Entropy functions

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For a category $\mathfrak{X}$, consider the category $\mathbf{Flow}_{\mathfrak{X}}$ of flows in $\mathfrak{X}$.

- Objects: (X, ϕ), where X is an object in $\mathfrak{X}$ and $\phi : X \to X$ a morphism in $\mathfrak{X}$.
- Morphisms: $u : (X, \phi) \to (Y, \psi)$, where $u : X \to Y$ in $\mathfrak{X}$ is such that $\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ Y & \xrightarrow[\psi]{} & Y \end{array}$ in $\mathfrak{X}$ commutes.

Problem

How to define an entropy function in $\mathbf{Flow}_{\mathfrak{X}}$*?*

We consider the case of $\mathbf{Flow}_R$, that is, $\mathbf{Flow}_{\mathfrak{X}}$ with $\mathfrak{X} = \mathbf{Mod}_R$, R unitary commutative ring.

- $\mathbf{Flow}_R \cong \mathbf{Mod}_{R[t]}$:
  $(M,\phi) \mapsto M_\phi$, with $t \cdot m = \phi(m)$ for every $m \in M_\phi$.

Let A be a unitary commutative ring (eventually, $A = R[t]$).

Definition

An entropy function h of $\mathbf{Mod}_A$ is $h : \mathbf{Mod}_A \to \mathbb{R}_+ \cup \{\infty\}$ such that:

(A1) $h(0) = 0$ and $h(M) = h(N)$ if $M \cong N$ in $\mathbf{Mod}_A$;

(A2) if $M \in \mathbf{Mod}_A$ and $N \leq M$, then
$h(M) = 0$ if and only if $h(N) = 0 = h(M/N)$;

(A3) if $M \in \mathbf{Mod}_A$ and $M = \varinjlim\{M_j \leq M : j \in J\}$, then
$h(M) = 0$ if and only if $h(M_j) = 0$ for every $j \in J$.

A functor $\mathbf{r} : \mathbf{Mod}_A \to \mathbf{Mod}_A$ is a *preradical* if $\mathbf{r}(M) \leq M$ for every $M \in \mathbf{Mod}_A$, and $f(\mathbf{r}(M)) \subseteq \mathbf{r}(N)$ for every $f : M \to N$ in $\mathbf{Mod}_A$.

- $\mathbf{r}$ *radical* if $\mathbf{r}(M/\mathbf{r}(M)) = 0$ for every $M \in \mathbf{Mod}_A$;
- $\mathbf{r}$ *hereditary* if $\mathbf{r}(N) = N \cap \mathbf{r}(M)$ for every $M \in \mathbf{Mod}_A$, $N \leq M$.

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h entropy function of $\mathbf{Mod}_A$.

Definition

The Pinsker radical $\mathbf{P}_h : \mathbf{Mod}_A \to \mathbf{Mod}_A$ is defined, for every $M \in \mathbf{Mod}_A$, by

$$\mathbf{P}_h(M) = \sum \{N_j \leq M : h(N_j) = 0\}.$$

- $h(\mathbf{P}_h(M)) = 0$ and
- $\mathbf{P}_h(M)$ is the greatest submodule of M with this property.

Theorem

$\mathbf{P}_h : \mathbf{Mod}_A \to \mathbf{Mod}_A$ *is a hereditary radical.*

h entropy function of $\mathbf{Flow}_R \cong \mathbf{Mod}_{R[t]}$.

(A0) $h(0_M) = 0$ and $h(1_M) = 0$ for every $M \in \mathbf{Mod}_R$.

(A2*) $h(M) = h(N) + h(M/N)$ for every $N \leq M$ in $\mathbf{Mod}_{R[t]}$.

(A3*) If $M = \varinjlim\{M_j \leq M : j \in J\}$, then $h(M) = \sup\{h(M_j) : j \in J\}$.

(A4) $h(\phi^n) = nh(\phi)$ for every $(M, \phi) \in \mathbf{Flow}_R$.

