Università degli Studi di Udine

Facoltà di Scienze Matematiche, Fisiche e Naturali

Corso di Dottorato di Ricerca in Matematica Ciclo XX

Tesi di Dottorato di Ricerca

Cardinal invariants of topological groups and applications to κ-pseudocompact subgroups

Relatore:
Prof. Dikran Dikranjan

Dottoranda:
Anna Giordano Bruno

14 febbraio 2008

Contents

Introduction

Pseudocompactness was introduced by Hewitt with the aim to weaken compactness in the spirit of Weierstraß theorem: a Tychonov topological space X is *pseudocompact* if every real valued continuous function of X is bounded [49]. Pseudocompactness coincides with compactness for metric spaces. Pseudocompact groups were characterized by Comfort and Ross [18, Theorem 4.1] (see Theorem 2.7). Moreover pseudocompact groups are *precompact*, that is their completion is compact [18, Theorem 1.1]. All topological groups in this thesis are Hausdorff.

A relevant problem involving pseudocompact groups is that of extremality, which was introduced and studied by Comfort and co-authors since 1982 [14, 20]. The following are the main two levels of extremality.

Definition 1. [7, 16, 29] *A pseudocompact group is:*

- s-extremal *if it has no proper dense pseudocompact subgroup;*
- r-extremal *if there exists no strictly finer pseudocompact group topology.*

It was immediately observed that every pseudocompact metrizable (so compact) group is s- and r-extremal. So arose the natural question of whether every pseudocompact group that is either s- or r-extremal is metrizable [14, 20]. This question, posed in 1982, turned out to be very difficult and many papers in the following twenty-five years proposed partial solutions [7, 9, 11, 12, 13, 14, 16, 20, 21, 22, 29, 43]. Recently Comfort and van Mill proved that the answer to this question is positive:

Theorem A. [22, Theorem 1.1] *For a pseudocompact abelian group G the following conditions are equivalent:*

(a) G is s-extremal;

(b) G is r-extremal;

(c) G is metrizable.

Studying this topic, we introduced singular groups in [29, Definition 1.2].

Definition 2. *A topological abelian group G is* singular *if there exists a positive integer m such that $w(mG) \leq \omega$.*

The condition given in [29, Definition 1.2] to define singular abelian groups G was

(1) there exists a positive integer m such that $G[m]$ is a G_δ-set of G,

and it was given for pseudocompact abelian groups. Anyway for pseudocompact abelian groups these properties are equivalent and they are equivalent also to a third one, that is

(2) G admits a closed torsion normal G_δ-subgroup.

The condition in (2) can be given also for non-necessarily abelian topological groups. The equivalence of these three conditions for pseudocompact abelian groups is proved by Lemma 3.35 with $\kappa = \omega$. So singularity has various aspects and this is a reason for which it is useful in different topics, as we are going to describe.

In [29] for example we proved Theorem A in the case when G is singular. Moreover we saw that singularity is a necessary condition for a pseudocompact abelian group to be either s- or r-extremal.

Recently singular groups turned out to be useful in another case: in Section 3.1 we show that a counterexample for a recent conjecture which was in a preliminary version of [21] (see http://atlas-conferences.com/cgi-bin/abstract/cats-72) can be found by making use of singular groups.

The form in (2) was already used in [32], where the problem of the characterization of compact groups admitting a proper dense subgroup with some compactness-like property was considered. We give a historical panoramic of the general problem and we focus our attention on the role of singular groups.

We begin recalling some definitions. A subgroup H of a topological group G is *strongly totally dense* if H densely intersects every closed subgroup of G, and it is *totally dense* if H densely intersects every closed normal subgroup of G [63]. These two concepts coincide in the abelian case. The totally dense subgroups of a compact group K are precisely those dense subgroups of K that satisfy the open mapping theorem [30, 31, 48] (see Theorem 1.28), according to the "total minimality criterion" (see Theorem 1.50). The groups with this property were introduced in [30] under the name totally minimal: a topological group G is *totally minimal* if for every topological group H and for every continuous surjective homomorphism $f : G \to H$, f is open.

A subgroup H of a topological group G is *essential* if H non-trivially intersects every non-trivial closed normal subgroup of G [59, 64]. A totally dense subgroup is necessarily dense and essential. A topological group G is *minimal* if there exists no strictly coarser group topology on G. A totally minimal group is minimal. A description of the dense minimal subgroups of compact groups was given in [59, 64] in terms of essential subgroups. According to the "minimality criterion" given in [31, 59, 64] (see Theorem 1.50) a dense subgroup H of a compact abelian group K is minimal if and only if H is essential in K.

A particular case of the previous mentioned problem has been largely studied, that is the description of compact groups admitting proper *totally dense* subgroups with some other compactness-like property. To better explain this problem and the known results about it, we recall some definitions of compactness-like properties which did not appear until here:

Definition 3. *A Tychonov topological space X is:*

- ω-bounded *if every countable subset of X is contained in a compact subset of X;*
- countably compact *if every countable open cover of X has a finite subcover;*
- strongly pseudocompact *if X contains a dense countably compact subspace* [2].

For topological groups we have the following chain of implications:

$$\text{compact} \Rightarrow \omega\text{-bounded} \Rightarrow \text{countably compact} \Rightarrow$$
$$\text{strongly pseudocompact} \Rightarrow \text{pseudocompact} \Rightarrow \text{precompact}.$$

It became clear that countable compactness and ω-compactness have to be immediately ruled out, as no compact group can contain a proper strongly totally dense *countably compact* subgroup [32, Theorem 1.4] (see also [25] for stronger results). So one has to limit the compactness-like property within (strong) pseudocompactness.

We study the problem of the existence of proper totally dense *pseudocompact* subgroups of compact abelian groups. In view of our previous observation and of Comfort and Ross theorem about pseudocompact groups (see Theorem 2.7), this is equivalent to look for proper totally minimal G_δ-dense subgroups of compact abelian groups.

This problem was studied for the first time by Comfort and Soundararajan [20], and they solved it in case K is a connected compact abelian group: the answer is if and only if K is non-metrizable.

A topological group admitting some dense pseudocompact subgroup is necessarily pseudocompact (see Corollary 2.14 with $\kappa = \omega$). A necessary condition for a topological group G to have a strongly totally dense pseudocompact subgroup was given in [32, Theorem 1.7]: G does not admit any closed G_δ-subgroup, that is G is *non-singular*. As noted above, this is the first time in which this concept appeared. In the case of compact abelian groups this condition, namely non-singularity, was proved to be also sufficient under the Lusin's Hypothesis, which states that $2^{\aleph_1} = 2^{\aleph_0}$ (it obviously negates the Continuum Hypothesis) [32, Theorem 1.8]. In [32, Problem 1.11] it was asked if it is possible to remove this set-theoretical condition.

In the same paper it was proved that the compact abelian groups K with non-metrizable connected component have the following *stronger* property TD_ω relaxing countable compactness: there exists a proper totally dense subgroup H of K that contains a dense ω-bounded subgroup of K [32, Theorem 1.9]. Obviously such an H is strongly pseudocompact, but need not be countably compact. So the property TD_ω appears to be stronger than having a proper totally dense pseudocompact subgroup and the question is if they are equivalent for compact abelian groups [32, Problem 1.12].

Inspired by the results in [32] and making advantage of [25], we use singular groups and projections onto special non-singular powers to get the final positive solution of the mentioned [32, Problems 1.11 and 1.12]. It is given by our following theorem, which is the first main result of this thesis.

Theorem B. *For a compact abelian group K the following conditions are equivalent:*

(a) K has a proper totally dense pseudocompact subgroup;

(b) K is non-singular;

(c) there exists a continuous surjective homomorphism of K onto S^{ω_1}, where S is a compact non-torsion abelian group;

(d) K has the property TD_ω.

The properties of singular compact abelian groups are crucial for the proof of this theorem. The main difficulty is to prove that every non-singular compact abelian group admits a projection onto an uncountable power of a compact non-torsion abelian group.

Another particular case of the general problem is that of compact abelian groups admitting some proper *essential* dense subgroup with some other compactness-like property.

It is known that there exist compact abelian groups with proper essential dense countably compact subgroups [33, 39]. Moreover the problem of the existence of connected compact abelian groups with proper essential dense countably compact subgroups is not decidable in ZFC (it is equivalent to that of the existence of measurable cardinals — see [38] and [24, Theorem 5.7]).

In analogy with Theorem B we decide to characterize compact abelian groups admitting some proper essential dense pseudocompact subgroup.

Definition 4. *A topological abelian group G is* super-singular *if there exist $p_1, \dots, p_n$ primes such that $w(p_1 \cdot \ldots \cdot p_n G) \leq \omega$.*

Each super-singular topological abelian group is singular.

In view of Theorem B, in order to characterize compact abelian groups admitting some proper essential dense pseudocompact subgroup it is sufficient to consider *only* the case when the group is singular. The following theorem gives necessary and sufficient conditions for a compact abelian group K to admit a proper essential dense pseudocompact subgroup. We give it in the "negative form" because in this way the statement is clearer.

Theorem C. *Let K be a compact abelian group. Then the following conditions are equivalent:*

(a) K admits no proper essential dense pseudocompact subgroup;

(b) *K is singular and $pT_p(K)$ is metrizable for every prime p;*

(c) *K is super-singular.*

This theorem can be considered also from another point of view, that we explain in what follows. The concept of r-extremal pseudocompact abelian group is "dual" to that of minimal pseudocompact group, in the sense that a pseudocompact abelian group G is minimal if there exists no strictly coarser pseudocompact group topology on G. As an immediate corollary of Theorem A it is possible to obtain that a proper dense pseudocompact subgroup of a topological abelian group G cannot be either s- or r-extremal: if H is a dense either s- or r-extremal pseudocompact subgroup of G, then H is metrizable and so compact by Theorem A; hence H is closed in G and so $H = G$ since H is also dense in G. In view of this consequence of Theorem A, it is natural to consider the problem of the characterization of pseudocompact abelian groups admitting some proper dense minimal pseudocompact subgroup. Theorem C solves this problem in the case of compact abelian groups, since a dense subgroup H of a compact abelian group K is minimal if and only if H is essential in K as noted before.

There is a very natural generalization of pseudocompactness given by Kennison.

Definition 5. [54] *Let κ be an infinite cardinal. A Tychonov topological space X is* κ-pseudocompact *if $f(X)$ is compact in $\mathbb{R}^\kappa$ for every continuous function $f : X \to \mathbb{R}^\kappa$.*

In Chapter 2 we study κ-pseudocompact groups, characterizing them in terms of how they are placed in their completion and also from the point of view of their "big" closed normal subgroups. We show that they have properties similar to those of pseudocompact groups. Moreover we introduce the P_κ-modification of a given topology (see Definition 2.2) and Theorem 2.27 is a new result of independent interest about the P_κ-modification of the Bohr topology.

Our next aim is to generalize for κ-pseudocompactness Theorems A, B and C, that is for each infinite cardinal κ we want to characterize

(A^κ) extremal κ-pseudocompact abelian groups,

(B^κ) compact abelian groups admitting some proper totally dense κ-pseudocompact subgroup, and

(C^κ) compact abelian groups admitting some proper essential dense κ-pseudocompact subgroup.

As singular groups and their properties were useful in working with these problems in the case $\kappa = \omega$, we want to analyze singularity in relation with κ-pseudocompactness and generalize it for every infinite cardinal κ. To this aim in Chapter 3 we introduce a new cardinal invariant of topological abelian groups, that is the *divisible weight* $w_d(-)$ (see Definition 3.10). This cardinal invariant measures singularity in a very precise way and using it we introduce counterparts of singularity and non-singularity: we define and study the groups for which this cardinal invariant is as big as possible (i.e., *w-divisible* groups — see Definition 3.26) and the abelian groups for which it is smaller than a given infinite cardinal κ (i.e., *κ-singular* groups — see Definition 3.32). The latter new notion allows us to get the required generalization of Theorem B(b). In order to have the right generalization also of Theorem C(c), we introduce another cardinal invariant of topological abelian groups, namely the *super-divisible weight* (see Definition 3.48), which is strictly related to the weight and have properties similar to those of the divisible weight. The super-divisible weight permits us to introduce *super-κ-singular* abelian groups (see Definition 3.51).

The main examples of w-divisible groups are *w-divisible products* (i.e., uncountable products of metrizable compact non-torsion abelian groups) and *w-divisible powers* (i.e., uncountable powers of metrizable compact non-torsion abelian groups) — see Example 3.29. It is onto these products and powers that we project the compact abelian groups of Chapter 4.

Chapter 4 is dedicated to special product-like groups in the above sense. Indeed it is well known that for every non-metrizable compact abelian group K there exists a continuous surjective homomorphism of K onto a product $\prod_{i\in I} K_i$ of non-trivial (metrizable) compact abelian groups with $|I| = w(K)$ (see Theorem 4.1). Since we would like to generalize Theorem B for all infinite cardinals κ and in particular we need to extend appropriately condition (c), motivated by these facts, we look for continuous surjective homomorphisms of a compact group K onto products $\prod_{i\in I} K_i$ of non-trivial metrizable compact abelian groups, asking the groups K_i to be also non-torsion. In case $|I|$ is uncountable, this means that we are looking for a projection of K onto a w-divisible product. Theorem D gives a necessary and sufficient condition for a non-singular compact abelian group K to have such a projection onto a w-divisible product $\prod_{i\in I} K_i$ with $|I| = w_d(K)$.

Theorem D. *Let K be a non-singular compact abelian group. There exists a continuous surjective homomorphism of K onto a w-divisible product $\prod_{i\in I} K_i$ if and only if $\omega < |I| \leq w_d(K)$. In particular every non-singular compact abelian group K admits a continuous surjective homomorphism onto a w-divisible product of weight $w_d(K)$.*

According to Corollary 4.12 of this theorem, a compact abelian group K admits such a projection, with $|I| = w(K) > \omega$, if and only if K is w-divisible.

In the rest of Chapter 4 we consider the particular case of this theorem when all K_i coincide, that is we want to know under which conditions there exists a continuous surjective homomorphism of a non-singular compact abelian group K onto a w-divisible *power* $S^{w_d(K)}$. Theorem 4.17 shows that for compact abelian groups K admitting a continuous surjective homomorphism onto a w-divisible power $S^{w_d(K)}$ there is a remarkable trichotomy. Moreover Corollary D* gives a necessary and sufficient condition that ensures the existence of such a projection:

Corollary D*. *Let κ be an infinite cardinal. A compact abelian group K is non-κ-singular if and only if there exists a continuous surjective homomorphism of K onto S^{κ^+}, where S is a compact non-torsion abelian group.*

Theorem D and Corollary D* apply to prove the main results of Chapters 5, 6 and 7.

In Chapter 5 there is a first application of the results of Chapter 4. We consider a problem related to the characterization of the abelian groups admitting pseudocompact group topologies, which in its full generality is still a hard open question [34, Problem 0.2] (see also [8, Problem 856] and [36, Problem 892]).

If X is a non-empty set and κ is an infinite cardinal, then a set $F \subseteq X^\kappa$ is *ω-dense* in X^κ, provided that for every countable set $A \subseteq \kappa$ and each function $\varphi \in X^A$ there exists $f \in F$ such that $f(\alpha) = \varphi(\alpha)$ for all $\alpha \in A$ [6] (see also [34, Definition 2.6]).

Definition 6. [34, Definition 2.6] *If λ and $\kappa \geq \omega$ are cardinals, then* $\mathrm{Ps}(\lambda, \kappa)$ *abbreviates the sentence "there exists an ω-dense set $F \subseteq \{0,1\}^\kappa$ with $|F| = \lambda$".*

Moreover $\mathbf{Ps}(\lambda)$ *denotes the sentence "*$\mathrm{Ps}(\lambda, \kappa)$ *holds for some infinite cardinal κ".*

This set-theoretical condition is closely related to the pseudocompact group topologies: $\mathrm{Ps}(\lambda, \kappa)$ holds for some cardinals λ and $\kappa \geq \omega$ if and only if there exists a group G of cardinality λ which admits a pseudocompact group topology of weight κ [15] (see also [34, Fact 2.12 and Theorem 3.3(i)]).

Definition 7. [34, Definition 3.1(i), Theorem 3.3(ii)] *An infinite cardinal κ is* admissible *if there exists a pseudocompact group G such that $|G| = \kappa$, i.e.,* $\mathbf{Ps}(\kappa)$ *holds true.*

Hence, if G is a pseudocompact group, then $\mathrm{Ps}(|G|, w(G))$ holds. But what about the free rank $r_0(G)$ of G? Does $\mathbf{Ps}(r_0(G))$ holds whenever G is a pseudocompact abelian group (i.e., is $r_0(G)$ admissible in case G is a pseudocompact abelian group)? This question is contained in [34, Problem 9.11]. As mentioned in [34], this problem seems to be an important step for the characterization of abelian groups admitting pseudocompact group topologies. If G is torsion, the problem makes no sense because $r_0(G) = 0$ and $\mathbf{Ps}(\kappa)$ is defined for infinite cardinals κ.

[34, Theorem 3.1] proved that if G is a non-trivial connected pseudocompact abelian group, then $\mathrm{Ps}(r_0(G), w(G))$ holds. The problem is open in general. It is possible to ask also whether connectedness is a necessary condition in order that $\mathrm{Ps}(r_0(G), w(G))$ holds. Applying Theorem D, in Theorem E we generalize [34, Theorem 3.1] to w-divisible groups, which are far from being connected (while connected pseudocompact groups are w-divisible):

Theorem E. *If G is a w-divisible pseudocompact abelian group, then* $\mathrm{Ps}(r_0(G), w(G))$ *holds.*

Moreover Example 5.15 shows that w-divisibility (and so also connectedness) is not a necessary condition in order that $\mathrm{Ps}(r_0(G), w(G))$ holds, and Corollary 5.16 provides the missing equivalence at a different level (namely that of pseudocompact topologization).

From Theorem E in Corollary 5.18 we deduce that $\mathbf{Ps}(r_0(G))$ holds whenever G is a pseudocompact non-torsion abelian group, and this is precisely the answer to the mentioned [34, Problem 9.11].

In the survey [7], exposing the history of the problem of the extremality of pseudocompact abelian groups, there is a wish of extending to more general cases Theorem A, that is the final solution of the problem. On one hand it is suggested to consider the non-abelian case. On the other hand it is proposed to consider, for any pair of topological classes $\mathcal{P}$ and $\mathcal{Q}$, the problem of understanding whether every topological group $G \in \mathcal{P}$ admits a dense subgroup and/or a strictly larger group topology in $\mathcal{Q}$. This problem is completely solved by Theorem A in the case $\mathcal{P} = \mathcal{Q} =$ {pseudocompact abelian groups}. In Chapter 6 we consider and completely solve the case $\mathcal{P} = \mathcal{Q} = \{\kappa$-pseudocompact abelian groups$\}$.

We need first to generalize to κ-pseudocompact groups the definitions of s- and r-extremality (for $\kappa = \omega$ we find exactly the definitions of s- and r-extremal pseudocompact group):

Definition 8. *Let κ be an infinite cardinal. A κ-pseudocompact group G is:*

- s_κ-extremal *if it has no proper dense κ-pseudocompact subgroup;*
- r_κ-extremal *if there exists no strictly finer κ-pseudocompact group topology on G.*

The main result of Chapter 6 is the next theorem, which completely solves the case

$$\mathcal{P} = \mathcal{Q} = \{\kappa\text{-pseudocompact abelian groups}\}.$$

In particular it generalizes Theorem A for every infinite cardinal κ.

Theorem A$^\kappa$. *Let κ be an infinite cardinal. For a κ-pseudocompact abelian group G the following conditions are equivalent:*

(a) G is s_κ-extremal;

(b) G is r_κ-extremal;

(c) $w(G) \leq \kappa$.

Our proof of Theorem 6.2 does not depend on the particular case $\kappa = \omega$, that is Theorem A. However many ideas used to prove it are taken from previous proofs in [12, 13, 15, 16, 21, 22, 29]. In particular we apply a set-theoretical lemma from [22] (see Lemma 6.24), which was also a fundamental step for the proof of Theorem A. In each of these cases we give references.

In Section 7.1 we generalize Theorem B for every infinite cardinal κ proving Theorem B$^\kappa$. We already have the generalization of pseudocompactness (i.e., κ-pseudocompactness) and of singularity (i.e., κ-singularity) and so we need to generalize also the remaining two properties: we give generalizations of ω-boundedness and introduce appropriate properties, denoted by TD_κ and TD^κ, extending the property TD_ω for every infinite cardinal κ. Corollary D* is applied in the proof of this theorem, since it gives exactly the equivalence of conditions (b) and (c).

Theorem B$^\kappa$. *Let κ be an infinite cardinal and let K be a compact abelian group. The following conditions are equivalent:*

(a) K has a proper totally dense κ-pseudocompact subgroup;

(b) K is non-κ-singular;

(c) there exists a continuous surjective homomorphism of K onto S^{κ^+}, where S is a compact non-torsion abelian group;

(d) K has the property TD_κ.

As noted after Theorem C for the case $\kappa = \omega$, the concept of r_κ-extremal κ-pseudocompact group is "dual" to that of minimal κ-pseudocompact group, in the sense that a κ-pseudocompact group is minimal if there exists no strictly coarser κ-pseudocompact group topology on G. Moreover, as an immediate corollary of Theorem A$^\kappa$, we obtain that a proper dense κ-pseudocompact subgroup of a topological abelian group cannot be either s_κ- or r_κ-extremal (see Corollary 7.16). This result suggests the problem of the characterization of κ-pseudocompact abelian groups admitting some proper dense minimal κ-pseudocompact subgroup. In the case of compact abelian groups this problem becomes to characterize those compact abelian groups admitting some proper essential dense κ-pseudocompact subgroup. Indeed in view of the foregoing observations a dense subgroup H of a compact abelian group K is minimal if and only if H is essential in K (see Theorem 1.50). Seen in this way, this problem is strictly related to that solved by Theorem B$^\kappa$. In Section 7.2 we study and completely solve it, giving for compact abelian groups properties equivalent to have some proper essential dense κ-pseudocompact subgroup:

Theorem C$^\kappa$. *Let κ be an infinite cardinal and let K be a compact abelian group. Then the following conditions are equivalent:*

(a) K admits no proper essential dense κ-pseudocompact subgroup;

(b) K is κ-singular and $w(pT_p(K)) \leq \kappa$ for every prime p;

(c) K is super-κ-singular.

This theorem generalizes Theorem C for every infinite cardinal κ.

In Theorem A$^\kappa$ we consider dense κ-pseudocompact subgroups of κ-pseudocompact abelian groups, while Theorem B$^\kappa$ is about totally dense κ-pseudocompact subgroups of compact abelian groups and Theorem C$^\kappa$ about essential dense κ-pseudocompact subgroups of compact abelian groups. So it is natural to ask when a κ-pseudocompact abelian group admits some proper totally dense or essential dense subgroup. We solve the compact case. Indeed in Theorem 7.22 we characterize compact abelian groups admitting some proper totally dense subgroup and as a corollary of Theorem C$^\kappa$ we find a characterization of compact abelian groups admitting some proper essential dense subgroup in Theorem 7.24.

In [15] Comfort and Robertson studied the compact groups K admitting some strongly totally dense pseudocompact subgroup H which is *small* (i.e., $|H| < |K|$). This problem is strictly related to Theorem B. [15, Theorem 6.2] showed that ZFC cannot decide whether there exists a compact group with small strongly totally dense pseudocompact subgroups. As a consequence of this theorem and the results in [15] it is possible to derive an analogous result for abelian groups, that is ZFC cannot decide whether there exists a totally disconnected compact *abelian* group with small totally dense pseudocompact subgroups (see Theorem 7.27).

In Theorem 7.45, also as a consequence of results from [4, 10, 15], we prove the counterpart of this theorem for essential dense subgroups, that is we prove that ZFC cannot decide whether there exists a totally disconnected compact abelian group K with small essential dense pseudocompact subgroups. But thanks to the properties of abelian groups, we see that the condition "dense" can be removed; indeed we prove:

Theorem F. *ZFC cannot decide whether there exists a compact abelian group K with small essential pseudocompact subgroups.*

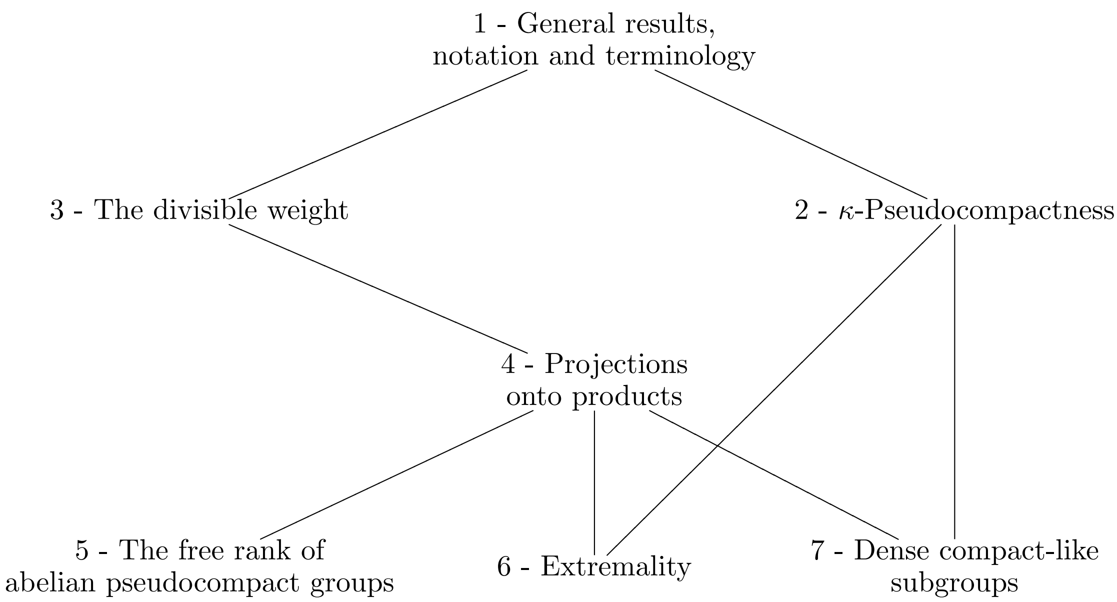
In order to prove this theorem, in Section 7.4.2 we introduce a new cardinal invariant, namely $E(-)$, that measures the minimal cardinality of essential subgroups of a topological group (see Definition 7.31). We introduce it in analogy with already existing cardinal invariants of the same nature, namely $ED(-)$ and $TD(-)$, which measure the minimal cardinality of essential dense subgroups and totally dense subgroups of a topological group respectively (see Definition 1.53). In the case of compact abelian groups this cardinal invariant $E(-)$, similarly to $ED(-)$ and $TD(-)$, turns out to be strictly related to purely algebraic cardinal invariants, that is the p-rank and the $\mathbb{Z}_p$-rank for p a prime; the definition and properties of these cardinal invariants are recalled in Section 1.2.1. The possibility to compute $E(-)$ allows us to prove in Theorem 7.39 that it coincides with $ED(-)$ and $TD(-)$ for compact abelian groups. This result plays a central role for the proof of Theorem F.

The non-abelian case is an open problem and we discuss it in [28].

In the next diagram we explain how the main results of this thesis, mentioned in the introduction, are related. Theorem A of Comfort and van Mill is a starting point. Also Theorems B and C, which are part of this thesis, have the same role. Indeed we generalize these three theorems for every infinite cardinal κ, obtaining Theorems A^κ, B^κ and C^κ respectively. A special role in this generalization is played by Theorem D and Corollary D^*. Indeed we apply them to prove Theorems A^κ and B^κ; then Theorem B^κ applies to prove Theorem C^κ. We do not give an explicit proof of Theorem B and C, since they are particular cases of Theorems B^κ and C^κ respectively. As an application of Theorem D we get also Theorem E. Theorem F is of a different nature but related to Theorem C.

$$\begin{tikzcd}
\mathrm{A} \arrow[dd, dashed] & \mathrm{B} \arrow[dd, dashed] & \mathrm{C} \arrow[dd, dashed] & \mathrm{F} \\
 & & & \mathrm{D}, \mathrm{D}^* \arrow[dlll] \arrow[dll] \arrow[d] \\
\mathrm{A}^\kappa & \mathrm{B}^\kappa \arrow[r] & \mathrm{C}^\kappa & \mathrm{E}
\end{tikzcd}$$

The next diagram explains the structure of this thesis and the relations among the chapters. Chapter 4 plays a central role: the results of the following chapters are applications of the theorems of Chapter 4. The results in Chapter 2 are about topological groups which are not necessarily abelian. Anyway the main theorems of this thesis are about abelian groups. The results in Chapters 2 and 6 are collected in [46] and those in Chapters 3, 4, 5 and 7 are collected in [26, 27, 28, 45].



Chapter 1

General results, notation and terminology

For eventually undefined terms see [40, 42, 50, 57].

For sets X and Y, we denote by X^Y the set of all functions $Y \to X$. Moreover we denote by id_X the identical function of X onto itself. Let I be a set of indices and for each $i \in I$ let X_i be a set. Then $\prod_{i\in I} X_i$ is the *direct product* of X_i, that is

$$\prod_{i\in I} X_i = \{(x_i)_{i\in I} : x_i \in X_i\}.$$

If all X_i are the same set X, then we simply write X^I and

$$\Delta X^I = \{\mathbf{x} = (x_i)_{i\in I} \in X : x_i = x_j \text{ for every } i, j \in I\}$$

is the *diagonal* subset of X^I.

1.1 Limit cardinals

We introduce some notations about cardinals following [57].

For a cardinal κ

$$\kappa^+ = \min\{\lambda : \lambda > \kappa\}, \quad \log\kappa = \min\{\lambda : 2^\lambda \geq \kappa\} \quad \text{and} \quad 2^{<\kappa} = \sup\{2^\lambda : \lambda < \kappa\}.$$

Furthermore κ is a *strong limit cardinal* if $2^\lambda < \kappa$ for every $\lambda < \kappa$. When at least one of the cardinals κ and λ is infinite we have

$$\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}.$$

The *cofinality* $\mathrm{cf}(\kappa)$ of κ is the minimal cardinality of a set I such that if A is a set with $|A| = \kappa$, there exists a family of sets $\{A_i\}_{i\in I}$ such that $|A_i| < \kappa$ for every $i \in I$ and $A = \bigcup_{i\in I} A_i$. Equivalently $\mathrm{cf}(\kappa)$ is the minimal cardinal γ such that there exists a sequence of cardinals $\{\kappa_\lambda : \lambda < \gamma\}$ with $\kappa = \sup_{\lambda<\gamma} \kappa_\lambda$ and $\kappa_\lambda < \kappa$ for every $\lambda < \gamma$. If $\mathrm{cf}(\kappa) = \kappa$, then κ is a *regular* cardinal. Every successor cardinal is regular.

As a consequence of König's lemma [55] we have the next condition.

Fact 1.1. *If κ is an infinite cardinal, then* $\mathrm{cf}(2^\kappa) > \kappa$.

The Generalized Continuum Hypothesis (briefly GCH) is equivalent to suppose that $2^\kappa = \kappa^+$ for every cardinal κ. Under GCH, if κ is a limit cardinal, then κ is a strong limit cardinal.

Claim 1.2. *Let κ be a cardinal of countable cofinality and $\lambda < \kappa$. Then $2^\lambda \leq \kappa$ implies $2^\lambda < \kappa$.*

Proof. If $\kappa = \omega$, it is clear that $2^\lambda < \kappa$ in case $\lambda < \kappa$. Suppose that $\kappa > \omega$. If $\lambda < \omega$, then $2^\lambda < \kappa$. So assume that $\omega \leq \lambda < \kappa$. If $2^\lambda = \kappa$, then $\mathrm{cf}(\kappa) = \mathrm{cf}(2^\lambda) > \lambda$ by Fact 1.1. This is not possible because $\lambda \geq \omega$ and so $\mathrm{cf}(\kappa) > \omega$, against the hypothesis that it is countable. $\square$

Claim 1.3. *For a cardinal κ the following conditions are equivalent:*

(a) *κ is a strong limit;*

(b) $\kappa = \log \kappa$.

If κ has countable cofinality, then the following condition is equivalent to (a) and (b):

(c) $2^{<\kappa} = \kappa$.

Proof. (a)$\Leftrightarrow$(b) A cardinal κ is a strong limit if and only if $2^\lambda < \kappa$ for every $\lambda < \kappa$. This is equivalent to $\kappa = \log \kappa$.

(a)$\Rightarrow$(c) Since $2^\lambda < \kappa$ for every $\lambda < \kappa$, it follows that $2^{<\kappa} = \sup_{\lambda<\kappa} 2^\lambda \leq \kappa$. Therefore $2^{<\kappa} = \kappa$.

(c)$\Rightarrow$(a) Since $2^{<\kappa} = \kappa$, then $2^\lambda \leq \kappa$ for every $\lambda < \kappa$. By Claim 1.2 this implies that $2^\lambda < \kappa$ for every $\lambda < \kappa$, that is κ is a strong limit. $\square$

In the second part of this claim the hypothesis that κ has countable cofinality is used only in (c)$\Rightarrow$(a), where Claim 1.2 applies.

The symbol $\mathfrak{c}$ stands for the cardinality of the continuum.

1.2 Abelian groups

We denote by $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{P}$, $\mathbb{N}$ and $\mathbb{N}_+$ respectively the field of rationals, the ring of integers, the set of primes, the set of natural numbers and the set of positive integers. For $m \in \mathbb{N}_+$, we use $\mathbb{Z}(m)$ for the finite cyclic group of order m. For $p \in \mathbb{P}$ the symbol $\mathbb{Z}_p$ indicates the ring of p-adic integers and $\mathbb{Z}(p^\infty)$ the Prüfer group.

For a group G we denote the neutral element by e_G. If G is abelian by 0.

Let G be an abelian group. An element $x \in G$ is *torsion* if there exists $n \in \mathbb{N}_+$ such that $nx = 0$. The subgroup of all torsion elements of G is $t(G)$. For $m \in \mathbb{N}_+$ let $G[m] = \{x \in G : mx = 0\}$. We say that G is *torsion free* if $t(G) = \{0\}$ and that G is *torsion* if it coincides with $t(G)$. Moreover G is *bounded torsion* if $G = G[m]$

for some $m \in \mathbb{N}_+$; equivalently in this case G is of finite *exponent* m. For $m \in \mathbb{N}$ let $mG = \{mx : x \in G\}$.

For any abelian group G and H let $\mathrm{Hom}(G, H)$ be the group of all homomorphisms from G to H. If G and H are algebraically isomorphic, then we write $G \cong H$. If S is a subset of G then $\langle S \rangle$ is the smallest subgroup of G containing S, i.e., the subgroup of G generated by S.

Let I be a set of indices and for each $i \in I$ let G_i be an abelian group. Then $\prod_{i \in I} G_i$ is the direct product and $\bigoplus G_i$ is the *direct sum* of the groups G_i, that is

$$\bigoplus_{i \in I} G_i = \left\{ \mathbf{x} \in \prod_{i \in I} G_i : |\mathrm{supp}(\mathbf{x})| \text{ is finite} \right\},$$

where $\mathrm{supp}(\mathbf{x}) = \{i \in I : x_i \neq 0\}$ is the *support* of $\mathbf{x} = (x_i)_{i \in I} \in \prod_{i \in I} G_i$. If I is empty then $\prod_{i \in I} G_i = \bigoplus_{i \in I} G_i = \{0\}$. For a cardinal κ we denote by G^κ the product and by $G^{(\kappa)}$ the direct sum of κ many copies of G, that is $\bigoplus_\kappa G$.

If κ is a cardinal $< |I|$ and $G = \prod_{i \in I} G_i$, let

$$\Sigma_\kappa G = \{\mathbf{x} \in G : |\mathrm{supp}(\mathbf{x})| \leq \kappa\}$$

be the *Σ_κ-product* of G. For $\kappa = \omega$, $\Sigma_\omega G$ is ΣG (i.e., the Σ-product centered at 0 of G). Moreover, if $G = H^I$, then ΔG is a subgroup of G, namely the diagonal subgroup of G. If $\omega \leq \kappa < |I|$, then $\Delta G \cap \Sigma_\kappa G = \{0\}$.

1.2.1 Ranks

Definition 1.4. *A subset S of an abelian group G is* independent *if $\sum_{i=1}^n z_i s_i = 0$ for $s_1, \dots, s_n \in S$, $z_1, \dots, z_n \in \mathbb{Z}$ and $n \in \mathbb{N}_+$, implies $z_1 = \ldots = z_n = 0$.*

Definition 1.5. *An abelian group F is* free *if there exists an independent subset S of F, such that $F = \langle S \rangle$.*

An abelian group F is free if and only if it is isomorphic to $\mathbb{Z}^{(\kappa)}$, where κ is a cardinal uniquely determined by F. Note that $|F| = \kappa \cdot \omega$.

If G is an abelian group, applying Zorn's lemma it is possible to prove the existence of a maximal independent subset of G. If S and T are maximal independent subsets of G, then $|S| = |T|$. This allows us to give the following definition.

Definition 1.6. *Let G be an abelian group. The* free rank *$r_0(G)$ of G is the cardinality of a maximal independent subset of G.*

Lemma 1.7. *Let G be an abelian group and N a subgroup of G. Then*

$$r_0(G) = r_0(G/N) + r_0(N).$$

Let G be an abelian group and let $p \in \mathbb{P}$. The *p-eth socle* of G is the subgroup $\mathrm{Soc}_p(G) = \{x \in G : o(x) = p\}$. It is a vector space over the field $\mathbb{F}_p$ of p many elements. Moreover the subgroup $\mathrm{Soc}(G) = \bigoplus_{p \in \mathbb{P}} \mathrm{Soc}_p(G)$ is the *socle* of G.

Definition 1.8. *Let G be an abelian group and let $p \in \mathbb{P}$. The* p-rank *of G is $r_p(G) = \dim_{\mathbb{F}_p} Soc_p(G)$.*

Note that $\mathrm{Soc}_p(G) \cong \bigoplus_{r_p(G)} \mathbb{Z}(p)$. Moreover $|\mathrm{Soc}_p(G)| = \omega \cdot r_p(G)$ whenever $\mathrm{Soc}_p(G)$ is infinite.

Lemma 1.9. *Let $p \in \mathbb{P}$, let G be an abelian group and N a subgroup of G. Then*

$$r_p(G) = r_p(G/N) + r_p(N).$$

Definition 1.10. *For an abelian group G the* rank *of G is $r(G) = r_0(G) + \sum_{p \in \mathbb{P}} r_p(G)$.*

If G is an infinite abelian group then $|G| = \omega \cdot r(G)$.

Lemma 1.11. *If G is an abelian group and N a subgroup of G, then*

$$r(G) = r(G/N) + r(N).$$

Since abelian groups are $\mathbb{Z}$-modules, in analogy with the free rank of abelian groups, it is possible to introduce the R-rank of R-modules, where R is a integral domain, i.e., a commutative unitary ring without zero-divisors.

