Convex Hull Algorithms

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Computational Geometry

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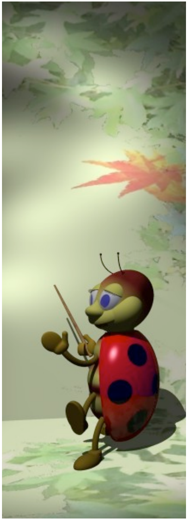
Outline

1. Incremental algorithm
   - degeneracies
   - correctness
   - computational costs
2. Divide-et-impera algorithm
   - recursive approach
   - corrrectness
   - computational costs
3. Randomized algorithm
   - conflict graph
   - corrrectness
   - computational costs



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- Easy to state…
- …and easy to solve, even efficiently
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From definition to operational definition

- Convex hull = $\bigcap \{C : P \subset C\}$, C convex region
- Convex hull = $\bigcap \{H : P \subset H\}$, H halfplane
- Convex hull = $\bigcap \{H_{pq} : p, q \in P \wedge P \subset H_{pq}\}$
- Directed segment *pq*: halfplane interior on its left
- Minimal (and finite!) set of halfplanes

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- pq bounds the convex hull $\Leftrightarrow P \setminus \{p, q\}$ on the left of pq
- `CGAL::left_turn( p1, p2, q )` $\rightarrow$ *code*
- When applied to a (small) set of *random* points it seems to work properly
- But what about the algorithm’s robustness?
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Degeneracies

- In a first stage it can be useful to ignore possible degenaracies
- Assuming some "general configuration" (e.g. no three colinear points)
- How to deal with colinear points on the boundary of the convex hull?
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- What is, exactly, the intended output representation?
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- Polygon sides in counterclockwise order
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Looking for improvements. . .

- Can this approach be improved?
- *Walking* along the convex hull boundary
- Starting vertex. . .
- Jarvis' March (1973)

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Is a better solution conceivable?

- How much room to improve performances?
- Convex hull sides must be ordered…
- Let us consider the following point set:

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- P's convex hull sorts $\{x_i : 1 \leq i \leq n\} \subset \mathbb{R}$!
- Lower bound to computational costs:

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Convex hull: Instrinsic complexity

- Jarvis' March is then interesting for $h = O(\log n)$
- But what about the (expected) number of sides?
- Clearly, it depends on point distribution
- See, e.g. (students' projects):
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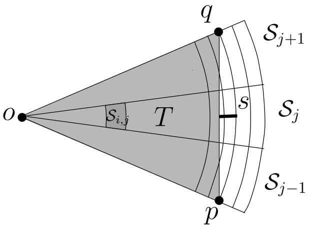

Theorem 2.3 *The expected number of vertices of the convex hull of n points, chosen uniformly and independently from the unit disk, is $O(n^{1/3})$.*

Proof: We claim that the expected area of the convex hull of n points, chosen uniformly and independently from the unit disk, is at least $\pi - O\left(n^{-2/3}\right)$.

Indeed, let D denote the unit disk, and assume without loss of generality, that $n = m^3$, where m is a positive integer. Partition D into m sectors, $\mathcal{S}_1, \ldots, \mathcal{S}_m$, by placing m equally spaced points on the boundary of D and connecting them to the origin. Let $D_1, \ldots, D_{m^2}$ denote the m^2 disks centered at the origin, such that (i) $D_1 = D$, and (ii) $Area(D_{i-1}) - Area(D_i) = \pi/m^2$, for $i = 2, \ldots, m^2$. Let r_i denote the radius of D_i, for $i = 1, \ldots, m^2$.

Let $S_{i,j} = (D_i \setminus D_{i+1}) \cap \mathcal{S}_j$, and $S_{m^2,j} = D_{m^2} \cap \mathcal{S}_j$, for $i = 1, \ldots, m^2 - 1$, $j = 1, \ldots, m$. The set $S_{i,j}$ is called the i-th *tile* of the sector $\mathcal{S}_j$, and its area is π/n, for $i = 1, \ldots, m^2$, $j = 1, \ldots, m$.

