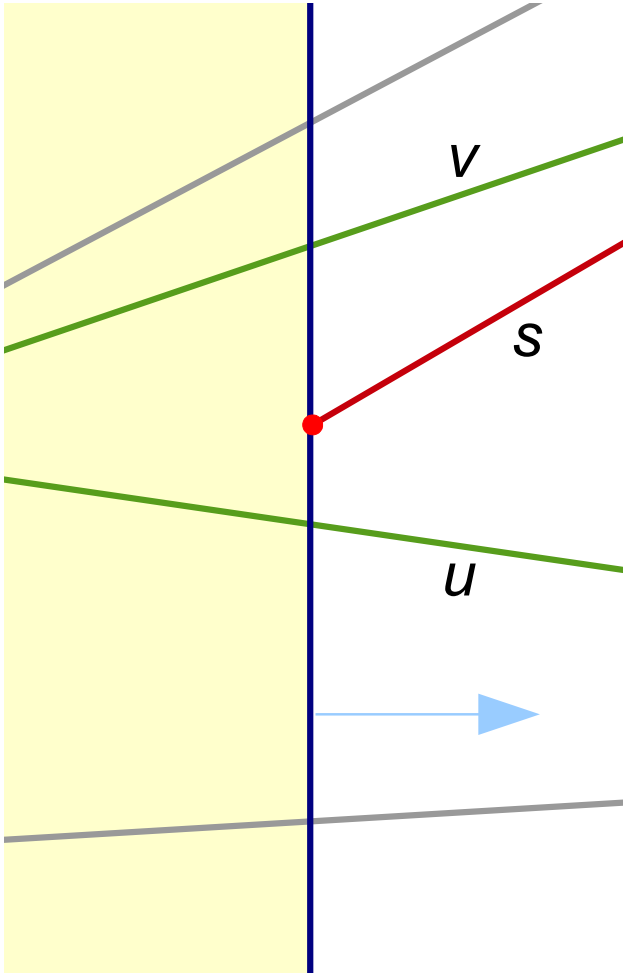

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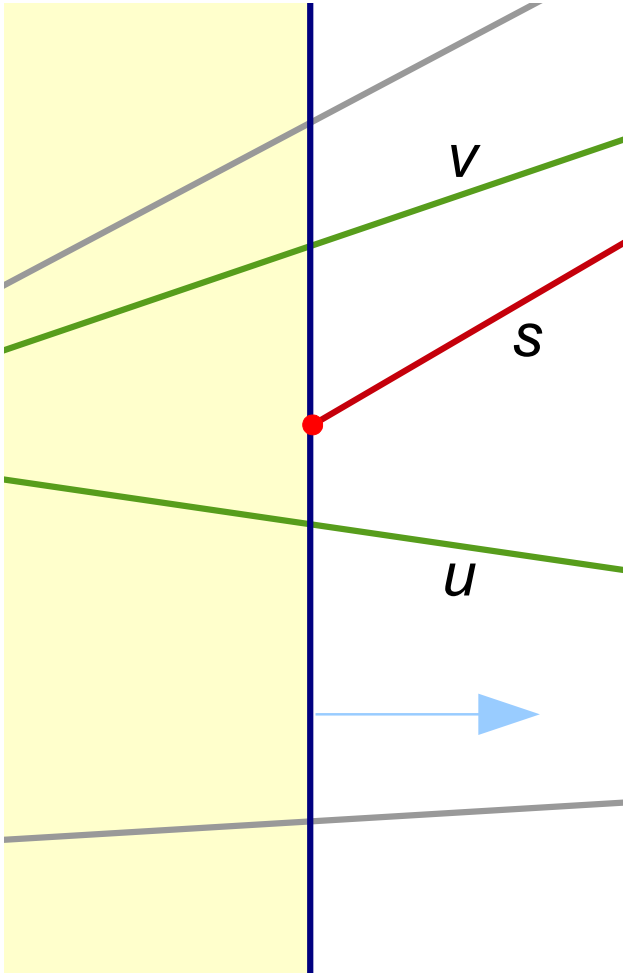

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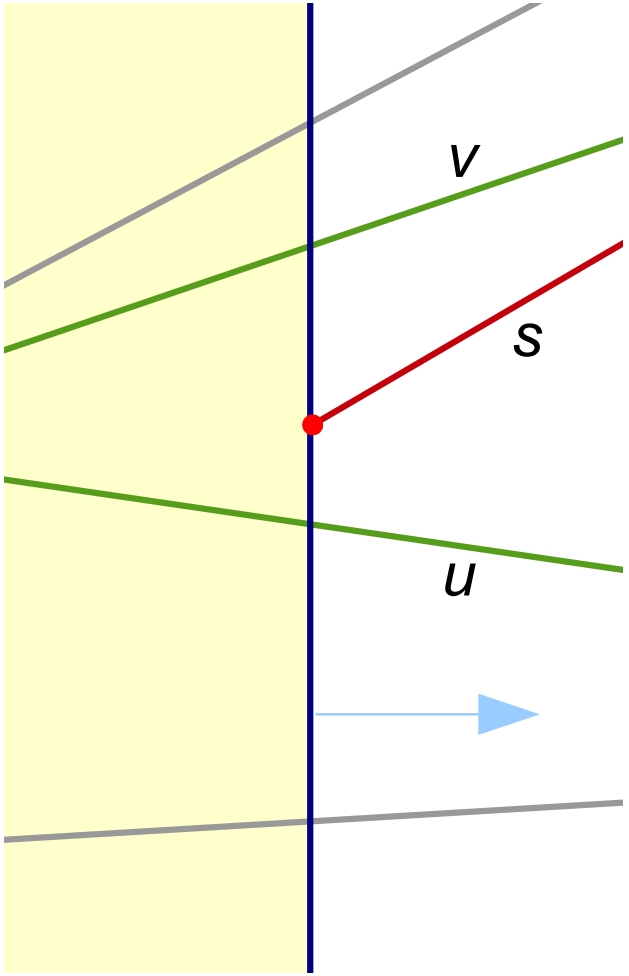

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EXIT event:

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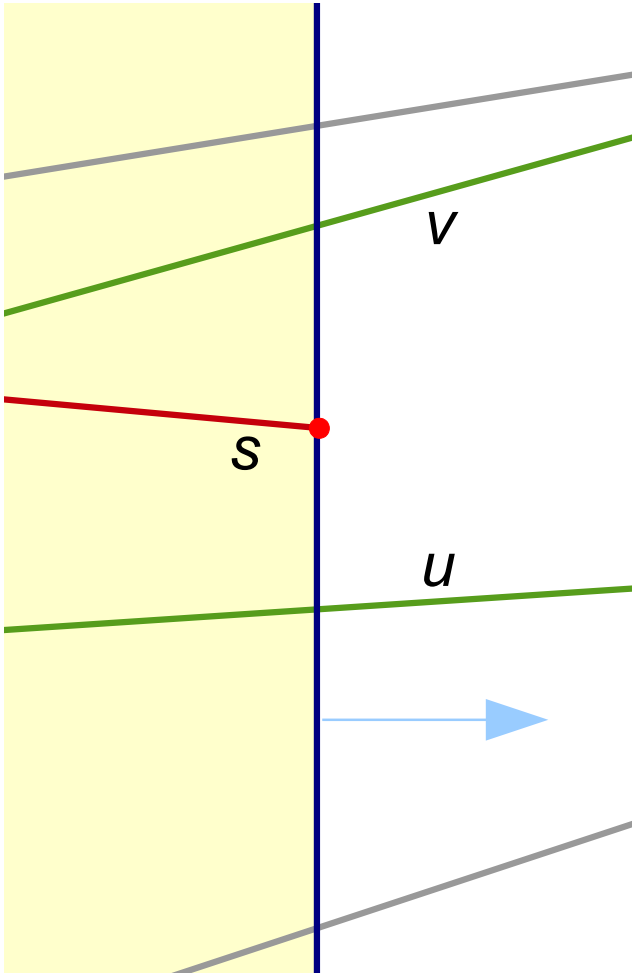