Definition 1.12. *Let R be an integral domain and let M be an R-module. A subset S of M is* R-independent *if $\sum_{i=1}^n r_i s_i = 0$ for $s_1, \dots, s_n \in S$, $r_1, \dots, r_n \in R$ and $n \in \mathbb{N}_+$, implies $r_1 = \ldots = r_s = 0$.*

As in the particular case of abelian groups, applying Zorn's lemma it is possible to prove the existence of a maximal R-independent subset of an R-module M. If S and T are maximal R-independent subsets of M, then $|S| = |T|$. This allows us to give the next definition.

Definition 1.13. *Let R be an integral domain and let M be an R-module. The* R-rank $\mathrm{rank}_R(G)$ *of M is the cardinality of a maximal R-independent subset of G.*

Lemma 1.14. *Let R be an integral domain and let M, N be R-modules. Let $\varphi : M \to N$ be a surjective homomorphism of R-modules. Then $\mathrm{rank}_R(M) \geq \mathrm{rank}_R(N)$.*

Proof. Let S be a maximal R-independent subset of N. Then $\mathrm{rank}_R(N) = |S|$. For every $y \in S$ let $x \in M$ be such that $\varphi(x) = y$. Then the subset T of all such element x has the same cardinality of S and it is R-independent. Indeed, if $\sum_{i=1}^n r_i t_i = 0$ for $t_1, \dots, t_n \in T$, $r_1, \dots, r_n \in R$ and $n \in \mathbb{N}_+$, then $\sum_{i=1}^n r_i \varphi(t_i) = 0$; since $\varphi(t_i) \in S$ for every $i \in \{1, \dots, n\}$, this implies that $r_1 = \ldots = r_n = 0$. Hence $\mathrm{rank}_R(M) \geq |T| = |S| = \mathrm{rank}_R(N)$. □

We are interested in the case $R = \mathbb{Z}_p$, that is in the $\mathbb{Z}_p$-rank of $\mathbb{Z}_p$-modules. We give a relation among the cardinality, the p-rank and the $\mathbb{Z}_p$-rank of an infinite $\mathbb{Z}_p$-module:

Lemma 1.15. *Let $p \in \mathbb{P}$. If M is an infinite $\mathbb{Z}_p$-module, then*

$$|M| = \begin{cases} \omega \cdot r_p(M) & \text{if } \mathrm{rank}_{\mathbb{Z}_p}(M) = 0 \\ \mathfrak{c} \cdot \mathrm{rank}_{\mathbb{Z}_p}(M) \cdot r_p(M) & \text{if } \mathrm{rank}_{\mathbb{Z}_p}(M) \geq 1. \end{cases}$$

Proof. We know that $|M| = |M/t(M)| \cdot |t(M)|$. Since M is infinite, at least one of $|M/t(M)|$ and $|t(M)|$ in infinite. If $\mathrm{rank}_{\mathbb{Z}_p}(M) = 0$, then $|t(M)|$ is infinite and $|M| = |t(M)| = \omega \cdot r_p(M)$. Suppose that $\mathrm{rank}_{\mathbb{Z}_p}(M) \geq 1$. Then $|M| \geq \mathfrak{c}$. Moreover $|M/t(M)| = \mathfrak{c} \cdot \mathrm{rank}_{\mathbb{Z}_p}(M)$ and $r_p(M)$ is equal either to $r_p(M)$ or to $\omega \cdot r_p(M)$. Therefore $|M| = |M/t(M)| \cdot |t(M)| = \mathfrak{c} \cdot \mathrm{rank}_{\mathbb{Z}_p}(M) \cdot r_p(M)$. □

Fact 1.16. *If G is an abelian group such that $r_0(G) \geq \mathfrak{c}$, then there exists a surjective homomorphism $G \to \mathbb{T}$.*

Proof. Since $|\mathbb{T}| = \mathfrak{c}$, there exists a surjective homomorphism f of the free abelian group $\mathbb{Z}^{(\mathfrak{c})}$ onto $\mathbb{T}$. Moreover there exists an injective homomorphism $\mathbb{Z}^{(c)} \to G$, because $r_0(G) \geq \mathfrak{c}$. Since $\mathbb{T}$ is divisible, f can be extended to a surjective homomorphism $G \to \mathbb{T}$. □

1.2.2 Abelian p-groups

Let $p \in \mathbb{P}$ and let G be an abelian group. An element $x \in G$ is *p-torsion* if there exists $n \in \mathbb{N}$ such that $p^n x = 0$. Let $t_p(G)$ be the subgroup of all p-torsion elements of G. We call an abelian group X *p-group* (or *p-torsion*) if all elements of X are p-torsion. We call it *bounded p-torsion* in case there exists $n \in \mathbb{N}$ such that $p^n x = 0$ for every $x \in X$; equivalently it has exponent p^n for some $n \in \mathbb{N}$.

Remark 1.17. Let $p \in \mathbb{P}$ and let X be an abelian p-group. According to [42, Theorem 32.3] X has a *basic* subgroup B_0, in other words there exist cardinals α_n, with $n \in \mathbb{N}_+$, such that:

$B_0 \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)}$,
B_0 is *pure* (i.e., $B_0 \cap p^n X = p^n B_0$ for every $n \in \mathbb{N}$) and
X/B_0 is divisible;

so $X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$ for some cardinal σ, because X/B_0 is a divisible abelian p-group [42, Theorem 23.1]. The definition of basic subgroup was given in [56].

Let $m \in \mathbb{N}_+$, and let

$$B_{1,m} = \bigoplus_{n=1}^{m} \mathbb{Z}(p^n)^{(\alpha_n)} \quad \text{and} \quad B_{2,m} = \bigoplus_{n=m+1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)}.$$

Then we prove that

$$X = X_{1,m} \oplus B_{1,m},$$

where $X_{1,m} = p^m X + B_{2,m}$. Indeed, $X = p^m X + B_0$ because X/B_0 is divisible. Moreover, this is a direct sum as $X_{1,m} \cap B_{1,m} = \{0\}$; in fact, if $z \in X_{1,m} \cap B_{1,m}$, then $z = b_1 \in B_{1,m}$ and $z = x + b_2$, where $x \in X_{1,m}$, $b_2 \in B_{2,m}$. It follows that $x = b_1 - b_2 \in B_0 \cap p^m X$. By the purity of B_0, we have $B_0 \cap p^m X = p^m B_0 \subseteq B_{2,m}$ and this yields $b_1 = 0$, since $B_{1,m} \cap B_{2,m} = \{0\}$. Moreover

$$X/B_0 \cong X_{1,m}/B_{2,m}.$$

In [62, Proposition 18.6] the condition $X = X_{1,m} \oplus B_{1,m}$ was proved to be also sufficient to be basic for a subgroup B which is direct sum of cyclic groups.

Let $p \in \mathbb{P}$. In [66] (see [42, §35]) the *final rank* of an abelian p-group X was defined as

$$\mathrm{fin}r(X) = \inf_{n\in\mathbb{N}} r_p(p^n X).$$

The next example shows which are all the abelian p-groups of finite final rank.

Example 1.18. Let $p \in \mathbb{P}$ and let X be an abelian p-group. Let $m \in \mathbb{N}$. Then

$$\mathrm{fin}r(X) = m \text{ if and only if } X \cong \mathbb{Z}(p^\infty)^m \oplus B_0, \text{ where } B_0 \text{ is bounded torsion.}$$

To prove this result assume first that $X \cong \mathbb{Z}(p^\infty)^m \oplus B_0$, where B_0 is bounded torsion. Since $\mathbb{Z}(p^\infty)^m$ is divisible, $\mathrm{fin}r(\mathbb{Z}(p^\infty)^m) = m$. Moreover, for the $k_0 \in \mathbb{N}$ such that p^{k_0} is the exponent of B_0, we have $r_p(p^k B_0) = 0$ for every $k \geq k_0$. Then $\mathrm{fin}r(\mathbb{Z}(p^\infty)^m \oplus B_0) = m$.

Suppose that $\mathrm{fin}r(X) = m$ for some $m \in \mathbb{N}$. Then there exists $k_1 \in \mathbb{N}$ such that $r_p(p^k X) = m$ for every $k \in \mathbb{N}$ with $k \geq k_1$. By Remark 1.17 there exists a basic subgroup $B_0 \cong \bigoplus_{n=1}^\infty \mathbb{Z}(p^n)^{(\alpha_n)}$ of X such that $X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$ for some cardinals α_n, with $n \in \mathbb{N}_+$, and σ. If $\alpha_n > 0$ for infinitely many $n \in \mathbb{N}_+$, then $r_p(p^k B_0) = \omega$ for every $k \in \mathbb{N}$. In this case $\mathrm{fin}r(X) = \omega$, but it is not possible in view of our hypothesis. Consequently B_0 is bounded torsion, i.e, there exists $k_0 \in \mathbb{N}$ such that $p^{k_0} B_0 = \{0\}$.

Let $k \in \mathbb{N}$ be such that $k \geq \max\{k_0, k_1\}$. Following the notation of Remark 1.17, let $B_{1,k} = \bigoplus_{n=1}^k \mathbb{Z}(p^n)^{(\alpha_n)}$, $B_{2,k} = \bigoplus_{n=k+1}^\infty \mathbb{Z}(p^n)^{(\alpha_n)}$ and $X_{1,k} = p^k X + B_{2,k}$. In our case

$$B_{2,k} = \{0\}$$

and so

$$p^k X = X_{1,k} \cong X_{1,k}/B_{2,k} \cong X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$$

in view of Remark 1.17. Since $r_p(p^k X) = m$, hence $\sigma = m$.

By Remark 1.17 we have $X = X_{1,k_0} \oplus B_{1,k_0}$. As noted before $B_{2,k_0} = \{0\}$. This implies $X_{1,k_0} \cong X_{1,k_0}/B_{2,k_0} \cong X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$ and $B_0 = B_{1,k_0}$. Hence

$$X = X_{1,k_0} \oplus B_{1,k_0} \cong \mathbb{Z}(p^\infty)^m \oplus B_0.$$

According to [62] for a basic group B of an abelian p-group X the final rank of X is a upper bound for the p-rank of X/B, in other words

$$r_p(X/B) \leq \mathrm{fin}r(X) = \inf_{n\in\mathbb{N}} r_p(p^n X).$$

In view of this result, a basic subgroup B of an abelian p-group X is called *infrabasic* if $r_p(X/B)$ is maximal. Moreover every abelian p-group X has an infrabasic subgroup, that is X has a basic subgroup B such that $r_p(X/B) = \mathrm{fin}r(X)$ [62, Theorem 20.4].

1.3 Topological spaces

If S is a subset of a topological space X then $\overline{S}^X$ denotes the *closure* of S in X. We write $\overline{S}$ when there is no possibility of confusion.

Definition 1.19. *Let X be a Tychonov topological space. The* Čech-Stone compactification βX *of X is the compact space βX with the dense topological embedding $i : X \to \beta X$, such that for every continuous function $f : X \to [0,1]$ there exists a continuous function $f' : \beta X \to [0,1]$ which extends f.*

Let $f : X \to Y$ be a function, where X and Y are topological spaces. We denote by Γ_f the *graph* of f, that is

$$\Gamma_f = \{(x, f(x)) : x \in X\}.$$

The following is the closed graph theorem.

Theorem 1.20. *Let X and Y be topological spaces, with Y Hausdorff, and let $f : X \to Y$ be a function.*

(a) If f is continuous then Γ_f is closed in $X \times Y$.

(b) If Y is compact, then f is continuous if and only if Γ_f is closed in $X \times Y$.

The hypothesis that the codomain has to be Hausdorff is necessary, as shown by the following example. For a set X let δ_X and ι_X denote the discrete and the indiscrete topology of X respectively.

Example 1.21. Let (X, τ) be a Hausdorff topological space with $|X| \geq 2$; for example take $\tau = \delta_X$. Then $\mathrm{id}_X : (X, \tau) \to (X, \iota_X)$ is continuous and $\Gamma_{\mathrm{id}_X} = \Delta X$. Since the topology on the codomain is indiscrete, Γ_{id_X} is dense in $(X, \tau) \times (X, \iota_X)$ and so it is not closed.

We recall the definitions of some cardinal invariants. For a topological space X,

- the *weight* $w(X)$ of X is the minimum cardinality of a base for the topology on X and
- the *density* $d(X)$ of X is the minimal cardinality of a dense subset of X.

Moreover, if $x \in X$,

- the *character* $\chi(x, X)$ *at* x of X is the minimal cardinality of a basis of the filter of the neighborhoods of x in X, and
- the *character* of X is $\chi(X) = \sup_{x \in X} \chi(x, X)$.

Analogously

- the *pseudocharacter* $\psi(x, X)$ *at* x of X is the minimal cardinality of a family $\mathcal{F}$ of neighborhoods of x in X such that $\bigcap_{U \in \mathcal{F}} U = \{x\}$, and
- the *pseudocharacter* of X is $\psi(X) = \sup_{x \in X} \psi(x, X)$.

In general for a Tychonov topological space X we have

$$\psi(X) \leq \chi(X) \leq w(X) \leq 2^{|X|} \quad \text{and} \quad |X| \leq 2^{w(X)}.$$

1.4 Topological groups

As said in the introduction, all topological groups in this thesis are supposed to be Hausdorff. If G and H are topological groups and they are topologically isomorphic, then we write $G \cong_{top} H$.

Lemma 1.22. *Let G and G_1 be topological abelian groups such that there exists a continuous surjective homomorphism $f : G \to G_1$, and let H be a subgroup of G_1.*

(a) If H is proper, then $f^{-1}(H)$ is proper.

If f is open, then:

(b) if H is dense in G_1, then $f^{-1}(H)$ is dense in G.

Proof. (a) The subgroup $f^{-1}(H)$ is proper in G since H is proper in G_1 and $f(f^{-1}(H)) = H$ since f is surjective.

(b) We have to prove that $\overline{f^{-1}(H)} = G$. Since $f^{-1}(H)$ contains $\ker f$, this is equivalent to

$$f(\overline{f^{-1}(H)}) = G_1.$$

Observe that

$$H = f(f^{-1}(H)) \subseteq f(\overline{f^{-1}(H)}) \subseteq \overline{f(f^{-1}(H))} = \overline{H} = G_1.$$

Since $\overline{f^{-1}(H)}$ is closed and contains $\ker f$, its image $f(\overline{f^{-1}(H)})$ is closed. Consequently $f(\overline{f^{-1}(H)}) = \overline{H} = G_1$. □

The cardinal invariants introduced in the previous section for topological spaces have a simpler description in the case of topological groups and have additional properties. Indeed the character has the same value at each point, since topological groups are homogeneous topological spaces. Then $\chi(G)$ is the minimal cardinality of a local base at e_G of the topology on a group G. Analogously for the pseudocharacter of a topological group, the value of which is that at e_G. Moreover we use often the following properties of the weight.

Fact 1.23. *If N is a subgroup of a topological group G, then*

$$w(G) = w(G/N) \cdot w(N).$$

If $G = \prod_{i\in I} G_i$ is a product of topological groups, then $w(G) = |I| \cdot \sup_{i\in I} w(G_i)$.

A net $\{g_\alpha\}_{\alpha\in A}$ in a topological group G is a *Cauchy net* if for every neighborhood U of e_G in G there exists $\alpha_0 \in A$ such that $g_\alpha^{-1} g_\beta \in U$ and $g_\beta g_\alpha^{-1} \in U$ for every $\alpha, \beta > \alpha_0$.

A Hausdorff topological group G is *complete (in the sense of Raikov)* if every Cauchy net in G converges in G.

Theorem 1.24. *For every Hausdorff topological group G there exists a complete topological group $\widetilde{G}$ and a topological embedding $i : G \to \widetilde{G}$ such that $i(G)$ is dense in $\widetilde{G}$. Moreover, if $f : G \to H$ is a continuous homomorphism and H is a complete topological group, then there is a unique continuous homomorphism $\widetilde{f} : \widetilde{G} \to H$ with $f = \widetilde{f} \circ i$.*

Therefore every Hausdorff topological group has a unique (up to topological isomorphisms) *(Raikov-)completion* $(\widetilde{G}, i)$, briefly denoted by $\widetilde{G}$, and we can assume that G is a dense subgroup of $\widetilde{G}$.

Proposition 1.25. *Let G, H be topological groups and let $f : G \to H$ be a continuous isomorphism. Then there exists a unique continuous homomorphism $\widetilde{f} : \widetilde{G} \to \widetilde{H}$ which extends f.*

Definition 1.26. *A topological group G is* locally compact *if there exists a compact neighborhood of e_G in G.*

Theorem 1.27 (Structure theorem of locally compact abelian groups)**.** *If L is a locally compact abelian group, then L is topologically isomorphic to $\mathbb{R}^n \times K$, where $n \in \mathbb{N}$ and K is a topological abelian group with a compact open subgroup K_0.*

A theorem of the following form is known as an open mapping theorem. We give it here for compact groups, but it has a more general version in which G is locally compact and σ-compact and H is locally compact. (A Tychonov topological space is *σ-compact* if it is the union of countably many compact subsets.)

Theorem 1.28. *Let G and H be topological groups. If G is compact and $f : G \to H$ is a continuous surjective homomorphism, then f is open.*

Theorem 1.29. *Let G be a topological group and let N be a dense subgroup of G. If L is another subgroup of G and $\pi : G \to G/L$ is the canonical projection, then $\pi \restriction_N : N \to \pi(N)$ is open if and only if $\overline{N \cap L} = L$.*

1.4.1 Pontryagin-van Kampen duality

For the results in this section we refer to [31, 50].

The circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathbb{Z}$ of the reals $\mathbb{R}$ and carries its usual topology.

A *character* of a topological abelian group G is a continuous homomorphism from G to $\mathbb{T}$. Define

$$\widehat{G} = \{\chi \in \mathrm{Hom}(G, \mathbb{T}) : \chi \text{ character of } G\};$$

it is the *dual group* of G if it is endowed with the *compact-open topology*: the family

$$\{W(K, U) : K \subseteq G \text{ is compact and } U \text{ is an open neighborhood of } 0\},$$

where $W(K, U) = \{\chi \in \widehat{G} : \chi(K) \subseteq U\}$, is a base of the neighborhoods of 0 in $\widehat{G}$.

For a subset S of G the *annihilator* of S in $\widehat{G}$ is $A(S) = \{\chi \in \widehat{G} : \chi(A) = \{0\}\}$ and for a subset T of $\widehat{G}$ the *annihilator* of T in G is $A(T) = \{x \in G : \chi(x) = 0 \text{ for every } \chi \in T\}$. We have that $A(A(S)) = S$.

Theorem 1.30 (Pontryagin duality theorem)**.** [50, 58] *Let L be a locally compact abelian group. Then $\widehat{L}$ is a locally compact abelian group and* *(a) the assignment $H \mapsto A(H)$ is an order-inverting bijection (i.e., a Galois correspondence) between the family of all closed subgroups of L and the family of all closed subgroups of $\widehat{L}$;*

(b) for every closed subgroup H of L the dual group $\widehat{H}$ is topologically isomorphic to $\widehat{L}/A(H)$;

(c) $\omega_L : L \to \widehat{\widehat{L}}$, defined by $\omega_L(x) : \widehat{L} \to \mathbb{T}$ for every $x \in L$, where $\omega_L(x)(\chi) = \chi(x)$ for every $\chi \in \widehat{L}$, is a canonical topological isomorphism;

(d) L is compact if and only if $\widehat{L}$ is discrete;

(e) $\widehat{L \times H}$ is topologically isomorphic to $\widehat{L} \times \widehat{H}$.

In view of the canonical topological isomorphism in (c) it is possible to identify L and $\widehat{\widehat{L}}$.

Lemma 1.31. *Let L be a locally compact abelian group and H a subgroup of L. Then $A(H)$ is topologically isomorphic to $\widehat{L/H}$.*

As a consequence of Theorem 1.30 and this lemma we have the following results.

Fact 1.32. *Let L be a locally compact abelian group. Then*

(a) $\widehat{L}[p] \cong_{top} \widehat{L/pL}$; and

(b) $c(L) = A(t(\widehat{L}))$.

Fact 1.33. *For a topological abelian group K, which is either compact or discrete, and for $m \in \mathbb{N}_+$, $\widehat{mK}$ is topologically isomorphic to $m\widehat{K}$.*

Proof. By Theorem 1.30(b) $\widehat{mK}$ is topologically isomorphic to $\widehat{K}/\widehat{K}[m]$, since $A(mK) = \widehat{K}[m]$. There exists a continuous isomorphism $\widehat{K}/\widehat{K}[m] \to m\widehat{K}$ and it is open; this is true by Theorem 1.28 in case K is compact and it is obvious if K is discrete. Hence $\widehat{mK} \cong_{top} m\widehat{K}$. □

If K is a topological abelian group which is either compact or discrete, then K is bounded torsion if and only if $\widehat{K}$ is bounded torsion.

Example 1.34. It is known that $\widehat{\mathbb{T}}$ is $\mathbb{Z}$, $\widehat{\mathbb{R}}$ is $\mathbb{R}$, $\widehat{\mathbb{Z}_p}$ is $\mathbb{Z}(p^\infty)$ for $p \in \mathbb{P}$, and $\widehat{\mathbb{Z}(m)}$ is $\mathbb{Z}(m)$ for $m \in \mathbb{N}_+$.

Also the next is a consequence of Theorem 1.30 and Lemma 1.31.

Lemma 1.35. *Let L be a locally compact abelian group and*

$$\{0\} \to H \to L \to L/H \to \{0\}$$

a short exact sequence. Then

$$0 \to \widehat{L/H} \cong A(H) \to \widehat{L} \to \widehat{H} \cong \widehat{L}/A(H) \to \{0\}$$

is a short exact sequence as well.

We often apply these results in the following chapters referring to them as Pontryagin duality in general, that is without a precise reference.

1.4.2 Pseudocompact and precompact groups

As a direct consequence of the definition continuous image of a pseudocompact is pseudocompact. The next is a useful property about the algebraic structure of pseudocompact abelian groups.

Fact 1.36. [16, 21, 34] *Let G be a pseudocompact abelian group. Then either G is bounded torsion or $r_0(G) \geq \mathfrak{c}$.*

The next theorem about the cardinality of pseudocompact groups is due to van Douwen.

Theorem 1.37. [67] *If G is an infinite pseudocompact abelian group, then*

- $|G| \geq \mathfrak{c}$, *and*
- *if $|G|$ is a strong limit,* $\mathrm{cf}(|G|) > \omega$.

From this theorem it follows that a pseudocompact abelian group is either finite or of size $\geq \mathfrak{c}$.

A topological group G is *precompact* if for every open neighborhood U of 0 in G there exists a finite subset F of G such that $FU = UF = G$. Furthermore G is precompact if and only if its completion $\widetilde{G}$ is compact [69].

Let G be an abelian group. Then $G^\#$ denotes G endowed with the *Bohr topology*, i.e., the initial topology of all characters in $\mathrm{Hom}(G, \mathbb{T})$. As all group topologies generated by characters, $G^\#$ is precompact. If $A \leq \mathrm{Hom}(G, \mathbb{T})$, let T_A denote the initial topology on G of all characters in A; it is proved in [17] that

$$w(G, T_A) = |A|. \tag{1.1}$$

Moreover from [53] it is known that

$$|\mathrm{Hom}(G, \mathbb{T})| = 2^{|G|}. \tag{1.2}$$

Claim 1.38. *If G is an abelian group, then $w(G^\#) = 2^{|G|}$.*

Proof. Since $G^\# = (G, T_G)$, from (1.1) and (1.2) it follows that $w(G^\#) = |\mathrm{Hom}(G, \mathbb{T})| = 2^{|G|}$. □

Every pseudocompact group is necessarily precompact [18, Theorem 1.1]. The converse implication is not true in general:

Claim 1.39. [19] *If G is an infinite abelian group, then $G^\#$ is non-pseudocompact.*

Proof. It can be proved that G admits a subgroup H such that $|G/H| = \omega$. In $G^\#$ every subgroup is closed; in particular H is closed. Then G/H is pseudocompact of size ω. This is in contradiction with Theorem 1.37. □

1.5 General properties of compact groups

For $p \in \mathbb{P}$ and for a subset $\pi \subseteq \mathbb{P}$ consider the metrizable compact abelian groups

$$\mathbb{G}_p = \prod_{n \in \mathbb{N}_+} \mathbb{Z}(p^n) \quad \text{and} \quad S_\pi = \prod_{q \in \pi} \mathbb{Z}(q).$$

Clearly, $\mathbb{G}_p$ is non-torsion, while S_π is non-torsion if and only if π is infinite. These groups play a special role in the main chapters of this thesis.

In the following fact we recall some general properties of compact abelian groups, which are used in this thesis (without giving explicit references).

Fact 1.40. [50, 51] *Let K be a compact infinite abelian group. Then:*

(a) $|K| = 2^{w(K)}$ *and* $w(K) = |\widehat{K}|$;

(b) $d(K) = \log w(K)$;

(c) $\psi(K) = \chi(K) = w(K)$;

(d) the following conditions are equivalent:

1. *K is connected;*
2. *K is divisible;*
3. *$\widehat{K}$ is torsion free.*

The first property in (a) implies that $|K| > w(K)$. This does not hold true in general for precompact groups. Indeed $w(G^\#) = 2^{|G|} > |G|$ for every infinite abelian group G by Claim 1.38.

For a topological group G we denote by $c(G)$ the *connected component* of the neutral element e_G in G. A topological group G is *connected* if $G = c(G)$, while G is said to be *totally disconnected* if $c(G)$ is trivial.

The following is a consequence of Pontryagin duality.

Fact 1.41. *Let K be a compact abelian group. Then there exists a continuous surjective homomorphism $K \to \mathbb{T}^{w(c(K))}$.*

Proof. In fact, if $X = \widehat{K}$, X has $X/t(X)$ as a quotient and $X/t(X) \cong \widehat{c(K)}$ by Fact 1.32(b) and Theorem 1.30(b). Consequently $X/t(X)$ is a torsion free group of cardinality $w(c(K))$ and so there exists an injective homomorphism $\mathbb{Z}^{(w(c(K)))} \to X/t(X)$. Therefore there exists an injective homomorphism $\mathbb{Z}^{(w(c(K)))} \to X$. By Lemma 1.35 there exists a continuous surjective homomorphism $K \to \mathbb{T}^{w(c(K))}$. □

We recall that totally disconnected compact abelian groups are precisely profinite abelian groups [61]. *Profinite* groups are topological groups isomorphic to inverse limits of an inverse system of finite groups. For a prime p, a group G is a *pro-p group* if it is the inverse limit of an inverse system of finite p-groups. Equivalently a pro-p group is a profinite group G such that G/N is a finite p-group for every open normal subgroup N of G. Moreover the compact $\mathbb{Z}_p$-modules are precisely the abelian pro-p groups.

The following are other applications of Pontryagin duality.

Fact 1.42. *If L is a locally compact abelian group, then:*

(a) L is profinite if and only if $\widehat{L}$ is torsion;

(b) L is a pro-p group if and only if $\widehat{L}$ is p-torsion.

Lemma 1.43. *Let K be a compact abelian group. Then there exists a totally disconnected closed subgroup N of K such that $K = N + c(K)$.*

Proof. Let $X = \widehat{K}$ and let S be a maximal independent subset of X. Put $Y = \langle S\rangle$ and $N = A(Y)$. Thus N is a closed subgroup of K. By Fact 1.42(a) N is totally disconnected, since $N \cong_{top} \widehat{X/Y}$ and X/Y is torsion. Recall that $c(K) = A(t(X))$ by Fact 1.32(b). Consequently $K = N + c(K)$, because $Y \cap t(X) = \{0\}$ and so $K = A(Y \cap t(X)) = A(Y) + A(t(X)) = N + c(K)$. □

Let p be a prime number and K a compact abelian group. The *topological p-component* of K is the subgroup $K_p = \{x \in K : p^n x \to 0 \text{ in } K, \text{ where } n \in \mathbb{N}\}$ [31]. The *p-component* $T_p(K)$ of K is the closure of K_p.

The subgroup K_p has a natural structure of $\mathbb{Z}_p$-module, which is very useful: identifying K with $\widehat{\widehat{K}}$ in view of Theorem 1.30(c), K_p can be viewed as $\mathrm{Hom}(\widehat{K}, \mathbb{Z}(p^\infty))$ (while K is $\mathrm{Hom}(\widehat{K}, \mathbb{T})$) [31, §4.1]. So K_p has its $\mathbb{Z}_p$-rank. This structure of $\mathbb{Z}_p$-module permits to have a better knowledge of K_p. For example, if $x \in K_p$, denoting with $\langle x\rangle_{\mathbb{Z}_p}$ the $\mathbb{Z}_p$-submodule of K_p generated by x, then

$$\overline{\langle x\rangle} = \langle x\rangle_{\mathbb{Z}_p} \cong_{top} \begin{cases} \mathbb{Z}_p \\ \mathbb{Z}(p^n) & \text{for some } n \in \mathbb{N}_+ \end{cases}$$

For example $K = \mathbb{T}$ is not a $\mathbb{Z}_p$-module, but for every $p \in \mathbb{P}$ $K_p = \mathbb{Z}(p^\infty)$ is a $\mathbb{Z}_p$-module.

For a compact abelian group K and $p \in \mathbb{P}$ we define

$$\rho_p(K) = \mathrm{rank}_{\mathbb{Z}_p}(K_p).$$

So this is the generalization of the $\mathbb{Z}_p$-rank to all compact abelian groups: the $\mathbb{Z}_p$-rank can be defined properly only for $\mathbb{Z}_p$-modules, but we have seen that, if K is a compact abelia group, K_p is a $\mathbb{Z}_p$-module for every $p \in \mathbb{P}$.

For the sake of easier reference we recall here also the following useful and well known property of the totally disconnected compact abelian groups.

Remark 1.44. Let K be a compact abelian group and $p \in \mathbb{P}$. Following [31] the p-component can be defined also as $T_p(K) = \bigcap\{nK : n \in \mathbb{N}_+, (n,p) = 1\}$. Thanks to this equivalent definition it is easy to see that $T_p(K) \supseteq c(K)$ for every $p \in \mathbb{P}$ and so $mT_p(K) \supseteq mc(K) = c(K)$ for every $m \in \mathbb{N}_+$ and for every $p \in \mathbb{P}$.

(a) If N is a closed subgroup of K then $T_p(N) = \overline{N_p} \subseteq \overline{K_p} = T_p(K)$.

(b) If K is totally disconnected, K_p is closed and so $T_p(K) = K_p$ for every $p \in \mathbb{P}$; each K_p is a compact $\mathbb{Z}_p$-module. Moreover $K \cong_{top} \prod_{p\in\mathbb{P}} K_p$ and every closed subgroup N of K is of the form $N \cong_{top} \prod_{p\in\mathbb{P}} N_p$, where each N_p is a closed subgroup of K_p [5],[31, Proposition 3.5.9].

(c) Let $L = K/c(K)$ and let π be the canonical projection of K onto L. Then $\pi\restriction_{T_p(K)}$: $T_p(K) \to T_p(L) = L_p$ is surjective and $T_p(K) = \pi^{-1}(L_p)$ [31, Proposition 4.1.5].

(d) If $w(c(K)) \leq \kappa$, where κ is an infinite cardinal, then:

  (d1) $w(T_p(K)) \leq \kappa$ if and only if $w(L_p) \leq \kappa$, and

  (d2) $w(pT_p(K)) \leq \kappa$ if and only if $w(pL_p) \leq \kappa$.

(e) If K is connected, then K_p is dense in K. Consequently, if $H \leq K$, then $H_p = K_p \cap H$.

(f) If $K_p = \{0\}$, then K is totally disconnected.

1.5.1 Properties of compact $\mathbb{Z}_p$-modules

Lemma 1.45. *Let $p \in \mathbb{P}$ and K be a compact $\mathbb{Z}_p$-module. Then K has a closed subgroup N such that:*

(a) $N \cong_{top} \mathbb{Z}_p^\sigma$ for some cardinal σ;

(b) $K/N \cong_{top} \prod_{n=1}^\infty \mathbb{Z}(p^n)^{\alpha_n}$, for some cardinals α_n, with $n \in \mathbb{N}_+$.

(c) $\pi(K[p^n]) = (K/N)[p^n]$ for every $n \in \mathbb{N}_+$, where $\pi : K \to K/N$ is the canonical projection.

In particular $0 \to N \to K \to K/N \to 0$ is a short exact sequence.

Proof. (a,b) The dual X of K is a p-group by Fact 1.42(b). Then it is possible to apply Remark 1.17 to find a basic subgroup $B_0 \cong \bigoplus_{n=1}^\infty \mathbb{Z}(p^n)^{(\alpha_n)}$ of X, for some cardinals α_n, $n \in \mathbb{N}_+$, with $X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$ for some cardinal σ. By Pontryagin duality we can identify K and $\widehat{X}$. Let $B = \widehat{B_0}$ and $N = A(B_0) \cong_{top} \widehat{X/B_0} \cong_{top} \mathbb{Z}_p^\sigma$ by Pontryagin duality. Moreover $K/N \cong_{top} B \cong_{top} \prod_{n=1}^\infty \mathbb{Z}(p^n)^{\alpha_n}$, again applying Pontryagin duality.

(c) It is clear that $\pi(K[p^n]) \subseteq (K/N)[p^n]$. To prove that $\pi(K[p^n]) \supseteq (K/N)[p^n]$, first note that for

$$W = \{x \in K : p^n x \in N\},$$

we have $W = \pi^{-1}((K/N)[p^n])$ (so in particular $\pi(W) = (K/N)[p^n]$). Then it is sufficient to prove that

$$W = N + K[p^n].$$

Since W and $N + K[p^n]$ are closed subgroups, to prove that they coincide it suffices to prove that their annihilators coincide. Since $W = \pi^{-1}((K/N)[p^n])$, obviously $W \supseteq$

$N + K[p^n]$. Consequently $A(W) \subseteq A(N) \cap A(K[p^n]) = B_0 \cap p^n X = p^n B_0$. The last equality follows from the purity of B_0 (see Remark 1.17). But we have also

$$A(W) = \{\chi \in X : (\forall x \in K)\ p^n x \in N \Rightarrow \chi(x) = 0\} \supseteq p^n A(N) = p^n B_0.$$

Therefore $A(W) = A(N) \cap A(K[p^n])$, that is $W = N + K[p^n]$. □

Lemma 1.46. *Let κ be an infinite cardinal, let $p \in \mathbb{P}$, and let K be a compact $\mathbb{Z}_p$-module. Suppose that there exists $m \in \mathbb{N}_+$ such that $w(p^m K) \leq \kappa$, but $w(p^{m-1}K) > \kappa$. Then there exist a compact $\mathbb{Z}_p$-module K_m of weight $\leq \kappa$ and a compact bounded torsion abelian group $B_m \cong_{top} \prod_{n=1}^m \mathbb{Z}(p^n)^{\alpha_n}$, for some cardinals α_n, with $n \in \mathbb{N}_+$, such that*

$$K \cong_{top} K' \times B$$

and $\alpha_m > \kappa$.

Proof. The dual X of K is a p-group by Fact 1.42. Then it is possible to apply Remark 1.17 to find a basic subgroup of X, namely $B_0 \cong \bigoplus_{n=1}^\infty \mathbb{Z}(p^n)^{(\alpha_n)}$, for some cardinals α_n, with $n \in \mathbb{N}_+$, such that $X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$ for some cardinal σ. Following the notations of Remark 1.17 let $B_{1,m} = \bigoplus_{n=1}^m \mathbb{Z}(p^n)^{(\alpha_n)}$ and $B_{2,m} = \bigoplus_{n=m+1}^\infty \mathbb{Z}(p^n)^{(\alpha_n)}$; then $X = X_{1,m} \oplus B_{1,m}$, where $X_{1,m} = p^m X + B_{2,m}$.

Let

$$K_m = \widehat{X_{1,m}} \text{ and } B_m = \widehat{B_{1,m}} \cong_{top} \prod_{n=1}^m \mathbb{Z}(p^n)^{\alpha_n}.$$

Thanks to Pontryagin duality we can write

$$K \cong_{top} K_m \times B_m.$$

By the hypothesis $w(p^{m-1}K) > \kappa$. Moreover

$$p^{m-1}K \cong_{top} p^{m-1}(K_m \times B_m) = p^{m-1}K_m \times p^{m-1}B_m.$$

Since $|X_{1,m}| \leq \kappa$, because $|p^m X| \leq \kappa$, it follows that $|B_{2,m}| \leq \kappa$. Consequently $w(K_m) = |X_{1,m}| \leq \kappa$. Hence $w(p^{m-1}B_m) > \kappa$, that is $\alpha_m > \kappa$. □

Under the hypotheses and notations of Lemma 1.46, since $X_{1,m}/B_{2,m} \cong X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$ in view of Remark 1.17, by Pontryagin duality K_m has a subgroup N_m isomorphic to $\mathbb{Z}_p^\sigma$ such that $K_m/N_m \cong_{top} \prod_{n=1}^\infty \mathbb{Z}(p^n)^{\alpha_n}$.

1.5.2 Essential dense and totally dense subgroups

Lemma 1.47. *Let G be a topological abelian group and let H be a subgroup of G.*

(a) If H is essential in G, then $H \supseteq \mathrm{Soc}(G)$.

(b) If H is totally dense in G, then $H \supseteq t(G)$.

Proof. (a) If $x \in \mathrm{Soc}(G)$, then $\langle x \rangle$ is a finite (so closed) subgroup of G of prime exponent; since H is essential in G, $H \cap \langle x \rangle \neq \{0\}$ and so $H \geq \langle x \rangle$.

(b) If $x \in t(G)$, then $\langle x \rangle$ is a finite (so closed) subgroup of G; since H is totally dense in G, $H \cap \langle x \rangle$ is dense in $\langle x \rangle$ and so $H \geq \langle x \rangle$. □

Example 1.48. The subgroup $t(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$ of $\mathbb{T}$ is totally dense in $\mathbb{T}$ and $|\mathbb{Q}/\mathbb{Z}| < |\mathbb{T}|$. In view of Lemma 1.47(b) a subgroup H of $\mathbb{T}$ is totally dense in $\mathbb{T}$ if and only if $H \supseteq \mathbb{Q}/\mathbb{Z}$.

Analogously $\mathrm{Soc}(\mathbb{T})$ is essential in $\mathbb{T}$, with $|\mathrm{Soc}(\mathbb{T})| < |\mathbb{T}|$, and in view of Lemma 1.47(a) a subgroup H of $\mathbb{T}$ is essential in $\mathbb{T}$ if and only if $H \supseteq \mathrm{Soc}(\mathbb{T})$.

The next lemma shows that essentiality and total density preserve by taking inverse images of continuous surjective homomorphisms.

Lemma 1.49. *Let K and K_1 be compact abelian groups such that there exists a continuous surjective homomorphism $f : K \to K_1$, and let H be a subgroup of K_1. Then:*

(a) if H is essential in K_1, then $f^{-1}(H)$ is essential in K;

(b) if H is totally dense in K_1, then $f^{-1}(H)$ is totally dense in K.

Proof. (a) Let N be a non-trivial closed subgroup of K. Since N is compact and f is continuous, $f(N)$ is compact and so $f(N)$ is closed in K_1. If $f(N)$ is non-trivial, then $H \cap f(N) \neq \{0\}$ and so there exists $h \neq 0$ in $H \cap f(N)$; hence $h = f(n)$ for some $n \neq 0$ in N and consequently $0 \neq n \in f^{-1}(H) \cap N$. If $f(N) = \{0\}$, then $N \subseteq \ker f \subseteq f^{-1}(H)$. This proves that $f^{-1}(H)$ is essential in K.

