Entropy on abelian groups
- The Bridge Theorem -

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X compact topological space, $\psi : X \to X$ continuous selfmap.
$\mathcal{U}, \mathcal{V}$ open covers of X.
$\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$.
$N(\mathcal{U})$ = the minimal cardinality of a subcover of $\mathcal{U}$.

- The entropy of $\mathcal{U}$ is $H(\mathcal{U}) = \log N(\mathcal{U})$.
- The topological entropy of ψ with respect to $\mathcal{U}$ is

$$H_{top}(\psi, \mathcal{U}) = \lim_{n\to\infty} \frac{H(\mathcal{U} \vee \psi^{-1}(\mathcal{U}) \vee \ldots \vee \psi^{-n+1}(\mathcal{U}))}{n}.$$

- The topological entropy of ψ is

$$h_{top}(\psi) = \sup\{H_{top}(\psi, \mathcal{U}) : \mathcal{U} \text{ open cover of } X\}.$$

[Adler-Konheim-McAndrew 1965]

For every compact abelian group K,

- $h_{top}(id_K) = 0$.
- The *left Bernoulli shift* ${}_K\beta : K^{\mathbb{N}} \to K^{\mathbb{N}}$ is defined by

$$ {}_K\beta(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots). $$

  Then $h_{top}({}_K\beta) = \log |K|$.

Let $f(x) = sx^n + a_1x^{n-1} + \ldots + a_n \in \mathbb{Z}[x]$ be a primitive polynomial, and let $\{\lambda_i : i = 1, \dots, n\}$ be the roots of $f(x)$.
The *Mahler measure* of $f(x)$ is

$$ m(f(x)) = \log |s| + \sum_{|\lambda_i|>1} \log |\lambda_i|. $$

Yuzvinski Formula: Let $n > 0$ and $\psi : \widehat{\mathbb{Q}}^n \to \widehat{\mathbb{Q}}^n$ a topological automorphism. Then

$$ h_{top}(\psi) = m(p_\psi(x)), $$

where $p_\psi(x)$ is the characteristic polynomial of ψ over $\mathbb{Z}$.

K compact abelian group, $\psi : K \to K$ continuous endomorphism.

Invariance under conjugation: $\psi : H \to H$ continuous endomorphism, $\xi : K \to H$ topological isomorphism and $\phi = \xi^{-1}\psi\xi$, then $h_{top}(\phi) = h_{top}(\psi)$.

Logarithmic law: $h_{top}(\phi^k) = k \cdot h_{top}(\phi)$ for every $k \geq 0$.

Continuity: $K = \varprojlim K/K_i$ with K_i closed ϕ-invariant subgroup, then $h(\phi) = \sup_{i \in I} h(\phi \restriction_{K_i})$.

Additivity for direct products: $K = K_1 \times K_2$, $\phi_i : K_i \to K_i$ endomorphism, $i = 1, 2$, then $h(\phi_1 \times \phi_2) = h(\phi_1) + h(\phi_2)$.

Addition Theorem: H closed ϕ-invariant subgroup of K, $\overline{\phi} : K/H \to K/H$ induced by ϕ. Then $h_{top}(\phi) = h_{top}(\phi \restriction_H) + h_{top}(\overline{\phi})$.

[Adler-Konheim-McAndrew 1965, Stojanov 1987]

G group, $\phi : G \to G$ endomorphism,
F a non-empty subset of G, $n > 0$.

- The *n-th ϕ-trajectory* of F is

$$T_n(\phi, F) = F \cdot \phi(F) \cdot \ldots \cdot \phi^{n-1}(F).$$

- The *algebraic entropy* of ϕ with respect to F is

$$H_{alg}(\phi, F) = \lim_{n\to\infty} \frac{\log |T_n(\phi, F)|}{n}.$$

- The algebraic entropy of ϕ is

$$h_{alg}(\phi) = \sup\{H_{alg}(\phi, F) : F \text{ non-empty finite subset of } G\}.$$

[Weiss 1974, Peters 1979, Dikranjan-GB 2009, 2011]

For every abelian group G,

- $h_{alg}(id_G) = 0$.
- The *right Bernoulli shift* $\beta_G : G^{(\mathbb{N})} \to G^{(\mathbb{N})}$ is defined by

$$\beta_G(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, \ldots).$$

  Then $h_{alg}(\beta_G) = \log |G|$.

Algebraic Yuzvinski Formula: Let $n > 0$ and $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ an endomorphism. Then

$$h_{alg}(\phi) = m(p_\phi(x)),$$

where $p_\phi(x)$ is the characteristic polynomial of ϕ over $\mathbb{Z}$.

[GB-Virili 2011]

G abelian group, $\phi : G \to G$ endomorphism.