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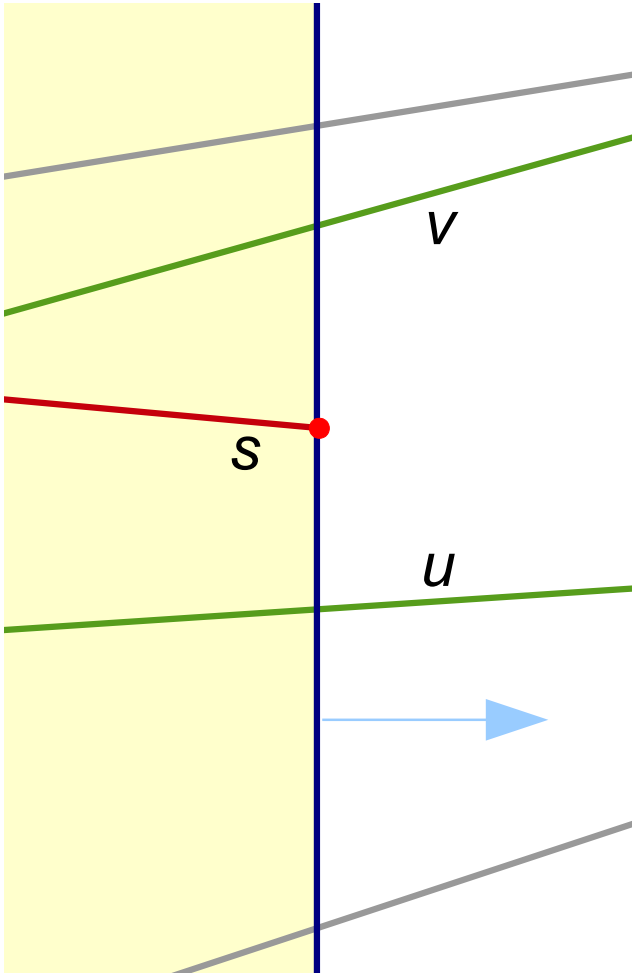

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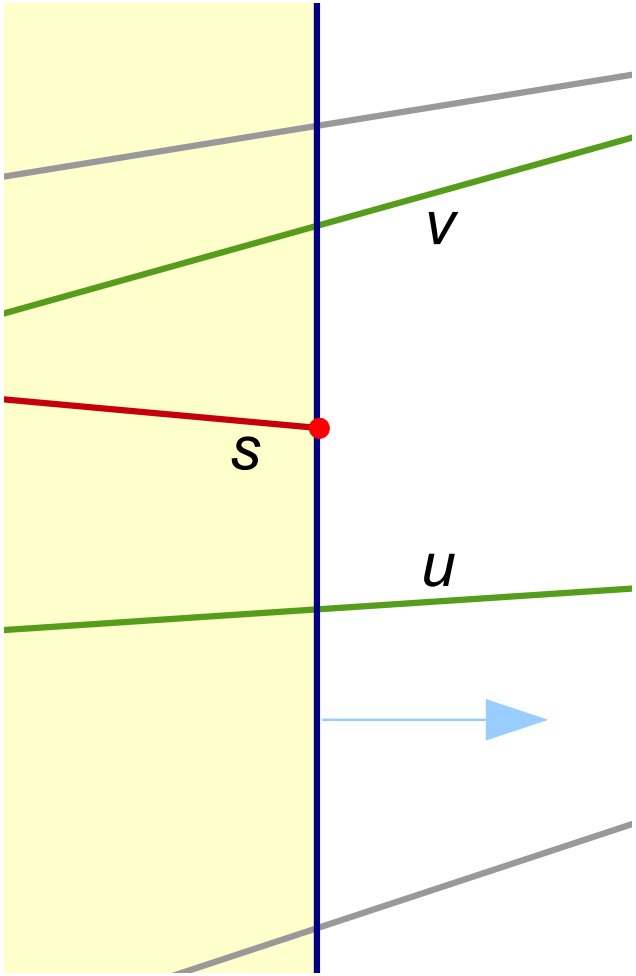

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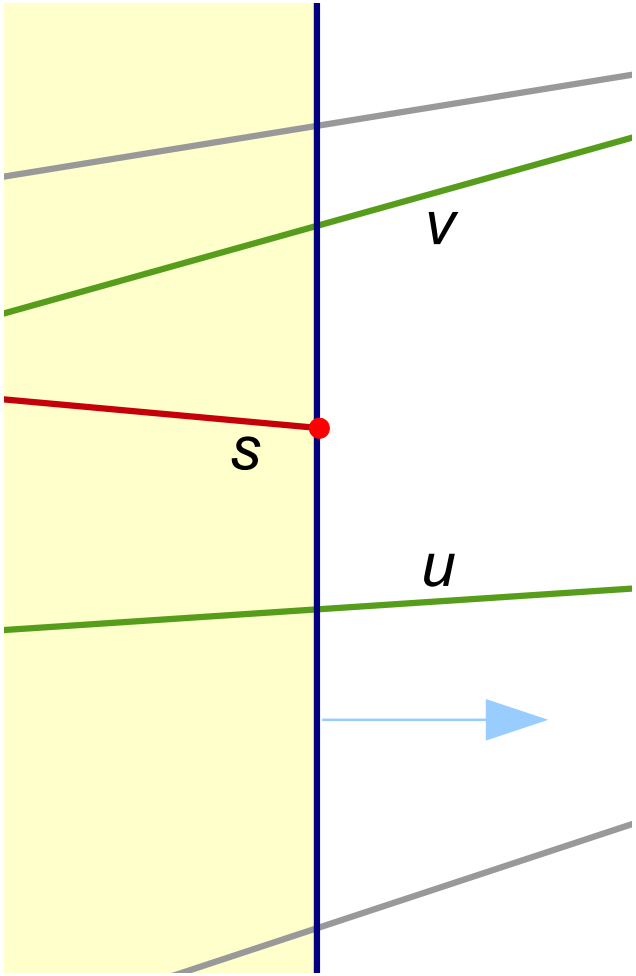

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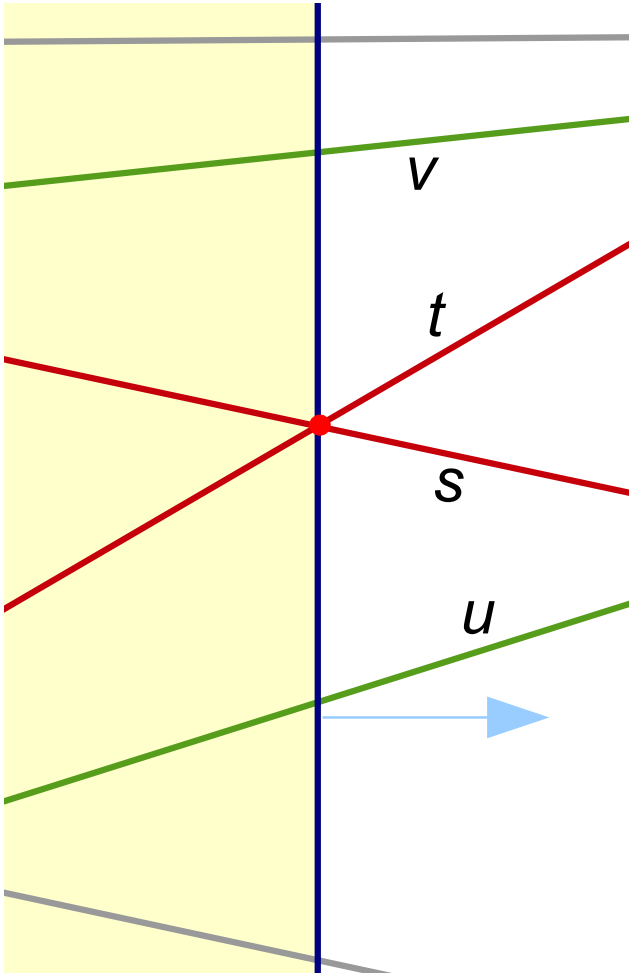