(b) Let N be a non-trivial closed subgroup of K. Since N is compact and f is continuous, $f(N)$ is compact and so $f(N)$ is closed in K_1. If $f(N)$ is non-trivial, then $H \cap f(N)$ is dense in $f(N)$ and so $f^{-1}(H) \cap N$ is dense in N. If $f(N) = \{0\}$, then $N \subseteq \ker f \subseteq f^{-1}(H)$. This proves that $f^{-1}(H)$ is totally dense in K. □

The next "(total) minimality criterion" shows that for dense subgroups of compact groups to be minimal is equivalent to be essential and to be totally minimal is equivalent to be totally dense. So it is possible to choose the point of view in studying them.

Theorem 1.50. [30, 31, 48, 59, 64] *A dense subgroup H of a compact group K is minimal (respectively, totally minimal) if and only if H is essential (respectively, totally dense) in G.*

The following theorem is a criterion for the (total) minimality of a subgroup of a compact abelian group. It follows from [31, Theorem 4.3.7] or [4, Theorem 3.1]. It shows that essentiality and total density can be studied locally. They depends on intersections with closed subgroups isomorphic either to $\mathbb{Z}(p)$ or to $\mathbb{Z}_p$.

Theorem 1.51. *Let K be a compact abelian group and H a subgroup of K. Then the following conditions are equivalent:*

(a) H is essential (respectively, totally dense);

(b) $H \cap N \neq \{0\}$ *(respectively,* $H \cap N \not\leq pN$*) for every* $p \in \mathbb{P}$ *and for every closed subgroup* N *of* K *such that* N *is isomorphic either to* $\mathbb{Z}(p)$ *or to* $\mathbb{Z}_p$*;*

(c) $\mathrm{Soc}_p(K) \subseteq H$ *and* $H \cap N \neq \{0\}$ *(respectively,* $t_p(K) \subseteq H$ *and* $H \cap N \not\leq pN$*) for every* $p \in \mathbb{P}$ *and for every closed subgroup* N *of* K *such that* N *is isomorphic to* $\mathbb{Z}_p$*.*

This theorem shows that the total density and the essentiality of a dense subgroup of a compact abelian group K can be verified in K_p for every $p \in \mathbb{P}$, as stated in the next result. Indeed a closed subgroup of K isomorphic either to $\mathbb{Z}_p$ or to $\mathbb{Z}(p)$ is contained in K_p.

Corollary 1.52. *Let* K *be a compact abelian group and* H *a subgroup of* K*. Then* H *is essential (respectively, totally dense) in* K *if and only if* $H \cap K_p$ *is essential (respectively, totally dense) in* K_p *for every* $p \in \mathbb{P}$*.*

We recall the definition of two cardinal invariants which measure the size of essential dense and totally dense subgroups respectively. We study them in the case of compact abelian groups, but they can be defined for arbitrary topological groups.

Definition 1.53. *For a topological group* G *let*

$$ED(G) = \min\{|H| : H \leq G \text{ essential dense}\}, \text{and}$$
$$TD(G) = \min\{|H| : H \leq G \text{ totally dense}\}.$$

These cardinal invariants were introduced for compact groups in [65] (see also [31, §5.3]) under the name $M(-)$ and $TM(-)$ respectively. Indeed, for a compact group K $M(K)$ was defined as the minimal cardinality of a dense minimal subgroup of K and $TM(K)$ as the minimal cardinality of a dense totally minimal subgroup of K. They are the same as $ED(K)$ and $TD(K)$ in view of Theorem 1.50.

Both $ED(-)$ and $TD(-)$ are monotone under taking closed subgroups. Moreover $TD(-)$ is monotone also under taking quotients by closed subgroups. In general, since a totally dense subgroup is necessarily essential dense, $ED(G) \leq TD(G)$. But for compact abelian groups they coincide:

Fact 1.54. [65, Proposition 3 or Theorem 4] *If* K *is a compact abelian group, then* $ED(K) = TD(K)$*.*

For a topological $\mathbb{Z}_p$-module, the $\mathbb{Z}_p$-rank and $ED(-)$ and $TD(-)$ are strictly related and if the $\mathbb{Z}_p$-rank is sufficiently large, they all coincide with the cardinality, as shown by Fact 1.56. When the $\mathbb{Z}_p$-rank is finite the situation is more complicated. We give an idea of this in the next example.

Example 1.55. [31, Proposition 3.5.11 and Corollary 3.5.12] Let σ be a cardinal and $p \in \mathbb{P}$. Then:

(a)
$$\mathrm{rank}_{\mathbb{Z}_p}(\mathbb{Z}_p^\sigma) = \begin{cases} 1 & \text{if } \sigma = 1 \\ \sigma & \text{if } 1 < \sigma < \omega \\ 2^\sigma & \text{if } \sigma \geq \omega \end{cases}$$

(b)
$$TD(\mathbb{Z}_p^\sigma) = ED(\mathbb{Z}_p^\sigma) = \begin{cases} \omega & \text{if } \sigma = 1 \\ \mathfrak{c} & \text{if } 1 < \sigma < \omega \\ 2^\sigma & \text{if } \sigma \geq \omega \end{cases}$$

So $\mathrm{rank}_{\mathbb{Z}_p}(\mathbb{Z}_p^\sigma) = ED(\mathbb{Z}_p^\sigma) = TD(\mathbb{Z}_p^\sigma)$ if and only if σ is infinite. Moreover this shows that

- $ED(\mathbb{Z}_p^\sigma) = |\mathbb{Z}_p^\sigma|$ if and only if $\sigma > 1$, and that
- $\mathrm{rank}_{\mathbb{Z}_p}(\mathbb{Z}_p^\sigma) = ED(\mathbb{Z}_p^\sigma) = |\mathbb{Z}_p^\sigma|$ if and only if $\sigma \geq \omega$.

The precise result about $\mathbb{Z}_p$-modules is the following.

Fact 1.56. [4, Lemma 3.7] *Let $p \in \mathbb{P}$ and let M be a topological $\mathbb{Z}_p$-module.*

(a) If either $\mathrm{rank}_{\mathbb{Z}_p}(M) \geq 2$ or $r_p(M) \geq \mathfrak{c}$, then $ED(M) = TD(M) = |M|$.

(b) If $\mathrm{rank}_{\mathbb{Z}_p}(M) \leq 1$ and $r_p(M) < \mathfrak{c}$, then $TD(M) \leq \omega \cdot r_p(M)$.

Following [4] we introduce the *class of Prodanov*

$$\mathcal{P} = \{G \text{ topological abelian group} : TD(G) \leq \omega\}.$$

The compact abelian groups $K \in \mathcal{P}$ have special behavior and structure and they are described in [60]; in particular they are metrizable by the following result.

Fact 1.57. [4, Lemma 2.4] *If K is a compact abelian group, then $ED(K) \geq w(K)$.*

As a direct consequence of Fact 1.56 we have the following characterization of $\mathbb{Z}_p$-modules belonging to $\mathcal{P}$ and a description of their cardinality. If K is a compact abelian group, then K_p is a topological $\mathbb{Z}_p$-module such that $K_p[p]$ is compact, that is K_p has the property requested in the next result.

Corollary 1.58. *Let $p \in \mathbb{P}$ and let M be a topological $\mathbb{Z}_p$-module such that $M[p]$ is compact. Then:*

(a) $M \in \mathcal{P}$ if and only if $\mathrm{rank}_{\mathbb{Z}_p}(M) \leq 1$ and $r_p(M)$ is finite;

(b) if $M \in \mathcal{P}$, then $|M| \leq \mathfrak{c}$;

(c) if $M \notin \mathcal{P}$ then $|M| = \mathfrak{c} \cdot \mathrm{rank}_{\mathbb{Z}_p}(M) \cdot r_p(M)$.

Proof. (a) Since $M[p]$ is compact, it is either finite or $|M[p]| \geq \mathfrak{c}$ by Theorem 1.37. Then apply Fact 1.56.

(b) Consider the short exact sequence $\{0\} \to t(M) \to M \to M/t(M) \to \{0\}$. By (a) $\mathrm{rank}_{\mathbb{Z}_p}(M) \leq 1$ and $r_p(M)$ is finite; so $|M/t_p(M)| \leq \mathfrak{c}$ and $|t_p(M)| \leq \omega$. Consequently $|M| = |M/t_p(M)| \cdot |t_p(M)| \leq \mathfrak{c}$.

(c) Since $M \notin \mathcal{P}$, by (a) either $\mathrm{rank}_{\mathbb{Z}_p}(M) \geq 2$ or $r_p(M)$ is infinite. If $\mathrm{rank}_{\mathbb{Z}_p}(M) \geq 2$, then apply Lemma 1.15. If $\mathrm{rank}_{\mathbb{Z}_p}(M) \leq 1$, necessarily $r_p(M)$ is infinite. Since $M[p]$ is compact, $|M[p]| \geq \mathfrak{c}$ by Theorem 1.37 and so $r_p(M) \geq \mathfrak{c}$. If $\mathrm{rank}_{\mathbb{Z}_p}(M) = 1$, then $|M| = \mathfrak{c} \cdot \mathrm{rank}_{\mathbb{Z}_p}(M) \cdot r_p(M)$ by Lemma 1.15. If $\mathrm{rank}_{\mathbb{Z}_p}(M) = 0$, then $|M| = r_p(M)$ again by Lemma 1.15 and so $|M| = \mathfrak{c} \cdot \mathrm{rank}_{\mathbb{Z}_p}(M) \cdot r_p(M)$ also in this case. □

In case the $\mathbb{Z}_p$-modules of this corollary are compact, this is a particular case of the general characterization of compact abelian groups belonging to $\mathcal{P}$ given in [60]:

Remark 1.59. Let K be a compact abelian group. Then

$$K \in \mathcal{P} \text{ if and only if } r_p(K) < \omega \text{ and } \rho_p(K) \leq 1 \text{ for every } p \in \mathbb{P} \text{ [60]},$$

i.e., if and only if $K_p \in \mathcal{P}$ for every $p \in \mathbb{P}$. Since each K_p is a topological $\mathbb{Z}_p$-module, Corollary 1.58(a) describes when $K_p \in \mathcal{P}$.

Let

$$\pi_{\mathcal{P}}(K) = \{p \in \mathbb{P} : K_p \notin \mathcal{P}\}.$$

Then $K \in \mathcal{P}$ if and only if $\pi_{\mathcal{P}}(K)$ is empty.

Thanks to this characterization we see that

$$\text{if } K \text{ is a compact abelian group and } K \notin \mathcal{P}, \text{ then } ED(K) = TD(K) \geq \mathfrak{c}.$$

Indeed, if $r_p(K)$ is infinite, then $r_p(K) \geq \mathfrak{c}$ and so $ED(K) \geq r_p(K) \geq \mathfrak{c}$ in view of Lemma 1.47(a); and if $\rho_p(K) \geq 2$, then K contains a subgroup isomorphic to $\mathbb{Z}_p^2$ and $ED(\mathbb{Z}_p^2) = \mathfrak{c}$ by Example 1.55. So $ED(K) \geq \mathfrak{c}$.

From now on we consider groups that do not belong to the already described class $\mathcal{P}$.

As a consequence of results in [4], Theorem 1.61 gives a precise value of $ED(-)$ and $TD(-)$ for compact abelian groups not belonging to $\mathcal{P}$. A fundamental step for the proof of this theorem is the connected case:

Fact 1.60. [4, Theorem 3.11] *If K is a connected compact abelian group such that $K \notin \mathcal{P}$, then $ED(K) = TD(K) = |K|$.*

This fact was proved for non-metrizable compact abelian groups in [20] and generalized for non-metrizable compact (non-necessarily abelian) groups in [15, Theorem 5.6].

Theorem 1.61. *Let K be a non-metrizable compact abelian group. Then*

$$TD(K) = \begin{cases} |K| & \text{if } w(K) = w(c(K)) \text{ or } \exists\, p \in \mathbb{P} \text{ such that } w(K) = w((K/c(K))_p) \\ 2^{<w(K)} & \text{if } w(K) > w(c(K)) \text{ and } w(K) \text{ is a limit cardinal.} \end{cases}$$

Proof. Suppose that $w(K) = w(c(K))$. By Fact 1.60

$$TD(c(K)) = |c(K)| = 2^{w(c(K))} = 2^{w(K)} = |K|.$$

Since $TD(K) \geq TD(c(K))$, it follows that $TD(K) = |K|$. If $w(K) = w((K/c(K))_p)$ for some $p \in \mathbb{P}$, then $(K/c(K))_p$ is non-metrizable and so by Fact 1.56

$$TD((K/c(K))_p) = |(K/c(K))_p| = 2^{w((K/c(K))_p)} = 2^{w(K)} = |K|.$$

Since $TD(K) \geq TD((K/c(K))_p)$, so $TD(K) = |K|$.

Suppose that $w(K) > w(c(K))$ and that $w(K)$ is a limit cardinal. Consequently $w(K/c(K)) = w(K)$ is a limit as well. In view of [4, Theorem 3.12] $TD(K/c(K)) = 2^{<w(K/c(K))} = 2^{<w(K)}$. Since [4, Theorem 3.14] says that

$$TD(K) = |c(K)| \cdot TD(K/c(K)),$$

we have

$$TD(K) = 2^{w(c(K))} \cdot 2^{<w(K)}.$$

Since $w(c(K)) < w(K)$, $2^{w(c(K))} \cdot 2^{<w(K)} = 2^{<w(K)}$ and so $TD(K) = 2^{<w(K)}$. $\square$

This theorem does not hold for K metrizable. For example for $\mathbb{Z}_p^2$ we have seen in Example 1.55 that $TD(\mathbb{Z}_p^2) = \mathfrak{c}$, while the theorem would imply that it was ω. This argument holds also for [4, Theorem 3.12], which is true with the hypothesis that K is non-metrizable.

Chapter 2

κ-Pseudocompactness

Since ω-pseudocompactness coincides with pseudocompactness [54, Theorem 2.1], κ-pseudocompactness is the natural generalization of pseudocompactness for all infinite cardinals κ. If $\kappa \geq \lambda$ are infinite cardinals, κ-pseudocompactness implies λ-pseudocompactness and in particular pseudocompactness.

As G_δ-sets and G_δ-density are fundamental to work with pseudocompact spaces, G_κ-sets and G_κ-density were introduced for κ-pseudocompactness:

Definition 2.1. [44] *Let κ be an infinite cardinal and let X be a Tychonov topological space.*

- *A G_κ-*set *S of X is the intersection of κ many open subsets of X.*
- *A subset D of X is G_κ-*dense *in X if D has non-empty intersection with every G_κ-set of X.*

The G_κ-sets for $\kappa = \omega$ are the G_δ-sets. A Tychonov topological space X has $\psi(X) \leq \kappa$ precisely when $\{x\}$ is a G_κ-set of X for every $x \in X$. So if D is a G_κ-dense subset of X and $\psi(X) \leq \kappa$, then $D = X$.

Here we introduce the P_κ-modification of a given topology (generalizing the P-modification):

Definition 2.2. *Let κ be an infinite cardinal and let (X, τ) be a Tychonov topological space. Then $P_\kappa\tau$ denotes the topology on X generated by the G_κ-sets of X, which is called P_κ-*topology.

Obviously $\tau \leq P_\kappa\tau$. If X is a topological space, we denote by $P_\kappa X$ the set X endowed with the P_κ-topology. Note that $Y \subseteq X$ is G_κ-dense in X if and only if Y is dense in $P_\kappa X$.

In this chapter we give properties of κ-pseudocompact groups and characterize them in terms of their completion. Then we introduce the family $\Lambda_\kappa(G, \tau)$ of "big" closed normal subgroups of a κ-pseudocompact group (G, τ), which turns out to be a local base at e_G of the P_κ-modification of τ. Moreover we study the P_κ-modification of the Bohr topology of topological abelian groups.

2.1 Characterization

Remark 2.3. Let κ be an infinite cardinal. An equivalent definition of κ-pseudocompactness is the following:

> a Tychonov topological space is κ-pseudocompact if for every continuous function $f : X \to Y$, where Y is a Tychonov topological space of weight $\leq \kappa$, $f(X)$ is compact.

Indeed every Y can be seen as a subspace of $[0,1]^\kappa \subseteq \mathbb{R}^\kappa$ by Tychonov theorem [40].

It immediately follows from this equivalent definition that a κ-pseudocompact space of weight $\leq \kappa$ is compact. We use often this fact.

In the next remark we show how κ-pseudocompactness can be introduced using a cardinal invariant.

Remark 2.4. Let X be a Tychonov topological space and let

$$psc(X) = \begin{cases} \infty & \text{if } X \text{ is compact} \\ \min\{\kappa : \exists \text{ continuous } f : X \to \mathbb{R}^\kappa : f(X) \text{ non-compact}\} & \text{otherwise} \end{cases}$$

be the *level of pseudocompactness*.

Let κ be an infinite cardinal. Then X is κ-pseudocompact if $psc(X) > \kappa$.

We give properties of G_κ-sets and G_κ-dense subsets:

Lemma 2.5. *Let κ be an infinite cardinal and let X be a Tychonov topological space. Let $Z \subseteq Y \subseteq X$.*

(a) If Z is a G_κ-set of Y and Y is a G_κ-set of X, then Z is a G_κ-set of X.

(b) If Z is G_κ-dense in Y and Y is G_κ-dense in X, then Z is G_κ-dense in X.

Proof. (a) By hypothesis $Y = \bigcap_{\lambda<\kappa} U_\lambda$ and $Z = \bigcap_{\lambda<\kappa} V_\lambda$, where U_λ is open in X and V_λ is open in Y for every $\lambda < \kappa$. For every $\lambda < \kappa$ there exists an open subset W_λ of X such that $V_\lambda = W_\lambda \cap Y$. Consequently $Z = \bigcap_{\lambda<\kappa} V_\lambda = \bigcap_{\lambda<\kappa} W_\lambda \cap Y = \bigcap_{\lambda<\kappa}(W_\lambda \cap U_\lambda)$. This proves that Z is a G_κ-set of X.

(b) Let W be a non-empty G_κ-set of X. Then $W \cap Y$ is a non-empty G_κ-set of Y, because Y is G_κ-dense in X. Since Z is G_κ-dense in Y, $W \cap Y \cap Z = W \cap Z$ is non-empty. This proves that Z is G_κ-dense in X. □

Fact 2.6. *Let κ be an infinite cardinal. Let X, Y be Tychonov topological spaces, $f : X \to Y$ a continuous surjective function and D a subset of X.*

(a) If D is dense in X, then $f(D)$ is dense in Y.

(b) If D is G_κ-dense in X, then $f(D)$ is G_κ-dense in Y.

(c) If D is κ-pseudocompact, then $f(D)$ is κ-pseudocompact.

Proof. (a) Let U be an open subset of Y. Then $f^{-1}(U)$ is an open subset of X. By the hypothesis $D \cap f^{-1}(U)$ is not empty. So there exists $x \in D \cap f^{-1}(U)$. Then $f(x) \in f(D) \cap U$, which is not empty. This proves that $f(D)$ is dense in Y.

(b) Let $W = \bigcup_{\lambda<\kappa} U_\lambda$ be a G_κ-set of Y. Then $f^{-1}(W) = \bigcup_{\lambda<\kappa} f^{-1}(U_\lambda)$ is a G_κ-set of X. By the hypothesis $D \cap f^{-1}(W)$ is not empty. So there exists $x \in D \cap f^{-1}(W)$. Then $f(x) \in f(D) \cap W$, which is not empty. This proves that $f(D)$ is G_κ-dense in Y.

(c) follows from the definition of κ-pseudocompact space. □

To work with κ-pseudocompact groups we need a characterization of them similar to that of pseudocompact groups given by Comfort and Ross theorem:

Theorem 2.7. [18, Theorem 4.1] *Let G be a precompact group. Then the following conditions are equivalent:*

(a) G is pseudocompact;

(b) G is G_δ-dense in $\widetilde{G}$;

(c) G is dense in $P\widetilde{G}$;

(d) every continuous function $f : G \to \mathbb{R}$ can be extended to $\widetilde{G}$;

(e) every continuous function $f : G \to \mathbb{R}$ is uniformly continuous;

(f) $\widetilde{G} = \beta G$.

We find the wanted characterization in Theorem 2.12 combining Theorem 2.7 with Corollary 2.11.

Definition 2.8. [44, Definition 1.1] *Let κ be an infinite cardinal. A subset B of a Tychonov topological space X is C_κ-*compact *in X if $f(B)$ is compact for every continuous function $f : X \to \mathbb{R}^\kappa$.*

Consequently a Tychonov topological space X is κ-pseudocompact if and only if X is C_κ-compact in X. Moreover we have the next property.

Claim 2.9. *Let κ be an infinite cardinal and let X be a Tychonov topological space. If $Z \subseteq Y \subseteq X$ and Z is C_κ-compact in Y, then Z is C_κ-compact in X.*

In particular, if Z is a κ-pseudocompact Tychonov topological space, then Z is C_κ-compact in X, for every Tychonov topological space X such that Z is a subspace of X.

The following is the main theorem about C_κ-compactness.

Theorem 2.10. [44, Theorem 1.2] *Let κ be an infinite cardinal, let X be a Tychonov topological space and B a subset of X. Then the following conditions are equivalent:*

(a) B is C_κ-compact;

(b) *$f(B)$ is compact for every continuous function $f : X \to Y$, where Y is a Tychonov topological space of weight $\leq \kappa$;*

(c) *B is G_κ-dense in $\overline{B}^{\beta X}$.*

As a consequence of Theorem 2.10 and using Claim 2.9, we have the next result, which generalizes for every infinite cardinal κ a result of Hewitt [49].

Corollary 2.11. *Let κ be an infinite cardinal. A Tychonov topological space X is κ-pseudocompact if and only if it is G_κ-dense in βX.*

The next theorem characterizes a κ-pseudocompact group in terms of its completion.

Theorem 2.12. *Let κ be an infinite cardinal and let G be a precompact group. Then the following conditions are equivalent:*

(a) *G is κ-pseudocompact;*

(b) *G is G_κ-dense in $\widetilde{G} = \beta G$;*

(c) *G is dense in $P_\kappa \widetilde{G}$.*

Proof. Both conditions (a) and (b) imply that G is pseudocompact. So in particular $\widetilde{G} = \beta G$ by Theorem 2.7. Then (a)$\Leftrightarrow$(b) is given precisely by Corollary 2.11.

(b)$\Leftrightarrow$(c) is obvious. $\square$

As a consequence of this theorem, we can give a fundamental example of non-compact κ-pseudocompact groups.

Example 2.13. Let κ be an infinite cardinal and let $G = \prod_{i\in I} G_i$ be a product of topological groups, such that $|\{i \in I : G_i \neq \{e_G\}\}| > \kappa$. The Σ_κ-product $\Sigma_\kappa G$ is G_κ-dense in G. Moreover it is proper, because $\Sigma_\kappa G \cap \Delta G = \{e_G\}$. If G is compact, then $\Sigma_\kappa G$ is non-compact and κ-pseudocompact in view of Theorem 2.12.

The next are other consequences of Theorem 2.12.

Corollary 2.14. *Let κ be an infinite cardinal. Let G be a topological group and D a dense subgroup of G. Then D is κ-pseudocompact if and only if D is G_κ-dense in G and G is κ-pseudocompact.*

Proof. Suppose that D is κ-pseudocompact. It follows that $\widetilde{D}$ is compact and D is G_κ-dense in $\widetilde{D}$ by Theorem 2.12. Since D is dense in G, so $\widetilde{D} = \widetilde{G}$ and hence D is G_κ-dense in G. Moreover $G \supseteq D$ and so G is G_κ-dense in $\widetilde{G}$. By Theorem 2.12 G is κ-pseudocompact.

Assume that G is κ-pseudocompact and that D is G_κ-dense in G. Since G is G_κ-dense in $\widetilde{G}$ by Theorem 2.12 and since D is G_κ-dense in G, it follows from Lemma 2.5(b) that D is G_κ-dense in $\widetilde{G} = \widetilde{D}$. Hence D is κ-pseudocompact by Theorem 2.12. $\square$

Lemma 2.15. *Let κ be an infinite cardinal. Let G be a topological group and H a κ-pseudocompact subgroup of G such that H has finite index in G. Then G is κ-pseudocompact.*

Proof. It suffices to note that each (of the finitely many) cosets xH is κ-pseudocompact. $\square$

The following lemma is the generalization to κ-pseudocompact groups of [16, Theorem 3.2].

Lemma 2.16. *Let κ be an infinite cardinal. If G is a κ-pseudocompact group and $\psi(G) \leq \kappa$, then G is compact and so $w(G) = \psi(G) \leq \kappa$.*

Proof. Since $\psi(G) \leq \kappa$, it follows that $\{e_G\} = \bigcap_{\lambda<\kappa} U_\lambda$ for neighborhoods U_λ of e_G in G and by the regularity of G it is possible to choose every U_λ closed in G. Let $K = \widetilde{G}$. Then $\bigcap_{\lambda<\kappa} \overline{U_\lambda}^K$ contains a non-empty G_κ-set W of K. Moreover $G \cap W \subseteq G \cap \bigcap_{\lambda<\kappa} \overline{U_\lambda}^K = \{e_G\}$ and $W \setminus \{e_G\}$ is a G_κ-set of K. Since G is G_κ-dense in K by Theorem 2.12, this is possible only if $W = \{e_G\}$. Then $\psi(K) \leq \kappa$ and this is equivalent to say that $P_\kappa K$ is discrete. Since G is dense in $P_\kappa K = (K, \delta_K)$ in view of Theorem 2.12, we can conclude that $G = K$. Then G is compact and $w(G) = \psi(G) \leq \kappa$. $\square$

2.2 The family $\Lambda_\kappa(G)$

Let G be a topological group and κ an infinite cardinal. We define

$$\Lambda_\kappa(G) = \{N \triangleleft G : N \text{ closed } G_\kappa\text{-subgroup}\}.$$

The family $\Lambda_\omega(G)$ is denoted by $\Lambda(G)$ (see [16, 29]). For $\kappa \geq \lambda$ infinite cardinals $\Lambda_\kappa(G) \supseteq \Lambda_\lambda(G)$.

In Theorem 2.20 we prove that for κ-pseudocompact groups the families in the following claim coincide.

Claim 2.17. *Let κ be an infinite cardinal and let G be a topological group. Then*

$$\Lambda_\kappa(G) \supseteq \{N \triangleleft G : \text{closed},\ \psi(G/N) \leq \kappa\} \supseteq \{N \triangleleft G : \text{closed},\ w(G/N) \leq \kappa\}.$$

Proof. Let N be a closed normal subgroup of G and suppose that $w(G/N) \leq \kappa$. It follows that $\psi(G/N) \leq \kappa$. So N is a G_κ-set of G and hence $N \in \Lambda_\kappa(G)$. $\square$

The following lemma shows in particular that $\Lambda_\kappa(G)$ is a local base at e_G of $P_\kappa G$, for a precompact group G. For $\kappa = \omega$, it is [16, Lemma 1.6], which is applied in the proof.

Lemma 2.18. *Let κ be an infinite cardinal. Let G be a precompact group and W a G_κ-set of G such that $e_G \in W$. Then W contains some $N \in \Lambda_\kappa(G)$ such that $\psi(G/N) \leq \kappa$.*

Proof. Let $W = \bigcap_{\lambda<\kappa} U_\lambda$, where U_λ is an open neighborhood of e_G in G for every $\lambda < \kappa$. Let $\lambda < \kappa$. Since U_λ is open and $e_G \in U_\lambda$, in particular U_λ is a G_δ-set of G containing e_G. Then there exists $N_\lambda \in \Lambda(G)$ such that $N_\lambda \subseteq U_\lambda$ and $\psi(G/N_\lambda) \le \omega$ [16, Lemma 1.6]. Let $N = \bigcap_{\lambda<\kappa} N_\lambda$. Then $N \in \Lambda_\kappa(G)$. Moreover $\psi(G/N) \le \kappa$ because there exists a continuous injective homomorphism $G/N \to \prod_{\lambda<\kappa} G/N_\lambda$ and so $\psi(G/N) \le \psi(\prod_{\lambda<\kappa} G/N_\lambda) \le \kappa$. □

A consequence of the next corollary is that $\{xN : x \in G, N \in \Lambda_\kappa(G)\}$ is a base of $P_\kappa G$, in case G is a precompact group.

Corollary 2.19. *Let κ be an infinite cardinal. If G is precompact and W is a G_κ-set of G, then there exist $a \in W$ and $N \in \Lambda_\kappa(G)$ such that $aN \subseteq W$. So a subset H of G is G_κ-dense in G if and only if $(xN) \cap H \neq \emptyset$ for every $x \in G$ and $N \in \Lambda_\kappa(G)$.*

The next theorem and the first statement of its corollary were proved in the case $\kappa = \omega$ in [16, Theorem 6.1 and Corollary 6.2].

Theorem 2.20. *Let κ be an infinite cardinal and let G be a precompact group. Then G is κ-pseudocompact if and only if $w(G/N) \le \kappa$ for every $N \in \Lambda_\kappa(G)$.*

Proof. Suppose that G is κ-pseudocompact and let $N \in \Lambda_\kappa(G)$. By Lemma 2.18 there exists $L \in \Lambda_\kappa(G)$ such that $L \le N$ and $\psi(G/L) \le \kappa$. Thanks to Lemma 2.16 we have that G/L is compact of weight $w(G/L) = \psi(G/L) \le \kappa$. Since G/N is continuous image of G/L, it follows that $w(G/N) \le \kappa$.

Suppose that $w(G/N) \le \kappa$ for every $N \in \Lambda_\kappa(G)$. By Theorem 2.12 and Corollary 2.19 it suffices to prove that $xM \cap G \neq \emptyset$ for every $x \in \widetilde{G}$ and every $M \in \Lambda_\kappa(\widetilde{G})$. So let $x \in \widetilde{G}$ and $M \in \Lambda_\kappa(\widetilde{G})$. Let $\widetilde{\pi} : \widetilde{G} \to \widetilde{G}/M$ be the canonical projection, $\pi = \widetilde{\pi} \restriction_G$ and $N = G \cap M$. Hence $N \in \Lambda_\kappa(G)$. By the hypothesis $w(G/N) \le \kappa$ and so G/N is compact. Since $\pi(G)$ is continuous image of $G/\ker\pi = G/N$, so $\pi(G)$ is compact as well. Since G is dense in $\widetilde{G}$, by Fact 2.6(a) $\pi(G)$ is dense in $\widetilde{G}/M$ and so $\pi(G) = \widetilde{G}/M$. Therefore $xM \in \pi(G) = \{gM : g \in G\}$ and hence $xM = gM$ for some $g \in G$, that is $g \in xM \cap G$, which consequently is non-empty. □

Corollary 2.21. *Let κ be an infinite cardinal. Let G be a κ-pseudocompact abelian group and let $N \in \Lambda_\kappa(G)$. Then:*

(a) if $L \in \Lambda_\kappa(N)$, then $L \in \Lambda_\kappa(G)$;

(b) N is κ-pseudocompact;

(c) if L is a closed subgroup of G such that $N \subseteq L$, then $L \in \Lambda_\kappa(G)$;

(d) if $w(G) > \kappa$, then $w(G) = w(N)$ for every $N \in \Lambda_\kappa(G)$.

Proof. (a) Since N is closed in G and L is closed in N, it follows that L is closed in G. Moreover L is a G_κ-set of G, because N is a G_κ-set of G and L is a G_κ-set of N and so Lemma 2.5(a) applies.

(b) Let $L \in \Lambda_\kappa(N)$. By (a) L is a G_κ-set of G and so there exists $M \in \Lambda_\kappa(G)$ such that $M \subseteq L$ by Lemma 2.18. By Theorem 2.20 $w(G/M) \leq \kappa$ and consequently $w(N/M) \leq w(G/M) \leq \kappa$. Since N/L is continuous image of N/M, it follows that $w(N/L) \leq w(N/M) \leq \kappa$. Hence N is κ-pseudocompact by Theorem 2.20.

(c) Since $w(G/N) \leq \kappa$ by Theorem 2.20 and G/L is continuous image of G/N, it follows that $w(G/L) \leq \kappa$. Hence $L \in \Lambda_\kappa(G)$ by Claim 2.17.

(d) Let $N \in \Lambda_\kappa(G)$. Since $w(G) = w(N) \cdot w(G/N)$ and $w(G/N) \leq \kappa$ by Theorem 2.20, then $w(G) = w(N)$. □

This corollary and the following lemmas were proved in the pseudocompact case in [29, Section 2].

Lemma 2.22. *Let κ be an infinite cardinal, G a κ-pseudocompact abelian group and D a subgroup of G. Then:*

(a) D is G_κ-dense in G if and only if $D + N = G$ for every $N \in \Lambda_\kappa(G)$;

(b) if D is G_κ-dense in G and $N \in \Lambda_\kappa(G)$, then $D \cap N$ is G_κ-dense in N and $G/D \cong N/(D \cap N)$.

Proof. (a) follows from Corollary 2.19 and (b) follows from (a) and Corollary 2.21. □

Lemma 2.23. *Let κ be an infinite cardinal, G a κ-pseudocompact abelian group, L a closed subgroup of G and $\pi : G \to G/L$ the canonical projection.*

(a) If $N \in \Lambda_\kappa(G)$ then $\pi(N) \in \Lambda_\kappa(G/L)$.

(b) If D is a G_κ-dense subgroup of G/L then $\pi^{-1}(D)$ is a G_κ-dense subgroup of G.

(c) If H is a κ-pseudocompact subgroup of G/L, then $\pi^{-1}(H)$ is a κ-pseudocompact subgroup of G.

Proof. (a) By Theorem 2.20 we have $w(G/N) \leq \kappa$. Since

$$(G/L)/\pi(N) = (G/L)/((N + L)/L)$$

and $(G/L)/((N + L)/L)$ is topologically isomorphic to $G/(N + L)$, it follows that $w((G/L)/\pi(N)) \leq w(G/(N + L)) \leq w(G/N) \leq \kappa$. Hence $\pi(N) \in \Lambda_\kappa(G/L)$ by Claim 2.17.

(b) Let $N \in \Lambda_\kappa(G)$. Since $\pi(N + \pi^{-1}(D)) = \pi(N) + D$ and by (a) $\pi(N) \in \Lambda_\kappa(G/L)$, Lemma 2.22(a) implies that $\pi(N) + D = G/L$. Then $\pi(N + \pi^{-1}(D)) = G/L$ and so $N + \pi^{-1}(D) = G$. By Lemma 2.22(a) $\pi^{-1}(D)$ is G_κ-dense in G.

(c) The restriction $\pi \restriction_{\pi^{-1}(\overline{H})}: \pi^{-1}(\overline{H}) \to \overline{H}$ is open. By Theorem 2.12 H is G_κ-dense in $\overline{H}$. Then $\pi^{-1}(H)$ is G_κ-dense in $\pi^{-1}(\overline{H})$ by (b). So $\pi^{-1}(\overline{H})$ is κ-pseudocompact by Theorem 2.12. □

The next remark explains the role of the graph of a homomorphism in relation to the topology of the domain (see [29, Remark 2.12] for more details).

Remark 2.24. Let (G,τ) and H be topological groups and $h:(G,\tau)\to H$ a homomorphism. Consider the map $j:G\to\Gamma_h$ such that $j(x)=(x,h(x))$ for every $x\in G$. Then j is an open isomorphism. Endow Γ_h with the group topology induced by the product $(G,\tau)\times H$. The topology τ_h is *the weakest group topology on G such that $\tau_h\geq\tau$ and for which j is continuous.* Then $j:(G,\tau_h)\to\Gamma_h$ is a homeomorphism. Moreover τ_h is the weakest group topology on G such that $\tau_h\geq\tau$ and for which h is continuous. Clearly h is τ-continuous if and only if $\tau_h=\tau$.

Thanks to this remark we can give an example which shows that the condition "open" in Lemmas 1.22(b), 1.49 and 2.23 cannot be removed.

Example 2.25. Let $K=\mathbb{T}^{\mathfrak{c}}$, and let τ be the usual product topology on K. If $H=H_1+H_2$, where $H_1=\Sigma K$ and $H_2=(\mathbb{Q}/\mathbb{Z})^{\mathfrak{c}}$, then H is G_δ-dense and totally dense in (K,τ).

We prove that
$$r_0(K/H)\geq\mathfrak{c}.$$
To see this note that $r_0(\Delta K)=\mathfrak{c}$ because $\Delta K\cong\mathbb{T}$. Let $\pi:K\to K/H$ be the canonical projection. Then $\pi(\Delta K)\cong\Delta K/(\Delta K\cap H)$. By Lemma 1.7 $r_0(\Delta K)=r_0(\Delta K/(\Delta K\cap H))+r_0(\Delta K\cap H)$. Since $r_0(\Delta K\cap H)=0$, because $\Delta K\cap H=\Delta(\mathbb{Q}/\mathbb{Z})^{\mathfrak{c}}$, it follows that $r_0(\pi(\Delta K))=\mathfrak{c}$. Therefore $r_0(\pi(\Delta K))=\mathfrak{c}$ and so $r_0(K/H)\geq\mathfrak{c}$.

Since $r_0(K/H)\geq\mathfrak{c}$, there exists a surjective homomorphism $\varphi:K/H\to\mathbb{T}$ by Fact 1.16. Let $h=\varphi\circ\pi:K\to\mathbb{T}$, and let τ_h be the weakest topology on K such that $\tau_h\geq\tau$ and h is continuous, as defined in Remark 2.24. Let $G=\ker h$. Since $H\subseteq G$, it follows that G is G_δ-dense in (K,τ). By Lemma 6.11(b) with $\kappa=\omega$, Γ_h is G_δ-dense in $(K,\tau)\times\mathbb{T}$ and so it is pseudocompact with the topology inherited from $(K,\tau)\times\mathbb{T}$ by Corollary 2.14 with $\kappa=\omega$. By Remark 2.24 (K,τ_h) is homeomorphic to Γ_h endowed with the topology inherited from $(K,\tau)\times\mathbb{T}$ and so τ_h is pseudocompact. Moreover $\tau_h>\tau$; indeed Γ_h is not closed in $(K,\tau)\times\mathbb{T}$ and so $h:(K,\tau)\to\mathbb{T}$ is not continuous by Theorem 1.20.

Consider now the non-open continuous isomorphism
$$\mathrm{id}_K:(K,\tau_h)\to(K,\tau).$$
We have that G is a G_δ-dense and totally dense subgroup of (K,τ). But G is proper and closed in (K,τ_h) and so G cannot be dense in (K,τ_h).

2.3 The P_κ-topology

The following lemma is the generalization to the P_κ-topology of [16, Theorem 5.16].