Golin & Sedgewick (1988)

- Uniform distribution of n points in a square
- Inner construction (quadrilateral) to remove most points from further consideration in $O(\, n \,)$
- Expected number of remaining candidates: $O(\, \sqrt{n} \,)$
- $h = O(\, \sqrt{n} \,) \Rightarrow$ Jarvis' March in $O(\, \sqrt{n} \cdot \sqrt{n} \,) = O(\, n \,)$
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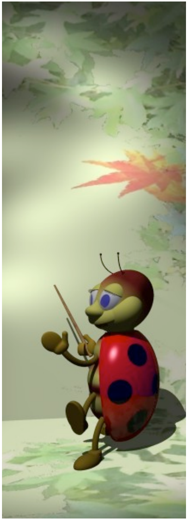

Graham's scan

- Lower hull first
- Adding points one by one, *sorted left-to-right*
- Lower hull is updated after each addition

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Scan step

- From *lower_hull*($\{\, p_1,\ p_2,\ p_3,\ \ldots,\ p_{i-1} \,\}$) ...
- ... to *lower_hull*($\{\, p_1,\ p_2,\ p_3,\ \ldots,\ p_{i-1},\ p_i \,\}$)
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Lower hull

- After the i-th step, p_1 and p_i belong to the lower hull
- p_1 and p_n belong to both the lower and upper hull
- Lower hull construction: Code

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Upper hull

- Upper hull: similar processing, by symmetry
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Degeneracies: Vertically aligned points

- *Lexicographic order*
- Geometric interpretation of lexicographic order
- “Symbolic perturbation”

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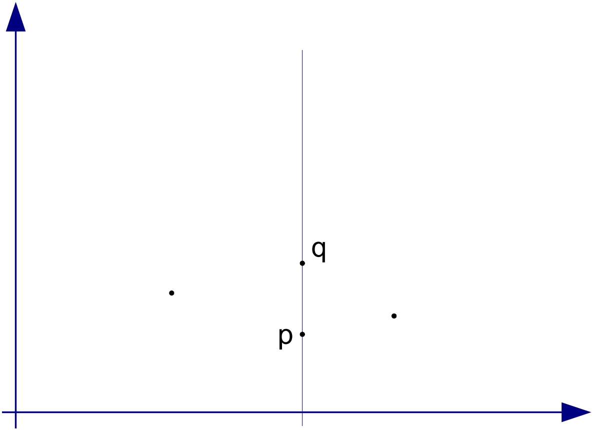

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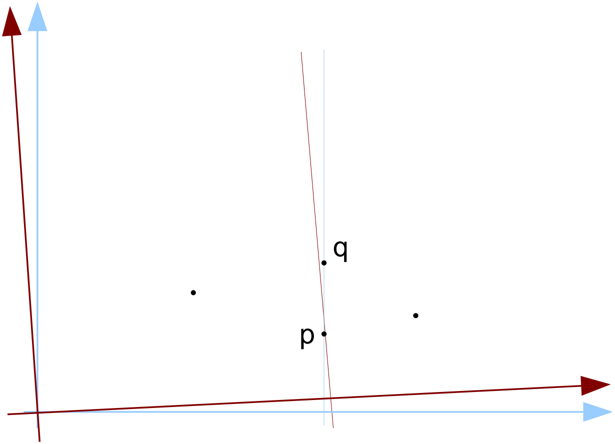

Degeneracies: Collinear hull vertices

- *Strict* left turn
- Which is the result?
- Animation (vertical/horizontal point alignments)
- Code details. . .

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Lower hull construction invariant:

- After i-th step: $P_i = \{ p_1, p_2, p_3, \ldots, p_{i-1}, p_i \}$
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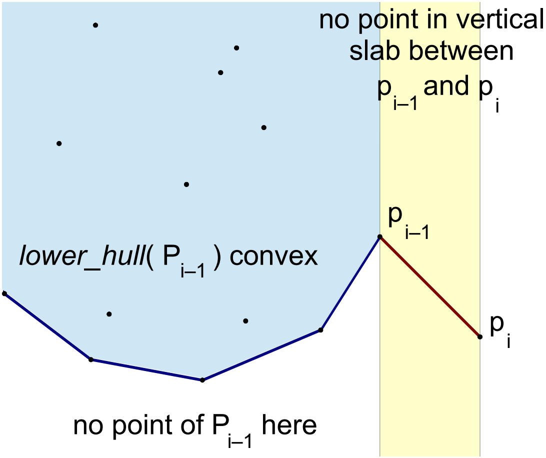

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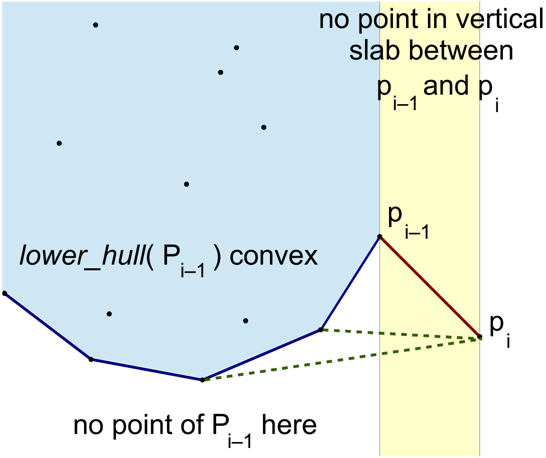

Computational costs

- Sorting points in lexicographic order: $O(\ n \log n\ )$
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- Updating lower/upper hull: $O(\ n\ )$ `while` iterations…
- … over *all* `for` iterations
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- *Sorting* $x_1, x_2, \ldots x_n$ can be reduced to *convex hull*...
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Variations on the theme. . .