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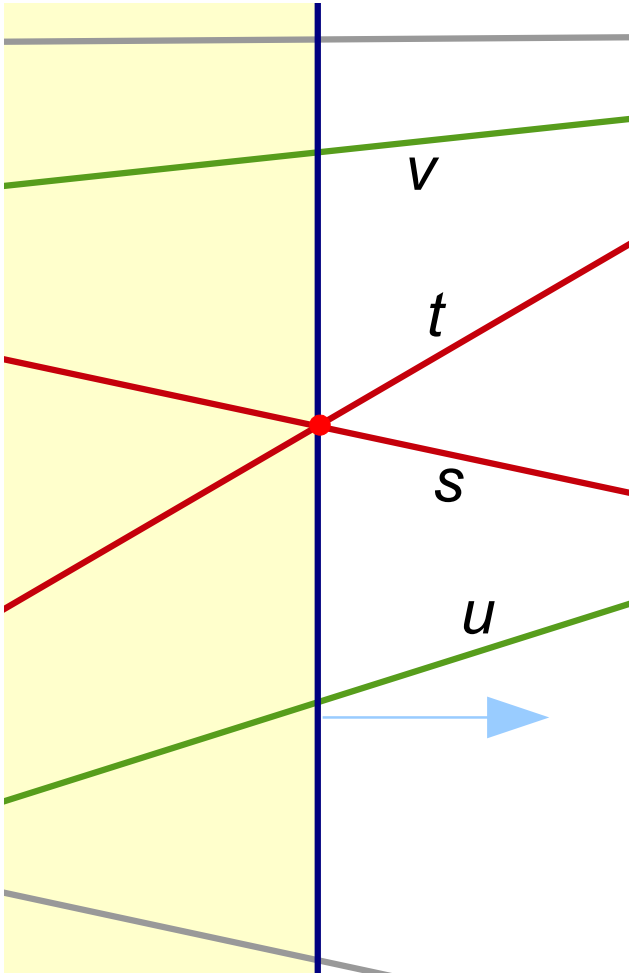

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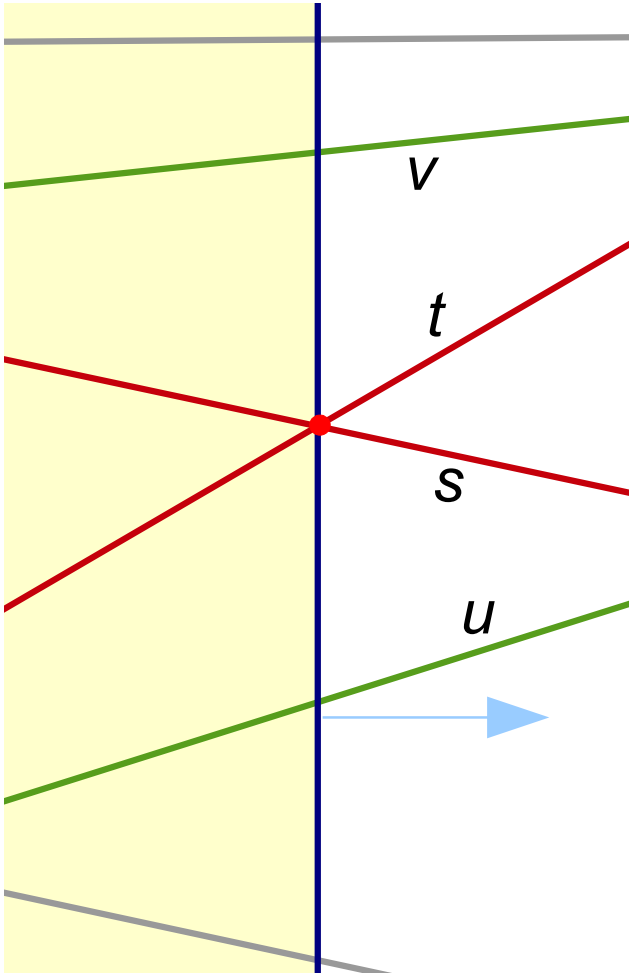

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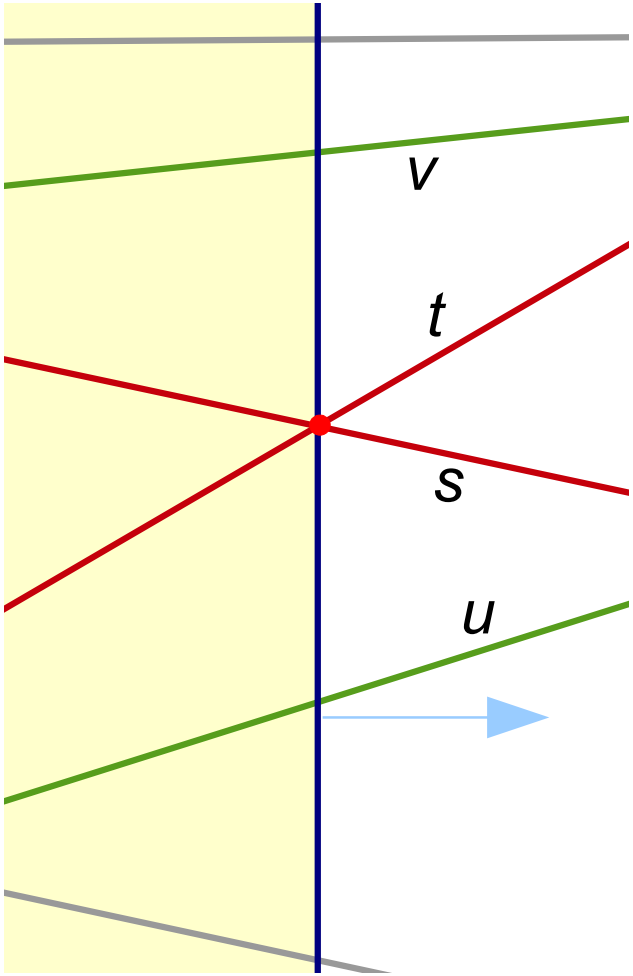

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- Segments do not overlap
- No three segments meet at a common point
- Endpoints are not shared between segments
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- Intersection events may be found more than once
- Only intersection events on the right side of the sweep line need to be considered
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- *Event queue*:
  - either priority queue or balanced search tree
  - insert / fetch in $O(\log |E|) = O(\log n^2) = O(\log n)$
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- Event queue initialization (all *ENTER*/*EXIT* events): $O(\ n \log n\ )$
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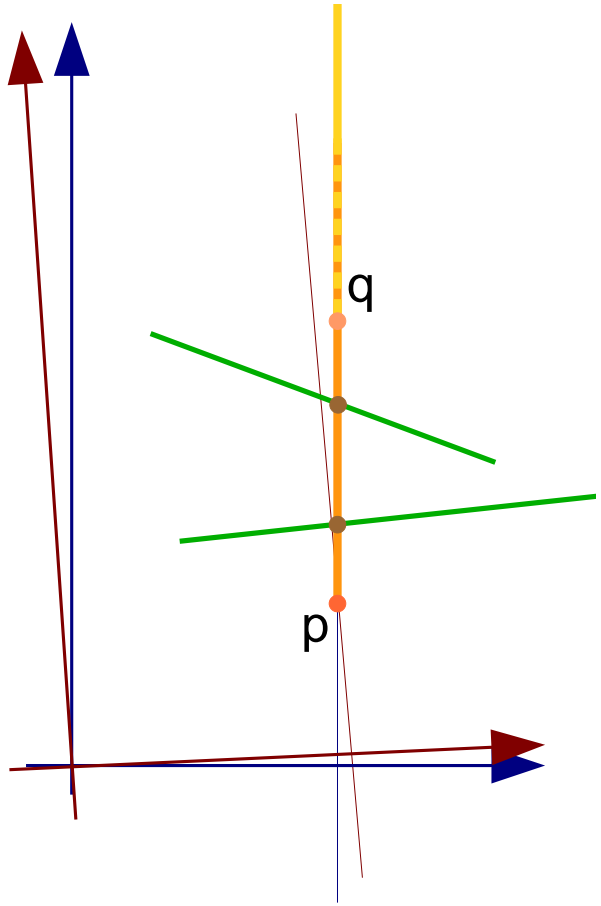