Lemma 2.26. *Let κ be an infinite cardinal and let (G,τ) be a precompact abelian group such that every $h\in\mathrm{Hom}(G,\mathbb{T})$ is $P_\kappa\tau$-continuous. Then $(G,P_\kappa\tau)=P_\kappa G^\#$.*

Proof. Let $\tau_G^\#$ be the Bohr topology on G, that is $G^\#=(G,\tau_G^\#)$. By hypothesis $\tau_G^\#\leq P_\kappa\tau$. Then $P_\kappa\tau_G^\#\leq P_\kappa P_\kappa\tau=P_\kappa\tau$. Moreover $\tau\leq\tau_G^\#$ yields $P_\kappa\tau\leq P_\kappa\tau_G^\#$. $\square$

A Tychonov topological space X is of *first category* if it can be written as the union of countably many nowhere dense subsets of X. (A subset Y of X is *nowhere dense* in Y if the interior of the closure of Y is empty.) Moreover X is of *second category (Baire)* if it is not of first category, that is if for every family $\{U_n\}_{n\in\mathbb{N}}$ of open dense subsets of X, also $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X.

The following theorem is a new interesting result about the Bohr topology. In the first part of it we give a more detailed description of the topology $P_\kappa G^\#$, while the equivalence of (a), (b) and (c) is the counterpart of [16, Theorem 5.17] for the P_κ-topology. Some ideas in the proof of this second part are similar to those in the proof of [16, Theorem 5.17], but our proof is shorter and simpler, thanks to the description in algebraic terms of the topology $P_\kappa G^\#$ given in the first part of the proof.

Theorem 2.27. *Let κ be an infinite cardinal and let G be an abelian group. Then $\Lambda'_\kappa(G^\#) = \{N \leq G : |G/N| \leq 2^\kappa\} \subseteq \Lambda_\kappa(G^\#)$ is a local base at 0 of $P_\kappa G^\#$. Consequently the following conditions are equivalent:*

- *(a) $|G| \leq 2^\kappa$;*
- *(b) $P_\kappa G^\#$ is discrete;*
- *(c) $P_\kappa G^\#$ is Baire.*

Proof. We prove first that

$$\Lambda''_\kappa(G^\#) = \left\{ \bigcap_{\lambda<\kappa} \ker \chi_\lambda : \chi_\lambda \in \mathrm{Hom}(G, \mathbb{T}) \right\} \subseteq \Lambda_\kappa(G^\#)$$

is a local base at 0 of $P_\kappa G^\#$ and then the equality

$$\Lambda'_\kappa(G^\#) = \Lambda''_\kappa(G^\#).$$

If W is a G_κ-set of G such that $0 \in W$, then $W = \bigcap_{\lambda<\kappa} U_\lambda$, where each U_λ is a neighborhood of 0 in $G^\#$ belonging to the base. This means that $U_\lambda = \chi_\lambda^{-1}(V_\lambda)$, where $\chi_\lambda \in \mathrm{Hom}(G, \mathbb{T})$ and V_λ is a neighborhood of 0 in $\mathbb{T}$. Therefore

$$W = \bigcap_{\lambda<\kappa} \chi_\lambda^{-1}(0) = \bigcap_{\lambda<\kappa} \ker \chi_\lambda.$$

Each $\chi_\lambda^{-1}(0)$ is a G_δ-set of $G^\#$, because $\{0\}$ is a G_δ-set of $\mathbb{T}$, and hence $\bigcap_{\lambda<\kappa} \chi_\lambda^{-1}(0)$ is a G_κ-set of $G^\#$. Until now we have proved that $\Lambda''_\kappa(G^\#)$ is a local base at 0 of $P_\kappa G^\#$. Moreover it is contained in $\Lambda_\kappa(G^\#)$, because each $\bigcap_{\lambda<\kappa} \ker \chi_\lambda$, where $\chi_\lambda \in \mathrm{Hom}(G, \mathbb{T})$, is a closed G_κ-subgroup of $G^\#$.

It remains to prove that $\Lambda'_\kappa(G^\#) = \Lambda''_\kappa(G^\#)$. Let $N = \bigcap_{\lambda<\kappa} \ker \chi_\lambda \in \Lambda''_\kappa(G^\#)$, where every $\chi_\lambda \in \mathrm{Hom}(G, \mathbb{T})$. Since for every $\lambda < \kappa$ there exists an injective homomorphism $G/\ker \chi_\lambda \to \mathbb{T}$, it follows that there exists an injective homomorphism $G/N \to \mathbb{T}^\kappa$. Then $|G/N| \leq 2^\kappa$ and so $N \in \Lambda'_\kappa(G^\#)$.

To prove the converse inclusion let $N \in \Lambda'_\kappa(G^\#)$. Then N is closed in $G^\#$, because every subgroup of G is closed in $G^\#$. Moreover, since $r(G/N) \leq 2^\kappa$, there exists an injective homomorphism $i : G/N \to \mathbb{T}^\kappa$; in fact, G/N has a subgroup $B(G/N)$ which non-trivially intersects each non-trivial subgroup of G/N (i.e., $B(G/N)$ is essential in G/N in algebraic sense) and

$$B(G/N) \cong \mathbb{Z}^{(r_0(G/N))} \oplus \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p)^{(r_p(G/N))}.$$

Since $r_0(\mathbb{T}^\kappa) = 2^\kappa$ and $r_p(\mathbb{T}^\kappa) = 2^\kappa$ for every $p \in \mathbb{P}$, there exists an injective homomorphism $B(G/N) \to \mathbb{T}^\kappa$ and by the divisibility of $\mathbb{T}^\kappa$ this homomorphism can be extended to G/N (the extended homomorphism is still injective by the algebraic essentiality of $B(G/N)$ in G/N). Let $\pi : G \to G/N$ and $\pi_\lambda : \mathbb{T}^\kappa \to \mathbb{T}$ be the canonical projections for every $\lambda < \kappa$. Then $\chi_\lambda = \pi_\lambda \restriction_{i(G)} \circ i \circ \pi : G \to \mathbb{T}$ is a homomorphism. Moreover $N = \bigcap_{\lambda<\kappa} \ker \chi_\lambda \in \Lambda''_\kappa(G^\#)$.

It is clear that (a)⇒(b) by the first part of the proof and that (b)⇒(c).

(c)⇒(a) Suppose for a contradiction that $|G| > 2^\kappa$. We prove that $P_\kappa G^\#$ is of first category. Let

$$D(G) = \mathbb{Q}^{(r_0(G))} \oplus \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{(r_p(G))}$$

be the divisible hull of G. Moreover G has a subgroup algebraically isomorphic to

$$B(G) = \mathbb{Z}^{(r_0(G))} \oplus \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p)^{(r_p(G))}.$$

We can think

$$B(G) \leq G \leq D(G) \leq \mathbb{T}^I, \quad \text{where} \quad |I| = r(G) = r_0(G) + \sum_{p\in\mathbb{P}} r_p(G),$$

because $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ are algebraically isomorphic to subgroups of $\mathbb{T}$ for every $p \in \mathbb{P}$. Since $|G| > 2^\kappa$, it follows that $|I| > 2^\kappa$.

For $x \in G$ let $s(x) = \{i \in I : x_i \neq 0\}$ and for $n \in \mathbb{N}$ set

$$A(n) = \{x \in G : |s(x)| \leq n\}.$$

Then $G = \bigcup_{n\in\mathbb{N}} A(n)$.

For every $n \in \mathbb{N}$ we have $A(n) \subseteq A(n + 1)$. We prove that $A(n)$ is closed in the topology τ induced on G by $\mathbb{T}^I$ for every $n \in \mathbb{N}$: it is obvious that $A(0)$ is compact and $A(1)$ is compact, because every open neighborhood of 0 in (G, τ) contains all but a finite number of elements of $A(1)$. Moreover, for every $n \in \mathbb{N}$ with $n > 1$, $A(n)$ is the sum of n copies of $A(1)$ and so it is compact.

To conclude the proof we have to show that for every $n \in \mathbb{N}$

$$A(n) \text{ has empty interior in } P_\kappa G^\#.$$

Since $A(n) \subseteq A(n+1)$ for every $n \in \mathbb{N}$, it suffices to prove that the interior $\mathrm{Int}_{P_\kappa G^\#}(A(n))$ is empty for sufficiently large $n \in \mathbb{N}$. We consider $n \in \mathbb{N}_+$. By the first part of the proof, it suffices to show that if $x \in G$ and N is a subgroup of G such that $|G/N| \leq 2^\kappa$, then $x \in x + N \not\subseteq A(n)$ for all $n \in \mathbb{N}_+$. Moreover we can suppose that $x = 0$. In fact, if there exist $x \in G$ and $N \leq G$ with $|G/N| \leq 2^\kappa$, such that $x+N \subseteq A(n)$, then $x = x+0 \in A(n)$ and $0 \in N \subseteq -x+A(n) \subseteq A(n)+A(n) \subseteq A(2n)$. So let $N \leq G$ be such that $|G/N| \leq 2^\kappa$. Let

$$\{I_\xi : \xi < (2^\kappa)^+\}$$

be a family of subsets of I such that $|I_\xi| = n$ and $I_\xi \cap I_{\xi'} = \emptyset$ for every $\xi < \xi' < (2^\kappa)^+$. For every $\xi < (2^\kappa)^+$ there exists $x_\xi \in B(G) \leq G$ such that $s(x_\xi) = I_\xi$. Let $\pi : G \to G/N$ be the canonical projection. Since

$$\left|\{x_\xi : \xi < (2^\kappa)^+\}\right| = (2^\kappa)^+ > 2^\kappa \geq |G/N|,$$

it follows that there exist $\xi < \xi' < (2^\kappa)^+$ such that $\pi(x_\xi) = \pi(x_{\xi'})$. Then $x_\xi - x_{\xi'} \in \ker \pi = N$. But $s(x_\xi - x_{\xi'}) = I_\xi \cup I_{\xi'}$ and $|I_\xi \cup I_{\xi'}| = 2n$. Hence $x_\xi - x_{\xi'} \notin A(n)$. □

The following theorem is the generalization to the P_κ-topology of [15, Lemma 2.4] for topological groups.

Theorem 2.28. *Let κ be an infinite cardinal and let G be a κ-pseudocompact group. Then $P_\kappa G$ is Baire.*

Proof. Let K be the completion of G, which is compact, and

$$\mathcal{B} = \{xN : x \in K, N \in \Lambda_\kappa(K)\}.$$

By Corollary 2.19 $\mathcal{B}$ is a base of $P_\kappa K$. Consider a family $\{U_n\}_{n\in\mathbb{N}}$ of open dense subsets of $P_\kappa K$. We can choose them so that $U_n \supseteq U_{n+1}$ for every $n \in \mathbb{N}$. Let $A \in P_\kappa \tau$, $A \neq \emptyset$. Then $A \cap U_0$ is a non-empty element of $P_\kappa \tau$. Therefore there exists $B_0 \in \mathcal{B}$ such that $B_0 \subseteq A \cap U_0$. We proceed by induction. If $B_n \in \mathcal{B}$ has been defined, then $B_n \cap U_{n+1}$ is a non-empty open set in $P_\kappa K$ and so there exists $B_{n+1} \in \mathcal{B}$ such that $B_{n+1} \subseteq B_n \cap V_{n+1}$. Then

$$A \cap \bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} (A \cap U_n) \supseteq \bigcap_{n\in\mathbb{N}} B_n.$$

Moreover $\bigcap_{n\in\mathbb{N}} B_n \neq \emptyset$, because $\{B_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of closed subsets of K, which is compact.

Being κ-pseudocompact, G is G_κ-dense in K by Theorem 2.12. Then G is G_κ-dense in $P_\kappa K$, which is Baire by the previous part of the proof. By [15, Lemma 2.4(b)] a G_δ-dense subspace of a Baire space is Baire, and so $P_\kappa G$ is Baire. □

Chapter 3

The divisible weight

In this chapter we first describe the positive aspects of singular abelian groups and their properties. We also find a singular abelian group, which is a counterexample of a recent conjecture. Then we introduce new cardinal invariants, namely the divisible weight and the super-divisible weight, which measure singularity of topological abelian groups with great precision. These cardinal invariants are strictly related to the weight of topological abelian groups. Using them we see singularity from another point of view and we introduce new notions, i.e., w-divisibility, κ-singularity and super-κ-singularity for κ an infinite cardinal. Moreover we describe these w-divisible, κ-singular and super-κ-singular abelian groups, which turn out to be useful for the problems studied in the following chapters.

3.1 The power of singularity

Singular groups were defined in [29, Definition 1.2] in the ambit of extremal groups. For example at the end of [29] we proved that for pseudocompact abelian groups singularity is equivalent to one important level of extremality, that is c-extremality. A topological group is *c-extremal* if $r_0(G/D) < \mathfrak{c}$ for every dense pseudocompact subgroup D of G; this definition was given in [29, Definition 1.1] for pseudocompact groups. One implication is quite easy to prove also in the general case of topological abelian groups which are non-necessarily pseudocompact:

Fact 3.1. *Every singular abelian group is c-extremal.*

Proof. Suppose that G is a singular abelian group. Then there exists $m \in \mathbb{N}_+$ such that $w(mG) \leq \omega$. Let D be a dense pseudocompact subgroup of G. By Corollary 2.14(a) with $\kappa = \omega$ the subgroup D is G_δ-dense in G; then mD is G_δ-dense in mG by Fact 2.6(b) with $\kappa = \omega$. Since mG is metrizable, $mD = mG$. Consequently $m(G/D) = \{0\}$ and, since $r_0(G/D) = r_0(m(G/D)) = 0$, it follows that G is c-extremal. $\square$

This fact generalizes [29, Proposition 4.7] for every topological abelian group non-necessarily pseudocompact. We give an example which shows that the converse implication does not hold for topological abelian groups in general; in other words we give an example of a non-singular c-extremal abelian group:

Example 3.2. Let $G = \mathbb{Z}^\#$. Then G is c-extremal because $|G| < \mathfrak{c}$. We show that G is non-singular. For every $m \in \mathbb{N}_+$, the subgroup $m\mathbb{Z}$ is dense in $\mathbb{Z}^\#$ and hence $w(m\mathbb{Z}^\#) = w(\mathbb{Z}^\#)$ for every $m \in \mathbb{N}_+$ and $w(\mathbb{Z}^\#) = \mathfrak{c}$ by Fact 1.38. This means that $\mathbb{Z}^\#$ is non-singular.

The converse implication of Fact 3.1 holds for pseudocompact abelian groups, even if the two properties involved have different nature. As observed at the end of [29], the equivalence of singularity and c-extremality for pseudocompact groups implies Theorem A of the introduction.

In the remaining part of this section we produce a singular group, which is a counterexample for Conjecture 3.6 below, that was formulated in a preliminary version of [23] (see http://atlas-conferences.com/cgi-bin/abstract/cats-72). This again shows that singular groups are useful.

Definition 3.3. *A pseudocompact abelian group G is* CvM-pseudocompact *if there exist D_0 and D_1 dense pseudocompact subgroups of G with trivial intersection.*

If G is a non-trivial CvM-pseudocompact abelian group, then G is not metrizable. Moreover we have the following.

Example 3.4. Let G be a metrizable (thus compact) pseudocompact abelian group. Then G is CvM-pseudocompact if and only if $G = \{0\}$.

The next lemma gives a necessary condition for a topological abelian group to be CvM-pseudocompact.

Lemma 3.5. *If G is a CvM-pseudocompact abelian group then mG is CvM-pseudocompact for every $m \in \mathbb{N}_+$. In particular, in case $mG \neq \{0\}$, mG is non-metrizable.*

Proof. There exist D_0 and D_1 dense pseudocompact subgroups of G with trivial intersection. Let $m \in \mathbb{N}_+$. Then mD_0 and mD_1 are dense pseudocompact subgroups of mG by Fact 2.6(a,c) with $\kappa = \omega$ and they have trivial intersection. $\square$

Conjecture 3.6. *Every non-metrizable pseudocompact non-torsion abelian group is CvM-pseudocompact.*

Lemma 3.7. *Let G be a pseudocompact non-torsion abelian group. If G is CvM-pseudocompact, then G is non-singular.*

Proof. Suppose for a contradiction that G is singular. There exists $m \in \mathbb{N}_+$ such that mG is metrizable. Let D_0 and D_1 be dense pseudocompact subgroups of G with trivial intersection. Then mD_0 and mD_1 are dense pseudocompact subgroups of mG by Fact 2.6(a,c) with $\kappa = \omega$. So mD_0 and mD_1 are G_δ-dense in mG by Corollary 2.14 with $\kappa = \omega$. Since mG is metrizable, $mD_0 = mD_1 = mG$. Hence $mG \subseteq D_0 \cap D_1 = \{0\}$ and G is torsion, against the hypothesis. $\square$

The previous lemma allows us to find a counterexample for Conjecture 3.6:

Example 3.8. Let $G = \mathbb{T} \times \mathbb{Z}(2)^{\mathfrak{c}}$ endowed with the product topology. Then G is non-torsion and singular. So G cannot be CvM-pseudocompact by Lemma 3.7.

The problem can be extended to all pseudocompact abelian groups:

Problem 3.9. *(a) Which non-singular pseudocompact abelian groups are CvM-pseudocompact?*

(b) Which torsion pseudocompact abelian groups are CvM-pseudocompact?

3.2 The divisible weight

Definition 3.10. *Let G be a topological abelian group. The* divisible weight *of G is*

$$w_d(G) = \inf_{m \in \mathbb{N}_+} w(mG).$$

Lemma 3.11. *Let $p \in \mathbb{P}$. If K is a compact $\mathbb{Z}_p$-module, then $w_d(K) = \inf_{n \in \mathbb{N}} w(p^n K)$.*

Proof. If $m = p^k m_1$, where m_1 is coprime to p, then $mK = p^k K$. □

Since the next remark shows that the divisible weight coincides with the final rank for discrete abelian p-groups ($p \in \mathbb{P}$) of infinite final rank, the divisible weight can be viewed as a natural generalization of the final rank to all "sufficiently large" abelian (topological) groups.

Remark 3.12. If K is a compact $\mathbb{Z}_p$-module with $w_d(K) \geq \omega$ and $X = \widehat{K}$, then

$$\mathrm{fin}r(X) = w_d(X) = w_d(K).$$

In fact, since by Fact 1.33 $\widehat{p^n K} \cong p^n X$ for every $n \in \mathbb{N}$, $w(p^n K) = |p^n X| = w(p^n X)$ for every $n \in \mathbb{N}$. Hence $w_d(K) = w_d(X)$. Being $w_d(K) \geq \omega$, $|p^n X|$ is infinite and so $|p^n X| = r_p(p^n X)$ for each $n \in \mathbb{N}$, that is $\mathrm{fin}r(X)$ is infinite. It follows from Lemma 3.11 that $\mathrm{fin}r(X) = w_d(X)$.

The equality $w_d(K) = w_d(X)$ of Remark 3.12 can be proved in general for a topological abelian group which is either compact or discrete:

Theorem 3.13. *Let K be a topological abelian group which is either compact or discrete. Then $w_d(K) = w_d(\widehat{K})$.*

Proof. By Fact 1.33 $\widehat{m!K} \cong m!\widehat{K}$, for every $m \in \mathbb{N}_+$. Then

$$w(m!K) = |\widehat{m!K}| = w(\widehat{m!K}) = w(m!\widehat{K})$$

for every $m \in \mathbb{N}_+$. This shows that $w_d(K) = w_d(\widehat{K})$. □

Corollary 3.14. *Let K be a topological abelian group which is either compact or discrete. Then K is w-divisible if and only if $\widehat{K}$ is w-divisible (in other words $|m\widehat{K}| = |\widehat{K}| > \omega$ for every $m \in \mathbb{N}_+$).*

Proof. By Theorem 3.13 $w_d(K) = w_d(\widehat{K})$. Moreover $w(K) = w(\widehat{K})$. Then $w_d(K) = w(K) > \omega$ if and only if $w_d(\widehat{K}) = w(\widehat{K}) > \omega$. This means that K is w-divisible if and only if $\widehat{K}$ is w-divisible. □

Since Theorem 3.13 holds for topological abelian groups which are either compact or discrete, we think that it remains true also in the more general case of locally compact abelian groups. The following question is open.

Question 3.15. *Does Theorem 3.13 hold true for locally compact abelian groups?*

We study the behavior of the divisible weight. Obviously it is monotone under taking subgroups.

Lemma 3.16. *Let G be a topological abelian group and let H be subgroup of G. Then:*

(a) $w_d(H) \leq w_d(G)$;

(b) if H is dense in G, then $w_d(H) = w_d(G)$.

Proof. (a) is obvious.

(b) Observe that $w(mH) = w(mG)$ for every $m \in \mathbb{N}_+$, because the homomorphism $G \to mG$, defined by the multiplication by m, is continuous and so mH is dense in mG for every $m \in \mathbb{N}_+$ by Fact 2.6(a). In particular $w_d(H) = w_d(G)$. □

Proposition 3.17. *Let G and L be abelian groups such there exists a continuous surjective homomorphism $f : G \to L$. If G is precompact, then $w_d(G) \geq w_d(L)$.*

Proof. For every $m \in \mathbb{N}_+$ there exists a continuous surjective homomorphism $mG \to mL$ and so $w(mG) \geq w(mL)$ because mG and mL are precompact. □

The next example shows that in the previous proposition the hypothesis of precompactness cannot be removed (as in the known case of the weight).

Example 3.18. Let $\mathrm{id}_{\mathbb{Z}} : (\mathbb{Z}, \delta_{\mathbb{Z}}) \to \mathbb{Z}^{\#}$. This is a continuous isomorphism. Moreover $w_d(\mathbb{Z}^{\#}) = \mathfrak{c}$ as showed in Example 3.2 and $w_d(\mathbb{Z}, \delta_{\mathbb{Z}}) = |\mathbb{Z}| = \omega$.

Using the same notation of Proposition 3.17 $G = (\mathbb{Z}, \delta_{\mathbb{Z}})$ and $L = (\mathbb{Z}, \tau)$. The difference between $w(G)$ and $w(L)$ is maximal for a countable group. In fact, in general $w(L) \leq 2^{|L|} \leq 2^{|G|} = 2^{\omega} = \mathfrak{c}$ and in our case $w(L) = \mathfrak{c}$.

Lemma 3.19. *If $n \in \mathbb{N}_+$, $G_1, \ldots, G_n$ are topological abelian groups and $G = G_1 \times \ldots \times G_n$, then*

$$w_d(G) = \max\{w_d(G_1), \ldots, w_d(G_n)\}.$$

Lemma 3.19 works with a finite number of groups, but it fails to be true in general, that is taking infinitely many groups:

Remark 3.20. Consider the group $K = \prod_{p\in\mathbb{P}} \mathbb{Z}(p)^{\omega_1}$ and observe that $K_p = \mathbb{Z}(p)^{\omega_1}$, so $w_d(K_p) = 1$, for each $p \in \mathbb{P}$. Moreover $K \cong_{top} (\prod_{p\in\mathbb{P}} \mathbb{Z}(p))^{\omega_1}$. Then $w_d(K) = w(K) = \omega_1$.

It follows from this remark that for a totally disconnected compact abelian group $K \cong_{top} \prod_{p\in\mathbb{P}} K_p$ (see Remark 1.44(b)) the equality $w_d(K) = \sup_{p\in\mathbb{P}} w_d(K_p)$ does not hold in general. So this relation holds for $w(-)$ but not for $w_d(-)$.

It is important to observe that for a compact abelian group K

$$w_d(K) \geq w(c(K)),$$

because $c(K)$ is divisible, being compact and connected, and so $c(K) \leq mK$ for every $m \in \mathbb{N}_+$.

Lemma 3.21. *Let G be a topological abelian group such that $c(G)$ is compact. Then*

$$w_d(G) = \max\{w(c(G)), w_d(G/c(G))\}.$$

Proof. The condition $w_d(G) > w(c(G))$ is equivalent to $w(mG) > w(c(G))$ for every $m \in \mathbb{N}_+$. Since $c(G)$ is connected and compact, $c(G)$ is divisible and so $c(G) \leq mG$ for every $m \in \mathbb{N}_+$. Then

$$w(mG) = w((mG)/c(G)) \cdot w(c(G))$$

and it follows that $w(mG) = w((mG)/c(G))$ for every $m \in \mathbb{N}_+$. Since $(mG)/c(G) = m(G/c(G))$ and $w_d(G) = \inf_{m\in\mathbb{N}_+} w(mG)$, this yields the equalities

$$w_d(G) = \inf_{m\in\mathbb{N}_+} w((mG)/c(G)) = \inf_{m\in\mathbb{N}_+} w(m(G/c(G))) = w_d(G/c(G)),$$

which complete the proof. □

This lemma can be improved for locally compact abelian groups:

Proposition 3.22. *Let L be a locally compact abelian group. Then*

$$w_d(L) = \max\{w(c(L)), w_d(L/c(L))\}.$$

Proof. By Theorem 1.27 L is topologically isomorphic to $\mathbb{R}^n \times K$, where $n \in \mathbb{N}$ and K has a compact open subgroup K_0. Then $w_d(L) = \omega \cdot w_d(K)$ by Lemma 3.19. We have

$$c(L) \cong_{top} \mathbb{R}^n \times c(K)$$

where $c(K) = c(K_0)$ is compact, since K_0 is open in K.

By Lemma 3.21 $w_d(K) = \max\{w(c(K)), w_d(K/c(K))\}$. Suppose that $w_d(L) > w(c(L))$. Then $w_d(L) = w_d(K)$ and consequently $w_d(K) > w(c(K))$, that is $w_d(K) = w_d(K/c(K))$. Then $w_d(L) = w_d(K/c(K))$. But $K/c(K)$ is topologically isomorphic to $L/c(L)$ and hence $w_d(L) = w_d(L/c(L))$. □

Since the minimal positive integer m_0 in the following lemma is uniquely determined by the group G, we denote it by $m_d(G)$.

Lemma 3.23. *Let G be a topological abelian group. Then there exists a minimal positive integer $m_0 \in \mathbb{N}_+$ such that if $H = m_0G$, then:*

(a) $w_d(H) = w(H) = w_d(G)$;

(b) $r_0(H) = r_0(G)$;

(c) if G is pseudocompact, then H is pseudocompact too.

Proof. Since $\{w(mG) : m \in \mathbb{N}_+\}$ is a set of cardinals, there exists $m_0 \in \mathbb{N}_+$ such that $w_d(G) = w(m_0G)$ and $w(m_0G) \leq w(kG)$ for every $k \in \mathbb{N}_+$. Let $H = m_0G$. Then $w(H) = w_d(G)$. Obviously $r_0(H) = r_0(G)$ and if G is pseudocompact, by Fact 2.6(b) H is pseudocompact as well being continuous image of G. □

Remark 3.24. Let G be a topological abelian group and let D be a dense subgroup of of G. As noted in the proof of Lemma 3.16, $w(mD) = w(mG)$ for every $m \in \mathbb{N}_+$. The sequences $\{w(mD) : m \in \mathbb{N}_+\}$ and $\{w(mG) : m \in \mathbb{N}_+\}$ have the same minimum and so $m_d(D) = m_d(G)$.

The counterpart of Lemma 3.19 for $m_d(-)$ fails to be true:

Example 3.25. Let $G_1 = \mathbb{Z}(2)^{\mathfrak{c}^+} \times \mathbb{T}^{\mathfrak{c}}$ and $G_2 = \mathbb{Z}(3)^{\mathfrak{c}} \times \mathbb{T}$. Then $2 = m_d(G_1) = m_d(G_1 \times G_2) < \max\{m_d(G_1), m_d(G_2)\} = m_d(G_2) = 3$.

For $G = G_1 \times \ldots \times G_n$, where $n \in \mathbb{N}_+$ and $G_1, \ldots, G_n$ are topological abelian groups, it is easy to see that $m_d(G) \geq \min\{m_d(G_1), \ldots, m_d(G_n)\}$.

3.3 w-Divisibility

Definition 3.26. *A topological abelian group G is* w-divisible *if $w_d(G) = w(G) > \omega$.*

Since a topological abelian group is w-divisible if and only if $w(mG) = w(G) > \omega$ for every $m \in \mathbb{N}_+$, this definition is justified by the fact that an abelian group G is *divisible* if and only if $G = mG$ for every integer $m > 0$ (see Example 3.29(a) for more details).

Fact 3.27. *If D is a dense subgroup of a topological abelian group G, then D is w-divisible if and only if G is w-divisible.*

Proof. Since D is dense in G, $w_d(D) = w_d(G)$ by Lemma 3.16(b). □

We give some examples of w-divisible groups.

Example 3.28. (a) Every topological divisible abelian group of uncountable weight is w-divisible.

(b) Connected compact abelian groups of uncountable weight are divisible and so w-divisible by (a).

(c) Connected precompact abelian groups G of uncountable weight are w-divisible; in fact, $K = \widetilde{G}$ is connected and so divisible (being compact), hence w-divisible by (b). Since G is dense in $\widetilde{G}$, Fact 3.27 applies to conclude that G is w-divisible.

(d) In general a connected abelian group of uncountable weight need not be w-divisible. Actually, there exist connected abelian groups of every prime exponent [3].

(e) In [29, Definition 1.3] a topological group G was defined to be *almost connected* whenever $c(G) \in \Lambda(G)$. Almost connected pseudocompact abelian groups of uncountable weight are examples of w-divisible groups.

(f) If L is a locally compact abelian group and it is connected, then L is divisible: by Theorem 1.27 $L \cong_{top} \mathbb{R}^n \times K$, where $n \in \mathbb{N}$ and K has a compact open subgroup K_0. Since L is connected, $K = c(K) \cong_{top} c(K_0)$ is connected and compact, so divisible. Consequently L is divisible too. By (a), if $w(L) > \omega$, then L is w-divisible.

As shown by (d), connectedness is not sufficient alone to have w-divisibility, but from (c) to (e) we weaken connectedness to almost connectedness and strengthen the compactness-like property from precompactness to pseudocompactness, obtaining two different sufficient conditions for w-divisibility.

Example 3.29. (a) Let $K = \prod_{i \in I} K_i$, where each K_i is a metrizable compact non-torsion abelian group and I is an uncountable set of indices. Then K is w-divisible of divisible weight $w_d(K) = |I|$, because $mK = \prod_{i \in I} mK_i$ has weight $|I|$ for every $m \in \mathbb{N}_+$, since $mK_i \neq \{0\}$ for every $i \in I$.

(b) A product of the form $K = \prod_{i \in I} K_i$, where each K_i is a non-trivial metrizable compact abelian group and I is an uncountable set of indices, can be w-divisible even if some of the metrizable compact abelian groups are torsion. For example $\mathbb{T}^{\omega_1} \times \mathbb{Z}(p)$, where $p \in \mathbb{P}$, is w-divisible of divisible weight ω_1.

In some case all the metrizable compact abelian groups K_i are torsion and the product K is still w-divisible. For example $\prod_{p \in \mathbb{P}} \mathbb{Z}(p)^{\omega_1}$ is w-divisible: it is isomorphic to $S_{\mathbb{P}}^{\omega_1}$, where $S_{\mathbb{P}}$ is compact metrizable non-torsion.

(c) We call *w-divisible product* a product $\prod_{i \in I} K_i$, where each K_i is a metrizable compact non-torsion abelian group and I is uncountable.

As noted in (b) these products are not the unique products that are w-divisible, but they are sufficient for what we do in the following chapter.

(d) A particular case of w-divisible products are *w-divisible powers*, that is powers S^κ, where S is a metrizable compact non-torsion abelian group and κ is an uncountable cardinal.

The w-divisible products and the w-divisible powers are the main examples of w-divisible groups and we make essential use of them in the following chapters.

Lemma 3.30. *Let $G = \prod_{i \in I} G_i$ where each G_i is a topological abelian group.*

(a) If G_i is w-divisible for every $i \in I$, then G is w-divisible.

(b) If I is finite, then G is w-divisible if and only if there exists $i \in I$ such that $w_d(G_i) = w(G) > \omega$ (in particular this G_i is w-divisible).

Proof. (a) For every $i \in I$ we have $w_d(G_i) = w(G_i) > \omega$. Then $w(G) > \omega$ and for every $m \in \mathbb{N}_+$

$$\begin{aligned} w(mG) = w\left(\prod_{i\in I} mG_i\right) = |I| \cdot \sup_{i\in I} w(mG_i) = \\ = |I| \cdot \sup_{i\in I} w_d(G_i) = |I| \cdot \sup_{i\in I} w(G_i) = w(G). \end{aligned}$$

(b) By Lemma 3.19 there exists $i \in I$ such that $w_d(G) = w_d(G_i)$. In case G is w-divisible, $w_d(G) = w(G) > \omega$ and so $w_d(G_i) = w(G) > \omega$. To prove the converse implication assume that there exists $i \in I$ such that $w_d(G_i) = w(G) > \omega$. In particular $w_d(G) = w(G) > \omega$ by Lemma 3.16(a), that is G is w-divisible. □

The converse implication of Lemma 3.30(a) does not hold in general, even in case I is finite:

Example 3.31. Let $G = G_1 \times G_2$, where $G_1 = \mathbb{T}^{\mathfrak{c}}$ and $G_2 = \mathbb{Z}(p)$. Then

$$\mathfrak{c} = w(G) = w_d(G) = w(G_1) = w_d(G_1)$$

and G_2 is far from being w-divisible.

So we have the monotonicity of w-divisibility for subgroups but not for quotients. Indeed, the quotient of a w-divisible group need not be w-divisible; e.g., for $p \in \mathbb{P}$ take $\mathbb{Z}_p^{\mathfrak{c}}$, which has $\mathbb{Z}(p)^{\mathfrak{c}}$ as a quotient.

3.4 κ-Singularity

While w-divisibility goes in the opposite direction with respect to singularity (w-divisible abelian groups are non-singular), we define different levels of singularity, one for each infinite cardinal κ.

Definition 3.32. *Let κ be an infinite cardinal. A topological abelian group G is* κ-singular *if $w_d(G) \leq \kappa$.*

Observe that a topological abelian group G is κ-singular if and only if there exists $m \in \mathbb{N}_+$ such that $w(mG) \leq \kappa$ (see Lemma 3.34) and so ω-singular abelian groups are precisely singular abelian groups. Moreover every topological abelian group G is $w(G)$-singular.

3.4.1 Characterization and properties

Example 3.33. Let $K = \prod_{i \in I} K_i$ be a w-divisible product with $|I| = \kappa > \omega$. Then K is w-divisible and non-λ-singular for every cardinal $\omega \leq \lambda < \kappa$.

In the following two lemmas we give some conditions equivalent to κ-singularity. In the second lemma we need the hypothesis that the topological abelian groups are κ-pseudocompact. For $\kappa = \omega$ we find [45, Lemma 2.5], which generalized [29, Lemma 4.1].

Lemma 3.34. *Let κ be an infinite cardinal and let G be a topological abelian group. Then the following conditions are equivalent:*

(a) G is κ-singular;

(b) there exists $m \in \mathbb{N}_+$ such that $w(mG) \leq \kappa$;

(c) $\widetilde{G}$ is κ-singular.

Proof. (a)$\Leftrightarrow$(b) is obvious.

(a)$\Leftrightarrow$(c) Since G is dense in $\widetilde{G}$, $w_d(G) = w_d(\widetilde{G})$ by Lemma 3.16(b). $\square$

The condition in (c) of the next lemma can be considered also for non-necessarily abelian topological groups, so that κ-singularity could be defined in the general case of topological groups.

Lemma 3.35. *Let κ be an infinite cardinal and let G be a κ-pseudocompact abelian group. Then the following conditions are equivalent:*

(a) G is κ-singular;

(b) there exists $m \in \mathbb{N}_+$ such that $G[m] \in \Lambda_\kappa(G)$;

(c) G has a closed torsion G_κ-subgroup;

(d) there exists $N \in \Lambda_\kappa(G)$ such that $N \subseteq t(G)$.

Proof. Let $m \in \mathbb{N}_+$ and let $\varphi_m : G \to G$ be the continuous homomorphism defined by $\varphi_m(x) = mx$ for every $x \in G$. Then $\ker \varphi_m = G[m]$ and $\varphi_m(G) = mG$. Let $i : G/G[m] \to mG$ be the continuous isomorphism such that $i \circ \pi = \varphi_m$, where $\pi : G \to G/G[m]$ is the canonical projection.

(a)$\Rightarrow$(b) By Lemma 3.34 there exists $m \in \mathbb{N}_+$ such that $w(mG) \leq \kappa$. Then $\psi(mG) \leq \kappa$. Since $i : G/G[m] \to mG$ is a continuous isomorphism, so $\psi(G/G[m]) \leq \kappa$. This implies that $G[m]$ is a G_κ-set of G.

(b)$\Rightarrow$(a) Suppose that $G[m] \in \Lambda_\kappa(G)$. Then the quotient $G/G[m]$ has weight $\leq \kappa$, hence it is compact. By Theorem 1.28 the isomorphism $i : G/G[m] \to mG$ is also open and consequently it is a topological isomorphism. Then $w(mG) \leq \kappa$. By Lemma 3.34 G is κ-singular.

(b)$\Rightarrow$(c) and (c)$\Leftrightarrow$(d) are obvious.

(d)$\Rightarrow$(b) By Corollary 2.21(b) N is κ-pseudocompact and so N is bounded torsion by Fact 1.36. Therefore there exists $m \in \mathbb{N}_+$ such that $mN = \{0\}$. Thus $N \subseteq G[m]$ and so $G[m] \in \Lambda_\kappa(G)$ by Corollary 2.21(c). □

The class of κ-singular groups is stable for subgroups. For dense subgroups we have the following:

Lemma 3.36. *Let κ be an infinite cardinal. Let D be a dense subgroup of a topological abelian group G. Then G is κ-singular if and only if D is κ-singular.*

Proof. By Lemma 3.16(b) $w_d(G) = w_d(D)$. Then $w_d(G) \leq \kappa$ if and only if $w_d(D) \leq \kappa$, i.e., G is κ-singular if and only if D is κ-singular. □

For $\kappa = \omega$ the following lemma is [26, Lemma 3.4].

Lemma 3.37. *Let κ be an infinite cardinal, let G be a precompact abelian group and let N be a closed subgroup of G.*

(a) If G is κ-singular, then N and G/N are κ-singular.

(b) If $G = K$ is compact, then K is κ-singular if and only if N and K/N are κ-singular.

Proof. (a) follows from Lemma 3.16(a) and Proposition 3.17.

(b) The necessity is (a). Let us prove the sufficiency. Suppose that both N and K/N are κ-singular. Let X and Y be the duals of K and K/N respectively. By Pontryagin duality

$$K \text{ is } \kappa\text{-singular if and only if } |kX| \leq \kappa \text{ for some } k \in \mathbb{N}_+. \tag{3.1}$$

By Pontryagin duality Y can be identified with a subgroup of X such that $X/Y \cong \widehat{N}$. Since N and K/N are κ-singular by the hypothesis, it follows that $|m_1(X/Y)| \leq \kappa$ for some $m_1 \in \mathbb{N}_+$ and $|m_2 Y| \leq \kappa$ for some $m_2 \in \mathbb{N}_+$. Thus there exists $m \in \mathbb{N}_+$ ($m \geq \max m_1, m_2$) such that $|m(X/Y)| \leq \kappa$ and $|mY| \leq \kappa$. Since $m(X/Y) = (mX+Y)/Y$ has cardinality $\leq \kappa$, mX is contained in a union $\bigcup_{i\in I}(z_i + Y)$, where $z_i \in mX$ and $|I| \leq \kappa$. As $|mY| \leq \kappa$, m^2X is contained in the union $\bigcup_{i\in I}(mz_i + mY)$, which has size $\leq \kappa$. So $|m^2X| \leq \kappa$, i.e., K is κ-singular by (3.1). □

In the following result we prove that the class of κ-singular abelian groups is closed also under finite products.

