Growth and entropy for group endomorphisms

Anna Giordano Bruno
(joint work with Pablo Spiga)

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Let G be a finitely generated group and
S a finite subset of generators of G, with $1 \notin S$ and $S = S^{-1}$.

For every $g \in G \setminus \{1\}$, let
$\ell_S(g)$ be the length of the shortest word representing g in S;
moreover, $\ell_S(1) = 0$.

For $n \geq 0$, let $B_S(n) = \{g \in G : \ell_S(g) \leq n\}$.

The growth function of G with respect to S is

$$\begin{aligned} \gamma_S \quad : \quad & \mathbb{N} \to \mathbb{N} \\ & n \mapsto |B_S(n)|. \end{aligned}$$

The growth rate of G with respect to S is

$$\lambda_S = \lim_{n\to\infty} \frac{\log \gamma_S(n)}{n}.$$

For two functions $\gamma, \gamma' : \mathbb{N} \to \mathbb{N}$,

- $\gamma \preceq \gamma'$ if $\exists\ n_0, C > 0$ such that $\gamma(n) \leq \gamma'(Cn)$, $\forall n \geq n_0$.
- $\gamma \sim \gamma'$ if $\gamma \preceq \gamma'$ and $\gamma' \preceq \gamma$.

For every $d, d' \in \mathbb{N}$, $n^d \sim n^{d'}$ if and only if $d = d'$;
for every $a, b \in \mathbb{R}_{>1}$, $a^n \sim b^n$.

Definition

A map $\gamma : \mathbb{N} \to \mathbb{N}$ is:

(a) polynomial if $\gamma(n) \preceq n^d$ for some $d \in \mathbb{N}_+$;

(b) exponential if $\gamma(n) \sim e^n$;

(c) intermediate if $\gamma(n) \succ n^d$ for every $d \in \mathbb{N}_+$ and $\gamma(n) \prec e^n$.

Definition

The finitely generated group $G = \langle S \rangle$ has:

(a) polynomial growth if γ_S is polynomial;

(b) exponential growth if γ_S is exponential;

(c) intermediate growth if γ_S is intermediate.

This definition does not depend on the choice of S; indeed, if $G = \langle S' \rangle$ then $\gamma_S \sim \gamma_{S'}$.

Properties:

- γ_S stabilizes if and only if G is finite;
- γ_S is at least polynomial if G is infinite;
- γ_S is at most exponential;
- γ_S is exponential if and only if $\lambda_S > 0$.

Problem (Milnor)

Let $G = \langle S \rangle$ be a finitely generated group.

(a) *Is γ_S either polynomial or exponential?*

(b) *Under which conditions G has polynomial growth?*

Answers:

- Grigorchuk's group of intermediate growth.

Theorem (Gromov)

A finitely generated group G has polynomial growth if and only if G is virtually nilpotent.

Let G be a group, $\phi : G \to G$ an endomorphism and $\mathcal{F}(G) = \{F \subseteq G : 1 \in F \neq \emptyset \text{ finite}\}$.

For $F \in \mathcal{F}(G)$ and $n > 0$, let $T_n(\phi, F) = F \cdot \phi(F) \cdot \ \ldots \ \cdot \phi^{n-1}(F)$.

The algebraic entropy of ϕ with respect to F is

$$H(\phi, F) = \lim_{n\to\infty} \frac{\log |T_n(\phi, F)|}{n};$$

[AKM, Weiss, Peters, Dikranjan] the algebraic entropy of ϕ is

$$h(\phi) = \sup_{F\in\mathcal{F}(G)} H(\phi, F).$$

Let $G = \langle S\rangle$ be a finitely generated group ($1 \notin S = S^{-1}$).

For $\phi = id$ and $F = S \cup \{1\}$,

$$T_n(id, F) = B_S(n) \quad \text{and} \quad H(id, F) = \lambda_S.$$

Let G be a group, $\phi : G \to G$ an endomorphism and $F \in \mathcal{F}(G)$.

The growth rate of ϕ with respect to F is

$$\begin{aligned} \gamma_{\phi,F} \quad : \quad & \mathbb{N}_+ \to \mathbb{N}_+ \\ & n \mapsto |T_n(\phi, F)|. \end{aligned}$$

Properties:

- $\gamma_{\phi,F}$ is at most exponential;
- $\gamma_{\phi,F}$ is exponential if and only if $H(\phi, F) > 0$.

If $G = \langle S \rangle$ is a finitely generated group ($1 \notin S = S^{-1}$), then

$$\boxed{\gamma_S = \gamma_{id,F}}$$

for $F = S \cup \{1\}$.

Problem

If also $G = \langle S' \rangle$, is it true that $\gamma_{\phi,S} \sim \gamma_{\phi,S'}$?

Definition

An endomorphism $\phi : G \to G$ of a group G has:

(a) polynomial growth if $\gamma_{\phi,F}$ is polynomial for every $F \in \mathcal{F}(G)$;

(b) exponential growth if $\exists\ F \in \mathcal{F}(G)$ such that $\gamma_{\phi,F}$ is exp.;

(c) intermediate growth otherwise.

This definition extends the classical one.

- ϕ has exponential growth if and only if $h(\phi) > 0$.

Definition

A group G has polynomial growth (resp., exp., intermediate) if id_G has polynomial growth (resp., exp., intermediate).

Theorem

A group G has polynomial growth if and only if every finitely generated subgroup of G is virtuallly nilpotent.

Problem

For which groups G every endomorphism $\phi : G \to G$ has either polynomial or exponential growth?

Eq., for which groups G, $h(\phi) = 0$ implies ϕ of polynomial growth?

Theorem

For G a virtually nilpotent group, no endomorphism $\phi : G \to G$ has intermediate growth.

Already known for abelian groups.

Theorem

For G a locally finite group, no endomorphism $\phi : G \to G$ has intermediate growth.

The problem remains open in general.

It is known that:

Theorem (Addition Theorem)

Let G be an abelian group, $\phi : G \to G$ an endomorphism and H a ϕ-invariant subgroup of G. Then

$$h(\phi) = h(\phi \restriction_H) + h(\phi_{G/H}),$$

where $\phi_{G/H} : G/H \to G/H$ is induced by ϕ.

The Addition Theorem does not hold in general:
consider $G = \mathbb{Z}^{(\mathbb{Z})} \rtimes_\beta \mathbb{Z}$ and $id_G : G \to G$;

- the group G has exponential growth and so $h(id_G) = \infty$;
- while $\mathbb{Z}^{(\mathbb{Z})}$ and $\mathbb{Z}$ are abelian and hence $h(id_{\mathbb{Z}^{(\mathbb{Z})}}) = 0 = h(id_{\mathbb{Z}})$.

Extending the Addition Theorem from the abelian case, we get:

Theorem

Let G be a nilpotent group, $\phi : G \to G$ an endomorphism, H a ϕ-invariant normal subgroup of G. Then

$$h(\phi) = h(\phi \restriction_H) + h(\phi_{G/H}),$$

where $\phi_{G/H} : G/H \to G/H$ is induced by ϕ.

Problem

For which classes of non-abelian groups,
does the Addition Theorem hold?

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END

Thank you!