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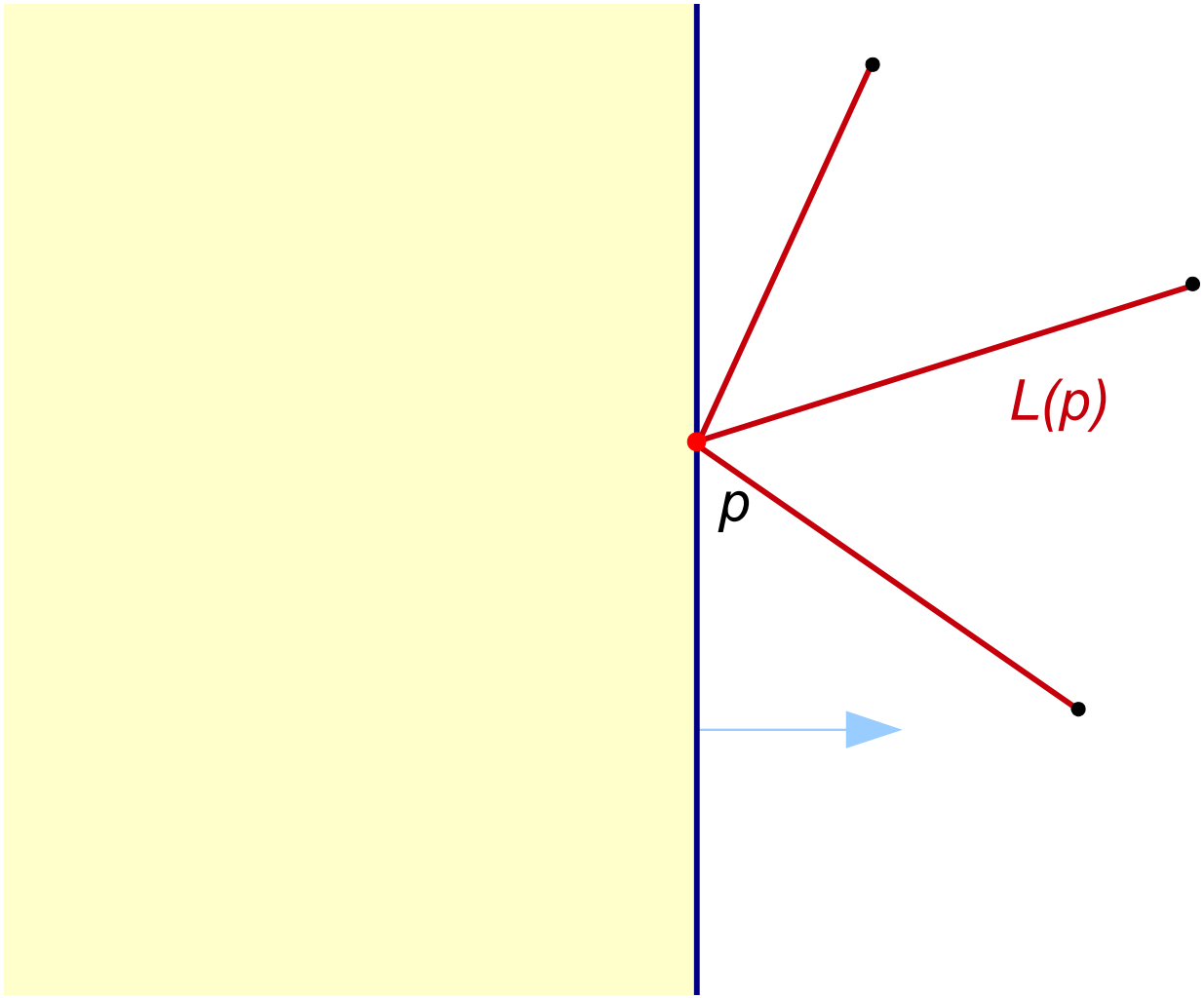

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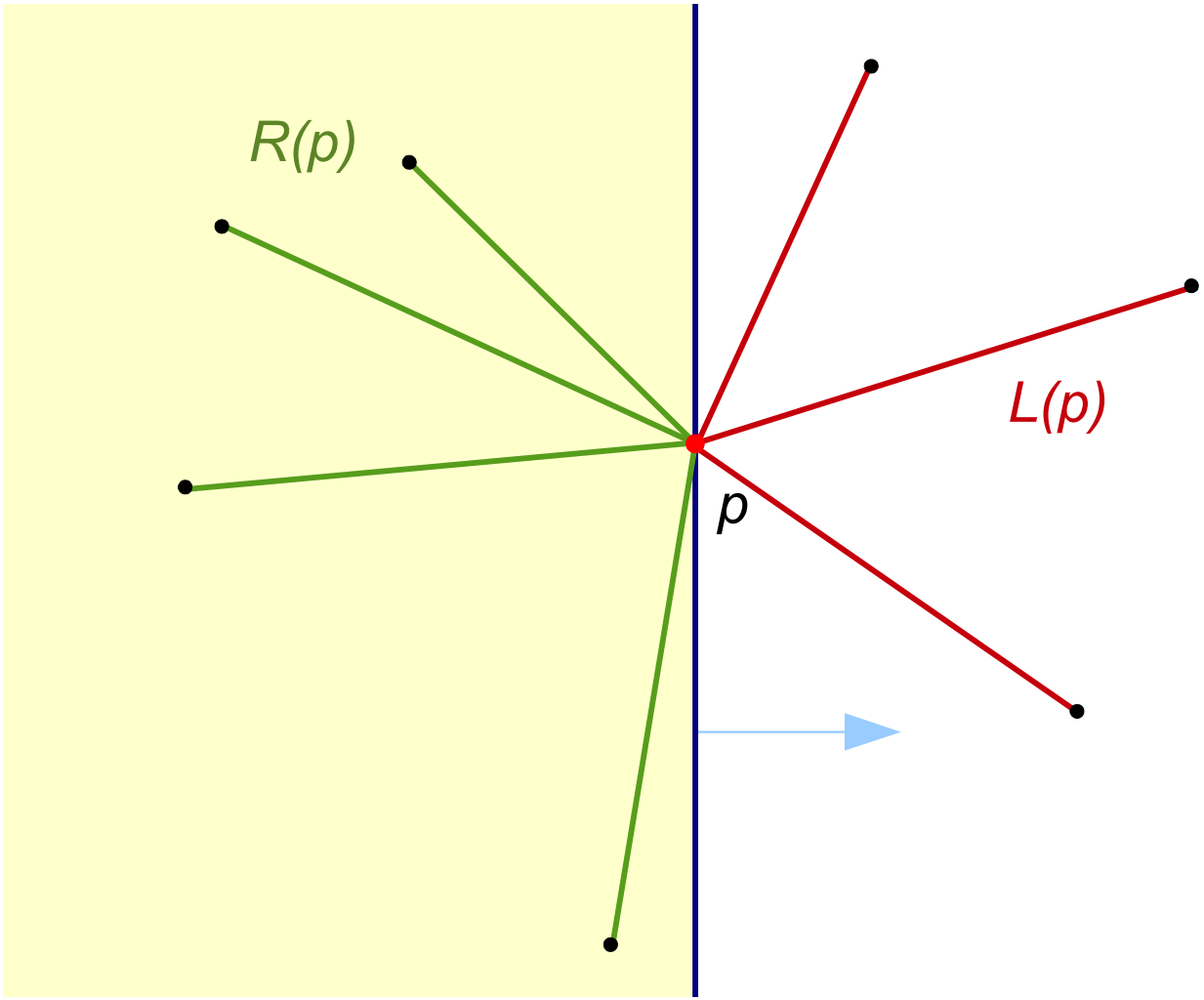