- Graham's scan for "angular" order of points (around a convex hull vertex)
- Animation . . .
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Outline

1. Incremental algorithm
   - degeneracies
   - correctness
   - computational costs
2. Divide-et-impera algorithm
   - recursive approach
   - corrrectness
   - computational costs
3. Randomized algorithm
   - conflict graph
   - corrrectness
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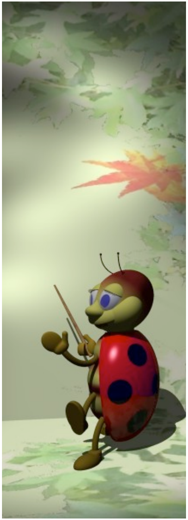

Preparata & Hong's recursive approach

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- Balanced bipartition through a vertical line
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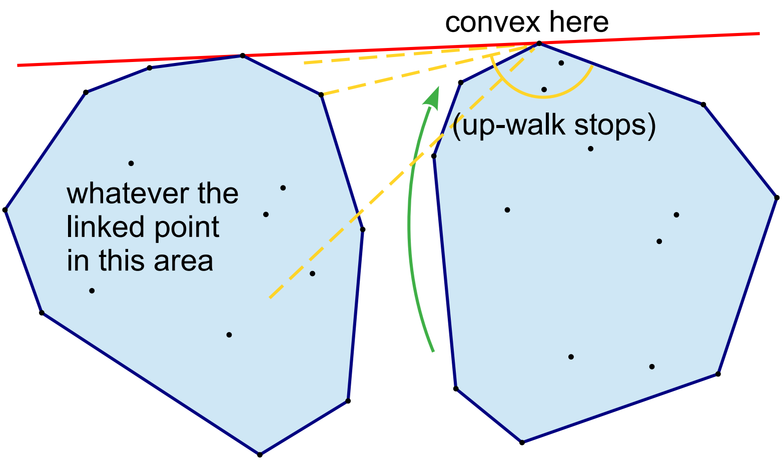

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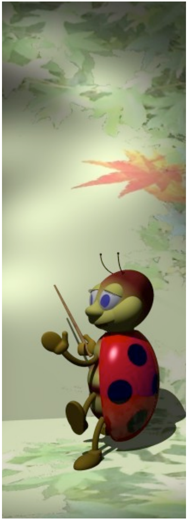

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Conflict-graph framework:

- *Objects*: problem input data — set S
- *Regions*: identified by $O(1)$ objects
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Result: Set of regions defined by the objects in S *without* conflicts with these objects

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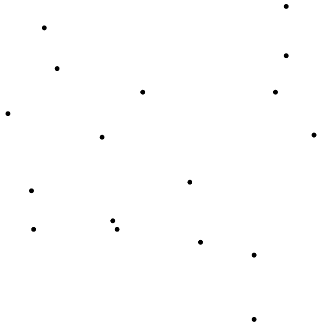

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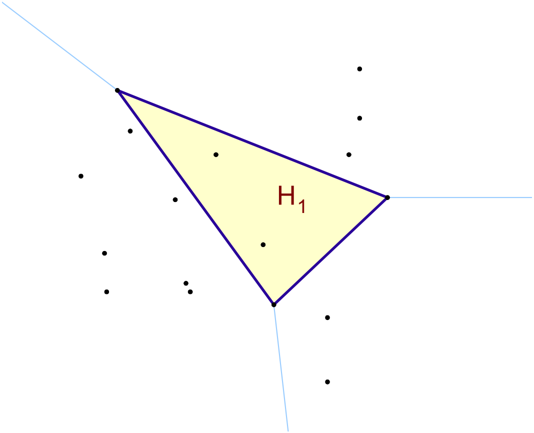