Invariance under conjugation: $\psi : H \to H$ endomorphism, $\xi : G \to H$ isomorphism and $\phi = \xi^{-1}\psi\xi$, then $h_{alg}(\phi) = h_{alg}(\psi)$.

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Continuity: $G = \varinjlim G_i$ with G_i ϕ-invariant subgroup, then $h_{alg}(\phi) = \sup_{i \in I} h_{alg}(\phi \restriction_{G_i})$.

Additivity for direct products: $G = G_1 \times G_2$, $\phi_i : G_i \to G_i$ endomorphism, $i = 1, 2$, then $h_{alg}(\phi_1 \times \phi_2) = h_{alg}(\phi_1) + h_{alg}(\phi_2)$.

Addition Theorem: H ϕ-invariant subgroup of G, $\overline{\phi} : G/H \to G/H$ induced by ϕ. Then $h_{alg}(\phi) = h_{alg}(\phi \restriction_H) + h_{alg}(\overline{\phi})$.

[Weiss 1974, Dikranjan-Goldsmith-Salce-Zanardo 2009: torsion case]
[Peters 1979, Dikranjan-GB 2009, 2011: general case]

Theorem (Bridge Theorem)

Let G be an abelian group and $\phi : G \to G$ an endomorphism. Denote by $\widehat{G}$ the Pontryagin dual of G and by $\widehat{\phi} : \widehat{G} \to \widehat{G}$ the dual endomorphism of ϕ. Then

$$h_{alg}(\phi) = h_{top}(\widehat{\phi}).$$

[Weiss 1974: torsion case.]
[Peters 1979: countable case, automorphisms.]

- The torsion case was proved by Weiss.
- Reduction to the torsion-free abelian groups.
  [Addition Theorems]
- Reduction to finite-rank torsion-free abelian groups.
  [Bernoulli shifts, continuity for direct/inverse limits]
- Reduction to divisible finite-rank torsion-free abelian groups, that is, $\mathbb{Q}^n$.
  [Addition Theorems]
- Reduction to injective endomorphisms $\Rightarrow$ surjective.
- $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ automorphism, $\widehat{\phi} : \widehat{\mathbb{Q}}^n \to \widehat{\mathbb{Q}}^n$ topological automorphism.
  [Algebraic Yuzvinski Formula and Yuzvinski Formula]

G locally compact group, μ Haar measure on G,
$\phi : G \to G$ continuous endomorphism; $n > 0$.
$\mathcal{C}(G)$ = the family of compact neighborhoods of e_G; $K \in \mathcal{C}(G)$.
The n-th ϕ-cotrajectory of K is
$C_n(\phi, K) = K \cap \phi^{-1}(K) \ldots \cap \phi^{-n+1}(K)$.

- The topological entropy of ϕ is

$$h_{top}(\phi) = \sup \left\{ \limsup_{n\to\infty} \frac{-\log \mu(C_n(\phi, K))}{n} : K \in \mathcal{C}(G) \right\}.$$

[Bowen 1971, Hood 1974]

The n-th ϕ-trajectory of K is $T_n(\phi, K) = K \cdot \phi(K) \cdot \ldots \cdot \phi^{n-1}(K)$.

- The algebraic entropy of ϕ is

$$h_{alg}(\phi) = \sup \left\{ \limsup_{n\to\infty} \frac{\log \mu(T_n(\phi, K))}{n} : K \in \mathcal{C}(G) \right\}.$$

[Peters 1981, Virili 2010, Dikranjan 2011]

Theorem (Bridge Theorem)

Let G be a totally disconnected locally compact abelian group such that $\widehat{G}$ is totally disconnected, and let $\phi : G \to G$ be a continuous endomorphism. Then

$$h_{top}(\phi) = h_{alg}(\widehat{\phi}).$$

Problem

Does the Bridge Theorem extend to all LCA groups?

K be a compact Hausdorff space, $\psi : K \to K$ homeomorphism.

- The topological Pinsker factor of (K, ψ) is the largest factor $\overline{\psi}$ of ψ with $h_{top}(\overline{\psi}) = 0$.

[Blanchard-Lacroix 1993]

G abelian group, $\phi : G \to G$ endomorphism.

- The Pinsker subgroup of G is the largest ϕ-invariant subgroup $\mathbf{P}(G, \phi)$ of G such that $h_{alg}(\phi \restriction_{\mathbf{P}(G,\phi)}) = 0$.