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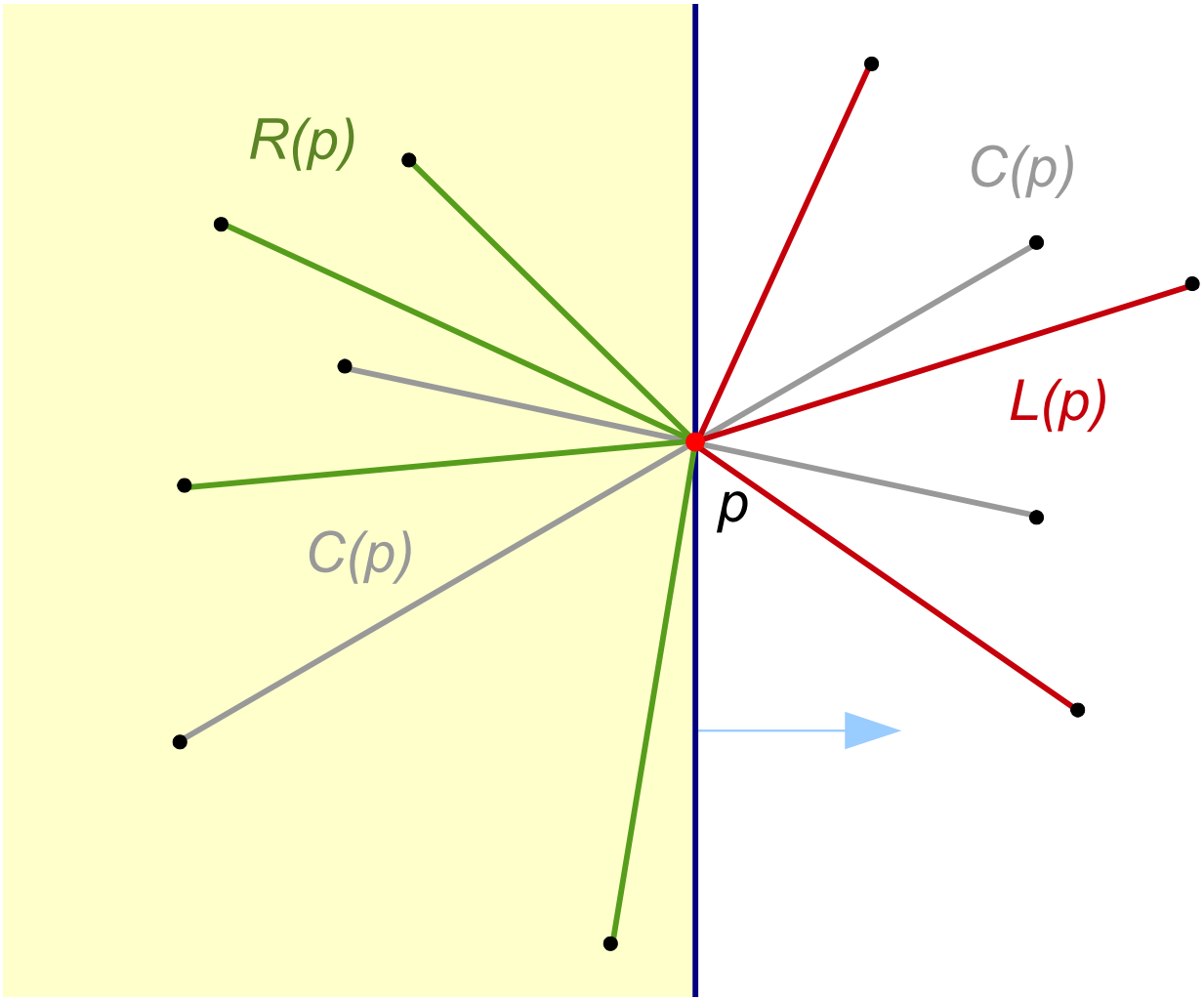

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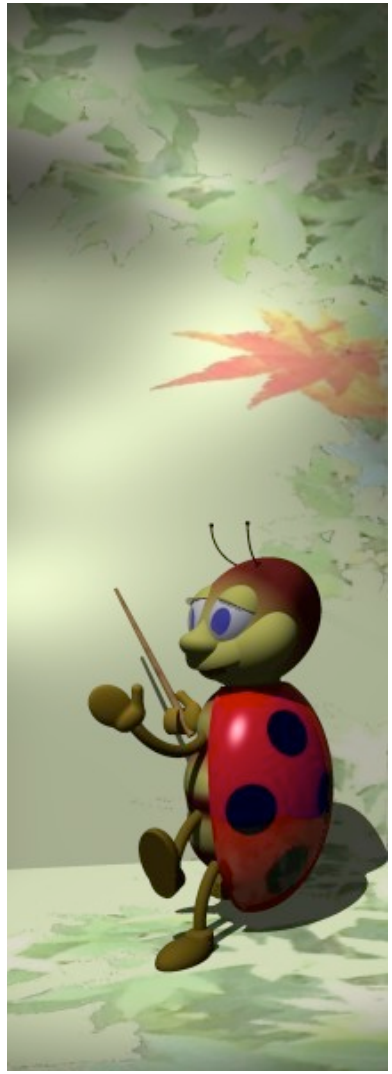