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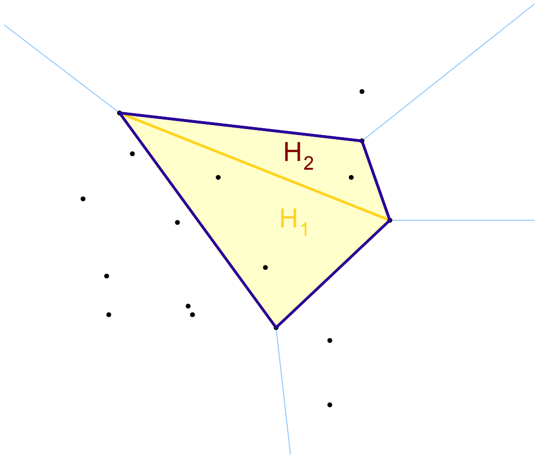

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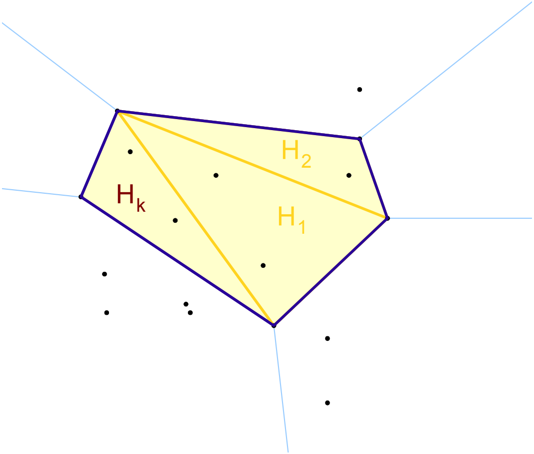

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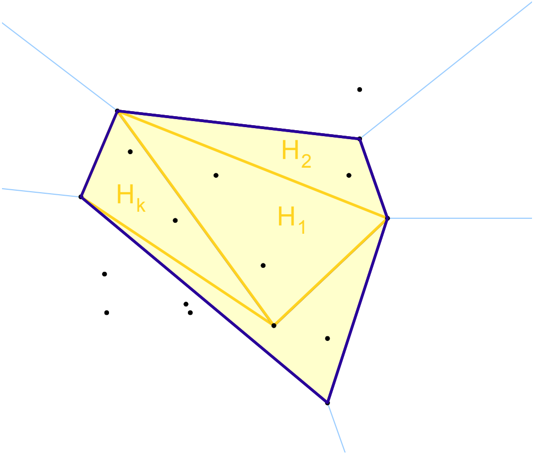

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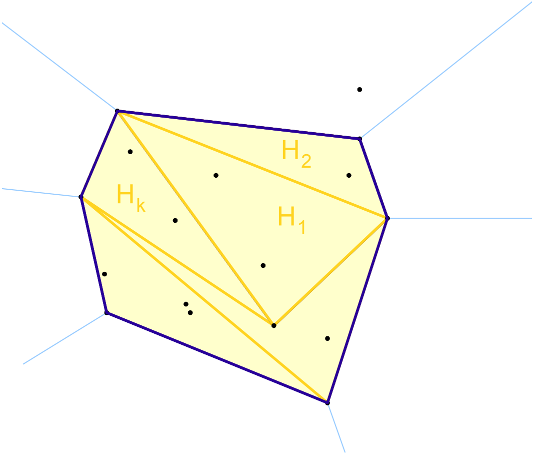

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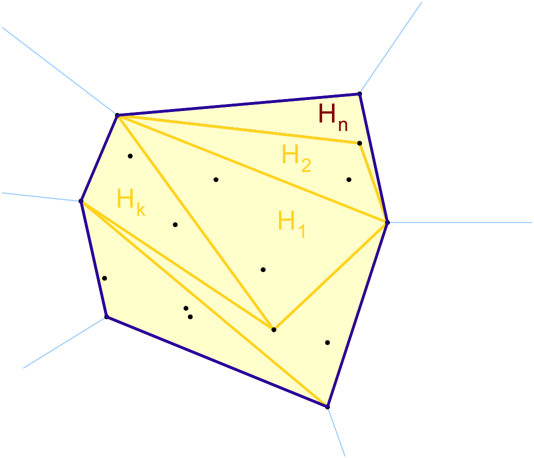

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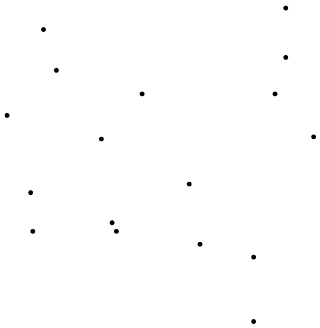