Lemma 3.38. *Let κ be an infinite cardinal. Finite products of κ-singular abelian groups are κ-singular.*

Proof. Let $G = G_1 \times \ldots \times G_n$ be such that $n \in \mathbb{N}_+$ and G_i is a κ-singular abelian group for every $i \in \{1, \ldots, n\}$, that is $w_d(G_i) \leq \kappa$ for every $i \in \{1, \ldots, n\}$. By Lemma 3.19 $w_d(G) = \max\{w_d(G_1), \ldots, w_d(G_n)\} \leq \kappa$. Then $w_d(G) \leq \kappa$ and G is κ-singular. □

Lemma 3.39. *Let κ be an infinite cardinal and let G be a κ-singular abelian group such that $c(G)$ is compact. Then $w(c(G)) \leq \kappa$.*

Proof. Since G is κ-singular, by Lemma 3.34 there exists $m \in \mathbb{N}_+$ such that $w(mG) \leq \kappa$. But $c(G)$ is divisible and so $c(G) = mc(G) \subseteq mG$. Therefore $w(c(G)) \leq \kappa$. $\square$

3.4.2 Measuring κ-singularity

Let K be a totally dense compact abelian group. Then $K \cong_{top} \prod_{p\in\mathbb{P}} K_p$ by Remark 1.44(b). We define

$$P_{s,\kappa}(K) = \{p \in \mathbb{P} : w_d(K_p) \leq \kappa\} \text{ and } P_{m,\kappa}(K) = \{p \in \mathbb{P} : w(K_p) \leq \kappa\}.$$

Observe that $P_{m,\kappa}(K) \subseteq P_{s,\kappa}(K) \subseteq \mathbb{P}$. Then $w(K) \leq \kappa$ if and only if $P_{m,\kappa}(K) = \mathbb{P}$. Furthermore $P_{s,\omega}(K) = P_s(K)$ and $P_{m,\omega}(K) = P_m(K)$, where $P_s(K)$ and $P_m(K)$ were introduced in [26].

Suppose that $w(K) > \kappa$. Let

$$K_{m,\kappa} = \prod_{p\in P_{m,\kappa}(K)} K_p, \quad K_{s,\kappa} = \prod_{p\in P_{s,\kappa}(K)\setminus P_{m,\kappa}(K)} K_p \quad \text{and} \quad K_{r,\kappa} = \prod_{p\in\mathbb{P}\setminus P_{s,\kappa}(K)} K_p.$$

If $P_{m,\kappa}(K)$ (respectively, $P_{s,\kappa}(K) \setminus P_{m,\kappa}(K)$ and $\mathbb{P} \setminus P_{m,\kappa}(K)$) is empty, put $K_{m,\kappa} = \{0\}$ (respectively, $K_{s,\kappa} = \{0\}$ and $K_{r,\kappa} = \{0\}$). Then

$$K \cong_{top} K_{m,\kappa} \times K_{s,\kappa} \times K_{r,\kappa}$$

and $w(K_{m,\kappa}) \leq \kappa$, while $K_{s,\kappa}$ is κ-singular.

Lemma 3.40. *Let $p \in \mathbb{P}$ and let K be a compact $\mathbb{Z}_p$-module.*

(a) If K/pK is finite, then $K \cong_{top} \mathbb{Z}_p^m \times F$, where $m \in \mathbb{N}$ and F is a finite p-group.

(b) If K/pK is infinite, then $w(K) = w(K/pK)$.

Proof. By Fact 1.42(b) $X = \widehat{K}$ is a p-group and $X[p]$ is the dual of K/pK by Fact 1.32(a).

(a) Assume that K/pK is finite. Then $X[p]$ is finite; hence $X[p]$ is isomorphic to a subgroup of $\mathbb{Z}(p^\infty)^n$, where $n = r_p(X)$. Since $\mathbb{Z}(p^\infty)^n$ is divisible, this immersion can be extended to $j : X \to \mathbb{Z}(p^\infty)^n$. Now j is injective because if $x \in \ker j$ and $x \neq 0$ then we can suppose without loss of generality that $px = 0$, that is $x \in X[p]$ and this is not possible. If $d(X)$ is the maximal divisible subgroup of X, then $d(X)$ is isomorphic to $\mathbb{Z}(p^\infty)^m$ with $m \leq n$. Thus $X \cong \mathbb{Z}(p^\infty)^m \oplus X_1$ where X_1 has no divisible subgroups (i.e., it is reduced) and so X_1 is finite because it has finite p-rank [42]. By Pontryagin duality $K \cong_{top} \widehat{X} \cong_{top} \mathbb{Z}_p^m \times F$, where $F = \widehat{X_1} \cong X_1$ is finite.

(b) follows from the fact that X is a p-group such that $|X| = |X[p]|$, when $X[p]$ is infinite. $\square$

For $\kappa = \omega$ the following proposition is [26, Proposition 4.3].

Proposition 3.41. *Let κ be an infinite cardinal and let K be a totally disconnected compact abelian group such that $\mathbb{P} \setminus P_{m,\kappa}(K)$ is infinite. Then there exists a continuous surjective homomorphism of K onto a w-divisible power S^{κ^+}.*

Proof. By Lemma 3.40 $\mathbb{P} \setminus P_{m,\kappa}(K) = \{p_n : n \in \mathbb{N}_+\} \subseteq \mathbb{P}$ is an infinite subset such that $w(K/p_nK) > \kappa$ for every $n \in \mathbb{N}_+$. Since, for $n \in \mathbb{N}_+$, K/p_nK is a compact abelian group of exponent p_n and weight $> \kappa$, it is topologically isomorphic to $\mathbb{Z}(p_n)^{w(K/p_nK)}$, where $w(K/p_nK) \geq \kappa^+$. This yields that for every $n \in \mathbb{N}_+$ there exists a continuous surjective homomorphism $f_n : K/p_nK \to \mathbb{Z}(p_n)^{\kappa^+}$. Moreover $K_q = p_nK_q \leq p_nK$ for every prime $q \neq p_n$. Therefore p_nK coincides with the subgroup $p_nK_{p_n} \times \prod_{q\in\mathbb{P}, q\neq p_n} K_q$ of K. So $K/p_nK \cong_{top} K_{p_n}/p_nK_{p_n}$ and consequently f_n can be identified with $f'_n : K_{p_n}/p_nK_{p_n} \to \mathbb{Z}(p_n)^{\kappa^+}$. Then

$$f = \prod_{n\in\mathbb{N}_+} f'_n : \prod_{n\in\mathbb{N}_+} K_{p_n}/p_nK \to \prod_{n\in\mathbb{N}_+} \mathbb{Z}(p_n)^{\kappa^+} \cong_{top} S_\pi^{\kappa^+}$$

is a continuous surjective homomorphism. Hence the composition of the continuous surjective homomorphism $K \cong_{top} \prod_{p\in\mathbb{P}} K_p \to \prod_{n\in\mathbb{N}_+}^{\infty} K_{p_n}/p_nK_{p_n}$ with f is a continuous surjective homomorphism $K \to S_\pi^{\kappa^+}$. □

In the next lemma we show that the sets $P_{m,\kappa}(K)$ and $P_{s,\kappa}(K)$ describe completely the κ-singularity of a compact abelian group K.

Lemma 3.42. *Let κ be an infinite cardinal and let K be a totally disconnected compact abelian group. Then K is κ-singular if and only if $P_{s,\kappa}(K) = \mathbb{P}$ and $\mathbb{P} \setminus P_{m,\kappa}(K)$ is finite.*

Proof. Suppose that $P_{s,\kappa}(K) = \mathbb{P}$ and $P_{s,\kappa}(K) \setminus P_{m,\kappa}(K)$ is finite. Then $K = K_{m,\kappa} \times K_{s,\kappa}$, where $K_{m,\kappa}$ is metrizable and $K_{s,\kappa}$ is κ-singular by Lemma 3.37. Then K is κ-singular by Lemma 3.37 again.

To prove the converse implication, suppose that $\mathbb{P} \setminus P_{m,\kappa}(K)$ is infinite. By Proposition 3.41 there exists a continuous surjective homomorphism of K onto a w-divisible power S^{κ^+}. Since S^{κ^+} is non-κ-singular, then K is non-κ-singular as well by Lemma 3.37. If $\mathbb{P} \neq P_{s,\kappa}(K)$ then again Lemma 3.37 implies that K is non-κ-singular. □

3.5 The stable weight

Lemma 3.21 splits the study of the divisible weight of a compact abelian group in two cases: the connected case is trivial as connected groups are already w-divisible; the more complicated totally disconnected case is analyzed in this section.

Let K be a totally disconnected compact abelian group. Then $K \cong_{top} \prod_{p\in\mathbb{P}} K_p$ by Remark 1.44(b) and in this section we identify the two groups, considering $K = \prod_{p\in\mathbb{P}} K_p$. Let

$$\kappa_p = w(K_p)$$

for each $p \in \mathbb{P}$. In particular

$$w(K) = \sup_{p\in\mathbb{P}} \kappa_p.$$

Definition 3.43. *The* stable weight *of a totally disconnected compact abelian group K is $w_s(K) = w_d(K)$, when $\mathbb{P} \setminus P_m(K)$ is finite, otherwise let*

$$w_s(K) = \inf_{n \in \mathbb{N}} w\left(\prod_{p \in \mathbb{P}, p > n} K_p\right).$$

From the definition it follows that in case $\mathbb{P} \setminus P_m(K)$ is infinite then

$$w_s(K) = \inf_{n \in \mathbb{N}} \sup_{p \in \mathbb{P}, p > n} \kappa_p > \omega.$$

Since $\sup_{p \in \mathbb{P}, p > n} \kappa_p$ is a decreasing sequence of cardinals, it stabilizes and so there exists $n_0 \in \mathbb{N}$ such that

$$w_s(K) = w\left(\prod_{p \in \mathbb{P}, p > n_0} K_p\right) = \sup_{p \in \mathbb{P}, p > n_0} \kappa_p.$$

In analogy with w-divisible abelian groups, which have maximal divisible weight, we introduce totally disconnected compact abelian groups that have maximal stable weight:

Definition 3.44. *A totally disconnected compact abelian group is* stable *if $w_s(K) = w(K) > \omega$.*

Lemma 3.45. *If K is a totally disconnected compact abelian group, then $w_s(K) \leq w_d(K)$. In particular, K stable implies K w-divisible.*

Proof. By the definition $w_s(K) = w_d(K)$ if $\mathbb{P} \setminus P_m(K)$ is finite. So suppose that $\mathbb{P} \setminus P_m(K)$ is infinite. As noted in the foregoing part of this section, there exists $n_0 \in \mathbb{N}$ such that

$$w_s(K) = \sup_{p \in \mathbb{P}, p > n_0} \kappa_p.$$

By Lemma 3.23 $w_d(K) = w(H)$, where $H = m_d(K)K$. Take $n_1 \in \mathbb{N}$ such that $n_1 \geq \max\{n_0, m_d(K)\}$. Consequently $\prod_{p \in \mathbb{P}, p > n_1} K_p \leq H$ and so

$$w_s(K) = w\left(\prod_{p \in \mathbb{P}, p > n_1} K_p\right) \leq w(H) = w_d(K).$$

$\square$

The inequality of this lemma can be strict only in case $w_d(K) = w_d(K_p)$ for some $p \in \mathbb{P}, p < n_0$: if $w_d(K) > w_s(K)$, then

$$w_d(K) = w(m_d(K)K) = w\left(\prod_{p \in \mathbb{P}, p \leq n_0} m_d(K)K_p\right) \cdot w_s(K) = w\left(\prod_{p \in \mathbb{P}, p \leq n_0} m_d(K)K_p\right)$$

by Lemma 3.19; since this is a finite product, $w_d(K) = w_d(K_p)$ for some prime $p \leq n_0$ again by Lemma 3.19.

Let K be a totally disconnected compact abelian group. By the definition $w_s(K) = w_d(K) \leq \omega$ when $\mathbb{P}\setminus P_m(K)$ is finite and in particular when K is metrizable. For a better understanding of $w_s(K)$ in the uncountable case assume that $\mathbb{P} \setminus P_m(K)$ is infinite and define the *d-spectrum* of K as

$$\Pi(K) = \{p \in \mathbb{P} \setminus P_m(K) : \kappa_p \leq w_s(K)\} = \{p \in \mathbb{P} : \omega < \kappa_p \leq w_s(K)\}.$$

The complement of $\Pi(K)$ in $\mathbb{P} \setminus P_m(K)$ is

$$\pi_f(K) = \{p \in \mathbb{P} \setminus P_m(K) : \kappa_p > w_s(K)\}.$$

Since $\pi_f(K)$ is finite by the definition of the stable weight, $\Pi(K)$ is infinite. Moreover we have the following partition:

$$\Pi(K) = \pi^*(K) \cup \pi(K),$$

where

$$\pi^*(K) = \{p \in \mathbb{P} \setminus P_m(K) : \kappa_p < w_s(K)\} \quad \text{and} \quad \pi(K) = \{p \in \mathbb{P} \setminus P_m(K) : \kappa_p = w_s(K)\}.$$

So we have the partition $\mathbb{P} = P_m(K) \cup \pi^*(K) \cup \pi(K) \cup \pi_f(K)$ and

$$K = \prod_{p\in P_m(K)} K_p \times \prod_{p\in\pi^*(K)} K_p \times \prod_{p\in\pi(K)} K_p \times \prod_{p\in\pi_f(K)} K_p.$$

Let

$$met(K) = \prod_{p\in P_m(K)} K_p, \quad sc(K) = \prod_{p\in\Pi(K)} K_p \text{ and } nst(K) = \prod_{p\in\pi_f(K)} K_p.$$

Then

$$K = met(K) \times sc(K) \times nst(K),$$

where $met(K)$ is metrizable, while $nst(K) = \prod_{p\in\pi_f(K)} K_p$ has no stable subgroups, because $nst(K)$ is a finite product and since every closed subgroup N of $nst(K)$ is of the form $N = \prod_{p\in\pi_f(K)} N_p$ by Remark 1.44(b). We see in Lemma 3.47(a) that the *stable core* $sc(K)$ of K is stable when $\Pi(K) \neq \emptyset$.

For sake of completeness, set

$$\Pi(K) = \emptyset \text{ and } \pi_f(K) = \mathbb{P} \setminus P_m(K), \quad \text{whenever } |\mathbb{P} \setminus P_m(K)| < \infty.$$

Claim 3.46. *A totally disconnected compact abelian group K is stable if and only if $\mathbb{P} \setminus P_m(K)$ is infinite and $\pi_f(K) = \emptyset$.*

Proof. Suppose that $p \in \pi_f(K)$. Then $w_s(K) < \kappa_p \leq w(K)$ and so K is not stable. If $\mathbb{P} \setminus P_m(K)$ is finite, then $w_s(K) \leq \omega$ and K is not stable.

Assume that $\mathbb{P} \setminus P_m(K)$ is infinite and $\pi_f(K)$ is empty. Then

$$w_s(K) = \sup_{p\in\mathbb{P}\setminus P_m(K)} \kappa_p = w(K) > \omega,$$

that is K is stable. □

Lemma 3.47. *Let K be a non-singular totally disconnected compact abelian group. Then:*

(a) if $\mathbb{P} \setminus P_m(K)$ is infinite, then $sc(K)$ is stable, so $w_s(sc(K)) = w(sc(K)) = w_s(K)$;

(b) either $w_d(K) = w_d(K_p)$ for some $p \in \mathbb{P}$ or $w_d(K) = w_s(K) = \sup_{p\in\Pi(K)} \kappa_p$.

Proof. (a) Since $\mathbb{P} \setminus P_m(K) = \mathbb{P} \setminus P_m(sc(K))$ is infinite and $\pi_f(sc(K)) = \emptyset$, Claim 3.46 applies.

(b) Suppose that $w_d(K) > w_d(K_p)$ for every $p \in \mathbb{P}$. Then $\mathbb{P} \setminus P_m(K)$ is infinite and so also $\Pi(K)$ is infinite.

By (a) $sc(K)$ is stable and

$$w_s(sc(K)) = w_d(sc(K)) = w(sc(K)) = \sup_{p\in\Pi(K)} \kappa_p = w_s(K).$$

Since

$$w_d(K) = \max\{w_d\,(nst(K)), w_d(sc(K))\}$$

by Lemma 3.19, and $w_d(K) > w_d(K_p)$ for all $p \in \mathbb{P}$ by our hypothesis, it follows that

$$w_d(K) = w_d(sc(K)).$$

Therefore $w_d(K) = w_d(sc(K)) = w(sc(K)) = w_s(K)$. □

We see in Chapter 4 that when $w_d(K) > w_d(nst(K))$, the stable core $sc(K)$ plays an essential role as far as projections on products are concerned.

3.6 Super-κ-singularity

For every infinite cardinal κ we introduce another level of singularity, which implies κ-singularity. Indeed in Theorem C we use super-singular groups, which are necessarily singular. Since we want to generalize Theorem C proving Theorem C$^\kappa$, we have to define the counterpart of this concept for every κ. To define it analogously as we do for κ-singularity, we first introduce another cardinal invariant based on the weight:

Definition 3.48. *Let G be a topological abelian group. The* super-divisible weight *of G is*

$$w_{sd}(G) = \inf\{w(p_1 \cdot \ldots \cdot p_n G) : p_1, \ldots, p_n \in \mathbb{P} \text{ are distinct},\ n \in \mathbb{N}_+\}.$$

Observe that $w_{sd}(G) = \inf\{w(mG) : m \in \mathbb{N}_+,\ m \text{ square free}\}$.

Remark 3.49. The super-divisible weight has properties analogous to those of the divisible weight. By the property of cardinal numbers there exist $p_1, \ldots, p_{n_0}$, with $n_0 \in \mathbb{N}_+$, distinct primes such that $w_{sd}(G) = w(p_1 \cdot \ldots \cdot p_{n_0} G)$. In analogy with what is done for $w_d(-)$ (see Lemma 3.23), we call $m_{sd}(G) = p_1 \cdot \ldots \cdot p_{n_0}$.

Another case in which $w_{sd}(-)$ behaves analogously to $w_d(-)$ is that of Lemma 3.16 and the proof is very similar. In fact, if G is a topological abelian group and H a subgroup of G, then:

(a) $w_{sd}(H) \leq w_{sd}(G)$;

(b) if H is dense in G, then $w_{sd}(H) = w_{sd}(G)$.

For G topological abelian group

$$w_{sd}(G) \geq w_d(G)$$

and the inequality can be strict:

Example 3.50. Let $K = \mathbb{Z}(p^2)^\kappa$, where κ is an infinite cardinal. Then $w_{sd}(K) = \kappa$, while $w_d(K) = 0$.

Using the super-divisible weight we introduce the following concept. For $\kappa = \omega$ we find precisely super-singular groups (see Lemma 3.52).

Definition 3.51. *Let κ be an infinite cardinal. A topological group G is* super-κ-singular *if there exist $p_1, \dots, p_n$, with $n \in \mathbb{N}_+$, distinct primes such that $w_{sd}(G) \leq \kappa$.*

The next lemmas give conditions equivalent to super-κ-singularity. They are the counterpart of Lemmas 3.34 and 3.35.

Lemma 3.52. *Let κ be an infinite cardinal and let G be a topological abelian group. Then the following conditions are equivalent:*

(a) G is super-κ-singular;

(b) there exist distinct primes $p_1, \dots, p_n$, with $n \in \mathbb{N}_+$, such that $w(p_1 \cdot \ldots \cdot p_n G) \leq \kappa$;

(c) $\widetilde{G}$ is super-κ-singular.

Proof. (a)$\Leftrightarrow$(b) is clear, because $m_{sd}(\widetilde{G}) = p_1 \cdot \ldots \cdot p_n$.

(b)$\Leftrightarrow$(c) Since G is dense in $\widetilde{G}$, it follows that $w_{sd}(G) = w_{sd}(\widetilde{G})$ as observed in Remark 3.49(b). $\square$

Lemma 3.53. *Let κ be an infinite cardinal and let G be a κ-pseudocompact abelian group. Then the following conditions are equivalent:*

(a) G is super-κ-singular;

(b) there exist distinct primes $p_1, \dots, p_n$, with $n \in \mathbb{N}_+$, such that $G[p_1 \cdot \ldots \cdot p_n] \in \Lambda_\kappa(G)$;

(c) G has a closed G_κ-subgroup of exponent $p_1 \cdot \ldots \cdot p_n$, where $p_1, \dots, p_n$, with $n \in \mathbb{N}_+$, are distinct primes;

(d) there exists $N \in \Lambda_\kappa(G)$ such that $N \subseteq \mathrm{Soc}(G)$.

Proof. Let $m \in \mathbb{N}_+$ and let $\varphi_m : G \to G$ be the continuous homomorphism defined by $\varphi_m(x) = mx$ for every $x \in G$. Then $\ker \varphi_m = G[m]$ and $\varphi_m(G) = mG$. Let $i : G/G[m] \to mG$ be the continuous isomorphism such that $i \circ \pi = \varphi_m$, where $\pi : G \to G/G[m]$ is the canonical projection.

(a)$\Rightarrow$(b) By Lemma 3.52 there exist $p_1, \dots, p_n \in \mathbb{P}$ distinct primes such that $w(p_1 \cdot \ldots \cdot p_n G) \leq \kappa$. Let $m = p_1 \cdot \ldots \cdot p_n$. Then $\psi(mG) \leq \kappa$. Since $i : G/G[m] \to mG$ is a continuous isomorphism $\psi(G/G[m]) \leq \kappa$. This implies that $G[m]$ is a G_κ-set of G.

(b)$\Rightarrow$(a) Let $m = p_1 \cdot \ldots \cdot p_n$. Suppose that $G[m] \in \Lambda_\kappa(G)$. Then the quotient $G/G[m]$ has weight $\leq \kappa$, hence it is compact. By Theorem 1.28 the isomorphism i is also open and consequently it is a topological isomorphism. Then $w(mG) \leq \kappa$ and so G is super-κ-singular by Lemma 3.52.

(b)$\Rightarrow$(c) and (c)$\Leftrightarrow$(d) are obvious.

(d)$\Rightarrow$(b) Let $N \in \Lambda_\kappa(G)$ be of exponent $p_1 \cdot \ldots \cdot p_n$, where $p_1, \dots, p_n$ are distinct primes. Thus $N \subseteq G[p_1 \cdot \ldots \cdot p_n]$ and so $G[p_1 \cdot \ldots \cdot p_n] \in \Lambda_\kappa(G)$ by Corollary 2.21(c). $\square$

Lemma 3.54. *Let κ be an infinite cardinal and let G be a topological abelian group.*

(a) If G is super-κ-singular, then every subgroup of G is κ-singular.

(b) If G is κ-pseudocompact and $N \in \Lambda_\kappa(G)$, then G is super-κ-singular if and only if N is super-κ-singular.

Proof. (a) immediately follows from Lemma 3.52.

(b) Assume that N is super-κ-singular. Then there exists $M \in \Lambda_\kappa(N)$ such that $M \subseteq \mathrm{Soc}(N)$ by Lemma 3.53. Therefore $M \subseteq \mathrm{Soc}(G)$. By Corollary 2.21(a) $M \in \Lambda_\kappa(G)$ and so G is super-κ-singular. $\square$

Chapter 4

Projection onto products

As noted in the introduction, in Theorem B the condition involving the projection of the compact abelian group onto an uncountable power of a compact non-torsion abelian group, that is a (non-singular) w-divisible power, was important in the proof of that theorem. Since we want to generalize Theorem B for every infinite cardinal κ, proving Theorem B$^\kappa$, in this chapter we study when it is possible for a compact abelian group to admit a continuous surjective homomorphism onto a w-divisible power which is non-κ-singular.

This problem of "projecting" a compact abelian group onto a special product or power has interest also on its own; indeed the following theorem is a well known result. It could be obtain also as a standard application of Pontryagin duality to [1, Theorem 1.1].

Theorem 4.1. *For every non-metrizable compact abelian group K there exists a continuous surjective homomorphism of K onto a product $\prod_{i\in I} K_i$ of non-trivial metrizable compact abelian groups with $|I| = w(K)$.*

Proof. Let $w(K) = \kappa > \omega$ and $X = \widehat{K}$. Then $|X| = \kappa > \omega$ and X contains a direct sum $\bigoplus_{i\in I} C_i$ of non-trivial cyclic subgroups C_i with $|I| = |X| = w(K)$. By Pontryagin duality there exists a continuous surjective homomorphism $K \to \prod_{i\in I} K_i$, where $K_i = \widehat{C_i} \neq \{0\}$ is either finite cyclic or $K_i \cong_{top} \mathbb{T}$. □

In this theorem the weight of the product is maximal. Indeed it cannot exceed the weight of K.

In the following remark we discuss the situation of Theorem 4.1 in case all K_i are the same group, that is we consider projections onto powers of maximal weight of non-trivial compact abelian groups.

Remark 4.2. Let K be a compact abelian group of weight $\kappa > \omega$ and let $X = \widehat{K}$. Since $r(X) = |X| = \kappa$, there are three cases.

(a) If $r_0(X) = \kappa$, then X has a subgroup isomorphic to $\mathbb{Z}^{(\kappa)}$ and so there exists a continuous surjective homomorphism $K \to \mathbb{T}^\kappa$ by Pontryagin duality.

(b) If there exists $p \in \mathbb{P}$ such that $r_p(X) = \kappa$, then X has a subgroup isomorphic to $\mathbb{Z}(p)^{(\kappa)}$ and so there exists a continuous surjective homomorphism $K \to \mathbb{Z}(p)^\kappa$.

(c) If $r_0(X) < \kappa$ and $r_p(X) < \kappa$ for every $p \in \mathbb{P}$, then $\sup_{p\in\mathbb{P}} r_p(X) = \kappa$ and X has a subgroup isomorphic to $\bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p)^{(r_p(X))}$. Then there exists a continuous surjective homomorphism $K \to \prod_{p\in\mathbb{P}} \mathbb{Z}(p)^{r_p(X)}$. Moreover $w\left(\prod_{p\in\mathbb{P}} \mathbb{Z}(p)^{r_p(X)}\right) = \sup_{p\in\mathbb{P}} r_p(X) = \kappa$.

In (a) and (b) also the converse implications hold.

Unlike the previous two cases, in the third one it is not possible to find a continuous surjective homomorphism onto a w-divisible power S^κ. Indeed suppose that this surjective continuous homomorphism exists. Since $r_0(X) < \kappa$, $|X| = |t(X)|$ and so $w(K/c(K)) = \kappa$ by Theorem 1.30 and Lemma 1.32(b). Hence we suppose without loss of generality that K is totally disconnected. By Remark 1.44(c) there exists a continuous surjective homomorphism $K_p \to S_p^\kappa$ for each $p \in \mathbb{P}$. This means that $X_p = \widehat{K_p} \geq \widehat{S_p}^{(\kappa)}$ and $\widehat{S_p}$ is p-torsion in view of Fact 1.42(b) for every $p \in \mathbb{P}$. Let $q \in \mathbb{P}$ be such that S_q is non-trivial. Then $r_q(\widehat{S_q}) > 0$ and so $r_q(X_q) \geq \kappa$. But this is not possible since $r_p(X) < \kappa$ for every $p \in \mathbb{P}$ by hypothesis.

In the metrizable case there is some exception:

Lemma 4.3. *A metrizable compact abelian group K admits no continuous surjective homomorphism onto a product $\prod_{i\in I} K_i$ of non-trivial metrizable compact abelian groups with $|I| = \omega$ if and only if $r_0(\widehat{K})$ and $r_p(\widehat{K})$ are finite for every $p \in \mathbb{P}$ and $r_p(X) = 0$ for all but finitely many $p \in \mathbb{P}$.*

Proof. Let $X = \widehat{K}$. If K admits a continuous surjective homomorphism onto a product $\prod_{i\in I} K_i$ of non-trivial metrizable compact abelian groups with $|I| = \omega$, this is equivalent to say that X has a subgroup of the form $\bigoplus_{i\in I} \widehat{K_i}$, where each $\widehat{K_i}$ is a non-trivial abelian group of cardinality $\leq \omega$ and $|I| = \omega$. This happens if and only if either $r_0(X) = \omega$, or $r_p(X) = \omega$ for some $p \in \mathbb{P}$, or $r_p(X) \neq 0$ for infinitely many $p \in \mathbb{P}$. □

In this chapter we consider first the problem of when a compact abelian group K admits a continuous surjective homomorphism onto a product $\prod_{i\in I} K_i$ of *non-torsion* metrizable compact abelian groups with $|I| = w(K)$ (we add "non-torsion" with respect to Theorem 4.1). Since $w_d(\prod_{i\in I} K_i) = w(\prod_{i\in I} K_i)$, and the divisible weight is monotone under continuous surjective homomorphisms of compact abelian groups (see Proposition 3.17), we obtain the restriction $|I| \leq w_d(K)$. Theorem 4.11 shows that this necessary condition is also sufficient in case K is non-metrizable.

Moreover Theorem 4.17 shows that for non-singular compact abelian groups K admitting a continuous surjective homomorphism onto a w-divisible power $S^{w_d(K)}$ there is a trichotomy, and its Corollary 4.18 gives a necessary and sufficient condition for the existence of such a projection.

Remark 4.4. Every compact non-torsion abelian group S admits a continuous surjective homomorphism onto a metrizable compact non-torsion abelian group S_0.

To prove this we consider the dual group X of S. Since S is not bounded torsion, X is not bounded torsion as well, by Pontryagin duality. Then for every $n \in \mathbb{N}_+$ there exists $x_n \in X$ such that $nx_n \neq 0$. The subgroup $X_0 = \langle x_n : n \in \mathbb{N}_+\rangle$ of X is not bounded torsion and countable. By Pontryagin duality there exists a continuous surjective homomorphism of S onto $S_0 = \widehat{X_0}$ and S_0 is compact, metrizable and non-torsion.

Let κ be an infinite cardinal.

As a consequence of the previous part of this remark, a compact abelian group K admitting a continuous surjective homomorphism onto a product $\prod_{i\in I} K_i$ of compact non-torsion abelian groups K_i with $|I| = \kappa$, admits also a continuous surjective homomorphism onto $\prod_{i\in I} K_{0,i}$, where each $K_{0,i}$ is a compact metrizable non-torsion abelian group.

Analogously if K is a compact abelian group admitting a continuous surjective homomorphism onto a power S^κ of a compact non-torsion abelian group S, then there exists a continuous surjective homomorphism of K onto S_0^κ, where S_0 is a compact metrizable non-torsion abelian group.

In particular this shows that a compact abelian group K admits a continuous surjective homomorphism onto a product $\prod_{i\in I} K_i$ of compact non-torsion abelian groups if and only if K admits a continuous surjective homomorphism onto a product $\prod_{i\in I} K_{0,i}$ of metrizable compact non-torsion abelian groups. And analogously K admits a continuous surjective homomorphism onto a power S^κ of a compact non-torsion abelian group S if and only if K admits a continuous surjective homomorphism onto a power S_0^κ of a metrizable compact non-torsion abelian group S_0.

4.1 The “local” case

Claim 4.5. [26, Claim 4.7] *Let $p \in \mathbb{P}$, let K be a compact $\mathbb{Z}_p$-module and N a closed subgroup of K isomorphic to $\mathbb{Z}_p^\sigma$, for some cardinal $\sigma > \omega$. Then there exists a continuous surjective homomorphism of K onto $\mathbb{G}_p^\sigma$.*

Proof. Let $N = \prod_{n=1}^\infty N_n$, where each $N_n \cong_{top} \mathbb{Z}_p^\sigma$, and $M = \prod_{n=1}^\infty p^n N_n$. Then N/M is topologically isomorphic to $\mathbb{G}_p^\sigma$. Let $K_0 = K/M$. Then $K_0[p^n] \supseteq (N/M)[p^n] \cong_{top} \mathbb{Z}(p^n)^\sigma$ for every $n \in \mathbb{N}_+$. Hence, since $p^{n-1}K_0[p^n]$ contains $\mathbb{Z}(p)^\sigma$,

$$w(p^{n-1}K_0[p^n]) \geq \sigma. \tag{4.1}$$

By Lemma 1.45 there exists a closed subgroup N_0 of K_0 such that $N_0 \cong_{top} \mathbb{Z}_p^{\sigma_1}$, $L_0 = K_0/N_0 \cong_{top} \prod_{n=1}^\infty \mathbb{Z}(p^n)^{\beta_n}$ for appropriate cardinals σ_1, β_n, with $n \in \mathbb{N}_+$, and the canonical projection $\pi : K_0 \to L_0$ satisfies $\pi(t(K_0)) = t(L_0)$. Since $K_0[p^n]$ is compact and trivially meets $N_0 = \ker \pi$, it follows that $\pi \restriction_{K_0[p^n]}: K_0[p^n] \to L_0[p^n]$ is a topological isomorphism. Consequently $p^{n-1}L_0[p^n]$ is topologically isomorphic to $p^{n-1}K_0[p^n]$ and hence $w(p^{n-1}L_0[p^n]) \geq \sigma$ for every $n \in \mathbb{N}_+$ by (4.1). Therefore $\sup_{n\in\mathbb{N},n\geq m} \beta_n \geq \sigma$ for every $m \in \mathbb{N}_+$.

Let us prove that there exists a continuous surjective homomorphism $f : L_0 \to \mathbb{G}_p^\sigma$. Then, combining it with π and with the canonical projection of K onto K_0, we are done. Infinitely many β_n are infinite. So it is possible to suppose without loss of generality that all β_n are infinite. If there are infinitely many β_n such that $\beta_n \geq \sigma$, it is immediately possible to find the wanted f. Otherwise there exists $n_0 \in \mathbb{N}_+$ such that $\omega \leq \beta_n < \sigma$ for every $n \geq n_0$, with $\sup_{n\in\mathbb{N},n\geq n_0} \beta_n = \sigma$. Take an increasing subsequence $\{\beta_{n_k}\}_{k\in\mathbb{N}_+}$ of $\{\beta_n\}_{n\in\mathbb{N}_+}$ such that $\sup_{k\in\mathbb{N}_+} \beta_{n_k} = \sigma$. Observe that

$$\prod_{k=1}^{\infty} \mathbb{Z}(p^{n_k})^{\beta_{n_k}} = \prod_{k=1}^{\infty} S_k^{\beta_{n_k}},$$

where $S_k = \prod_{i=k}^{\infty} \mathbb{Z}(p^{n_i})$ is a metrizable compact non-torsion abelian group. For every $k \in \mathbb{N}_+$ there exists a continuous surjective homomorphism of S_k onto $\mathbb{G}_p$ and so we have the continuous surjective homomorphism $f : L_0 \to \mathbb{G}_p^\sigma$ which is the composition $L_0 \to \prod_{k=1}^{\infty} S_k^{\beta_{n_k}} \to \mathbb{G}_p^\sigma$. □

Lemma 4.6. *Let $p \in \mathbb{P}$ and let K be a non-singular compact $\mathbb{Z}_p$-module. Then there exists a continuous surjective homomorphism of K onto $\mathbb{G}_p^{w_d(K)}$.*

Proof. Let us reduce to the case when K is w-divisible. By Lemma 3.23 the subgroup $H = m_d(K)K$ of K is such that $w(H) = w_d(K)$ and it is w-divisible because K is non-singular. Consider the continuous surjective homomorphism $\varphi_{m_d(K)} : K \to H$ given by the multiplication by $m_d(K)$. Clearly every continuous surjective homomorphism of H onto $\mathbb{G}_p^{w_d(K)}$ composed with $\varphi_{m_d(K)}$ gives rise to a continuous surjective homomorphism of K onto $\mathbb{G}_p^{w_d(K)}$. This is why we suppose without loss of generality that K itself is w-divisible.

Let $X = \widehat{K}$. Observe that $|X| = w(K) > \omega$. By Remark 1.17 there exists a basic subgroup B_0 of X such that, for some cardinals α_n, with $n \in \mathbb{N}_+$, and σ,

$$B_0 \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)} \quad \text{and} \quad X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma')};$$

put $|X/B_0| = \sigma$ and note that $\sigma = \sigma'$ in case $\sigma > \omega$. As in Remark 1.17, for every $m \in \mathbb{N}_+$ let

$$B_{1,m} = \bigoplus_{n=1}^{m} \mathbb{Z}(p^n)^{(\alpha_n)} \quad \text{and} \quad B_{2,m} = \bigoplus_{n=m+1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)}.$$

Then $X = X_{1,m} \oplus B_{1,m}$, where

$$X_{1,m} = p^m X + B_{2,m} \quad \text{and} \quad X_{1,m}/B_{2,m} \cong X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma')}.$$

By Corollary 3.14 $|X| = |mX|$ for every $m \in \mathbb{N}_+$. Moreover $|X| = |mX_{1,n}|$ for every $m, n \in \mathbb{N}_+$; indeed

$$|mX_{1,n}| = |mp^n X + mB_{2,n}| = |X|$$

by Corollary 3.14 and our hypothesis on K. Consider, for every $n \in \mathbb{N}_+$, the sequence of cardinals $\{\beta_n : n \in \mathbb{N}_+\}$ where $\beta_n = \sup\{\alpha_m : m \in \mathbb{N}, m \geq n\}$. By the property of cardinals, there exists $n_0 \in \mathbb{N}_+$ such that $\beta_n = \beta_{n_0} = \beta$ for every $n \geq n_0$. Thus $|X| = \sigma \cdot \beta$, in fact

$$|X| = |X_{1,n_0}| = |X_{1,n_0}/B_{2,n_0}| \cdot |B_{2,n_0}| = \sigma \cdot \beta.$$

If $|X| = \sigma$, then $\sigma > \omega$ because $|X| = w(K) > \omega$. So in this case $\sigma = \sigma'$ and $X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$. By Pontryagin duality K has a subgroup topologically isomorphic to $\mathbb{Z}_p^\sigma$. Claim 4.5 applies to conclude that there exists a continuous surjective homomorphism of K onto $\mathbb{G}_p^\sigma$.