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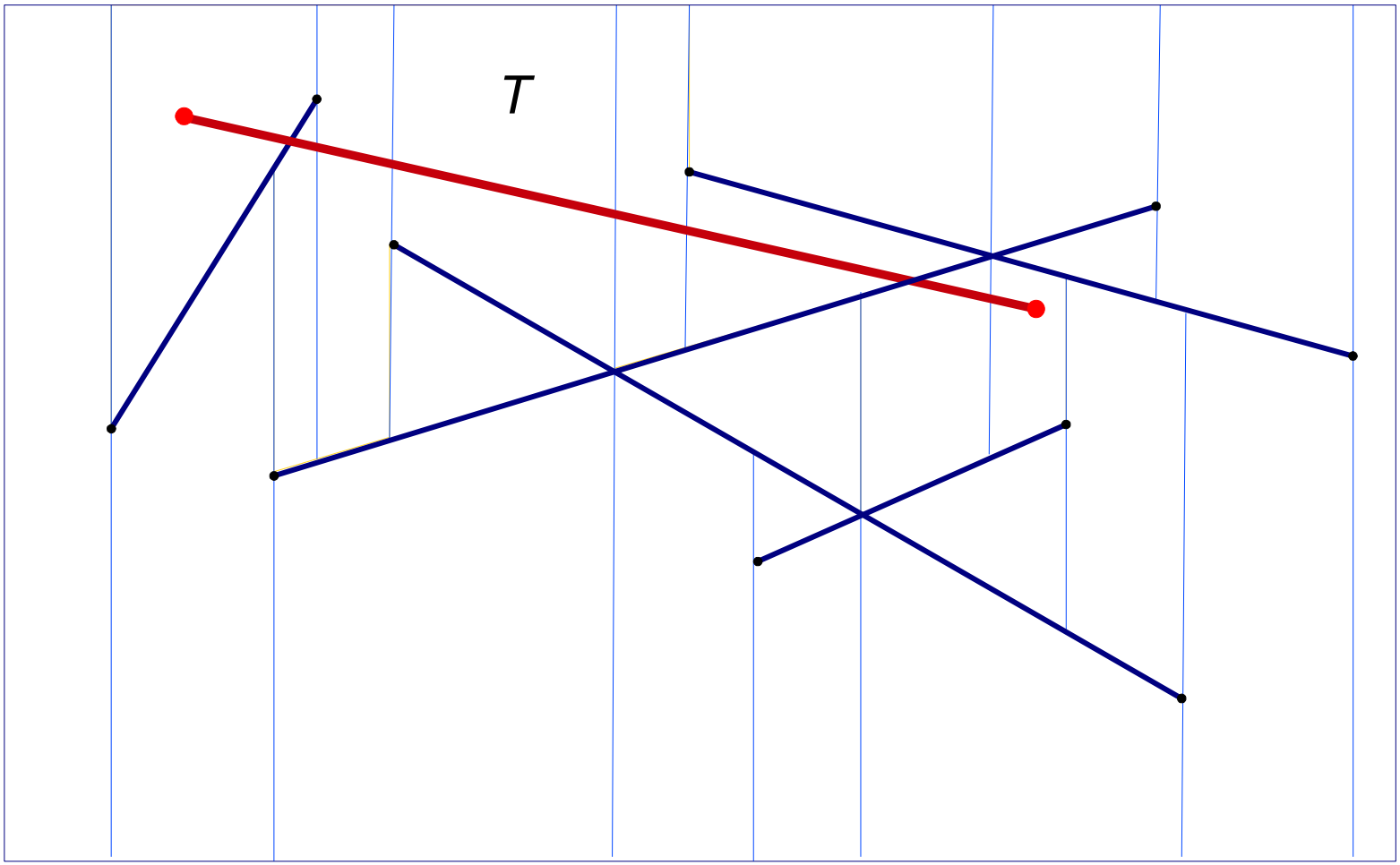
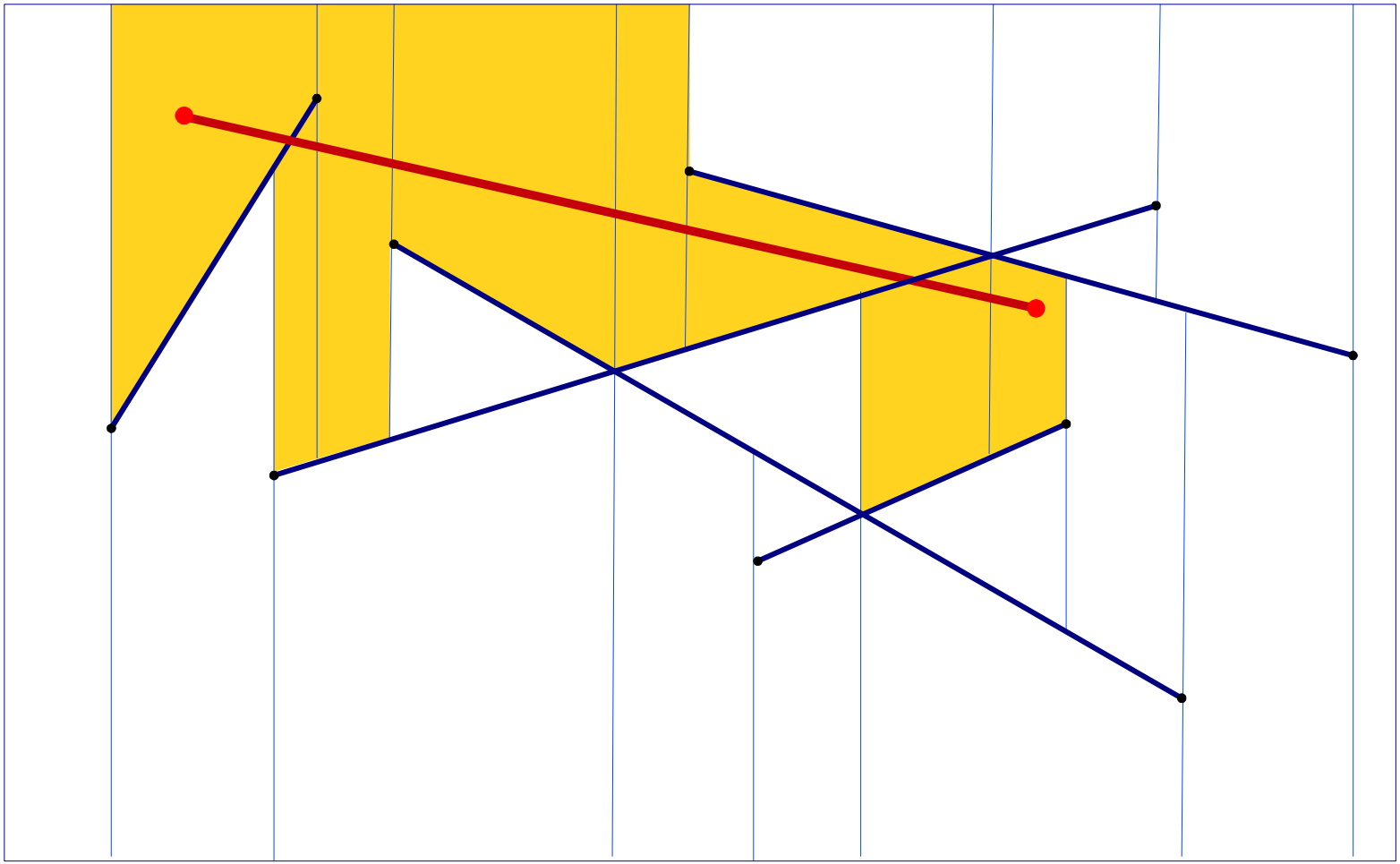
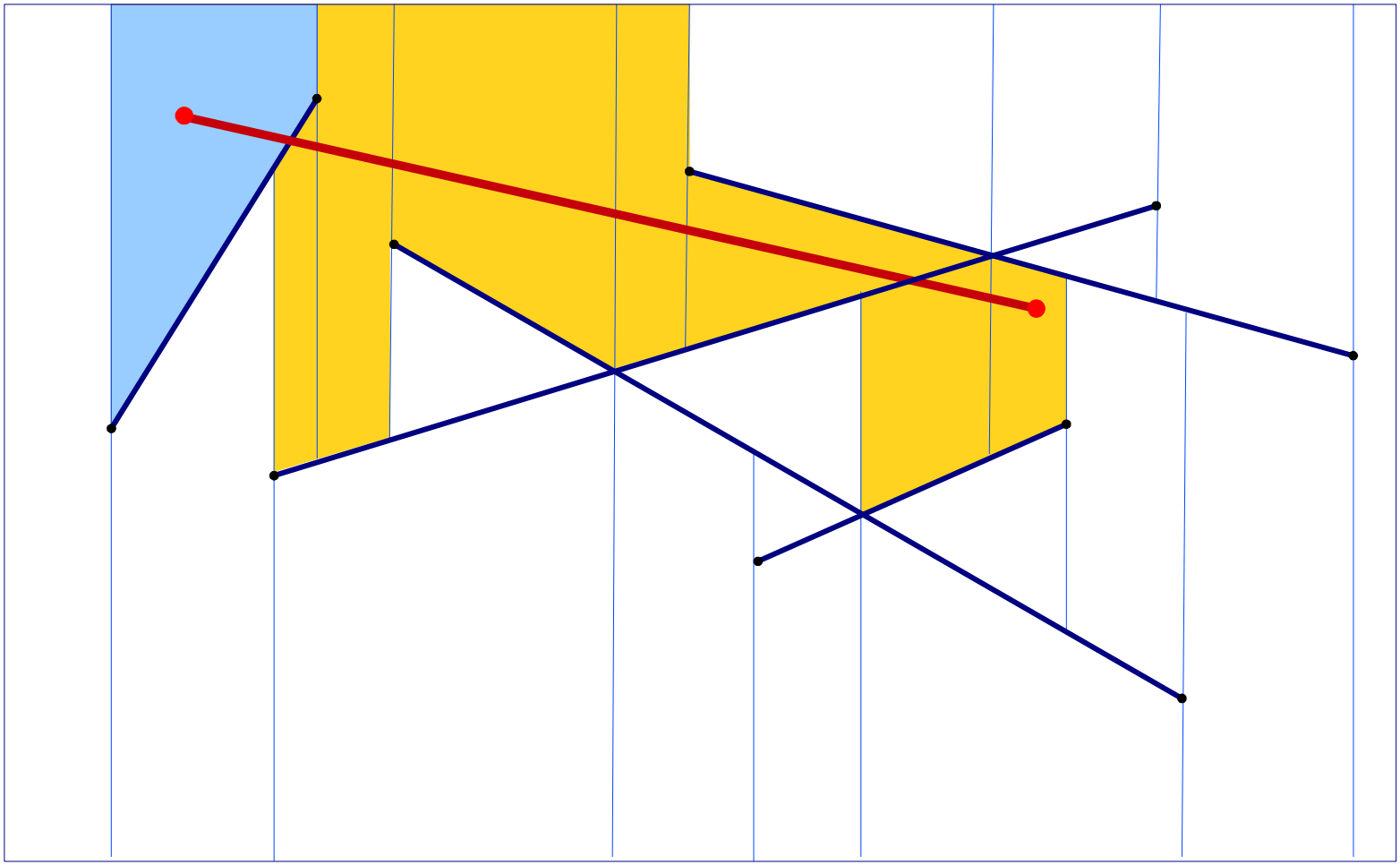