For $M \in \mathbf{Mod}_R$, the *right Bernoulli shift* is $\beta_M : M^{(\mathbb{N})} \to M^{(\mathbb{N})}$ defined by $(x_0, \ldots, x_n, \ldots) \mapsto (0, x_0, \ldots, x_n, \ldots)$.
I ideal of R; $((R/I)^{(\mathbb{N})}, \beta_{R/I}) \in \mathbf{Flow}_R \mapsto (R/I)[t] \in \mathbf{Mod}_{R[t]}$.

(A5) $h((R/I)[t]) = r_I$ (i.e., $h(\beta_{R/I}) = r_I$), for every ideal I of R.

(A5*) $h(R[t]/J) = r_J$, for every ideal J of $R[t]$.

(R commutative Noetherian), (A1), (A2*), (A3*), (A5*) for prime ideals $\Longrightarrow$ Uniqueness of h. [Vámos]

Definition (Peters)

G abelian group, $\phi : G \to G$ endomorphism,
F non-empty finite subset of G, n positive integer.

- The *n-th ϕ-trajectory* of F is
$$T_n(\phi, F) = F + \phi(F) + \ldots + \phi^{n-1}(F).$$
- The *algebraic entropy of ϕ with respect to F* is
$$H_a(\phi, F) = \lim_{n\to\infty} \frac{\log |T_n(\phi, F)|}{n}.$$
- The algebraic entropy of ϕ is
$$h_a(\phi) = \sup\{H_a(\phi, F) : F \text{ non-empty finite subset of } G\}.$$

G abelian group, H ϕ-invariant,

$$\begin{array}{ccc} H & \xrightarrow{\phi\restriction_H} & H \\ \downarrow & & \downarrow \\ G & \xrightarrow{\phi} & G \\ \downarrow & & \downarrow \\ G/H & \xrightarrow{\overline{\phi}} & G/H \end{array}$$

(A0) $h_a(0_G) = h_a(id_G) = 0$.

(A1) $\xi : G \to H$ isomorphism, $\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ \xi\downarrow & & \downarrow\xi \\ H & \xrightarrow{\psi} & H \end{array}$ commutes, $h_a(\phi) = h_a(\psi)$.

(A2*) $h_a(\phi) = h_a(\phi\restriction_H) + h_a(\overline{\phi})$.

(A3*) $G = \varinjlim\{G_i : i \in I\}$, G_i ϕ-inv, then $h_a(\phi) = \sup_{i\in I} h_a(\phi\restriction_{G_i})$.

(A4) $h_a(\phi^k) = k \cdot h_a(\phi)$ for every $k \geq 0$.

(A5) $h_a(\beta_{\mathbb{Z}(p)}) = \log p$ for every prime p.

h_a is an entropy function of $\mathbf{Flow}_{\mathbb{Z}} \cong \mathbf{Mod}_{\mathbb{Z}[t]}$.

Uniqueness of the algebraic entropy:
h_a is the unique collection $h_a = \{h_G : G \text{ abelian group}\}$ of functions $h_G : \mathrm{End}(G) \to \mathbb{R}_+ \cup \{\infty\}$ satisfying (A1), (A2*), (A3*), (A5) and the following:

Theorem (Algebraic Yuzvinski Formula)

For $n \in \mathbb{N}$ and $\phi \in \mathrm{Aut}(\mathbb{Q}^n)$ described by $A \in GL_n(\mathbb{Q})$,

$$h_a(\phi) = \log s + \sum_{|\lambda_i|>1} \log |\lambda_i|,$$

where λ_i are the eigenvalues of A and s is the least common multiple of the denominators of the coefficients of the (monic) characteristic polynomial of A.

X compact space, $\psi : X \to X$ continuous function.