If $|X| > \sigma$, then $|X| = |B_{2,n_0}| = \beta > \omega$. We prove that

$$B_{2,n_0} \geq \bigoplus_{n \in \mathbb{N}_+} \mathbb{Z}(p^n)^{(\beta)}. \tag{4.2}$$

Let

$$A = \{n \in \mathbb{N} : n \geq n_0, \alpha_n = \beta\}.$$

If A is infinite, then $\bigoplus_{n \in A} \mathbb{Z}(p^n)^{(\alpha_n)} = \bigoplus_{n \in A} \mathbb{Z}(p^n)^{(\beta)}$ contains a subgroup topologically isomorphic to $\bigoplus_{n \in \mathbb{N}_+} \mathbb{Z}(p^n)^{(\beta)}$, so (4.2) holds true. Assume that A is finite. Then there exists an appropriate $n_1 \geq n_0$ such that $\beta > \alpha_n$ for every $n \geq n_1$. Still $\beta = \sup\{\alpha_m : m \geq n_1\}$ holds true, because $|X| = |mX|$ for every $m \in \mathbb{N}_+$ as noted before, and $|X| = |B_{2,n_0}|$ by our hypothesis $|X| > \sigma$. There exists an infinite subset I of $\mathbb{N}_+$ such that $\{\alpha_n : n \in I\}$ is strictly increasing with $\beta = \sup\{\alpha_m : m \in I\}$ and $n \geq n_1$ for all $n \in I$. Clearly $\beta = \sup_{m \in I'} \alpha_m$ holds true also for every infinite subset I' of I. We can write $I = \bigcup_{n \in \mathbb{N}_+} I_n$, where each I_n is infinite and $I_n \cap I_m = \emptyset$ for every $n \neq m$. Then for every $n \in \mathbb{N}_+$

$$\bigoplus_{m \in I_n} \mathbb{Z}(p^m)^{(\alpha_m)} \geq \bigoplus_{m \in I_n, m \geq n} \mathbb{Z}(p^n)^{(\alpha_m)} \cong \mathbb{Z}(p^n)^{(\beta)}.$$

Since $B_{2,n_0} \geq \bigoplus_{n \geq n_1} \mathbb{Z}(p^n)^{(\alpha_n)}$, we see that B_{2,n_0} contains also $\bigoplus_{n \in I} \bigoplus_{m \in I_n} \mathbb{Z}(p^m)^{(\alpha_m)}$ and the latter group contains a subgroup topologically isomorphic to $\bigoplus_{n \geq n_1} \mathbb{Z}(p^n)^{(\beta)}$. Thus (4.2) holds also in this case. By Pontryagin duality there exists a continuous surjective homomorphism of K onto the dual of the latter set, that is topologically isomorphic to $\mathbb{G}_p^\beta$. □

4.2 Projection onto w-divisible products

Fact 4.7. *A metrizable compact non-torsion abelian group S admits as a quotient one of the following four groups: either $\mathbb{T}$, or $\mathbb{G}_p$, or $\mathbb{Z}_p$ for some $p \in \mathbb{P}$, or S_π for some infinite $\pi \subseteq \mathbb{P}$.*

Proof. Let $X = \widehat{S}$. Then X is a countable discrete abelian group, which is not bounded torsion by Pontryagin duality. If X contains an isomorphic copy of $\mathbb{Z}$, then S admits $\mathbb{T}$

as a quotient. If $\pi = \{p \in \mathbb{P} : r_p(X) > 0\}$ is infinite, since X has a subgroup isomorphic to $\bigoplus_{p\in\pi} \mathbb{Z}(p)$, it follows that S admits S_π as a quotient. If π is finite, there exists $p \in \mathbb{P}$ such that $t_p(X)$ is infinite. So we can assume without loss of generality that X is an infinite p-group. By Remark 1.17 there exists a basic subgroup $B_0 \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)}$ of X, for some cardinals $\alpha_n \leq \omega$, with $n \in \mathbb{N}_+$, such that $X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$ for some cardinal σ. If there exists a sequence $\{n_k\}_{k\in\mathbb{N}_+}$ of positive integers such that $n_k \to \infty$ and $\alpha_{n_k} > 0$ for every $k \in \mathbb{N}_+$, then B_0 contains a subgroup isomorphic to $\bigoplus_{k=1}^{\infty} \mathbb{Z}(p^{n_k})$. Since the latter group obviously contains a copy of the group $\bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)$, we conclude that in this case S admits $\mathbb{G}_p$ as a quotient by Pontryagin duality. If this subsequence does not exist, B_0 is bounded torsion, that is, there exists $n \in \mathbb{N}_+$ such that $p^n B_0 = \{0\}$. By Remark 1.17 $X \cong p^n X \oplus B_0$, where $X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$. Note that $\sigma > 0$, otherwise X would be bounded torsion and consequently S would be torsion by Pontryagin duality. So X contains a copy of the group $\mathbb{Z}(p^\infty)$, therefore S admits $\mathbb{Z}_p$ as a quotient again by Pontryagin duality. □

Example 4.8. Let $p \in \mathbb{P}$. There exists a continuous surjective homomorphism $\mathbb{G}_p \to \mathbb{G}_p^\omega$.

To see this consider for every $n \in \mathbb{N}_+$ an increasing sequence of natural numbers $\{m_{n,k}\}_{k\in\mathbb{N}_+}$. Then $\mathbb{G}_p \cong_{top} \prod_{n=1}^{\infty}\prod_{k=1}^{\infty} \mathbb{Z}(p^{m_{n,k}})$, which admits a continuous surjective homomorphism onto $\mathbb{G}_p^\omega$.

In the following remark we show the cases in which a singular compact abelian group admits a projection onto a product $\prod_{i\in I} K_i$, where each K_i is a metrizable compact non-torsion abelian group and $|I| = \omega$. The condition $w_d(K) = \omega$ is necessary to have this projection. Indeed, if $w_d(K) < \omega$, then K is bounded torsion and it admits no continuous surjective homomorphism onto a product of a metrizable compact non-torsion abelian group S.

Remark 4.9. A singular compact non-torsion abelian group K admits a continuous surjective homomorphism onto $\prod_{n=1}^{\infty} K_n$, where each K_n is a metrizable compact non-torsion abelian group, precisely when some of the following occurs:

(a) there exists a continuous surjective homomorphism $f : K \to \mathbb{T}^\omega$ if and only if $r_0(\widehat{K}) = \omega$;

(b) for some $p \in \mathbb{P}$, there exists a continuous surjective homomorphism $f : K \to \mathbb{G}_p^\omega$ if and only if $r_p(p^m \widehat{K}) = \omega$ for every $m \in \mathbb{N}$ (i.e., $w_d((K/c(K))_p) = \omega$);

(c) there exists a continuous surjective homomorphism $f : K \to \prod_{n=0}^{\infty} S_{\pi_n}$, where each π_n is an infinite subset of $\mathbb{P}$, if and only if there exists an infinite subset π of $\mathbb{P}$ such that $r_p(\widehat{K}) \neq 0$ for every $p \in \pi$.

Proof. Assume that K admits a continuous surjective homomorphism onto $\prod_{n=1}^{\infty} K_n$, where each K_n is a metrizable compact non-torsion abelian group. Since K is singular and non-torsion, there exists $m \in \mathbb{N}_+$ such that $w(mK) = \omega$. Moreover there exists a continuous surjective homomorphism of mK onto $\prod_{n=1}^{\infty} mK_n$, where each mK_n is a metrizable compact non-torsion abelian group. So we can assume without loss of generality that K is metrizable. Then $X = \widehat{K}$ is countable.

By Pontryagin duality there exists such a continuous surjective homomorphism if and only if X contains $\bigoplus_{n=1}^{\infty} X_n$, where each X_n is not bounded torsion (otherwise K_n would be torsion by Pontryagin duality). Moreover by Fact 4.7 each K_n admits as a quotient one of the following four types of groups: either $\mathbb{T}$ or $\mathbb{G}_p$ or $\mathbb{Z}_p$ for some $p \in \mathbb{P}$ or S_π for some infinite $\pi \subseteq \mathbb{P}$.

(a) Suppose that there exists a continuous surjective homomorphism $K \to \mathbb{T}^\omega$. Equivalently X contains a subgroup isomorphic to $\mathbb{Z}^{(\omega)}$, that is $r_0(X) = \omega$.

(b) If there exists a continuous surjective homomorphism $K \to \mathbb{G}_p^\omega$, then X has a subgroup isomorphic to $\bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\omega)}$. Consequently $r_p(p^n X) = \omega$ for every $n \in \mathbb{N}$.

Suppose that $r_p(p^n X) = \omega$ for every $n \in \mathbb{N}$. By Remark 1.17 there exists a basic subgroup B_0 of X, that is $B_0 \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)}$ and $X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$ for some cardinals α_n, with $n \in \mathbb{N}_+$, and σ.

If $\alpha_n > 0$ for infinitely many $n \in \mathbb{N}_+$, then X contains a subgroup isomorphic to $\bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)$. By Pontryagin duality there exists a continuous surjective homomorphism $K \to \mathbb{G}_p$. By Example 4.8 there exists a continuous surjective homomorphism $\mathbb{G}_p \to \mathbb{G}_p^\omega$.

If $\alpha_n = 0$ for all but finitely many $n \in \mathbb{N}_+$, it means that $p^m B_0 = \{0\}$ for some $m \in \mathbb{N}$. Therefore, following the notations of Remark 1.17, $B_{2,m} = \{0\}$ and so $X_{1,m} = p^m X + B_{2,m} = p^m X$ and

$$X_{1,m} = X_{1,m}/B_{2,m} \cong X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}.$$

Since by hypothesis $r_p(p^m X) = \omega$, it follows that $r_p(\mathbb{Z}(p^\infty)^{(\sigma)}) = \omega$. Therefore $\sigma = \omega$. Since $p^m X$ is a subgroup of X, by Pontryagin duality there exists a continuous surjective homomorphism $K \to \mathbb{Z}_p^\omega$ and $\mathbb{Z}_p^\omega$ admits a continuous surjective homomorphism onto $\mathbb{G}_p^\omega$.

(c) Suppose that there exists a continuous surjective homomorphism $K \to \prod_{n=1}^{\infty} S_{\pi_n}$ where each π_n is an infinite subset of $\mathbb{P}$. Therefore $X \supseteq \bigoplus_{n=1}^{\infty} \bigoplus_{p \in \pi_n} \mathbb{Z}(p)$. Then $\pi = \bigcup_{n=1}^{\infty} \pi_n$ is an infinite subset of of $\mathbb{P}$ such that $r_p(X) \neq 0$ for every $p \in \pi$.

Conversely, if there exists an infinite subset π of $\mathbb{P}$ such that $r_p(X) \neq 0$ for every $p \in \pi$, then there exist countably many infinite subsets π_n pairwise with trivial intersection, where $n \in \mathbb{N}_+$, of π such that $\bigcup_{n=1}^{\infty} \pi_n = \pi$. Then

$$X \supseteq \bigoplus_{p \in \pi} \mathbb{Z}(p) = \bigoplus_{n=1}^{\infty} \bigoplus_{p \in \pi_n} \mathbb{Z}(p).$$

By Pontryagin duality there exists a continuous surjective homomorphism of K onto $\prod_{n=1}^{\infty} S_{\pi_n}$. $\square$

The next claim is the totally disconnected case of Theorem 4.11 and it is applied in the proof of that theorem.

Claim 4.10. *If K is a non-singular totally disconnected compact abelian group, then there exists a continuous surjective homomorphism of K onto a w-divisible product of weight $w_d(K)$.*

Proof. As K is totally disconnected, we can write $K \cong_{top} \prod_{p\in\mathbb{P}} K_p$ by Remark 1.44(b). Let $\kappa_p = w(K_p)$ for every $p \in \mathbb{P}$. As K is non-metrizable, $\mathbb{P} \setminus P_m(K) \neq \emptyset$.

If $\mathbb{P} \setminus P_m(K)$ is finite, then by Lemma 3.19

$$w_d(K) = \max\{w_d(K_p) : p \in \mathbb{P} \setminus P_m(K)\}.$$

So there exists $p \in \mathbb{P} \setminus P_m(K)$ such that $w_d(K) = w_d(K_p)$. We can apply Lemma 4.6 to the non-singular $\mathbb{Z}_p$-module K_p to find a continuous surjective homomorphism from K_p to $S^{w_d(K_p)} = {\mathbb{G}_p}^{w_d(K)}$. Take the composition of this homomorphism with the canonical projection $K \to K_p$. We can argue in the same way when $w_d(K) = w_d(K_p)$ for some $p \in \mathbb{P}$. Therefore from now on we assume that $w_d(K) > w_d(K_p)$ for all $p \in \mathbb{P}$ and this implies that $\mathbb{P} \setminus P_m(K)$ is infinite. By Lemma 3.47(b) we have $w_d(K) = w_s(K)$.

For $p \in \mathbb{P} \setminus P_m(K)$ consider the quotient K_p/pK_p. If $X = \widehat{K_p}$, by Fact 1.42(b) X is a p-group. By Fact 1.32(a) we know that $X[p] \cong_{top} \widehat{K_p/pK_p}$. Moreover, since $|X|$ is not countable, $|X[p]| = |X| = \kappa_p$. By Pontryagin duality $K_p/pK_p \cong_{top} \mathbb{Z}(p)^{\kappa_p}$. Consequently there exists a continuous surjective homomorphism of K_p onto $\mathbb{Z}(p)^{\kappa_p}$.

Since $\mathbb{P} \setminus P_m(K)$ is infinite, $\Pi(K)$ is not empty and so infinite as well. We have two cases. If $\pi(K)$ is infinite, then there exists a continuous surjective homomorphism

$$K \to \prod_{p\in\pi(K)} \mathbb{Z}(p)^{w_d(K)} \cong_{top} S^{w_d(K)}_{\pi(K)},$$

as $\kappa_p = w_s(K) = w_d(K)$ for all $p \in \pi(K)$. Otherwise $\pi(K)$ is finite and so $\pi^*(K)$ is infinite, because $\Pi(K)$ is infinite and $\Pi(K) = \pi^*(K) \cup \pi(K)$ is a partition of $\Pi(K)$. So we can suppose without loss of generality that $\Pi(K) = \pi^*(K)$. By Lemma 3.47(b) $w_s(K) = \sup_{p\in\Pi(K)} \kappa_p$. Moreover $w_d(K) = w_s(K) > \kappa_p$ for all $p \in \Pi(K)$. Order the set $\{\kappa_p : p \in \Pi(K)\}$ so that $\kappa_{p_1} < \kappa_{p_2} < \ldots < \kappa_{p_n} < \ldots$ and note that the inclusion $\{p_n : n \in \mathbb{N}_+\} \subseteq \Pi(K)$ could be proper. Nevertheless $w_s(K) = \sup_{n\in\mathbb{N}_+} \kappa_{p_n}$. Let $C = \prod_{n=1}^\infty \mathbb{Z}(p_n)^{\kappa_{p_n}}$. Then $w(C) = w_s(K) = w_d(K)$ and

$$C = \prod_{n=1}^\infty \mathbb{Z}(p_n)^{\kappa_{p_n}} = \prod_{n=1}^\infty \prod_{i=1}^n \mathbb{Z}(p_n)^{\kappa_{p_i}} = \prod_{i=1}^\infty \prod_{n=i}^\infty \mathbb{Z}(p_n)^{\kappa_{p_i}} = \prod_{i=1}^\infty S^{\kappa_{p_i}}_{\pi_i},$$

where $\pi_i = \{p_n : n \in \mathbb{N}_+, n \geq i\}$ is infinite for every $i \in \mathbb{N}_+$. To end up the proof note that C is a w-divisible product of weight $w_d(K)$. □

The next result is Theorem D of the introduction.

Theorem 4.11. *Let K be a non-singular compact abelian group. There exists a continuous surjective homomorphism of K onto a w-divisible product $\prod_{i\in I} K_i$ if and only if $\omega < |I| \leq w_d(K)$.*

In particular every non-singular compact abelian group K admits a continuous surjective homomorphism onto a w-divisible product of weight $w_d(K)$.

Proof. Since K is non-singular, $\kappa = w_d(K) > \omega$. According to Lemma 3.21

$$w_d(K) = \max\{w(c(K)), w_d(K/c(K))\}.$$

If $w_d(K) = w(c(K))$, there exists a continuous surjective homomorphism of K onto $\mathbb{T}^\kappa$ by Fact 1.41. So it is possible to suppose that $w_d(K) > w(c(K))$ and then $w_d(K) = w_d(K/c(K))$. By Claim 4.10 applied to $K/c(K)$ there exists a continuous surjective homomorphism of $K/c(K)$ onto the w-divisible product $\prod_{i\in I} K_i$ such that $|I| = \kappa$. It remains to take the composition $K \to K/c(K) \to \prod_{i\in I} K_i$.

To prove the opposite implication, suppose that there exists a continuous surjective homomorphism of K onto a w-divisible product $\prod_{i\in I} K_i$. Then $|I| > \omega$. Since the divisible weight is monotone by Proposition 3.17, so $w_d(K) \geq |I|$. □

Corollary 4.12. *A compact abelian group K is w-divisible if and only if there exists a continuous surjective homomorphism of K onto a w-divisible product $\prod_{i\in I} K_i$ such that $|I| = w(K) > \omega$.*

In Corollary 4.18 the following result can be improved, taking the K_i all equal.

Corollary 4.13. *Let κ be an infinite cardinal. A compact abelian group K is non-κ-singular if and only if there exists a continuous surjective homomorphism of K onto a w-divisible product $\prod_{i\in I} K_i$, where $|I| = w_d(K) > \kappa$.*

Here is a corollary of Lemma 3.23 and Theorem 4.11. It is not used in our proofs, but we give it in order to emphasize the analogy between w-divisible and connected/divisible compact groups, since $r_0(K) = |K| = 2^{w(K)}$ for every divisible compact abelian group K [31].

Corollary 4.14. *If K is a non-singular compact abelian group, then $r_0(K) = 2^{w_d(K)}$. In particular $r_0(K) = 2^{w(K)}$, whenever K is w-divisible.*

Proof. Let $\sigma = w_d(K)$ and $H = m_d(K)K$. By Lemma 3.23 $\sigma = w(H)$ and H is w-divisible because $w_d(H) = w(H) = \sigma > \omega$ since K is non-singular. By Theorem 4.11 there exists a continuous surjective homomorphism of H onto the w-divisible product $\prod_{i\in I} K_i$ with $|I| = \sigma$. By Fact 1.36 and since $|K_i| = \mathfrak{c}$, $r_0(K_i) = \mathfrak{c}$. Consequently $r_0(\prod_{i\in I} K_i) = 2^\sigma$ and $r_0(H) \geq 2^\sigma$. But $|H| = 2^\sigma$ and so $r_0(H) = 2^\sigma$. Hence $r_0(K) = r_0(H) = 2^\sigma$. □

4.3 Projection onto w-divisible powers

As we do in Remark 4.9 for products, in the following remark we describe the cases in which a singular compact abelian group admits a projection onto S^ω, where S is a metrizable compact non-torsion abelian group. As in Remark 4.9 $w_d(K) = \omega$ is a necessary condition to have these projection.

Remark 4.15. A singular compact non-torsion abelian group K admits a continuous surjective homomorphism onto S^ω for some metrizable compact non-torsion abelian group S precisely when some of the following occurs:

(a) there exists a continuous surjective homomorphism $f : K \to \mathbb{T}^\omega$ if and only if $r_0(\widehat{K}) = \omega$;

(b) for some $p \in \mathbb{P}$, there exists a continuous surjective homomorphism $f : K \to \mathbb{G}_p^\omega$ if and only if $r_p(p^m \widehat{K}) = \omega$ for every $m \in \mathbb{N}$ (i.e., $w_d((K/c(K))_p) = \omega$);

(c) for π an infinite subset of $\mathbb{P}$ there exists a continuous surjective homomorphism $f : K \to S_\pi^\omega$ if and only if $r_p(\widehat{K}) = \omega$ for every $p \in \pi$.

Proof. Let $X = \widehat{K}$. Assume that K admits a continuous surjective homomorphism onto S^ω for some metrizable compact non-torsion abelian group S. By Fact 4.7 such a group S admits as a quotient one of the following four groups: either $\mathbb{T}$, or $\mathbb{G}_p$, or $\mathbb{Z}_p$ for some $p \in \mathbb{P}$, or S_π for some infinite $\pi \subseteq \mathbb{P}$.

(a) and (b) are proved in Remark 4.9.

(c) Suppose that there exists a continuous surjective homomorphism $K \to S_\pi^\omega$. Equivalently, by Pontryagin duality, $X \supseteq \bigoplus_{p\in\pi} \mathbb{Z}(p)^{(\omega)}$. This means that $r_p(X) = \omega$ for every $p \in \pi$. □

The following claim, which is used to prove Theorem 4.17, analyzes when it is possible to project a totally disconnected compact abelian group onto a power of a metrizable compact non-torsion abelian group.

Claim 4.16. *Let K be a totally disconnected compact abelian group, $\kappa_p = w(K_p)$ for each $p \in \mathbb{P}$ and I an uncountable set of indices.*

(a) If $\pi \subseteq \mathbb{P}$ and $|I| \leq \kappa_p$ for all $p \in \pi$, then there exists a continuous surjective homomorphism $f : K \to S_\pi^I$.

(b) If $w_d(K_p) < |I|$ for all $p \in \mathbb{P}$ and there exists a continuous surjective homomorphism $f : K \to S^I$, where S is a metrizable compact non-torsion abelian group, then $|I| \leq \kappa_p$ for all $p \in \pi$ for some infinite $\pi \subseteq \mathbb{P}$.

Proof. (a) For every $p \in \pi$ the inequality $|I| \leq \kappa_p$ yields that κ_p is uncountable, so $w(K_p/pK_p) = \kappa_p$ by Lemma 3.40. Hence K_p/pK_p is isomorphic to $\mathbb{Z}(p)^{\kappa_p}$. Moreover, there exists a continuous surjective homomorphism $\mathbb{Z}(p)^{\kappa_p} \to \mathbb{Z}(p)^I$, since $|I| \leq \kappa_p$. Therefore there exists a continuous surjective homomorphism

$$K \cong_{top} \prod_{p\in\mathbb{P}} K_p \to \prod_{p\in\pi} \mathbb{Z}(p)^I \cong_{top} \left(\prod_{p\in\pi} \mathbb{Z}(p)\right)^I = S_\pi^I.$$

(b) Suppose that $w_d(K_p) < |I|$ for all $p \in \mathbb{P}$ and that there exists a continuous surjective homomorphism $f : K \to S^I$, where S is a metrizable compact non-torsion abelian group. Then S is totally disconnected and compact, so $S \cong_{top} \prod_{p\in\mathbb{P}} S_p$ by Remark 1.44(b). Since $w_d(K_p) < |I|$ for all $p \in \mathbb{P}$, it follows that S_p is torsion for all $p \in \mathbb{P}$. Indeed, if S_p were not torsion, since there exists the surjective homomorphism $f_p = f \restriction_{K_p}\colon K_p \to S_p^I$ by Remark 1.44(c), Proposition 3.17 would imply that $w_d(K_p) \geq$

$w_d(S_p^I) = |I|$. Hence S_p is a bounded p-torsion group for every $p \in \mathbb{P}$, being p-torsion and compact and in view of Fact 1.36. Since S is non-torsion, S_p has to be non-trivial for infinitely many $p \in \mathbb{P}$, and so $r_p(S) = r_p(S_p) > 0$ for infinitely many $p \in \mathbb{P}$. Let $p \in \mathbb{P}$ be such that $r_p(S_p) > 0$. By Pontryagin duality there exists a continuous injective homomorphism $\bigoplus_I \widehat{S_p} \to \widehat{K_p}$. Since S_p is a bounded p-torsion abelian group, $\widehat{S_p}$ is a bounded p-torsion abelian group as well by Fact 1.42(b) and by Pontryagin duality. Therefore $r_p(\widehat{S_p}) > 0$ and it follows that $r_p(\widehat{K_p}) \geq |I|$. Hence $\kappa_p = w(K_p) \geq |I|$. □

Theorem 4.17. *A non-singular compact abelian group K admits a continuous surjective homomorphism onto a w-divisible power $S^{w_d(K)}$ precisely when some of the following occurs:*

(a) there exists a continuous surjective homomorphism $f : K \to \mathbb{T}^{w_d(K)}$ if and only if $w_d(K) = w(c(K))$;

(b) for some $p \in \mathbb{P}$, there exists a continuous surjective homomorphism $f : K \to \mathbb{G}_p^{w_d(K)}$ if and only if $w_d(K) = w_d((K/c(K))_p)$;

(c) if π is an infinite subset of $\{p \in \mathbb{P} : p > m_d(K)\}$, then there exists a continuous surjective homomorphism $f : K \to S_\pi^{w_d(K)}$ if and only if $w_d(K) = w((K/c(K))_p)$ for every $p \in \pi$.

Moreover every compact abelian group K with $\mathrm{cf}(w_d(K)) > \omega$ admits a continuous surjective homomorphism onto a w-divisible power $S^{w_d(K)}$.

Proof. Assume that K admits a continuous surjective homomorphism onto a w-divisible power $S^{w_d(K)}$. By Fact 4.7 S, being a metrizable compact non-torsion abelian group, admits as a quotient one of the following four groups: either $\mathbb{T}$, or $\mathbb{G}_p$, or $\mathbb{Z}_p$ for some $p \in \mathbb{P}$, or S_π for some infinite $\pi \subseteq \mathbb{P}$.

Depending on which of these four cases occurs we have either (a) or (b) or (c).

(a) Assume that there exists a continuous surjective homomorphism $f : K \to \mathbb{T}^{w_d(K)}$. Then the restriction of f to the connected component $c(K)$ gives rise to a surjective continuous homomorphism $f\restriction_{c(K)}: c(K) \to \mathbb{T}^{w_d(K)}$. This yields $w(c(K)) \geq w_d(K)$, while the inequality $w_d(K) \geq w(c(K))$ is always available. This proves that $w_d(K) = w(c(K))$.

On the other hand, if $w_d(K) = w(c(K))$, then there exists a surjective continuous homomorphism $K \to \mathbb{T}^{w(c(K))} = \mathbb{T}^{w_d(K)}$ by Fact 1.41.

(b) Assume that there exists a continuous surjective homomorphism $f : K \to \mathbb{Z}_p^{w_d(K)}$. From $w_d(K) > \omega$, we conclude that $\mathbb{Z}_p^{w_d(K)}$ admits $\mathbb{G}_p^{w_d(K)}$ as a quotient. So we can suppose that there exists a continuous surjective homomorphism $f : K \to \mathbb{G}_p^{w_d(K)}$ for some $p \in \mathbb{P}$. Since $\mathbb{G}_p^{w_d(K)}$ is totally disconnected, $f(c(K)) = \{0\}$ and so f factorizes through the projection $K \to K/c(K)$. This produces a continuous surjective homomorphism $K/c(K) \to \mathbb{G}_p^{w_d(K)}$. By Remark 1.44(c) there exists a continuous surjective homomorphism $(K/c(K))_p \to \mathbb{G}_p^{w_d(K)}$, hence $w_d(K) \leq w_d((K/c(K))_p)$ by Proposition

3.17. The other inequality is always available by Proposition 3.17 again. This proves that $w_d(K) = w_d((K/c(K))_p)$.

On the other hand, if $w_d(K) = w_d((K/c(K))_p)$, then there exists a continuous surjective homomorphism $(K/c(K))_p \to \mathbb{G}_p^{w_d(K)}$ by Lemma 4.6. It remains to compose with the canonical projections $K \to K/c(K) \to (K/c(K))_p$.

(c) Assume that there exists a continuous surjective homomorphism $f : K \to S_\pi^{w_d(K)}$. Since for every $p \in \pi$ there exists a continuous surjective homomorphism $\phi_p : S_\pi \to S_\pi/pS_\pi \cong \mathbb{Z}(p)$, we obtain a continuous surjective homomorphism $f_p : K \to \mathbb{Z}(p)^{w_d(K)}$. Since every f_p factorizes through the canonical projection $K \to (K/c(K))_p$ (as noted in the proof of (b)) applying Remark 1.44(c) we get a surjective continuous homomorphism $l : (K/c(K))_p \to \mathbb{Z}(p)^{w_d(K)}$. This proves that $w((K/c(K))_p) \geq w_d(K)$ for every $p \in \pi$ by Proposition 3.17. The converse inequality holds, because by hypothesis for every $p \in \pi$, $p > m_d(K)$ and so $m_d(K)(K/c(K))_p = (K/c(K))_p$. Hence $w_p((K/c(K))_p) \leq w(m_d(K)(K/c(K))) \leq w_d(K)$ for every $p \in \pi$.

Assume that $w_d(K) = w((K/c(K))_p)$ for every $p \in \pi$. Since in general $w_d(K) \geq w_d(K/c(K)) \geq w((K/c(K))_p)$ for every $p \in \pi$ by Proposition 3.17, we obtain

$$w_d(K/c(K)) = w((K/c(K))_p) \text{ for every } p \in \pi.$$

Apply Claim 4.16 to the totally disconnected compact abelian group $K/c(K)$ to find a continuous surjective homomorphism $g : K/c(K) \to S_\pi^{w_d(K)}$. Then take the composition of g with the canonical projection $K \to K/c(K)$.

To finish the proof we have to see that if K is a non-singular compact abelian group such that there exists no continuous surjective homomorphism $f : K \to S^{w_d(K)}$, where $S^{w_d(K)}$ is a w-divisible power, then $\mathrm{cf}(w_d(K)) = \omega$. By (a) $w_d(K) > w(c(K))$ and so $w_d(K) = w_d(K/c(K))$ by Lemma 3.21. By (b) and (c) $w_d(K) = w_d(K/c(K)) > w_d((K/c(K))_p)$ for all $p \in \mathbb{P}$. In view of Lemma 3.47(b)

$$w_d(K) = w_d(K/c(K)) = w_s(K/c(K)) = \sup_{p \in \Pi(K/c(K))} w((K/c(K))_p).$$

This proves that $\mathrm{cf}(w_d(K)) = \omega$. □

The next result is Corollary D* of the introduction. It improves in some sense Corollary 4.13 and gives exactly the condition needed for a crucial part of Theorem B$^\kappa$ of the introduction.

Corollary 4.18. *Let κ be an infinite cardinal. A compact abelian group K is non-κ-singular if and only if there exists a continuous surjective homomorphism of K onto a w-divisible power S^{κ^+}.*

Proof. If there exists a continuous surjective homomorphism of K onto a w-divisible power S^{κ^+}, then $w_d(K) > \kappa$ by Lemma 3.17 and so K is non-κ-singular.

Suppose that K is non-κ-singular. We have to consider two cases. If there exists a continuous surjective homomorphism of K onto a w-divisible power $S^{w_d(K)}$, we are done since $w_d(K) \geq \kappa^+$.

Assume that such a homomorphism is not available. By Theorem 4.17 this means that $\mathrm{cf}(w_d(K)) = \omega$ and $w_d(K) > w(c(K))$. The condition $\mathrm{cf}(w_d(K)) = \omega$, together with $w_d(K) > \kappa$, implies $w_d(K) > \kappa^+$. The condition $w_d(K) > w(c(K))$ implies $w_d(K) = w_d(K/c(K))$ by Lemma 3.21. So assume without loss of generality that K is totally disconnected. According to Theorem 4.17 our hypothesis yields $w_d(K) > w_d(K_p)$ for all $p \in \mathbb{P}$. By Lemma 3.47(b) $w_d(K) = w_s(K) = \sup_{p\in\Pi(K)} w(K_p)$. Since $\Pi(K)$ is infinite, either $\Pi(K) = \pi^*(K)$ is infinite or $\pi(K)$ is infinite. In both cases $w(K_p) \geq \kappa^+$ for infinitely many $p \in \mathbb{P}$. By Claim 4.16 there exists a continuous surjective homomorphism $K \to S^{\kappa^+}$. $\square$

4.4 The non-abelian case

The connected compact groups behaves "nicely" even in the non-abelian case. Indeed in [68] it is proved that if K is a connected compact group, then $K/Z(K)$ is topologically isomorphic to a product of metrizable groups. This result was improved in the next one. A *Lie group* is a topological group locally homeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}_+$.

Theorem 4.19. [15, Theorem 4.2] *If K is a non-trivial connected compact group, then $K/Z(K)$ is topologically isomorphic to $\prod_{i\in i} K_i$, where each K_i is a non-trivial connected compact non-abelian simple Lie group.*

In particular there exists a continuous surjective homomorphism of K onto a product $\prod_{i\in I} K_i$ of non-trivial Lie groups with $|I| = w(K)$.

The problem is open in general:

Problem 4.20. *Does every compact group admit a continuous surjective homomorphism onto a product of metrizable compact non-torsion groups?*

Since the answer is positive for connected compact groups by Theorem 4.19, it is convenient to consider first the opposite case:

Problem 4.21. *Does every totally disconnected compact group admit a continuous surjective homomorphism onto a product of metrizable compact non-torsion groups?*

Chapter 5

The free rank of abelian pseudocompact groups

In [34] the authors proved the following theorem and left open in general the problem of the admissibility of the free rank of a pseudocompact abelian group:

Theorem 5.1. [34, Theorem 3.21] *If G is a non-trivial connected pseudocompact abelian group, then* $\mathrm{Ps}(r_0(G), w(G))$ *holds.*

In particular this implies that $\mathbf{Ps}(r_0(G))$ holds, i.e., $r_0(G)$ is admissible, for every non-trivial connected pseudocompact abelian group.

Problem 5.2. [34, Problem 9.11] *Is* $\mathbf{Ps}(r_0(G))$ *a necessary condition for the existence of a pseudocompact group topology on a non-torsion abelian group G?*

As mentioned in [34] this problem seems to be important for the characterization of the abelian groups admitting pseudocompact group topologies [34, Problem 0.2].

In this chapter, in Theorem 5.13 we prove the counterpart of Theorem 5.1 weakening the hypothesis from connected to w-divisible. Furthermore we answer positively to Problem 5.2 in Corollary 5.18.

5.1 Measuring dense pseudocompact subgroups

In the introduction we have defined $\mathrm{Ps}(\lambda, \kappa)$ for cardinals λ, κ with κ infinite. For a given infinite cardinal λ, the set

$$A_\lambda = \{\kappa \text{ infinite cardinal} : \mathrm{Ps}(\lambda, \kappa) \text{ holds}\}$$

is not empty because $2^\lambda \in A_\lambda$. Then, for the properties of cardinal numbers, A_λ admits a minimal element. So we can give the definition of the cardinal function $m(-)$, which is strictly related to $\mathrm{Ps}(-, -)$.

Definition 5.3. [34, Definition 2.6] *Let κ be an infinite cardinal. Then $m(\kappa)$ is the minimal cardinal λ such that* $\mathrm{Ps}(\lambda, \kappa)$ *holds.*

This cardinal function was defined in [15] in terms of compact groups: if K is a compact group of weight κ, then $m(\kappa)$ denotes the minimum cardinality of a dense pseudocompact subgroup of K. The function $m(-)$ is defined as a function of the weight because it depends only on it [15]. A consequence of this fact, of Theorem 2.7 and of the following theorem is that these two definitions of $m(-)$ are equivalent.

Theorem 5.4. [15] (see also [34, Fact 2.12 and Theorem 3.3(i)]) *Let λ and $\kappa \geq \omega$ be cardinals. Then $\mathrm{Ps}(\lambda,\kappa)$ holds if and only if there exists a group G of cardinality λ which admits a pseudocompact group topology of weight κ.*

Fact 5.5. [6] (see also [15, Theorem 2.7]) *Let κ be an infinite cardinal. Then:*

(a) $m(\kappa) \geq 2^\omega$ *and* $\mathrm{cf}(m(K)) > \omega$*;*

(b) $\log(\kappa) \leq m(\kappa) \leq (\log\kappa)^\omega$.

Some useful properties of the condition $\mathrm{Ps}(\lambda,\kappa)$ are collected in the next proposition; (a) and (b) are part of [34, Lemma 2.7] and (d) is a particular case of [34, Lemma 3.4(i)].

Proposition 5.6. *(a)* $\mathrm{Ps}(\mathfrak{c},\omega)$ *holds and moreover* $m(\omega) = \mathfrak{c}$.

(b) If $\mathrm{Ps}(\lambda,\kappa)$ *holds for some cardinals* $\lambda,\kappa \geq \omega$*, then* $\mathrm{Ps}(\lambda',\kappa)$ *holds for every cardinal* λ' *such that* $\lambda \leq \lambda' \leq 2^\kappa$.

(c) For cardinals λ, $\kappa \geq \omega$, $\mathrm{Ps}(\lambda,\kappa)$ *holds if and only if* $m(\kappa) \leq \lambda \leq 2^\kappa$.

(d) $\mathrm{Ps}\left(2^\kappa, 2^{2^\kappa}\right)$ *holds for every infinite cardinal* κ.

Proof. (a) For the first part it suffices to note that $\{0,1\}^\omega$ has cardinality $\mathfrak{c}$. (Using Theorem 5.4, for example $\mathbb{T}$ is metrizable and has cardinality $\mathfrak{c}$.) For the second part we need to observe that every ω-dense subset S of $\{0,1\}^\omega$ has to coincide with $\{0,1\}^\omega$ and in particular $|S| = c$.

(b) If F is an ω-dense subset of $\{0,1\}^\kappa$ of cardinality λ, then every subset F' of $\{0,1\}^\kappa$ such that $F \subseteq F'$ and F' is ω-dense in $\{0,1\}^\kappa$. Moreover the cardinality of F' cannot be bigger than $|\{0,1\}^\kappa| = 2^\kappa$.

(c) Suppose that $\mathrm{Ps}(\lambda,\kappa)$ holds for some cardinals λ, $\kappa \geq \omega$. Then $\lambda \geq m(\kappa)$ by the definition of $m(-)$. Moreover a subset of $\{0,1\}^\kappa$ cannot have size bigger that 2^κ and so $\lambda \leq 2^\kappa$. Conversely, suppose that $m(\kappa) \leq \lambda \leq 2^\kappa$ for some cardinals λ, $\kappa \geq \omega$. The $\mathrm{Ps}(\lambda,\kappa)$ holds, because $\mathrm{Ps}(m(\kappa),\kappa)$ holds and so it is possible to apply (b).

(d) Since $\log 2^{2^\kappa} \leq 2^\kappa$ and by Fact 5.5(a),

$$m\left(2^{2^\kappa}\right) \leq \left(\log 2^{2^\kappa}\right)^\omega \leq (2^\kappa)^\omega = 2^\kappa.$$

Consequently $m(2^{2^\kappa}) \leq 2^\kappa \leq 2^{2^\kappa}$ and hence $\mathrm{Ps}\left(2^\kappa, 2^{2^\kappa}\right)$ holds by (b). □

We give some results about $m(-)$ concerning cardinals κ for which it is possible that $m(\kappa) < 2^\kappa$.