Mulmuley: Conflicts



Mulmuley: Point location



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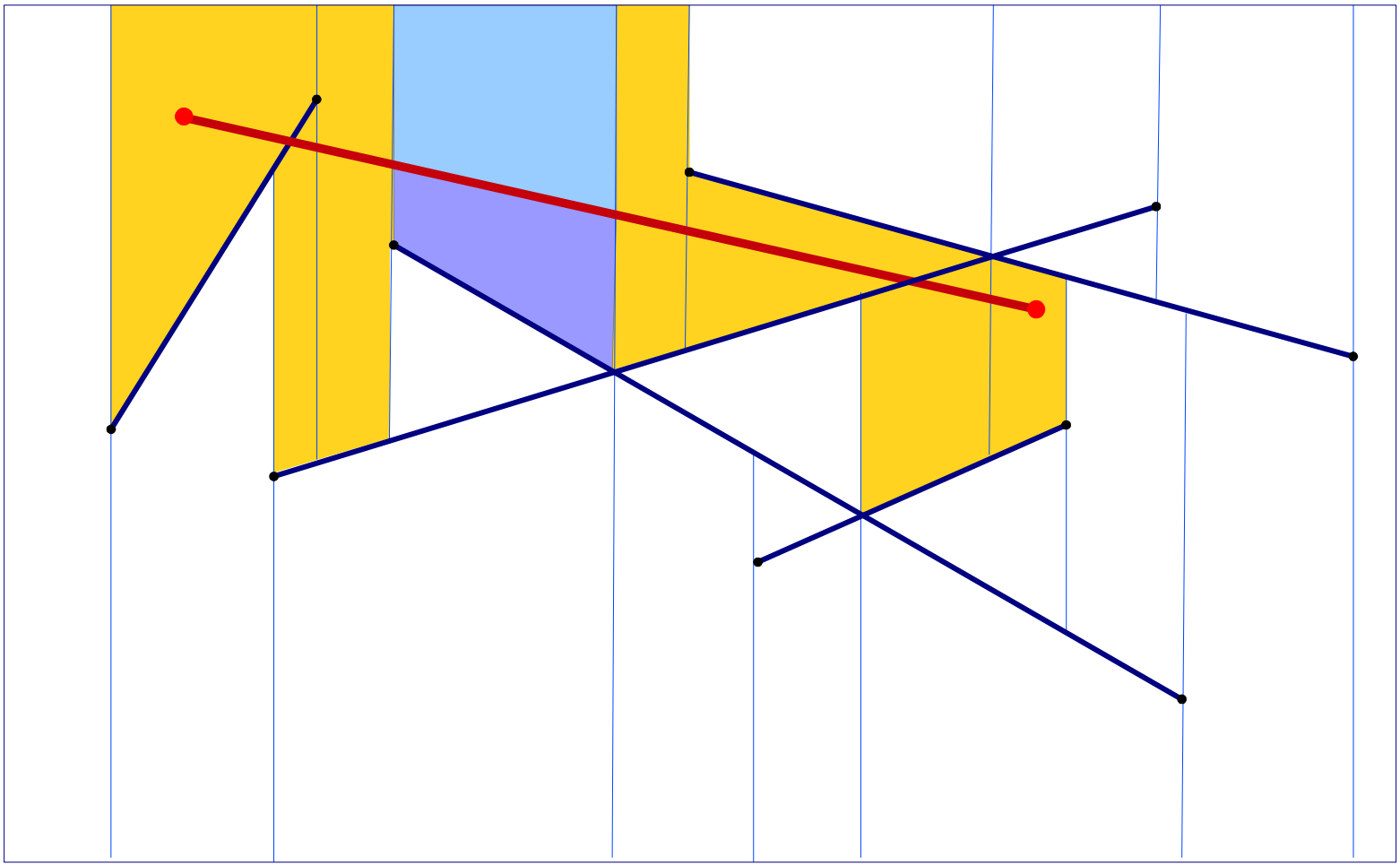

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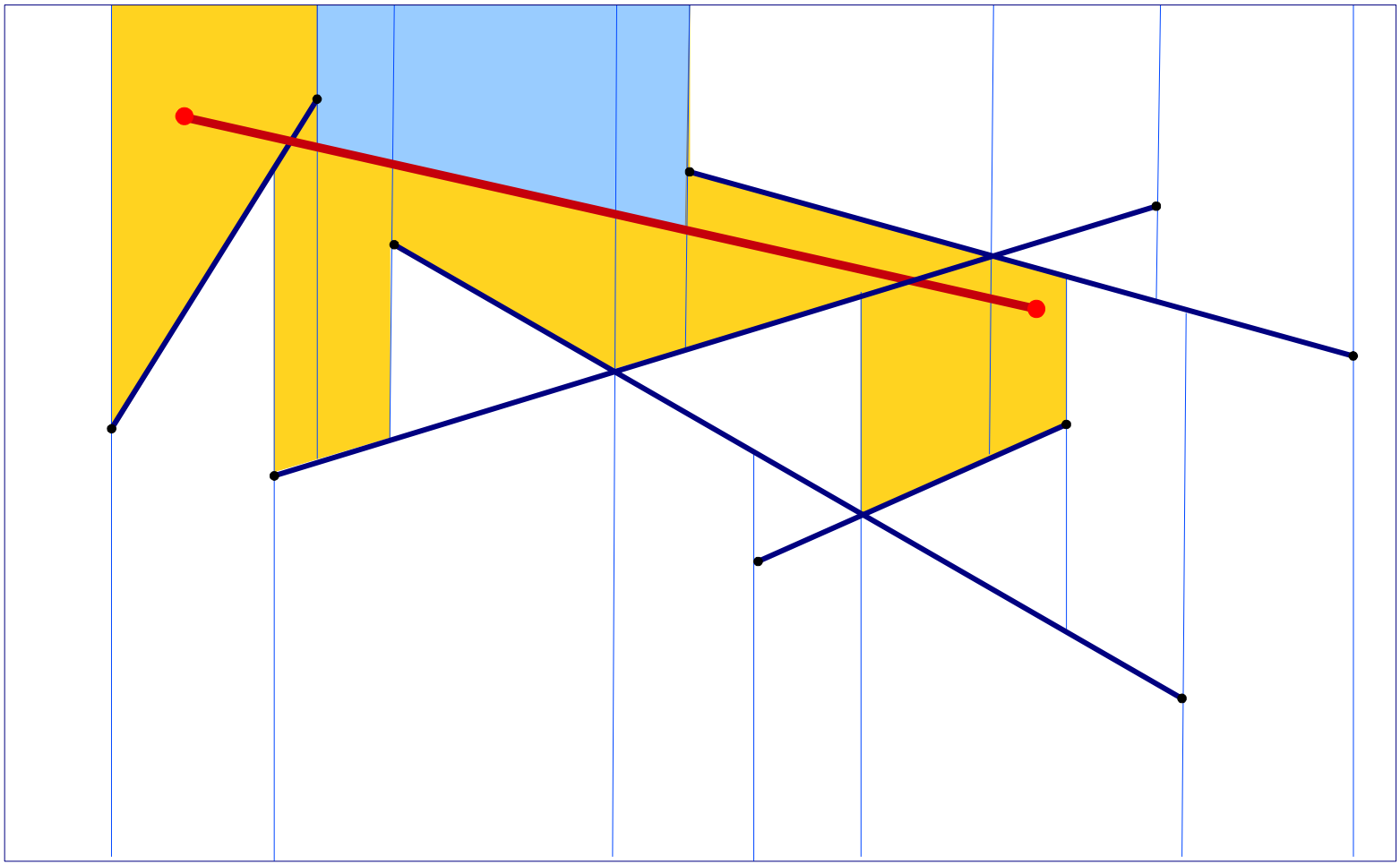

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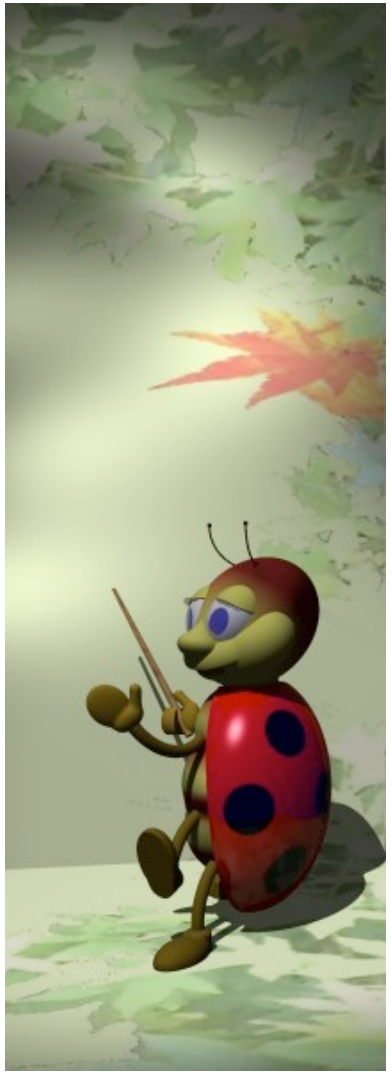