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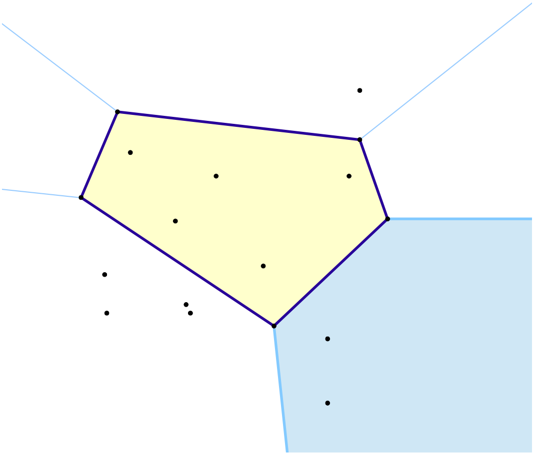

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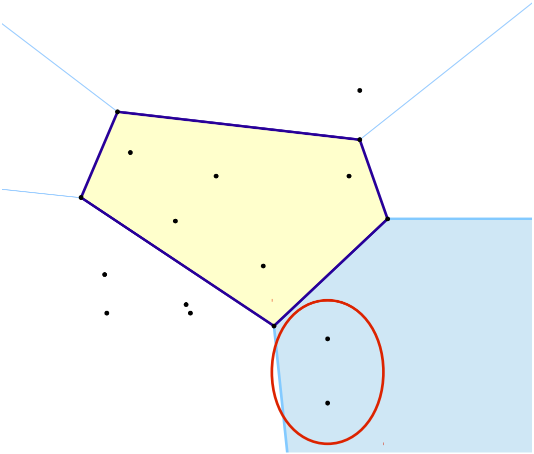

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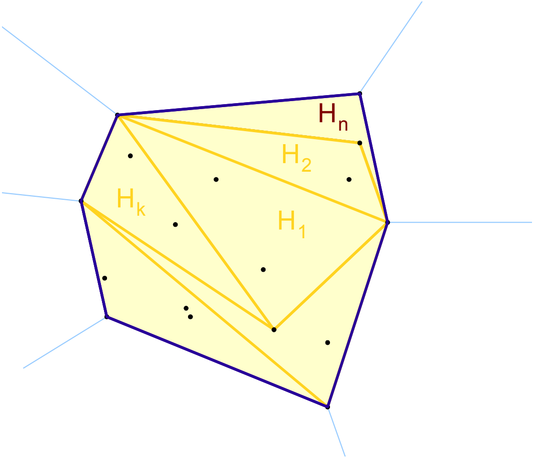

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- Expected cost
- Approach: backward analysis
- Points which do not fall outside H_k at some stage will no longer be taken into account
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- Because of the randomization, each of the k points in P_k may be the last added with equal probability $1/k$
- Let p be this last added point
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- Then, the expected cost of the k-th stage is

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since the conflict lists contain $n - k \leq n$ points overall[1]

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Backward analysis

- … the cost of re-arranging links at stage k is $O(l' + l'')$, where l', l'' are the sizes of the conflict lists of e', e''
- Then, the expected cost of the k-th stage is

$$\sum_{p \in P_k} \frac{1}{k} \, O(l' + l'') \quad = \quad \frac{1}{k} \sum_{e \in H_k} 2 \, O(l) \quad \leq \quad \frac{2}{k} \, O(n)$$

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- Expected cost of the k-th stage: $O(n)/k$
- Notice that $O(n)/k$ does not depend on the specific P_k, but the result would be the same for any $P_k \subset P$ of size k
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Log trend of the harmonic series: Why?

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Log trend of the harmonic series: Why?

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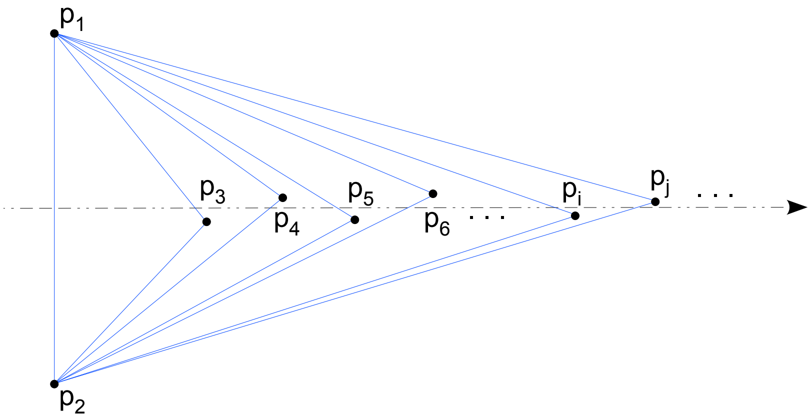
Incremental algorithm: Worst-case point sequence