Lemma 5.7. *If κ is a cardinal of countable cofinality such that $2^{<\kappa} < 2^\kappa$, then $m(\kappa) = 2^\kappa$ implies that κ is a strong limit.*

Proof. Suppose that κ is not a strong limit. We prove that $m(\kappa) < 2^\kappa$. Since κ has countable cofinality, there exists an increasing sequence $\{\kappa_n\}_{n\in\mathbb{N}}$ of cardinals such that $\kappa = \sup_{n\in\mathbb{N}} \kappa_n$ and $\kappa_n < \kappa$ for every $n \in \mathbb{N}$. By Claim 1.3 there exists $n \in \mathbb{N}$ such that $2^{\kappa_n} \geq \kappa$. Hence $\log \kappa \leq \kappa_n$. Therefore

$$(\log \kappa)^\omega \leq \kappa_n^\omega \leq 2^{\kappa_n} \leq 2^{<\kappa}.$$

By Fact 5.5(a) $m(\kappa) \leq (\log \kappa)^\omega$ and so $m(\kappa) \leq 2^{<\kappa}$. Finally $2^{<\kappa} < 2^\kappa$ by hypothesis. □

Corollary 5.8. *Let K be a compact abelian group such that $w(K) = \kappa$ is an infinite cardinal of countable cofinality and $2^{<\kappa} < 2^\kappa$. Then K admits a small dense pseudocompact subgroup D whenever κ is not a strong limit. In such a case D can be chosen of size any κ with $2^{<\kappa} \leq \lambda < 2^\kappa$.*

Proof. The first statement is a direct consequence of Lemma 5.7. For the second part it is sufficient to prove that $m(\kappa) \leq 2^{<\kappa}$. Since $\mathrm{cf}(\kappa) = \omega$, $\kappa > \omega$ and there exists an increasing sequence of infinite cardinals $\{\kappa_n\}_{n\in\mathbb{N}}$ such that $\kappa = \sup_{n\in\mathbb{N}} \kappa_n$ and $\kappa_n < \kappa$. Since κ is not a strong limit, there exists $n \in \mathbb{N}$ such that $2^{\kappa_n} \geq \kappa$ and so $\kappa_n \geq \log \kappa$. By Fact 5.5(a) $m(\kappa) \leq (\log \kappa)^\omega$. Therefore $m(\kappa) \leq (\kappa_n)^\omega \leq 2^{\kappa_n}$. Hence $m(\kappa) \leq 2^{<\kappa}$. □

Since under GCH every limit cardinal is a strong limit, the following are consequences of Lemma 5.7.

Corollary 5.9. *Under GCH, if κ is an infinite cardinal of countable cofinality such that $2^{<\kappa} < 2^\kappa$, then $m(\kappa) = 2^\kappa$.*

Proof. Since $\mathrm{cf}(\kappa) = \omega$, κ is a limit cardinal and then κ is a strong limit because we are assuming GCH. Then $\log \kappa = \kappa$ by Claim 1.3. By Fact 5.5 $m(\kappa) \geq \log \kappa$ and $m(\kappa)$ cannot have countable cofinality. So $m(\kappa) > \kappa$ and hence $m(\kappa) = \kappa^+ = 2^\kappa$. □

The following is a direct consequence of Corollary 5.9.

Corollary 5.10. *Under GCH a compact abelian group K such that $w(K) = \kappa$ is an infinite cardinal of countable cofinality and $2^{<\kappa} < 2^\kappa$ admits no small dense pseudocompact subgroup.*

Remark 5.11. Under GCH every limit cardinal is a strong limit and so the opposite implication of Lemma 5.7 holds as shown by Corollary 5.9, but it does not hold in general. In fact in [47] Gitik and Shelah showed that it is possible that there exists a strong limit cardinal κ with $\mathrm{cf}(\kappa) = \omega$ and $m(\kappa) = \kappa^+ < 2^\kappa$.

5.2 The case of w-divisible pseudocompact abelian groups

It is immediate to weaken the hypothesis of Theorem 5.1 from connected to almost connected:

Corollary 5.12. *If G is a non-trivial almost connected pseudocompact abelian group, then* $\mathrm{Ps}(r_0(G), w(G))$ *holds.*

Proof. Since $c(G) \in \Lambda(G)$, by Corollary 2.21(b,d) with $\kappa = \omega$, $c(G)$ is a non-trivial connected pseudocompact abelian group such that $w(c(G)) = w(G)$. By Theorem 5.1 $\mathrm{Ps}(r_0(c(G)), w(c(G)))$ holds. Consequently it remains to prove that $r_0(c(G)) = r_0(G)$. This holds because $r_0(G) = r_0(c(G)) + r_0(G/c(G))$ by Lemma 1.7, where $r_0(c(G)) \geq \mathfrak{c}$ by Fact 1.36 and, since $G/c(G)$ is metrizable by Theorem 2.20, $r_0(G/c(G)) \leq |G/c(G)| \leq \mathfrak{c}$. □

One can ask also whether connectedness is a necessary condition in order that $\mathrm{Ps}(r_0(G), w(G))$ holds for a pseudocompact abelian group G. Using a technique similar to that of the proof of Theorem 5.1 and applying Theorem 4.11 we prove the following result, that generalizes Theorem 5.1 to w-divisible pseudocompact abelian groups, which are far from being connected (while connected pseudocompact abelian groups are w-divisible). This result is Theorem E of the introduction.

Theorem 5.13. *If G is a w-divisible pseudocompact abelian group, then* $\mathrm{Ps}(r_0(G), w(G))$ *holds.*

Proof. Let $w(G) = \kappa > \omega$ and $K = \widetilde{G}$. Then K is a w-divisible compact abelian group of weight κ in view of Lemma 3.27. By Theorem 4.11 there exists a continuous surjective homomorphism $f : K \to \prod_{i\in I} K_i$, where $\prod_{i\in I} K_i$ is a w-divisible product of weight $|I| = \kappa > \omega$. By Fact 1.36 $r_0(K) \geq \mathfrak{c}$.

Let $\varphi : \prod_{i\in I} K_i \to \prod_{i\in I} K_i/t(K_i)$ be the product of the canonical projections $K_i \to K_i/t(K_i)$. For $A \subseteq I$ let

$$\varphi_A = \varphi \restriction_{\prod_{i\in A} K_i} : \prod_{i\in A} K_i \to \prod_{i\in A} K_i/t(K_i).$$

Moreover

$$\pi_A : \prod_{i\in I} K_i \to \prod_{i\in A} K_i \quad \text{and} \quad \bar{\pi}_A : \prod_{i\in I} K_i/t(K_i) \to \prod_{i\in A} K_i/t(K_i)$$

are the canonical projections. Let

$$H = f(G) \subseteq \prod_{i\in I} K_i \text{ and } \bar{H} = \varphi(H) \subseteq \prod_{i\in I} K_i/t(K_i),$$

while

$$i : H \to \prod_{i\in I} K_i \quad \text{and} \quad \bar{i} : \bar{H} \to \prod_{i\in I} K_i/t(K_i)$$

are the inclusion maps. Finally $\tilde{\varphi} = \varphi \restriction_H \colon H \to \bar{H}$.

Let $i \in I$. Then $|K_i/t(K_i)| = \mathfrak{c}$, because $K_i/t(K_i)$ is torsion free, K_i is metrizable compact non-torsion and $r_0(K_i/t(K_i)) = r_0(K_i) = \mathfrak{c}$ by Fact 1.36. Then there exists a bijection $\xi_i : K_i/t(K_i) \to X$, where X is a set of cardinality $\mathfrak{c}$. Consequently

$$\xi = \prod_{i\in I} \xi_i : \prod_{i\in I} K_i/t(K_i) \to X^I \cong X^\kappa,$$

defined by $\xi((k_i)_{i\in I}) = (\xi_i(k_i))_{i\in I}$ for every $(k_i)_{i\in I} \in \prod_{i\in I} K_i/t(K_i)$, is a bijection. Define $\bar{\bar{H}} = \xi(\bar{H})$ and $\tilde{\xi} = \xi \restriction_{\bar{H}} \colon \bar{H} \to \bar{\bar{H}}$, let $\bar{\bar{i}} : \bar{\bar{H}} \to X^I$ be the inclusion map and $\bar{\bar{\pi}}_A : X^I \to X^A$ the canonical projection. Moreover let

$$\tilde{\chi} = \tilde{\xi} \circ \tilde{\varphi}, \quad \xi_A = \xi \restriction_{\prod_{i\in A} K_i/t(K_i)} \quad \text{and} \quad \chi_A = \xi_A \circ \varphi_A$$

and define

$$\omega_A = \pi_A \circ i \quad \text{and} \quad \bar{\bar{\omega}}_A = \bar{\bar{\pi}}_A \circ \bar{\bar{i}}.$$

This gives the following commutative diagram:

$$\begin{tikzcd}
H \arrow[r, hook, "i"] \arrow[rr, bend left, "\omega_A"] \arrow[d, "\tilde{\varphi}"] \arrow[dd, bend right, "\tilde{\chi}"'] & \prod_{i\in I} K_i \arrow[r, two heads, "\pi_A"] \arrow[d, "\varphi"] & \prod_{i\in A} K_i \arrow[d, "\varphi_A"] \arrow[dd, bend left, "\chi_A"] \\
\bar{H} \arrow[r, hook, "\bar{i}"] \arrow[d, "\tilde{\xi}"] & \prod_{i\in I} K_i/t(K_i) \arrow[r, two heads, "\bar{\pi}_A"] \arrow[d, "\xi"] & \prod_{i\in A} K_i/t(K_i) \arrow[d, "\xi_A"] \\
\bar{\bar{H}} \arrow[r, hook, "\bar{\bar{i}}"] \arrow[rr, bend right, "\bar{\bar{\omega}}_A"'] & X^I \arrow[r, two heads, "\bar{\bar{\pi}}_A"] & X^A
\end{tikzcd} \tag{5.1}$$

We want to prove that

$$\operatorname{Ps}\big(|\bar{H}|, \kappa\big) \text{ holds.}$$

To this aim we prove that $\bar{\bar{H}} = \xi(\bar{H})$ is ω-dense in X^I.

Let A be a countable subset of I. Since G is a dense pseudocompact subgroup of K, H is a dense pseudocompact subgroup of $\prod_{i\in A} K_i$ by Fact 2.6(a,c). Therefore $\omega_A : H \to \prod_{i\in A} K_i$ is surjective. In fact each K_i is metrizable, so $\prod_{i\in A} K_i$ is metrizable as well; since $\omega_A(H)$ is a pseudocompact subgroup of the metrizable group $\prod_{i\in A} K_i$, it is compact; being also dense, it coincides with $\prod_{i\in A} K_i$. Also $\chi_A : \prod_{i\in A} K_i \to X^A$ is a surjection and so $\chi_A \circ \omega_A$ is surjective as well. But

$$\chi_A \circ \omega_A = \bar{\bar{\omega}}_A \circ \tilde{\chi}$$

and hence $\bar{\bar{\omega}}_A \circ \tilde{\chi}$ is surjective; thus $\bar{\bar{\omega}}_A$ is surjective too. Since A is an arbitrary countable subset of I, this proves that $\bar{\bar{H}}$ is ω-dense in X^I. Therefore $|\bar{\bar{H}}| > \omega$. Since ξ is a bijection, $|\bar{H}| = |\bar{\bar{H}}| > \omega$. This yields that $\operatorname{Ps}\big(|\bar{H}|, \kappa\big)$ holds.

Since there exists a surjective homomorphism of G onto $\bar{H}$, $r_0(G) \geq r_0(\bar{H}) = |\bar{H}|$. (The last equality is due to $|\bar{\bar{H}}| = |\bar{H}| > \omega$ and the fact that $\bar{H}$ is torsion free as a subgroup of the the torsion free group $\prod_{i \in I} K_i/t(K_i)$.) Moreover $r_0(G) \leq |K| = 2^\kappa$. Since $\mathrm{Ps}\left(|\bar{H}|, \kappa\right)$ holds, $\mathrm{Ps}(r_0(G), \kappa)$ holds as well by Proposition 5.6(b). $\square$

Lemma 5.14. *Let G be a non-singular pseudocompact abelian group. If* $\mathrm{Ps}(r_0(G), w(G))$ *holds, then* $w(G) \leq 2^{2^{w_d(G)}}$.

Proof. To begin with, by Proposition 5.6(b) $\mathrm{Ps}(r_0(G), w(G))$ yields

$$w(G) \leq 2^{r_0(G)}. \tag{5.2}$$

By Lemma 3.23 there exists a pseudocompact subgroup H of G such that $w(H) = w_d(H) = w_d(G)$ and $r_0(H) = r_0(G)$. Since G is non-singular, $w_d(H) = w(H) > \omega$ and so H is w-divisible. In view of Theorem 5.13 $\mathrm{Ps}(r_0(H), w(H))$ holds and so $r_0(H) \leq 2^{w(H)}$, that is

$$r_0(G) \leq 2^{w_d(G)}. \tag{5.3}$$

Equations (5.2) and (5.3) together give $w(G) \leq 2^{2^{w_d(G)}}$. $\square$

The upper bound given by this lemma can be reached for singular groups: the following example furnishes an example of a singular pseudocompact abelian group G for which $\mathrm{Ps}(r_0(G), w(G))$ and $w(G) = 2^{2^{w_d(G)}}$ hold.

Example 5.15. For every infinite cardinal κ there exists a compact abelian group H_κ such that:

- $r_0(H_\kappa) = 2^\kappa$;
- $w_d(H_\kappa) = \kappa$;
- $w(H_\kappa) = 2^{2^\kappa}$;
- $\mathrm{Ps}(r_0(H_\kappa), w(H_\kappa))$ holds.

The group $H_\kappa = \{0,1\}^{2^\kappa} \times \mathbb{T}^\kappa$ has the requested properties. Indeed $\mathrm{Ps}(r_0(H_\kappa), w(H_\kappa))$ is $\mathrm{Ps}(2^\kappa, 2^{2^\kappa})$, which holds by Proposition 5.6(d).

Every H_κ is not w-divisible and H_ω is singular.

This example shows also that the converse implication of Theorem 5.13 does not hold. This means that w-divisibility (and so also connectedness) is not a necessary condition in order that $\mathrm{Ps}(r_0(G), w(G))$ holds for a pseudocompact abelian group G. Nevertheless, we have the following:

Corollary 5.16. *For an infinite abelian group G and a cardinal $\kappa > \omega$ the following conditions are equivalent:*

(a) G admits a connected pseudocompact group topology of weight κ;

(b) G admits a w-divisible pseudocompact group topology of weight κ;

(c) $\mathrm{Ps}(r_0(G), \kappa)$ *and* $|G| \leq 2^\kappa$ *hold.*

Proof. (a)⇔(c) is proved in [34, Theorem 7.1], (a)⇒(b) is obvious and (b)⇒(c) follows from Theorem 5.13. □

In this corollary we suppose that κ is uncountable, otherwise condition (b) makes no sense, in view of the definition of w-divisible abelian group. But conditions (a) and (c) can be considered also for $\kappa = \omega$: (a) becomes G admits a metrizable connected compact group topology, so in particular $r_0(G) = |G| = \mathfrak{c}$ by Fact 1.36, while (b) becomes equivalent to $r_0(G) = \mathfrak{c}$. In fact $\mathrm{Ps}(r_0(G), \omega)$ and $|G| \leq 2^\omega$ hold if and only if $\mathfrak{c} = m(\omega) \leq r_0(G) \leq |G| \leq \mathfrak{c}$, that is $r_0(G) = \mathfrak{c}$, in view of Proposition 5.6(a,c). So in the countable case (a)⇒(c). But the converse implication does not hold true: the group $G = \mathbb{Z}^\omega$ does not admit any connected compact group topology of weight ω being not divisible, while it has $r_0(G) = \mathfrak{c}$. More in general G admits no compact group topology at all [35, Example 13.3].

5.3 The general case

The following result can be easily deduced from Theorem 5.13. An alternative way to prove it is adopted in [34].

Theorem 5.17. *The condition* $\mathrm{Ps}(r_0(G), w_d(G))$ *holds, whenever G is a pseudocompact non-torsion abelian group.*

Proof. By Lemma 3.23 there exists a pseudocompact subgroup H of G such that $w(H) = w_d(H) = w_d(G)$ and $r_0(H) = r_0(G)$. If G is non-singular then H is w-divisible and Theorem 5.13 applies to H to conclude that $\mathrm{Ps}(r_0(G), w_d(G))$ holds. If G is singular, by Lemma 3.34 with $\kappa = \omega$ there exists $m \in \mathbb{N}_+$ such that $w(mG) \leq \omega$, so mG is compact; therefore $r_0(G) = r_0(mG) \leq |mG| \leq \mathfrak{c}$. Since G is non-torsion, $r_0(G) \geq \mathfrak{c}$ by Fact 1.36. Hence $r_0(G) = \mathfrak{c}$ and $w_d(G) = \omega$ because G is non-torsion. By Proposition 5.6(a) $\mathrm{Ps}(\mathfrak{c}, \omega)$ holds. □

The following immediate corollary of Theorem 5.17 is precisely the answer to Problem 5.2.

Corollary 5.18. *If G is a pseudocompact non-torsion abelian group, then* $\mathbf{Ps}(r_0(G))$ *holds.*

In Corollary 5.16 we considered the problem of the characterization of the abelian groups admitting pseudocompact group topologies in the case of w-divisible topologies, which go closer to connected ones. We conclude with the case of singular topologies, which are closer to the "opposite end", namely the torsion pseudocompact groups (that are always totally disconnected). Here we offer only the following:

Conjecture 5.19. *For an infinite abelian group G the following conditions are equivalent:*

(a) G admits a singular pseudocompact group topology;

(b) there exists $m \in \mathbb{N}_+$ such that $G[m]$ admits a pseudocompact group topology and mG admits a metrizable compact group topology.

If the conjecture was true, this case could be reduced to those of pseudocompact group topologies on torsion abelian groups ($G[m]$) and of metrizable compact group topologies on abelian groups (mG). In [34, §6] is given a clear criterion of when a torsion abelian group admits a pseudocompact group topology, while in [35] the groups which admit a metrizable compact group topology are well characterized.

We intend to study whether the results of this chapter can be extended to κ-pseudocompact abelian groups. The first step is to have a better knowledge of the structure of κ-pseudocompact abelian groups, which seem to have properties similar to those of pseudocompact groups — for instance see Theorem 2.12. Apparently here the counterparts of $\mathrm{Ps}(-,-)$ and $\mathbf{Ps}(-)$ are needed for every infinite cardinal κ.

Chapter 6

Extremality

As we have done in the introduction for s- and r-extremality (see Definition 8), we generalize for an infinite cardinal κ the definitions of the other different levels of extremality given in [29] for pseudocmpact groups (for $\kappa = \omega$ we find exactly the definitions of d-, c- and weakly-extremal group).

Definition 6.1. *Let κ be an infinite cardinal. A topological group G is:*

- d_κ-extremal *if G/D is divisible for every dense κ-pseudocompact subgroup D of G;*
- c_κ-extremal *if $r_0(G/D) < 2^\kappa$ for every dense κ-pseudocompact subgroup D of G;*
- κ-extremal *if it is both d_κ- and c_κ-extremal.*

If $\lambda \leq \kappa$ are infinite cardinals, then s_λ- (respectively, r_λ-, d_λ-, c_λ-, λ-) extremality yields s_κ- (respectively, r_κ-, d_κ-, c_κ-, κ-) extremality. Immediate examples of d_κ- and c_κ-extremal groups are divisible and torsion topological groups respectively.

In the following diagram we give an idea of the relations among these levels of extremality for κ-pseudocompact abelian groups. The non-obvious implications in the diagram are proved in Proposition 6.6, Theorem 6.13 and Lemma 6.21.

$$\begin{tikzcd}
 & w(G) \leq \kappa \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] \arrow[ddrr, Rightarrow, bend left] & & \\
s_\kappa\text{-extremal} \arrow[dr, Rightarrow] & & r_\kappa\text{-extremal} \arrow[dl, Rightarrow] & \\
 & \kappa\text{-extremal} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \kappa\text{-singular} \arrow[dl, Rightarrow] \\
d_\kappa\text{-extremal} & & c_\kappa\text{-extremal} &
\end{tikzcd} \tag{6.1}$$

The obvious symmetry of this diagram is "violated" by κ-singularity; but Corollary 6.4 shows that it is equivalent to c_κ-extremality.

The main theorem of this chapter, from which Theorem A^κ of the introduction immediately follows, shows that four of the remaining properties in the diagram coincide:

Theorem 6.2. *Let κ be an infinite cardinal. For a κ-pseudocompact abelian group G the following conditions are equivalent:*

(a) G is κ-extremal;

(b) G is either s_κ- or r_κ-extremal;

(c) $w(G) \leq \kappa$.

As the proof of Theorem 6.2 does not make use of Theorem A of the introduction, in this way we obtain also a completely self-contained proof of that theorem.

Example 6.25 shows that in general d_κ- and c_κ-extremality do not coincide with the other levels of extremality.

To prove Theorem 6.2 we show in Theorem 6.10 that c_κ-extremal κ-pseudocompact abelian groups have "small" free rank. Moreover Theorem 6.20 proves Theorem 6.2 in the torsion case. Then, as a consequence of Corollary 4.18, we have that for compact abelian groups κ-singularity is equivalent to c_κ-extremality and to a third property of a completely different nature:

Theorem 6.3. *Let κ be an infinite cardinal. For a compact abelian group K the following conditions are equivalent:*

(a) K is c_κ-extremal;

(b) K is κ-singular;

(c) there exists no continuous surjective homomorphism of K onto a w-divisible power S^I with $|I| > \kappa$.

Using (c) we prove in Proposition 6.22 that the free rank of non-κ-singular κ-pseudocompact abelian groups is "large". This allows us to extend the equivalence of (a) and (b) to the more general case of κ-pseudocompact groups:

Corollary 6.4. *Let κ be an infinite cardinal. Every κ-pseudocompact abelian group is c_κ-extremal if and only if it is κ-singular.*

The last stage in the proof of Theorem 6.2 is to show that every κ-singular κ-extremal group has weight $\leq \kappa$, applying the torsion case of the theorem, that is Theorem 6.20.

6.1 First results

The next proposition shows the stability under taking quotients of d_κ- and c_κ-extremality for κ-pseudocompact abelian groups.

Proposition 6.5. *Let κ be an infinite cardinal. Let G be a κ-pseudocompact abelian group and let L be a closed subgroup of G. If G is d_κ- (respectively, c_κ-) extremal, then G/L is d_κ- (respectively, c_κ-) extremal.*

Proof. Let $\pi : G \to G/L$ be the canonical projection. If D is a dense κ-pseudocompact subgroup of G/L, by Lemmas 1.22(a) and 2.23(b) $\pi^{-1}(D)$ is a dense κ-pseudocompact subgroup of G. Moreover $G/\pi^{-1}(D) \cong (G/L)/D$.

Suppose that G/L is not d_κ-extremal. Then there exists a dense κ-pseudocompact subgroup D of G/L such that $(G/L)/D$ is not divisible. Therefore $G/\pi^{-1}(D)$ is not divisible, hence G is not d_κ-extremal. If G/L is not c_κ-extremal. Then there exists a dense κ-pseudocompact subgroup D of G/L such that $r_0((G/L)/D) \geq 2^\kappa$. Consequently $r_0(G/\pi^{-1}(D)) \geq 2^\kappa$, so G is not c_κ-extremal. □

In this proposition we consider only d_κ- and c_κ-extremality. Indeed Theorem 6.2 and Example 6.25 prove that, for κ-pseudocompact abelian groups, these are the only levels of extremality that are not equivalent to having weight $\leq \kappa$.

The next proposition covers the implication (c)⇒(b) of Theorem 6.2, even for non-necessarily abelian groups.

Proposition 6.6. *Let κ be an infinite cardinal and let (G, τ) be a compact group of weight $\leq \kappa$. Then (G, τ) is s_κ- and r_κ-extremal.*

Proof. First we prove that (G, τ) is s_κ-extremal. Let D be a dense κ-pseudocompact subgroup of (G, τ). Then $w(D) \leq \kappa$ and so D is compact. So D is closed in (G, τ) and therefore $D = G$, because D is dense in (G, τ).

Now we prove that (G, τ) is r_κ-extremal. Let τ' be a κ-pseudocompact group topology on G such that $\tau' \geq \tau$. Since $\psi(G, \tau) \leq \kappa$, it follows that also $\psi(G, \tau') \leq \kappa$. By Lemma 2.16 (G, τ') is compact. Then $\tau' = \tau$ by Theorem 1.28. □

6.2 Construction of G_κ-dense subgroups

The following lemma is a generalization to κ-pseudocompact abelian groups of [21, Lemma 2.13]. The construction is very similar.

Lemma 6.7. *Let κ be an infinite cardinal. Let G be a κ-pseudocompact abelian group and $G = \bigcup_{n\in\mathbb{N}} A_n$, where all A_n are subgroups of G. Then there exist $n \in \mathbb{N}$ and $N \in \Lambda_\kappa(G)$ such that $A_n \cap N$ is G_κ-dense in N.*

Proof. Since $(G, P_\kappa\tau)$ is Baire by Theorem 2.28 and since $G = \bigcup_{n\in\mathbb{N}} \overline{A_n}^{P_\kappa\tau}$, there exists $n \in \mathbb{N}$ such that $\mathrm{Int}_{P_\kappa\tau}\overline{A_n}^{P_\kappa\tau} \neq \emptyset$. The family $\{x + N : x \in G,\ N \in \Lambda_\kappa(G)\}$ is a base of $P_\kappa\tau$ by Corollary 2.19; consequently there exist $x \in G$ and $N \in \Lambda_\kappa(G)$ such that $x + N \subseteq \overline{A_n}^{P_\kappa\tau}$. Since $x + N$ is open and closed in $P_\kappa\tau$, then

$$\overline{A_n \cap (x + N)}^{P_\kappa\tau} = x + N,$$

that is $A_n \cap (x+N)$ is G_κ-dense in $x+N$.

We can suppose without loss of generality that $x \in A_n$, because we can choose $a \in A_n$ such that $a+N = x+N$. In fact, since $A_n \cap (x+N) \neq \emptyset$, because $A_n \cap (x+N)$ is G_κ-dense in $x+N$, it follows that there exists $a \in A_n \cap (x+N)$. In particular $a \in x+N$ and so $a+N = x+N$.

We can choose $x = 0$ because all $A_n \leq G$: since $A_n \cap (x+N)$ is G_κ-dense in $x+N$, it follows that $(A_n - x) \cap N$ is G_κ-dense in N and $A_n - x = A_n$ since $x \in A_n$. □

For $\kappa = \omega$ the next lemma is [13, Lemma 4.1(b)].

Lemma 6.8. *Let κ be an infinite cardinal and let G be a κ-pseudocompact abelian group. If $N \in \Lambda_\kappa(G)$ and D is G_κ-dense in N, then there exists a subgroup E of G such that $|E| \leq 2^\kappa$ and $D+E$ is G_κ-dense in G.*

Proof. Since D is G_κ-dense in $N \in \Lambda_\kappa(G)$, it follows that $x+D$ is G_κ-dense in $x+N$ for every $x \in G$. By Theorem 2.20 G/N is compact of weight κ and so $|G/N| \leq 2^\kappa$, i.e., there exists $X \subseteq G$ with $|X| \leq 2^\kappa$ such that $G/N = \{x+N : x \in X\}$. We set $E = \langle X \rangle$; then $|E| \leq 2^\kappa$ and $D+E$ is G_κ-dense in G. □

Lemmas 6.7 and 6.8 imply that in case G is a κ-pseudocompact abelian group such that $G = \bigcup_{n\in\mathbb{N}} A_n$, where all A_n are subgroups of G, then there exist $n \in \mathbb{N}$, $N \in \Lambda_\kappa(G)$ and $E \leq G$ with $|E| \leq 2^\kappa$ such that $(A_n \cap N) + E$ is G_κ-dense in G. In particular we have the following useful result.

Corollary 6.9. *Let κ be an infinite cardinal. Let G be a κ-pseudocompact abelian group such that $G = \bigcup_{n\in\mathbb{N}} A_n$, where all $A_n \leq G$. Then there exist $n \in \mathbb{N}$ and a subgroup E of G such that $|E| \leq 2^\kappa$ and $A_n + E$ is G_κ-dense in G.*

Thanks to the previous results we prove the following theorem, which gives a first restriction for extremal κ-pseudocompact groups, that is the free rank cannot be too big. The case $\kappa = \omega$ of this theorem is [29, Theorem 3.6]. That theorem was inspired by [12, Theorem 5.10 (b)] and used ideas from the proof of [13, Proposition 4.4].

Theorem 6.10. *Let κ be an infinite cardinal and let G be a κ-pseudocompact abelian group. If G is c_κ-extremal, then $r_0(G) \leq 2^\kappa$.*

Proof. Let S be a maximal independent subset of G. Then $|S| = r_0(G)$ and there exists a partition $S = \bigcup_{n\in\mathbb{N}_+} S_n$ such that $|S_n| = r_0(G)$ for each $n \in \mathbb{N}_+$. Let $U_n = \langle S_n \rangle$, $V_n = U_1 \oplus \ldots \oplus U_n$ and $A_n = \{x \in G : n!x \in V_n\}$ for every $n \in \mathbb{N}_+$. Then $G = \bigcup_{n\in\mathbb{N}_+} A_n$. By Corollary 6.9 there exist $n \in \mathbb{N}_+$ and a subgroup E of G such that $D = A_n + E$ is G_κ-dense in G and $|E| \leq 2^\kappa$. Hence $|E/(A_n \cap E)| \leq 2^\kappa$. Since $D/A_n = (A_n+E)/A_n$ is algebraically isomorphic to $E/(A_n \cap E)$, it follows that $|D/A_n| \leq |E| \leq 2^\kappa$. Since G is c_κ-extremal, it follows that $r_0(G/D) < 2^\kappa$ and so $r_0(G/A_n) \leq 2^\kappa$ because $(G/A_n)/(D/A_n)$ is algebraically isomorphic to G/D. On the other hand, $r_0(G/A_n) \geq r_0(G)$, as U_n embeds into G/A_n. Hence $r_0(G) \leq 2^\kappa$. □

6.3 The dense graph theorem

For $\kappa = \omega$ the following is [29, Lemma 3.7].

Lemma 6.11. *Let κ be an infinite cardinal. Let G be a topological abelian group and H a compact abelian group with $|H| > 1$. Let $h : G \to H$ be a homomorphism.*

(a) If $\ker h$ is G_κ-dense in G and $h(G)$ is G_κ-dense in H, then Γ_h is G_κ-dense in $G \times H$.

(b) Suppose that $w(H) \leq \kappa$. Then $\ker h$ is G_κ-dense in G and h is surjective if and only if Γ_h is G_κ-dense in $G \times H$.

Proof. (a) Suppose that $\ker h$ is G_κ-dense in G and that $h(G)$ is G_κ-dense in H. Every non-empty G_κ-set of $G \times H$ contains a G_κ-set of $G \times H$ of the form $W \times O$, where W is a non-empty G_κ-set of G and O is a non-empty G_κ-set of H. Since $h(G)$ is G_κ-dense in H, there exists $z \in G$ such that $y = h(z) \in O$. Also $z + \ker h$ is G_κ-dense in G and so $W \cap (z + \ker h) \neq \emptyset$. Consequently there exists $x \in W \cap (z + \ker h)$. From $x \in z + \ker h$ it follows that $x - z \in \ker h$. Therefore $h(x - z) = 0$ and so $h(x) = h(z) = y$. Since $x \in W$, it follows that $(x, y) \in (W \times O) \cap \Gamma_h$. This proves that

$$(W \times O) \cap \Gamma_h \neq \emptyset.$$

Hence Γ_h is G_κ-dense in $G \times H$.

(b) One implication follows from (a).

Suppose that Γ_h is G_κ-dense in $G \times H$. Let W be a non-empty G_κ-set of G. Since $W \times \{0\}$ is a G_κ-set of $G \times H$, because $w(H) = \psi(H) \leq \kappa$, it follows that $\Gamma_h \cap (W \times \{0\}) \neq \emptyset$. But

$$\Gamma_h \cap (W \times \{0\}) = (W \cap \ker h) \times \{0\}$$

and so

$$W \cap \ker h \neq \emptyset.$$

This proves that $\ker h$ is G_κ-dense in G. Moreover consider the canonical projection $p_2 : G \times H \to H$. Then $p_2(\Gamma_h) = h(G)$ is G_κ-dense in H by Fact 2.6(b). Since $w(H) \leq \kappa$, it follows that $h(G) = H$, that is h is surjective. $\square$

From the hypothesis of this lemma that $h(G)$ is G_κ-dense in H and H is not trivial, it follows that $\ker h$ is proper in G.

The following theorem gives a necessary condition for a κ-pseudocompact group to be either s_κ- or r_κ-extremal. We call it "dense graph theorem" because of the nature of this necessary condition. Moreover it is the generalization to κ-pseudocompact groups of [16, Theorem 4.1].

Theorem 6.12. *Let κ be an infinite cardinal and let (G, τ) be a non-metrizable κ-pseudocompact group such that there exists a homomorphism $h : G \to H$ where H is a κ-pseudocompact abelian group with $|H| > 1$ and Γ_h is G_κ-dense in $(G, \tau) \times H$. Then:*

(a) there exists a κ-pseudocompact group topology $\tau' > \tau$ on G such that $w(G,\tau') = w(G,\tau)$;

(b) there exists a proper G_κ-dense subgroup D of (G,τ) such that $w(D) = w(G,\tau)$.

Proof. We first prove that H can be chosen compact metrizable and that in such a case h is surjective. There exists a continuous character $\chi : H \to \mathbb{T}$ such that $\chi(H) \neq \{0\}$. Let $H' = \chi(H) \subseteq \mathbb{T}$ and $h' = \chi \circ h$. Then H' is compact and metrizable. So H' is either $\mathbb{T}$ or $\mathbb{Z}(n) \leq \mathbb{T}$ for some integer $n > 1$. Since

$$\mathrm{id}_G \times \chi : (G,\tau) \times H \to (G,\tau) \times H'$$

is a continuous surjective homomorphism such that $(\mathrm{id}_G \times \chi)(\Gamma_h) = \Gamma_{h'}$ and Γ_h is G_κ-dense in $(G,\tau) \times H$, it follows that $\Gamma_{h'}$ is G_κ-dense in $(G,\tau) \times H'$ by Fact 2.6(b). Let $p_2 : G \times H' \to H'$ be the canonical projection. Then $p_2(\Gamma_{h'}) = h'(G)$ is G_κ-dense in H' by Fact 2.6(b). Since H' is metrizable, $h'(G) = H'$ and h' is surjective.

(a) Since G is G_κ-dense in $\widetilde{(G,\tau)}$ by Theorem 2.12 and since Γ_h is G_κ-dense in $(G,\tau) \times H$, it follows that Γ_h is G_κ-dense in $\widetilde{(G,\tau)} \times H$. Consequently Γ_h with the topology inherited from $\widetilde{(G,\tau)} \times H$ is κ-pseudocompact in view of Corollary 2.14. As in Remark 2.24 let τ_h be the coarsest group topology on G such that $\tau_h \geq \tau$ and h is τ_h-continuous; then (G,τ_h) is homeomorphic to Γ_h and so it is κ-pseudocompact. If $\tau_h = \tau$, then h is continuous and Theorem 1.20 yields that Γ_h is closed in $(G,\tau) \times H$. This is not possible because Γ_h is dense in $(G,\tau) \times H$ by hypothesis. Hence $\tau_h \gneq \tau$. By hypothesis $w(G,\tau) > \omega$ and since H is metrizable, then

$$w(G,\tau_h) = w(\Gamma_h) = w((G,\tau) \times H) = w(G,\tau) \cdot w(H) = w(G,\tau).$$

(b) Let $D = \ker h$. By Lemma 6.11 D is G_κ-dense in (G,τ). Moreover D is proper in G. Clearly $w(D) = w(G,\tau)$. □

The next theorem shows that κ-extremality "puts together" s_κ- and r_κ-extremality. It is the generalization to κ-pseudocompact abelian groups of [29, Theorem 3.12].

Theorem 6.13. *Let κ be an infinite cardinal and let G be a κ-pseudocompact abelian group which is either s_κ- or r_κ-extremal. Then G is κ-extremal.*

Proof. Suppose looking for a contradiction that G is not κ-extremal. Then there exists a dense κ-pseudocompact subgroup D of G such that either G/D is not divisible or $r_0(G/D) \geq 2^\kappa$. In both cases D has to be a proper dense κ-pseudocompact subgroup of G. Then G is not s_κ-extremal. We prove that G is not r_κ-extremal as well. The subgroup D is G_κ-dense in G by Corollary 2.14. Let $\pi : G \to G/D$ be the canonical projection.

We build a surjective homomorphism $h : G/D \to H$, where H is compact $|H| > 1$ and $\ker h$ is G_κ-dense in G. By assumption D is a G_κ-dense subgroup of G such that either G/D is not divisible or $r_0(G/D) \geq 2^\kappa$. In the first case G/D admits a non-trivial finite quotient H, while in the second case we can find a surjective homomorphism

$G/D \to \mathbb{T} = H$, in view of Fact 1.16, as $|\mathbb{T}| \leq r_0(G/D)$. Since $\ker h$ contains D in both cases, $\ker h$ is G_κ-dense in G. Apply Theorem 6.12 and Corollary 2.14 to conclude that G is not r_κ-extremal. □

The following proposition and lemma are the generalizations to the κ-pseudocompact case of [16, Theorems 5.8 and 5.9] respectively. The ideas used in the proofs are similar. The next claim is needed in the proofs of both.

Claim 6.14. *Let $p \in \mathbb{P}$, let G be an abelian group of exponent p and $h : G \to \mathbb{Z}(p) \leq \mathbb{T}$ a continuous surjective homomorphism. Then Γ_h has index p in $G \times \mathbb{Z}(p)$.*

Proof. Consider $\xi : G\times\mathbb{Z}(p) \to \mathbb{Z}(p)$, defined by $\xi(g, y) = h(g)-y$ for all $(g, y) \in G\times\mathbb{Z}(p)$. Then ξ is surjective and $\ker \xi = \Gamma_h$. Therefore $G \times \mathbb{Z}(p)/\ker \xi = G \times \mathbb{Z}(p)/\Gamma_h$ is algebraically isomorphic to $\mathbb{Z}(p)$ and so they have the same cardinality p. □

The following proposition shows that for κ-pseudocompact abelian groups of prime exponent s_κ-extremality is equivalent to r_κ-extremality.

Proposition 6.15. *Let κ be an infinite cardinal and let (G, τ) be a κ-pseudocompact abelian group of exponent $p \in \mathbb{P}$. Then the following conditions are equivalent:*

(a) there exist a κ-pseudocompact abelian group H with $|H| > 1$ and a homomorphism $h : G \to H$ such that Γ_h is G_κ-dense in $(G, \tau) \times H$;

(b) (G, τ) is not s_κ-extremal;

(c) (G, τ) is not r_κ-extremal.

Proof. (a)⇒(b) and (a)⇒(c) follow from Theorem 6.12.

(b)⇒(c) Suppose that (G, τ) is not s_κ-extremal. Then there exists a dense κ-pseudocompact subgroup D of (G, τ). We can suppose without loss of generality that D is maximal and so that $|G/D| = p$. Let τ' be the coarsest group topology such that

$$\tau' \supseteq \tau \cup \{x + D : x \in G\}.$$

Since $D \notin \tau$ but $D \in \tau'$, so $\tau' > \tau$. Since $(D, \tau' \restriction_D) = (D, \tau \restriction_D)$ and D is a κ-pseudocompact subgroup of (G, τ), it follows that D is a κ-pseudocompact subgroup of (G, τ'). Hence (G, τ') is κ-pseudocompact by Lemma 2.15.

(c)⇒(a) Suppose that G is not r_κ-extremal. Then there exists a κ-pseudocompact group topology τ' on G such that $\tau' > \tau$. Since both topologies are precompact, there exists an homomorphism $h : G \to \mathbb{T}$ such that h is τ'-continuous but not τ-continuous. Note that $h(G) \neq \{0\}$. Being G of exponent p, so $h(G) = \mathbb{Z}(p) \leq \mathbb{T}$. Let

$$H = \mathbb{Z}(p).$$

Since h is not τ-continuous, by Theorem 1.20 Γ_h is not closed in $(G, \tau) \times \mathbb{Z}(p)$. Moreover $|(G, \tau) \times \mathbb{Z}(p)/\Gamma_h| = p$ by Claim 6.14. Since Γ_h is a subgroup of index p in $(G, \tau) \times \mathbb{Z}(p)$ and it is not closed, then Γ_h is dense in $(G, \tau) \times \mathbb{Z}(p)$.