[Dikranjan-GB 2010]

Let G be an abelian group, $\phi : G \to G$ an endomorphism, $K = \widehat{G}$ and $\psi = \widehat{\phi}$; $\mathbf{P} = \mathbf{P}(G, \phi)$, $\mathbf{E} = \mathbf{E}(G, \psi) := \mathbf{P}^\perp$.

$$\begin{array}{ccccc}
\mathbf{P} & \hookrightarrow & G & \twoheadrightarrow & G/\mathbf{P} \\
\downarrow {\scriptstyle \phi\restriction_{\mathbf{P}}} & & \downarrow {\scriptstyle \phi} & & \downarrow {\scriptstyle \overline{\phi}} \\
\mathbf{P} & \hookrightarrow & G & \twoheadrightarrow & G/\mathbf{P}
\end{array}$$

$$\begin{array}{ccccccccc}
\widehat{\mathbf{P}} & = & K/\mathbf{E} & \twoheadleftarrow & K & \hookleftarrow & \mathbf{E} & = & \widehat{G/\mathbf{P}} \\
\uparrow {\scriptstyle \widehat{\phi\restriction_{\mathbf{P}}}} & \cong & \uparrow {\scriptstyle \overline{\psi}} & & \uparrow {\scriptstyle \psi} & & \uparrow {\scriptstyle \psi\restriction_{\mathbf{E}}} & \cong & \uparrow {\scriptstyle \widehat{\overline{\phi}}} \\
\widehat{\mathbf{P}} & = & K/\mathbf{E} & \twoheadleftarrow & K & \hookleftarrow & \mathbf{E} & = & \widehat{G/\mathbf{P}}
\end{array}$$

$\overline{\psi} : K/\mathbf{E} \to K/\mathbf{E}$ is the topological Pinsker factor of (K, ψ).

G group, $\phi : G \to G$ endomorphism,
F non-empty finite subset of G;

$$\tau_{\phi,F} : \mathbb{N}_+ \to \mathbb{R}_{\geq 0} \text{ by } n \mapsto \tau_{\phi,F}(n) = |T_n(\phi, F)|.$$

For every $n \in \mathbb{N}_+$, $\tau_{\phi,F}(n) \leq |F|^n$.

- $\tau_{\phi,F}$ has *exponential growth* if there exists $b \in \mathbb{R}$, $b > 1$, such that $\tau_{\phi,F}(n) \geq b^n$ for every $n \in \mathbb{N}_+$.

$\tau_{\phi,F}$ has exponential growth $\Leftrightarrow H_{alg}(\phi, F) > 0$.

- $\tau_{\phi,F}$ has *polynomial growth* if there exists $P_F(X) \in \mathbb{Z}[X]$ such that $\tau_{\phi,F}(n) \leq P_F(n)$ for every $n \in \mathbb{N}_+$.

$\tau_{\phi,F}$ has polynomial growth $\Rightarrow H_{alg}(\phi, F) = 0$.

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- $\tau_{\phi,F}$ has *polynomial growth* if there exists $P_F(X) \in \mathbb{Z}[X]$ such that $\tau_{\phi,F}(n) \leq P_F(n)$ for every $n \in \mathbb{N}_+$.

$\tau_{\phi,F}$ has polynomial growth $\Rightarrow H_{alg}(\phi, F) = 0$. ($\Leftarrow$?)

The abelian case

G abelian group, $\phi : G \to G$ endomorphism.

- The Pinsker subgroup of G is the largest ϕ-invariant subgroup $\mathbf{P}(G, \phi)$ of G such that $h_{alg}(\phi \restriction_{\mathbf{P}(G,\phi)}) = 0$;
- $\mathrm{Pol}(G, \phi)$ is the largest ϕ-invariant subgroup of G such that $\tau_{\phi\restriction_{\mathrm{Pol}(G,\phi)},F}$ has polynomial growth for every non-empty finite subset F of $\mathrm{Pol}(G, \phi)$;

Theorem

$\mathrm{Pol}(G, \phi) = \mathbf{P}(G, \phi)$.

Theorem (Dichotomy Theorem)

- $H_{alg}(\phi, F) = 0$ *if and only if* $\tau_{\phi,F}$ *has polynomial growth;*
- $H_{alg}(\phi, F) > 0$ *if and only if* $\tau_{\phi,F}$ *has exponential growth.*

The general case

G finitely generated group, S (symmetric) set of generators.
$\tau_{id_G,S}$ is the growth of G; it does not depend on the choice of S.
[Schwarzc 1965, Milnor 1968]

Problem (Milnor 1968)

Does G have either polynomial or exponential growth?

[Wolf 1968, Tits 1972, Bass 1972, ...]

Theorem (Gromov 1981)

G has polynomial growth if and only if G is virtually nilpotent.

Example (Grigorchuk 1983)

There exists G with intermediate growth.

Problem

Study the growth of $\tau_{\phi,F}$ in the general case.

The Pinsker subgroup does not always exist in the general case.

Problem

For which groups does the Pinsker subgroup exist?

Problem

Extend the results on the Pinsker subgroup and the growth from the discrete abelian case to the general case of continuous endomorphisms of locally compact abelian groups.