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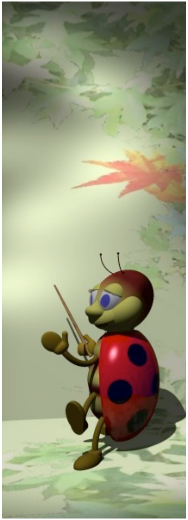

Static vs. semi-dynamic algorithms

- Conflict graph $\quad\rightarrow\quad$ influence graph
- *Influence graph*: tree-like, incrementally updated structure
- *static* algorithm $\quad\rightarrow\quad$ *semi-dynamic* algorithm
- Same computational trend, for *random* input data, provided the cost of each graph update is $O(\log n)$

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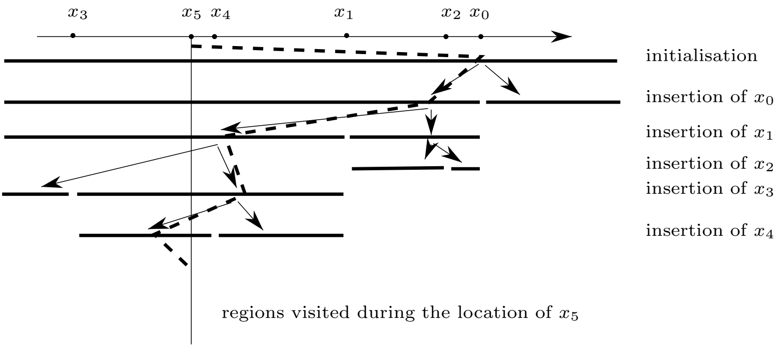
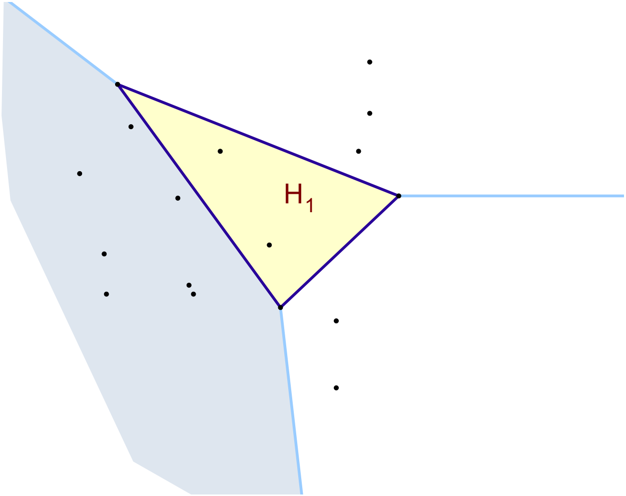
Devillers (1996)



Semi-dynamic convex hull algorithm: Embedded tree



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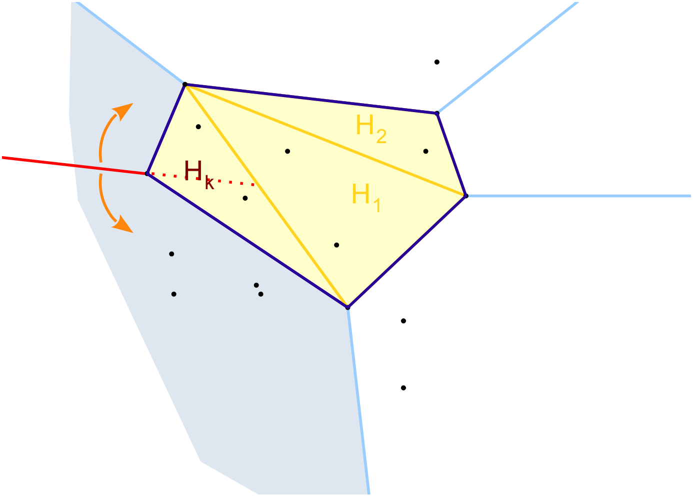