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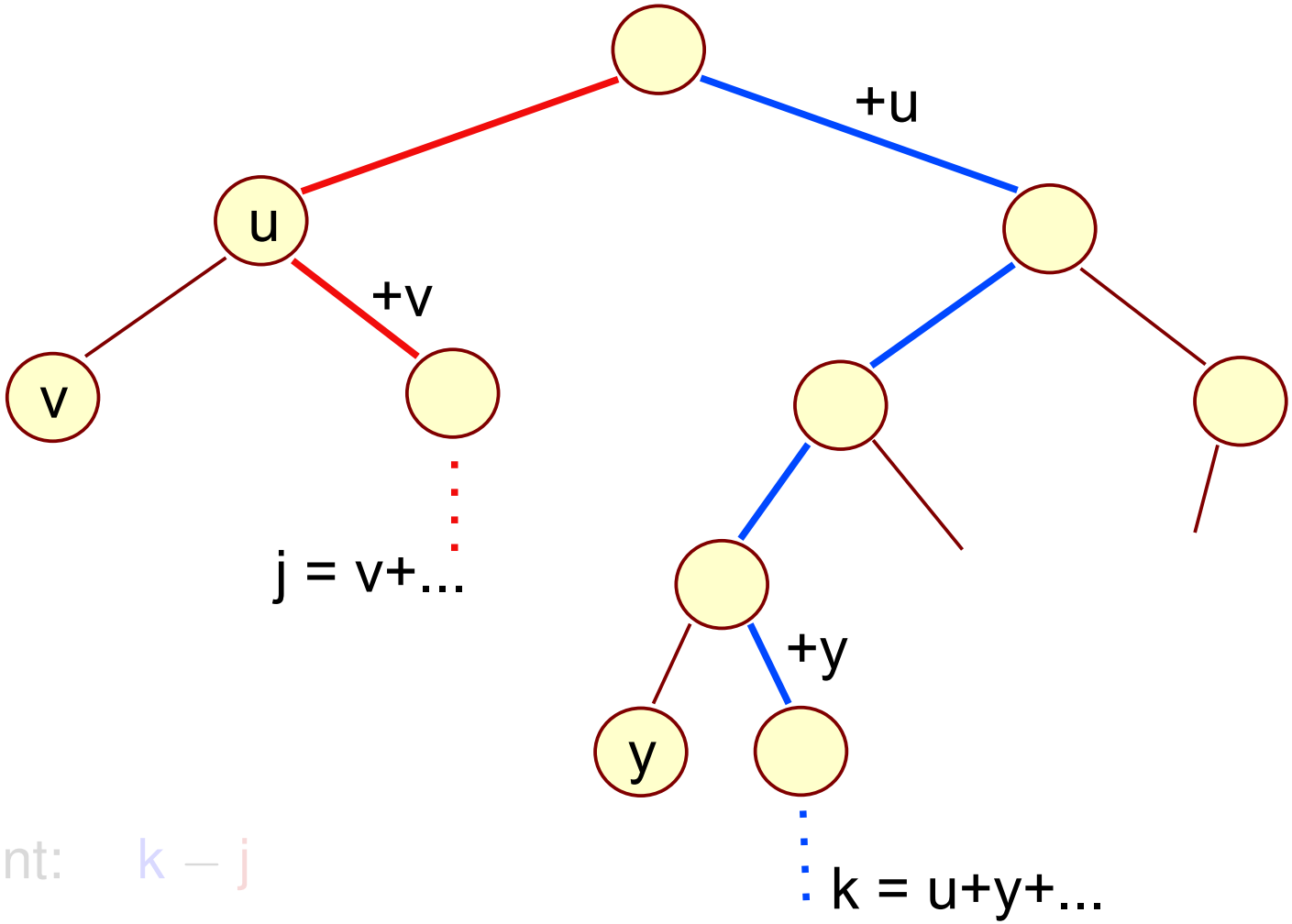

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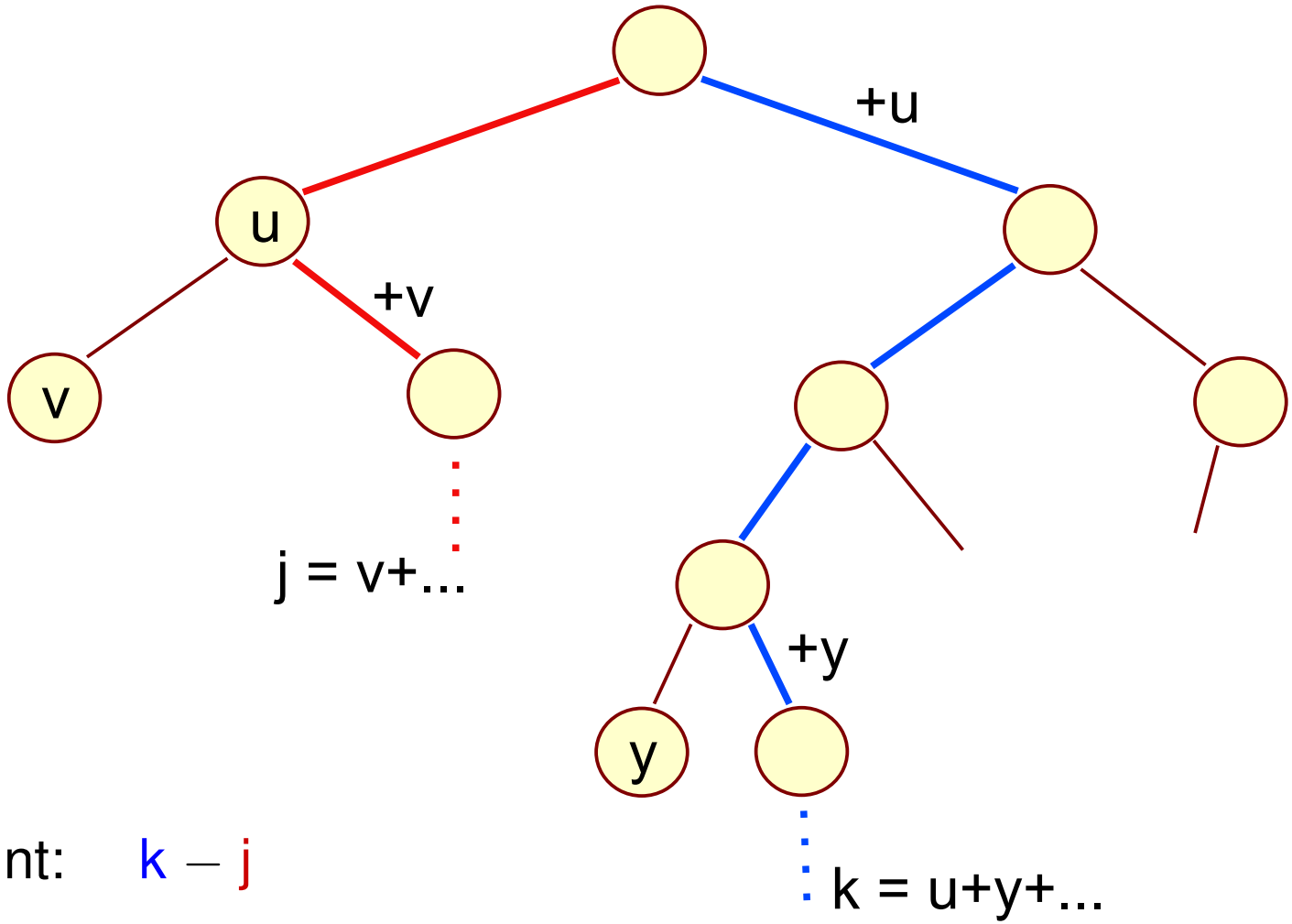

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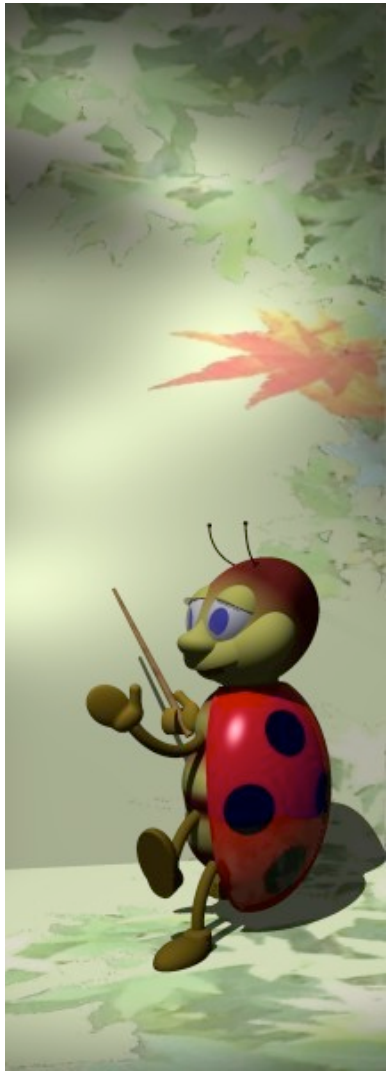

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