Definition (Adler, Konheim, Mc Andrew)

For $\mathcal{U}$ open cover of X, let $N(\mathcal{U}) = \min\{|\mathcal{V}| : \mathcal{V}$ subcover of $\mathcal{U}\}$.
For $\mathcal{U}$ and $\mathcal{V}$ open covers of X, $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$.
The *topological entropy of ψ with respect to $\mathcal{U}$* is

$$H_{top}(\psi,\mathcal{U}) = \lim_{n\to\infty} \frac{\log N(\mathcal{U} \vee \psi^{-1}(\mathcal{U}) \vee \ldots \vee \psi^{-n+1}(\mathcal{U}))}{n},$$

and the topological entropy of ψ is

$$h_{top}(\psi) = \sup\{H_{top}(\psi,\mathcal{U}) : \mathcal{U} \text{ open cover of } X\}.$$

Theorem

Let G be an abelian group and $\phi : G \to G$ an endomorphism, $\widehat{G}$ the Pontryagin dual of G and $\widehat{\phi} : \widehat{G} \to \widehat{G}$ the adjoint of ϕ. Then $h_a(\phi) = h_{top}(\widehat{\phi})$.

G abelian group, $\phi : G \to G$ endomorphism.

The Pinsker subgroup of G is the greatest ϕ-invariant subgroup $\mathbf{P}(G, \phi)$ of G such that $h_a(\phi \restriction_{\mathbf{P}(G,\phi)}) = 0$.

- $\mathbf{P}(G, \phi) = \mathbf{P}_{h_a}(G, \phi)$ for every (G, ϕ).

X compact space, $\psi : X \to X$ continuous function.

A *factor* $(\pi, (Y, \eta))$ of (X, ψ) is a compact space Y with $\eta : Y \to Y$ a continuous map and $\pi : X \to Y$ a continuous surjective map such that $\pi \circ \psi = \eta \circ \pi$.

- (X, ψ) admits a largest factor with zero topological entropy, called Pinsker factor. [Blanchard e Lacroix]

K compact abelian group, $\psi : K \to K$ continuous endomorphism,
$G = \widehat{K}$ and $\phi = \widehat{\psi}$,
$N \leq K$ closed ψ-invariant, $H = N^\perp \leq G$ ϕ-invariant ($N = H^\perp$).

$$
\begin{array}{ccccc}
K/N & \twoheadleftarrow & K & \hookleftarrow & N \\
\uparrow \overline{\psi} & & \uparrow \psi & & \uparrow \psi\restriction_N \\
K/N & \twoheadleftarrow & K & \hookleftarrow & N
\end{array}
\qquad
\begin{array}{ccccc}
H & \hookrightarrow & G & \twoheadrightarrow & G/H \\
\downarrow \phi\restriction_H & & \downarrow \phi & & \downarrow \overline{\phi} \\
H & \hookrightarrow & G & \twoheadrightarrow & G/H
\end{array}
$$

$\mathcal{E}(G, \psi) := \mathbf{P}(\widehat{K}, \widehat{\phi})^\perp$
(for $N = \mathcal{E}(G, \psi)$ and $H = \mathbf{P}(\widehat{K}, \widehat{\phi})$)

Theorem

$(K/\mathcal{E}(G, \psi), \overline{\psi})$ *is the topological Pinsker factor of* (K, ψ).

Definition

Let $\mathfrak{X} = \mathbf{CompGrp}$. A *contravariant entropy function* of $\mathbf{Flow}_{\mathfrak{X}}$ is a function $h : \mathbf{Flow}_{\mathfrak{X}} \to \mathbb{R}_+ \cup \{\infty\}$ such that

(C1) $h(X,\phi) = h(Y,\psi)$ if $(X,\phi) \cong (Y,\psi)$ in $\mathbf{Flow}_{\mathfrak{X}}$;

(C2) if $(X,\phi) \in \mathfrak{X}$ and (Y,ψ) is a subobject of (X,ϕ), then $h(X,\phi) = 0$ if and only if $h(Y,\phi\restriction_Y) = 0 = h(X/Y,\overline{\phi})$, where $\overline{\phi} : X/Y \to X/Y$ is the endomorphism induced by ϕ;

(C3) if $(X,\phi) \in \mathbf{Flow}_{\mathfrak{X}}$ and $X = \varprojlim\{X_j : j \in J\}$, X_j ϕ-invariant subobject of X, then $h(X,\phi) = 0$ if and only if $h(X_j,\phi\restriction_{X_j}) = 0$ for every $j \in J$.

Problem

Develop the theory of (covariant and) contravariant entropy functions in a possibly more general setting.