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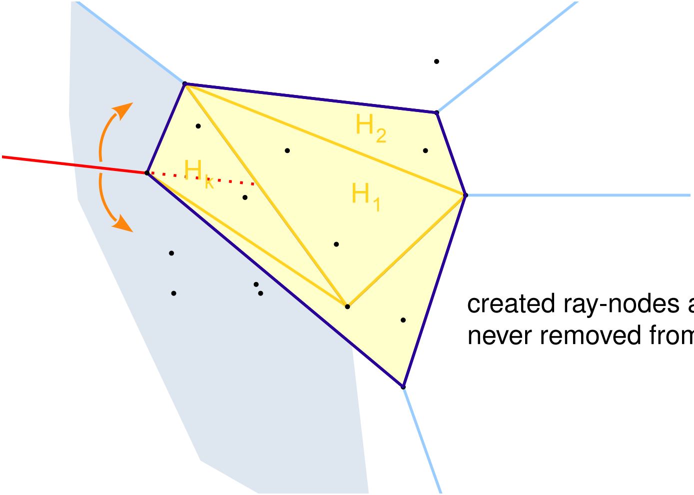

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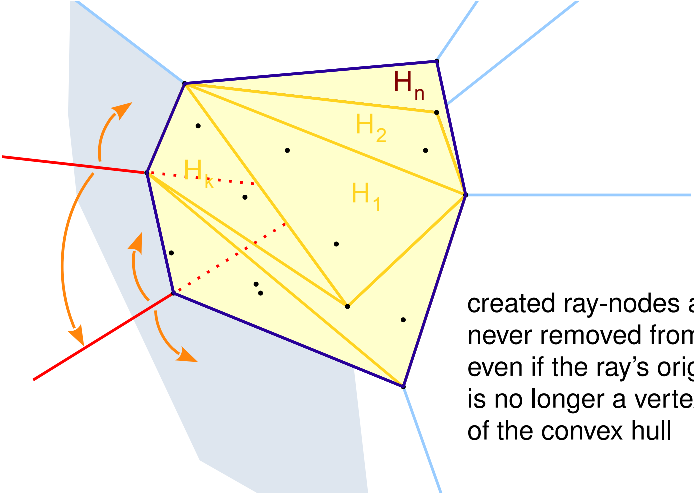
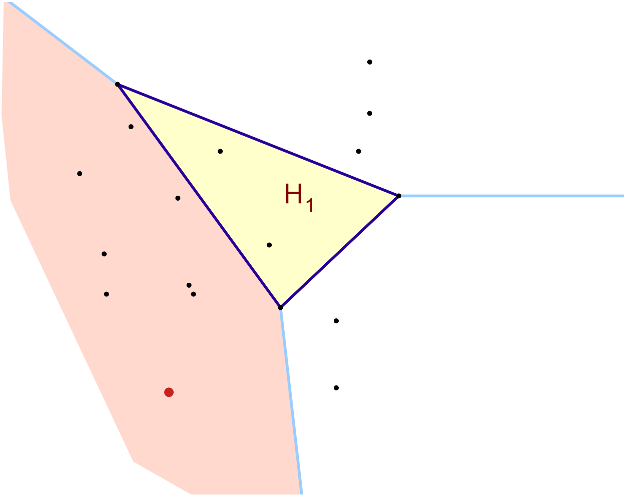


created ray-nodes are never removed from trees even if the ray's origin is no longer a vertex of the convex hull

Semi-dynamic algorithm: Point location



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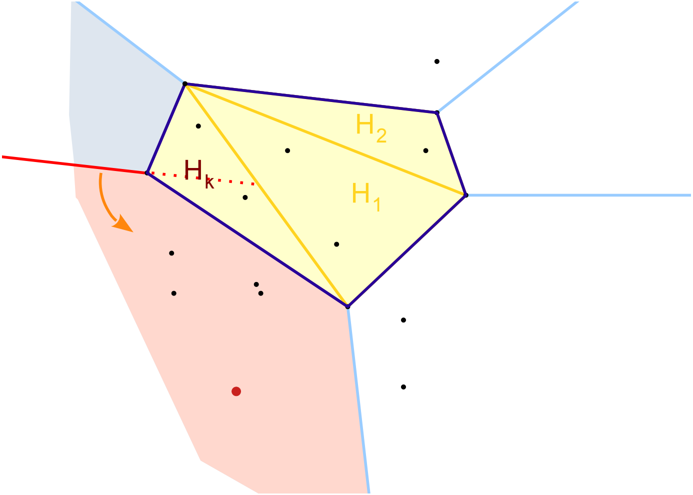

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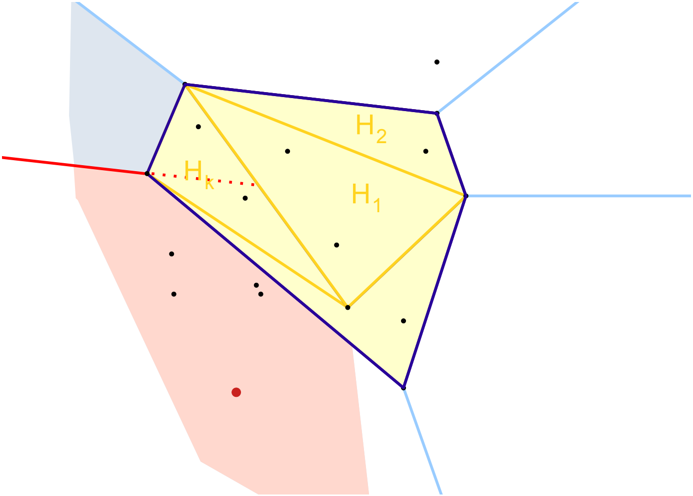

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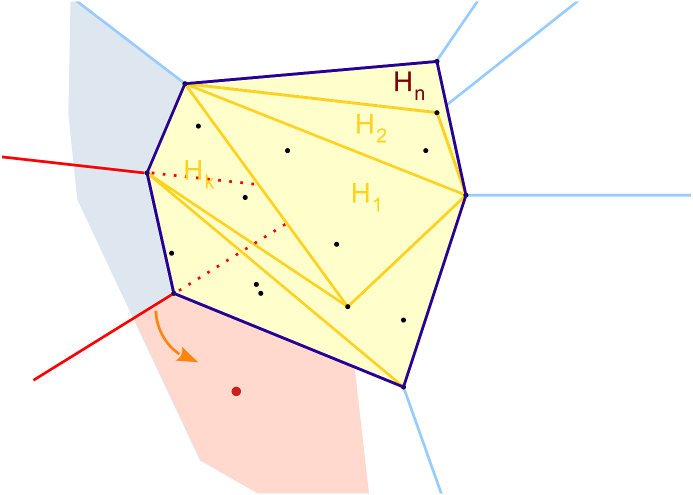

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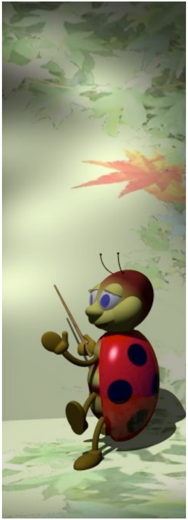

Convex hull in 3D

- No straightforward 3D generalization of Graham's scan
- Divide-et-impera approach:
  Preparata & Hong (1977), $O(n \log n)$
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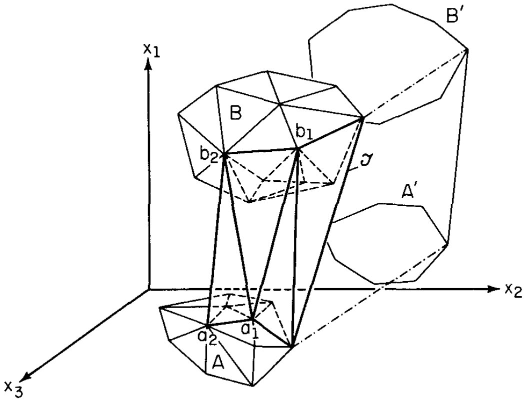
Preparata & Hong (1977)

Fig. 4. Merging two convex hulls. Construction of $\mathfrak{J}$.



Remark: Euler's formula for planar graphs

- Relation between # of vertices (V), edges (E), faces (F), and connected components (C) of a planar graph
- $$V + F = E + C + 1$$
- *Proof*: adding vertex or edge connecting two vertices
- Base case (1 vertex): $V = 1,\ F = 1,\ E = 0,\ C = 1$

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- Adding a disconnected vertex:

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- No disconnected vertices: $V \leq 2E$
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  i.e., $2E$ halfedges and ≥ 3 halfedges per face
- Hence: $E = V + F - 2 < V + 2E/3$
- $\Rightarrow$ $E < 3V$ and $F < 2V$
- To sum up:

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More about convex hull algorithms

- Jarvis' march (2D) / gift wrapping (3D)
  - simple to code
  - running time: $O(\ nh\ )$
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- Optimal output sensitive algorithms
  - E.g., Chan (1996)
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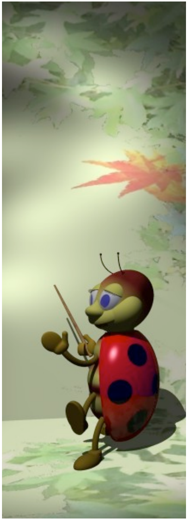

Epilogue. . .

"Convex hull is the favorite paradigm of computational geometers.

Although the description of the problem is fairly simple, its solution takes into account all aspects of computational geometry."

Olivier Devillers (1996)

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