Endow G with the topology τ_h, that is the coarsest group topology on G such that $\tau_h \geq \tau$ and h is τ_h-continuous (see Remark 2.24). Then (G, τ_h) is κ-pseudocompact, because h is τ'-continuous and so $\tau_h \leq \tau'$ and τ' is κ-pseudocompact. By Remark 2.24 Γ_h is homeomorphic to (G, τ_h) and so Γ_h is κ-pseudocompact. Since Γ_h is dense and κ-pseudocompact in $(G, \tau) \times \mathbb{Z}(p)$, Corollary 2.14 yields that Γ_h is G_κ-dense in $(G, \tau) \times \mathbb{Z}(p) = (G, \tau) \times H$. □

Lemma 6.16. *Let κ be an infinite cardinal. Let (G, τ) be a κ-pseudocompact abelian group of exponent $p \in \mathbb{P}$ such that (G, τ) is either s_κ- or r_κ-extremal. Then every $h \in \mathrm{Hom}(G, \mathbb{T})$ is $P_\kappa\tau$-continuous (i.e., $\mathrm{Hom}(G, \mathbb{T}) \subseteq \widehat{(G, P_\kappa\tau)}$).*

Proof. If $h \equiv 0$, then h is $P_\kappa\tau$-continuous. Suppose that $h \not\equiv 0$. Then $h(G) = \mathbb{Z}(p) \leq \mathbb{T}$. Since G is either s_κ- or r_κ-extremal, by Proposition 6.15 Γ_h is not G_κ-dense in $(G, \tau) \times \mathbb{Z}(p)$, that is Γ_h is not dense in

$$P_\kappa((G, \tau) \times \mathbb{Z}(p)) = P_\kappa(G, \tau) \times P_\kappa\mathbb{Z}(p) = (G, P_\kappa\tau) \times \mathbb{Z}(p).$$

Claim 6.14 implies that $|G \times \mathbb{Z}(p)/\Gamma_h| = p$. Since Γ_h is not dense and is of prime index in $(G, P_\kappa\tau) \times \mathbb{Z}(p)$, it follows that Γ_h is closed in $(G, P_\kappa\tau) \times \mathbb{Z}(p)$. By Theorem 1.20 h is $P_\kappa\tau$-continuous. □

6.4 Torsion abelian groups and extremality

The following definition and two lemmas are the generalization to the κ-pseudocompact case of [16, Notation 5.10, Theorem 5.11 and Lemma 5.13]. The constructions are almost the same.

Definition 6.17. *Let κ be an infinite cardinal, let X be a topological space and $Y \subseteq X$. The* κ-closure *of Y in X is*

$$\kappa\text{-cl}_X(Y) = \bigcup \left\{ \overline{Y}^X : A \subseteq Y,\ |A| \leq \kappa \right\}.$$

For $Y \subseteq X$, the set $\kappa\text{-cl}_X(Y)$ is *κ-closed* in X, i.e., $\kappa\text{-cl}_X(\kappa\text{-cl}_X(Y)) = \kappa\text{-cl}_X(Y)$.

Lemma 6.18. *Let κ be an infinite cardinal. Let G be a κ-pseudocompact abelian group and let $h \in \mathrm{Hom}(G, \mathbb{T})$. Then the following conditions are equivalent:*

(a) $h \in \kappa\text{-cl}_{\mathrm{Hom}(G,\mathbb{T})}\widehat{G}$;

(b) there exists $N \in \Lambda_\kappa(G)$ such that $N \subseteq \ker h$.

Proof. (a)⇒(b) Suppose that $h \in \kappa\text{-cl}_{\mathrm{Hom}(G,\mathbb{T})}\widehat{G}$. Let $A \subseteq \widehat{G}$ such that $|A| \leq \kappa$ and $h \in \overline{A}^{\mathrm{Hom}(G,\mathbb{T})}$. We set $N = \bigcap\{\ker f : f \in A\}$. Then $N \in \Lambda_\kappa(G)$. Moreover $N \subseteq \ker h$ as $h \in \overline{A}^{\mathrm{Hom}(G,\mathbb{T})}$.

(b)⇒(a) Let $N \in \Lambda_\kappa(G)$ and let $\pi : G \to G/N$ be the canonical projection. The group G/N is compact of weight $\leq \kappa$ and so

$$|\widehat{G/N}| = w(G/N) \leq \kappa$$

by Fact 1.40(c). We enumerate the elements of $\widehat{G/N}$ as $\widehat{G/N} = \{\chi_\lambda : \lambda < \kappa\}$ and define

$$A = \{\chi_\lambda \circ \pi : \lambda < \kappa\} \leq \widehat{G}.$$

We prove that $h \in \overline{A}^{\mathrm{Hom}(G,\mathbb{T})}$. Suppose that $h \notin \overline{A}^{\mathrm{Hom}(G,\mathbb{T})}$. Since A is a closed subgroup of the compact group $\mathrm{Hom}(G,\mathbb{T})$, there exists $\xi \in \widehat{\mathrm{Hom}(G,\mathbb{T})}$ such that $\xi(h) \neq 0$ and $\xi(f) = 0$ for every $f \in A$. By Pontryagin duality there exists $x \in G$ such that $f(x) = \xi(f)$ for every $f \in \widehat{G}$. Then

$$\chi_\lambda(\pi(x)) = \xi(\chi_\lambda \circ \pi) = 0$$

for every $\chi_\lambda \in \widehat{G/N}$. Then $\pi(x) = x + N = N$ and so $x \in N \subseteq \ker h$, i.e., $h(x) = 0$. But $h(x) = \xi(h) \neq 0$, a contradiction. □

Lemma 6.19. *Let κ be an infinite cardinal. Let (G, τ) be a κ-pseudocompact abelian group of exponent $p \in \mathbb{P}$ such that G is either s_κ- or r_κ-extremal and $|G| = \lambda \geq \kappa$. Then for the completion K of (G, τ):*

- *(a) for every $h \in \mathrm{Hom}(G,\mathbb{T})$ there exists $h' \in \mathrm{Hom}(K,\mathbb{T})$ such that $h' \restriction_G = h$ and h' is $P_\kappa K$-continuous;*
- *(b) $\mathrm{Hom}(G,\mathbb{T}) \subseteq \kappa\text{-cl}_{\mathrm{Hom}(G,\mathbb{T})}\widehat{(G,\tau)}$;*
- *(c) $\psi(G,\tau) \leq \kappa \cdot \log \lambda$.*

Proof. (a) By Theorem 2.12 G is dense in $P_\kappa K$. By Lemma 6.16 every $h \in \mathrm{Hom}(G,\mathbb{T})$ is $P_\kappa\tau$-continuous. Therefore h can be extended to $h' \in \mathrm{Hom}(K,\mathbb{T})$, such that h' is $P_\kappa K$-continuous, because $(G, P_\kappa\tau)$ is dense in $P_\kappa K$.

(b) Since $h \in \mathrm{Hom}(G,\mathbb{T})$, by (a) there exists $h' \in \mathrm{Hom}(K,\mathbb{T})$ such that $h' \restriction_G = h$ and h' is $P_\kappa K$-continuous. By Lemma 6.18 $h' \in \kappa\text{-cl}_{\mathrm{Hom}(K,\mathbb{T})}\widehat{K}$. Therefore

$$h \in \kappa\text{-cl}_{\mathrm{Hom}(G,\mathbb{T})}\widehat{(G,\tau)}.$$

Indeed, let $A' \subseteq \widehat{K}$ be such that $|A'| \leq \kappa$ and $h' \in \overline{A'}^{\mathrm{Hom}(K,\mathbb{T})}$. For $f' \in A'$ we set

$$f = f' \restriction_G \in \widehat{(G,\tau)} \quad \text{and} \quad A = \{f' \restriction_G : f' \in A'\}.$$

There exists a net $\{f'_\lambda\}_\lambda$ in A' such that $f'_\lambda \to h'$ in $\mathrm{Hom}(K,\mathbb{T})$; since the topology on $\mathrm{Hom}(K,\mathbb{T})$ is the point-wise convergence topology, this means that $f'_\lambda(x) \to h'(x)$ for every $x \in K$. Then $f_\lambda(x) \to h(x)$ for every $x \in G$. Hence $f_\lambda \to h$ in $\mathrm{Hom}(G,\mathbb{T})$ and so

$$h \in \overline{A}^{\mathrm{Hom}(G,\mathbb{T})} \subseteq \kappa\text{-cl}_{\mathrm{Hom}(G,\mathbb{T})}\widehat{(G,\tau)}.$$

(c) By Fact 1.40(a,c) $d(\mathrm{Hom}(G,\mathbb{T})) = \log w(\mathrm{Hom}(G,\mathbb{T})) \leq \log\lambda$. Then there exists a dense subset S of $\mathrm{Hom}(G,\mathbb{T})$ such that $|S| \leq \log\lambda$. By (b) for every $h \in S$ there exists $A(h) \subseteq \widehat{(G,\tau)}$ such that $|A(h)| \leq \kappa$ and $h \in \overline{A(h)}^{\mathrm{Hom}(G,\mathbb{T})}$. Then $A = \bigcup\{A(h) : h \in S\}$ is dense in $\mathrm{Hom}(G,\mathbb{T})$, because $S \subseteq \overline{A}^{\mathrm{Hom}(G,\mathbb{T})}$. Moreover

$$A \subseteq \widehat{(G,\tau)} \quad \text{and} \quad |A| \leq \kappa \cdot \log\lambda.$$

Let $x \in G \setminus \{0\}$ and let $\{V_n : n \in \mathbb{N}\}$ be a local base at 0 of $\mathbb{T}$. Since A is dense in $\mathrm{Hom}(G,\mathbb{T})$, it separates the points of G and so there exists $f \in A$ such that $f(x) \neq 0$. Then there exists $n \in \mathbb{N}$ such that $f(x) \notin V_n$. Therefore $\bigcap_{n\in\mathbb{N}, f\in A} f^{-1}(V_n) = \{0\}$ and hence

$$\psi(G,\tau) \leq |A| \leq \kappa \cdot \log\lambda,$$

that concludes the proof. □

Now we prove Theorem 6.2 in the torsion case. For $\kappa = \omega$ it implies [16, Corollary 7.5] and the proof is inspired by this result. Since every torsion κ-pseudocompact group is c_κ-extremal, we observe that a torsion κ-pseudocompact abelian group is κ-extremal if and only if it is d_κ-extremal.

Theorem 6.20. *Let κ be an infinite cardinal and let G be a κ-pseudocompact torsion abelian group. Then G is κ-extremal if and only if $w(G) \leq \kappa$.*

Proof. If $w(G) \leq \kappa$, then G is κ-extremal by Proposition 6.6 and Theorem 6.13.

Suppose that $w(G) > \kappa$. We prove that there exists $p \in \mathbb{P}$ such that $w(G/\overline{pG}) > \kappa$. Since G is torsion, then it is bounded torsion by Fact 1.36. Therefore $K = \widetilde{G}$ is bounded torsion and $K \cong_{top} \prod_{p\in\mathbb{P}} t_p(K)$, where $t_p(K) \neq \{0\}$ for finitely many $p \in \mathbb{P}$ (see Remark 1.44(b)). Since

$$w(K) = \max_{p\in\mathbb{P}} w(t_p(K)) \quad \text{and} \quad w(K) = w(G) > \kappa,$$

there exists $p \in \mathbb{P}$ such that

$$w(t_p(K)) > \kappa.$$

Moreover for this p we have $w(t_p(K)) = w(K_{(p)})$, where $K_{(p)} = t_p(K)/pt_p(K)$, by Lemma 3.40(b). Consider the composition φ_p of the canonical projections $K \to t_p(K)$ and $t_p(K) \to K_{(p)}$. Since G is dense in K, it follows that $\varphi_p(G)$ is dense in $K_{(p)}$ by Fact 2.6(a) and so

$$w(\varphi_p(G)) = w(K_{(p)}) > \kappa.$$

Moreover there exists a continuous isomorphism $G/(\ker\varphi_p \cap G) \to \varphi_p(G)$. Since $\ker\varphi_p = pK$ and $pK \cap G = \overline{pG}$, there exists a continuous isomorphism $G/\overline{pG} \to \varphi_p(G)$. Hence

$$w\left(G/\overline{pG}\right) \geq w(\varphi_p(G)) > \kappa.$$

Let $G_1 = G/\overline{pG}$. We prove that G_1 is not s_κ-extremal. Suppose for a contradiction that G_1 is s_κ-extremal. By Lemma 6.16 every $h \in \mathrm{Hom}(G_1,\mathbb{T})$ is $P_\kappa\tau$-continuous, and so $P_\kappa G_1 = P_\kappa G_1^\#$ by Lemma 2.26. By Theorem 2.28 $P_\kappa G_1$ is Baire, hence $|G_1| \leq 2^\kappa$ by Theorem 2.27. By Lemma 6.19(c)

$$\psi(G_1) \leq \kappa \cdot \log 2^\kappa = \kappa$$

and so Lemma 2.16 implies

$$w(G_1) = \psi(G_1) \leq \kappa;$$

this contradicts our assumption.

Then there exists a proper dense κ-pseudocompact subgroup D of G_1. By Corollary 2.14 D is G_κ-dense in G_1. Let $\pi : G \to G_1$ be the canonical projection. By Lemma 2.23(b) $\pi^{-1}(D)$ is a proper G_κ-dense subgroup of G, then dense κ-pseudocompact in G by Corollary 2.14. Since $G/\pi^{-1}(D)$ is algebraically isomorphic to G_1/D, it follows that $G/\pi^{-1}(D)$ is of exponent p and hence not divisible. Therefore G is not d_κ-extremal and so not κ-extremal. □

The following lemma proves one implication of Corollary 6.4 and it is the generalization of Fact 3.1. This result holds for non-necessarily κ-pseudocompact abelian groups.

Lemma 6.21. *Let κ be an infinite cardinal and let G be a κ-singular abelian group. Then $r_0(G/D) = 0$ for every dense κ-pseudocompact subgroup D of G. In particular G is c_κ-extremal.*

Proof. By definition there exists a positive integer m such that $w(mG) \leq \kappa$. Let D be a dense κ-pseudocompact subgroup of G. By Corollary 2.14 D is G_κ-dense in G. Since mD is a G_κ-dense subgroup of mG by Fact 2.6(b) and $w(mG) \leq \kappa$, so $mD = mG$. Therefore $mG \leq D$ and hence the quotient G/D is bounded torsion. In particular $r_0(G/D) = 0$. □

Proof of Theorem 6.3. (a)⇒(c) Suppose that there exists a continuous surjective homomorphism φ of K onto a w-divisible power S^λ with $\lambda > \kappa$. Then φ is also open by Theorem 1.28. Note that $S^\lambda \cong_{top} (S^\kappa)^\lambda$. Let $T = S^\kappa$. Then $D = \Sigma_\kappa T^\lambda$ is a G_κ-dense subgroup of T^λ, so dense κ-pseudocompact by Corollary 2.14, and D trivially intersects ΔT^λ, which is topologically isomorphic to $T = S^\kappa$. Consequently

$$r_0(T^\lambda/D) \geq 2^\kappa.$$

Therefore T^λ is not c_κ-extremal and K is not c_κ-extremal as well by Proposition 6.5.

(b)⇒(a) is Lemma 6.21 and (b)⇔(c) is Corollary 4.18. □

6.5 Proof of the main theorem

Proposition 6.22. *Let κ be an infinite cardinal. If G is a non-κ-singular κ-pseudocompact abelian group, then N is non-κ-singular and $r_0(N) = r_0(G) \geq 2^\kappa$ for every $N \in \Lambda_\kappa(G)$.*

Proof. First we prove that $r_0(G) \geq 2^\kappa$. Since G is non-κ-singular, then $K = \widetilde{G}$ is non-κ-singular as well by Lemma 3.34. By Theorem 6.3 there exists a continuous surjective homomorphism $\varphi : K \to S^I$, where S is a metrizable compact non-torsion abelian group and $|I| > \kappa$. Let J be a subset of I of cardinality κ and consider the composition ϕ of φ with the canonical projection $S^I \to S^J$. The group S^J has weight κ and free rank 2^κ. Since G is G_κ-dense in K by Theorem 2.12, it follows that $\phi(G)$ is G_κ-dense in S^J by Fact 2.6(b). Since $w(S^J) \leq \kappa$, $\phi(G) = S^J$ is compact. Hence $r_0(G) \geq 2^\kappa$.

Let $N \in \Lambda_\kappa(G)$. Then N is κ-pseudocompact by Corollary 2.21(b). Since G is non-κ-singular, so N is non-κ-singular by Lemma 3.37(a).

By the first part of the proof, $r_0(N) \geq 2^\kappa$. By Theorem 2.20 $w(G/N) \leq \kappa$ and so G/N is compact with $|G/N| \leq 2^\kappa$. Since $r_0(G) = r_0(G/N) + r_0(N)$ by Lemma 1.7, we conclude that $r_0(G) = r_0(N)$. □

The next lemma shows that for κ-pseudocompact abelian groups c_κ-extremality is hereditary for κ-pseudocompact subgroups that are sufficiently large. This is the generalization to κ-pseudocompact abelian groups of [29, Theorem 4.11].

Lemma 6.23. *Let κ be an infinite cardinal and let G be a c_κ-extremal κ-pseudocompact abelian group. Then every κ-pseudocompact subgroup of G of index $\leq 2^\kappa$ is c_κ-extremal. In particular, every $N \in \Lambda_\kappa(G)$ is c_κ-extremal.*

Proof. Aiming for a contradiction, assume that there exists a κ-pseudocompact subgroup N of G with $|G/N| \leq 2^\kappa$ such that N is not c_κ-extremal. Then there exists a dense κ-pseudocompact subgroup D of N with $r_0(N/D) \geq 2^\kappa$. Therefore $|G/N| \leq r_0(N/D)$. By [29, Corollary 4.9] there exists a subgroup L of G/D such that $L + N/D = G/D$ and $r_0((G/D)/L) \geq r_0(N/D)$. Let $\pi : G \to G/D$ be the canonical projection and $D_1 = \pi^{-1}(L)$. Then $N + D_1 = G$. Since D is G_κ-dense in N by Corollary 2.14, so $\overline{D_1}^{P_\kappa G} \supseteq N + D_1 = G$ and so D_1 is G_κ-dense in G; equivalently D_1 is dense κ-pseudocompact in G by Corollary 2.14. Since G/D_1 is algebraically isomorphic to $(G/D)/L$, it follows that

$$r_0(G/D_1) = r_0((G/D)/L) \geq r_0(N/D) \geq 2^\kappa.$$

We have produced a G_κ-dense (so dense κ-pseudocompact by Corollary 2.14) subgroup D_1 of G with $r_0(G/D_1) \geq 2^\kappa$, a contradiction.

If $N \in \Lambda_\kappa(G)$ then $w(G/N) \leq \kappa$ by Theorem 2.20 and so G/N is compact. Hence $|G/N| \leq 2^\kappa$. So N is c_κ-extremal by the previous part of the proof. □

Lemma 6.24. [22, Lemma 3.2] *Let κ be an infinite cardinal and suppose that $\mathcal{A}$ is a family of subsets of 2^κ such that:*

(1) for $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| \leq \kappa$, $\bigcap_{B \in \mathcal{B}} B \in \mathcal{A}$ and

(2) each element of $\mathcal{A}$ has cardinality 2^κ.

Then there exists a countable infinite family $\mathcal{B}$ of subsets of 2^κ such that:

(a) $B_1 \cap B_2 = \emptyset$ for every $B_1, B_2 \in \mathcal{B}$ and

(b) if $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $|A \cap B| = 2^\kappa$.

Now we can prove our main results.

Proof of Corollary 6.4. If G is κ-singular then G is c_κ-extremal by Lemma 6.21.

Suppose that G is c_κ-extremal and assume for a contradiction that G not κ-singular. By Theorem 6.10 $r_0(G) \leq 2^\kappa$ and by Proposition 6.22 $r_0(G) \geq 2^\kappa$. Hence $r_0(G) = 2^\kappa$. Let $D(G_1) = \mathbb{Q}^{(S)}$, with $|S| = 2^\kappa$, be the divisible hull of the torsion free quotient $G_1 = G/t(G)$. Let $\pi : G \to D(G_1)$ be the composition of the canonical projection $G \to G_1$ and the inclusion map $G_1 \to D(G_1)$.

For a subset A of S let

$$G(A) = \pi^{-1}\left(\mathbb{Q}^{(A)}\right) \text{ and } \mathcal{A} = \{A \subseteq S : G(A) \supseteq N \in \Lambda_\kappa(G)\}.$$

Then $\mathcal{A}$ has the property that for $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| \leq \kappa$, $\bigcap_{B\in\mathcal{B}} B \in \mathcal{A}$; and $|A| = 2^\kappa$ for all $A \in \mathcal{A}$, as $r_0(N) = 2^\kappa$ for every $N \in \Lambda_\kappa(G)$ by Proposition 6.22 and $r_0(G) = 2^\kappa$. By Lemma 6.24 there exists a partition $\{P_n\}_{n\in\mathbb{N}}$ of S such that $|A \cap P_n| = 2^\kappa$ for every $A \in \mathcal{A}$ and for every $n \in \mathbb{N}$. Define

$$V_n = G(P_0 \cup \ldots \cup P_n)$$

for every $n \in \mathbb{N}$ and note that $G = \bigcup_{n\in\mathbb{N}} V_n$. By Lemma 6.7 there exist $m \in \mathbb{N}$ and $N \in \Lambda_\kappa(G)$ such that $D = V_m \cap N$ is G_κ-dense in N, so dense κ-pseudocompact in N by Corollary 2.14. By Lemma 6.23 to get a contradiction it suffices to show that $r_0(N/D) = 2^\kappa$.

Let F be a torsion free subgroup of N such that $F \cap D = \{0\}$ and maximal with this property. Suppose for a contradiction that $|F| = r_0(N/D) < 2^\kappa$. So $\pi(F) \subseteq \mathbb{Q}^{(S_1)}$ for some $S_1 \subseteq S$ with $|S_1| < 2^\kappa$ and $W = P_0 \cup \ldots \cup P_m \cup S_1$ has $|W \cap P_{m+1}| < 2^\kappa$. Consequently $W \notin \mathcal{A}$ and so $N \not\subseteq G(W)$. Take $x \in N \setminus G(W)$. Since $G/G(W)$ is torsion free,

$$\langle x \rangle \cap G(W) = \{0\}$$

and x has infinite order. But $D + F \subseteq G(W)$ and so $\langle x \rangle \cap (D + F) = \{0\}$, that is

$$(F + \langle x \rangle) \cap D = \{0\}.$$

This contradicts the maximality of F. □

Proof of Theorem 6.2. (a)⇒(c) If G is κ-extremal, in particular it is c_κ-extremal and so κ-singular by Corollary 6.4.

Suppose that $w(G) > \kappa$. Since G is κ-singular, by Lemma 3.34 there exists $m \in \mathbb{N}_+$ such that $w(mG) \leq \kappa$; in particular mG is compact and so closed in G. Since $w(mG) \leq \kappa$ and $w(G) = w(G/mG) \cdot w(mG)$, it follows that $w(G/mG) = w(G) > \kappa$. Then G/mG is not κ-extremal by Theorem 6.20 and so G is not κ-extremal by Proposition 6.5.

(c)⇒(b) is Proposition 6.6 and (b)⇒(a) is Theorem 6.13. □

The following example shows that d_κ- and c_κ-extremality cannot be equivalent conditions in Theorem 6.2. Item (a) shows that κ-singular κ-pseudocompact abelian groups need not be d_κ-extremal and also that there exists a c_κ-extremal (non-compact) κ-pseudocompact abelian group of weight $> \kappa$, which is not d_κ-extremal. It is the analogous of [29, Example 4.4]. In item (b) we give an example of a non-c_κ-extremal d_κ-extremal κ-pseudocompact abelian group of weight $> \kappa$.

Example 6.25. Let κ be an infinite cardinal.

(a) Let $p \in \mathbb{P}$ and let $G = \mathbb{Z}(p)^{2^\kappa}$. Then $H = \Sigma_\kappa \mathbb{Z}(p)^{2^\kappa}$ is κ-pseudocompact by Corollary 2.14, because it is G_κ-dense in $\mathbb{Z}(p)^{2^\kappa}$. The group $G = \mathbb{T}^\kappa \times H$ is a κ-singular κ-pseudocompact abelian group with $r_0(G) = 2^\kappa$ and $w(G) = 2^\kappa > \kappa$. In particular G is c_κ-extremal by Lemma 6.21. Thus G is not d_κ-extremal by Theorem 6.2.

(b) Let $G = \mathbb{T}^{2^\kappa}$. Then G is a κ-pseudocompact divisible abelian group G of weight $> \kappa$. So G is d_κ-extremal of weight $> \kappa$. We can prove that G is not c_κ-extremal as in the proof of Theorem 6.3: $G = (\mathbb{T}^\kappa)^{2^\kappa} = T^{2^\kappa}$ and $\Sigma_\kappa T^{2^\kappa}$ is a G_κ-dense (so dense κ-pseudocompact by Corollary 2.14) subgroup of T^{2^κ} such that $r_0(T^{2^\kappa}/\Sigma T^{2^\kappa}) \geq 2^\kappa$. That G is not c_κ-extremal follows also from Theorem 6.10, because G has free rank $> 2^\kappa$.

Chapter 7

Dense compact-like subgroups

7.1 Totally dense κ-pseudocompact subgroups

In this section we generalize Theorem B of the introduction for every infinite cardinal κ, proving Theorem B$^\kappa$. Indeed we characterize compact abelian groups admitting some proper totally dense κ-pseudocompact subgroup, solving the following problem.

Problem 7.1. *For an infinite cardinal κ, determine when a compact abelian group admits some proper totally dense κ-pseudocompact subgroup.*

In order to generalize Theorem B proving Theorem B$^\kappa$, from previous chapters we already have κ-pseudocompactness, κ-singularity and the projections onto non-κ-singular w-divisible products for every infinite cardinal κ. So we need to generalize also the property TD_ω. To this aim we introduce appropriate notions generalizing ω-boundedness and the property TD_ω for every infinite cardinal κ.

7.1.1 κ-Boundedness

Definition 7.2. *Let κ be an infinite cardinal. A Tychonov topological space X is:*

- weakly κ-bounded *if every subset of X of cardinality $< \kappa$ is contained in a compact subset of X;*
- κ-bounded *if every subset of X of cardinality at most κ is contained in a compact subset of X.*

Obviously every topological group is weakly ω-bounded. Moreover these two notions are related as follows: weakly κ-bounded coincides with the conjunction of λ-bounded for all $\lambda < \kappa$; in particular, κ-bounded coincides with weakly κ^+-bounded).

Examples of non-compact weakly κ-bounded groups of weight κ are given in Example 7.5.

Lemma 7.3. *For an infinite cardinal κ, every κ-bounded Tychonov space X is κ-pseudocompact.*

Proof. Let $f : X \to \mathbb{R}^\kappa$ be a continuous function and let $Y = f(X)$. In particular $w(Y) \leq \kappa$. There exists a dense subset D of Y such that $|D| \leq \kappa$. There exists a subset D_1 of X such that $f\restriction_{D_1}: D_1 \to D$ is bijective. Then $|D_1| \leq \kappa$. Since X is κ-bounded, $\overline{D}_1$ is compact. Therefore $f(\overline{D}_1)$ is compact. But $f(\overline{D}_1) \supseteq D$, D is dense in Y, and so $f(\overline{D}_1) = Y$ is compact. □

Lemma 7.4. *Let G and G_1 be topological abelian groups such that there exists an open continuous surjective homomorphism $f : G \to G_1$, and let H be a subgroup of G_1. If H is κ-bounded, then $f^{-1}(H)$ is κ-bounded.*

Proof. Let X be a subset of $f^{-1}(H)$ of cardinality $\leq \kappa$. Then $f(X)$ is a subset of H of cardinality $\leq \kappa$ and it is contained in a compact subset Y of H. Then $X \subseteq f^{-1}(Y)$, which is compact. □

We give examples of non-compact weakly $w(G)$-bounded groups G and of non-compact λ-bounded groups G for $\lambda < w(G)$.

Example 7.5. Let κ be an uncountable cardinal and let $K = \prod_{i\in I} K_i$ be a w-divisible product with $|I| = \kappa$.

(a) Let $\lambda < \kappa$. We show that

$$\Sigma_\lambda K = \{x \in K : |\mathrm{supp}(x)| \leq \lambda\}$$

is λ-bounded (non-compact). Take $A \subseteq \Sigma_\lambda K$ with $|A| \leq \lambda$. If $a \in A$, then $a \in \prod_{i\in L_a} K_i$, where $L_a \subseteq I$ and $|L_a| \leq \lambda$. Define $L = \bigcup_{a\in A} L_a$. Thus $A \subseteq \prod_{i\in L} K_i$ and $|L| = |A| \cdot \sup_{a\in A} |L_a| \leq \lambda$. Moreover $\prod_{i\in L} K_i$ is contained in $\Sigma_\lambda K$. Furthermore $\Sigma_\lambda K$ is proper and dense in K, so it is non-compact and of the same weight of K.

(b) Suppose that κ is regular and consider the following proper subgroup of K:

$$S = \bigcup_{\lambda<\kappa} \Sigma_\lambda K;$$

in other words $S = \{x \in K : |\mathrm{supp}(x)| < \kappa\}$. Clearly S is dense in K, hence S is not compact and $w(S) = w(K) = \kappa$. Let us see that S is weakly κ-bounded (so λ-bounded for every cardinal $\lambda < \kappa$). Take $A \subseteq S$ with $|A| < \kappa$. If $a \in A$, then $a \in \prod_{i\in L_a} K_i$, where $L_a \subseteq I$ and $|L_a| < \kappa$. Define $L = \bigcup_{a\in A} L_a$. Thus $A \subseteq \prod_{i\in L} K_i$ and $|L| = |A| \cdot \sup_{a\in A} |L_a| < \kappa$, as κ is regular. Moreover $\prod_{i\in L} K_i$ is contained in S.

Lemma 7.6. *Let κ be an infinite cardinal. A weakly κ-bounded group G with $w(G) < \kappa$ is necessarily compact, so every $w(G)$-bounded group G is compact.*

Proof. Let X be a dense subset of G of size $\leq w(G)$. As $w(G) < \kappa$ and G is weakly κ-bounded, X is contained in a compact subset C of G. The density of X in G yields the density of C in G. Hence $C = G$. This proves that G is compact. □

7.1.2 The property TD_κ

In analogy with the property TD_ω, for every infinite cardinal κ a compact abelian group K:

- *has the property* TD_κ (briefly, $K \in TD_\kappa$) if K has a proper totally dense subgroup that contains a dense κ-bounded subgroup;

- *has the property* TD^κ (briefly, $K \in TD^\kappa$) if K has a proper totally dense subgroup that contains a dense weakly κ-bounded subgroup.

The properties TD_κ and TD^κ have properties analogous to those of TD_ω. Obviously TD_κ coincides with TD^{κ^+}, in particular TD_κ implies TD^κ and TD^{ω_1} coincides with TD_ω. Nevertheless, for a limit cardinal κ the property TD^κ needs not coincide with the conjunction of all TD_λ for $\lambda < \kappa$ (see the comments after Theorem 7.15).

The property TD_κ seems to be stronger than to admit a proper totally dense κ-pseudocompact subgroup, but in Theorem 7.15 we show that these properties are equivalent for compact abelian groups.

The following lemma shows that for each infinite cardinal κ the properties TD_κ and TD^κ are stable under taking inverse images. It generalizes [25, Lemma 3.12] and [26, Lemma 2.6], the proof remains almost the same. It easily follows from Lemmas 1.49 and 7.4.

Lemma 7.7. *Let $f : K \to L$ be a continuous surjective homomorphism of compact abelian groups. Let κ be an infinite cardinal. If L has the property TD_κ (respectively, TD^κ), then K has the property TD_κ (respectively, TD^κ) too.*

Remark 7.8. Let κ be an infinite cardinal and let K be a compact abelian group. Moreover let $H = m_d(K)K$. Then $K \in TD_\kappa$ (respectively, $K \in TD^\kappa$) if and only if $H \in TD_\kappa$ (respectively, $H \in TD^\kappa$) — see Lemma 3.23.

In the next example we explicitly construct a compact abelian group K which has the property TD_κ and in particular is a w-divisible power.

Example 7.9. Let κ be an uncountable cardinal, $p \in \mathbb{P}$ and $K_p = \mathbb{Z}(p)^{\kappa^+}$. Define $K = \prod_{p\in\mathbb{P}} K_p$; then $K \cong_{top} S_{\mathbb{P}}^{\kappa^+}$, so that we can identify these groups. The subgroup $H = \Sigma_\kappa K + t(K)$ is totally dense in K. Indeed $t(K) = \bigoplus_{p\in\mathbb{P}} K_p$ is totally dense in K because each closed subgroup N of K is of the form $N = \prod_{p\in\mathbb{P}} N_p$ by Remark 1.44(b). Moreover H contains $\Sigma_\kappa K$, which is κ-bounded as proved in Example 7.5(a). Finally we show that H is proper; let $\mathbf{x} = (x_i)_{i<\kappa^+} \in K = S_{\mathbb{P}}^{\kappa^+}$ be such that $x_i = x = (y_p)_{p\in\mathbb{P}} \in S_{\mathbb{P}}$ for every $i < \kappa^+$ and each $y_p \in \mathbb{Z}(p) \setminus \{0\}$. Then $\mathbf{x} \in \Delta K \setminus t(K)$ and hence $\mathbf{x} \notin H$, because $\Delta K \cap \Sigma_\kappa K = \{0\}$. This proves that H is a proper totally dense subgroup of K which contains a dense κ-bounded subgroup of K, namely $\Sigma_\kappa K$, that is K has the property TD_κ.

In Proposition 7.12 we produce a compact abelian group with the property TD_κ for a given infinite cardinal κ. To prove it we need Lemma 7.11, which makes use of the next one.

Lemma 7.10. *Let K be a compact abelian group, C a torsion free subgroup of K and B a subgroup of K maximal with the property $B \cap C = \{0\}$. Then $B \cap mN = m(B \cap N)$ for every $m \in \mathbb{Z} \setminus \{0\}$ and for every subgroup N of K. In particular, if $p \in \mathbb{P}$, then $B \cap N \not\subseteq pN$ for every subgroup N of K topologically isomorphic to $\mathbb{Z}_p$, whenever $r_0(C) < \mathfrak{c}$.*

Proof. Let $p \in \mathbb{P}$ and let N be a subgroup of K such that $N \cong_{top} \mathbb{Z}_p$. If $m \in \mathbb{Z} \setminus \{0\}$ and $x \in B \cap mN$, then $x = mz$, with $z \in N$. Assume that $z \notin B$. It follows that $B_1 = B + \langle z \rangle$ contains properly B. So there exists $y \in B_1 \cap C$, $y \neq 0$, that is $y = b + kz \in C$, where $b \in B$ and $k \in \mathbb{Z}$. Then $my = mb + kmz \in B \cap C = \{0\}$ and hence $my = 0$. As C is torsion free, we conclude that $y = 0$, finding a contradiction.

Suppose that $r_0(C) < \mathfrak{c}$ and assume for a contradiction that N is a closed subgroup of K isomorphic to $\mathbb{Z}_p$ with $B \cap N \subseteq pN$. Then $B \cap N = B \cap pN$. Applying the first part we have also $B \cap pN = p(B \cap N) = p(B \cap pN)$. By induction $B \cap pN = p^n(B \cap pN)$ for every $n \in \mathbb{N}$. Hence

$$B \cap pN = \bigcap_{n=0}^{\infty} p^n(B \cap pN),$$

but $\bigcap_{n=0}^{\infty} p^n(B \cap N) \subseteq \bigcap_{n=0}^{\infty} p^n N = \{0\}$. So $B \cap N = \{0\}$. If $\pi : K \to K/B$ is the canonical projection, this yields that $\pi \restriction_N$ is injective and consequently $r_0(K/B) \geq r_0(N) = \mathfrak{c}$.

To get a contradiction we prove that

$$r_0(K/B) \leq r_0(C) < \mathfrak{c}.$$

Suppose that $r_0(K/B) > r_0(C)$. Then $\pi(C)$ is a torsion free subgroup of K/B such that $r_0((K/B)/\pi(C)) \geq 1$. So there exists an infinite cyclic subgroup C_1 of K/B such that $C_1 \cap \pi(C) = \{0\}$. Since $\pi^{-1}(C_1) \cap C = B \cap C = \{0\}$ and $\pi^{-1}(C_1) \supsetneq \ker \pi = B$, contradicting the maximality of B. □

Thanks to this lemma we prove the following, that produces a totally dense subgroup containing a given subgroup. It was announced without a proof in [25, Lemma 3.16].

Lemma 7.11. *Let K be a compact abelian group that admits a subgroup B such that $r_0(K/B) \geq 1$. Then K has a proper totally dense subgroup H that contains B.*

Proof. Since $r_0(K/B) \geq 1$, there exists a cyclic infinite subgroup C of K such that $B \cap C = \{0\}$. Let $B_1 = B + t(K)$; then $B_1 \cap C = \{0\}$. By Zorn's lemma there exists a subgroup H of K such that $H \supseteq B_1$, $H \cap C = \{0\}$ and H is maximal with respect to these two properties. It immediately follows that $H \supseteq B$ and that H is a proper subgroup of K. Moreover $t(K) \subseteq H$ and by Lemma 7.10 $H \cap N \not\subseteq pN$ for every subgroup N of K isomorphic to $\mathbb{Z}_p$ and for every $p \in \mathbb{P}$. Apply Theorem 1.51 to conclude that H is totally dense in K. □

Then next proposition generalizes [26, Proposition 2.4].

Proposition 7.12. *Let κ be an uncountable cardinal and $K = \prod_{i \in I} K_i$ a w-divisible product with $|I| = \kappa$. Then K has the property TD_λ for every $\omega \leq \lambda < \kappa$.*

Proof. Let λ be an infinite cardinal $< \kappa$. As shown in Example 7.5(a) $B = \Sigma_\lambda K$ is a λ-bounded subgroup of K. For every $i \in I$ let c_i be a non-torsion element of K_i. Defining $C = \langle (c_i)_{i \in I} \rangle$ we have $B \cap C = \{0\}$ and so $r_0(K/B) \geq 1$. Apply Lemma 7.11 to conclude the proof. □

7.1.3 Main theorems

The regularity of κ is essential in Example 7.5(b). Indeed we have the following theorem characterizing the regularity of uncountable cardinals κ in terms of the topological property TD^κ.

Theorem 7.13. *Let κ be an uncountable cardinal. Then the following conditions are equivalent:*

- *(a) κ is regular;*
- *(b) every w-divisible product of the form $\prod_{i \in I} K_i$ with $|I| = \kappa$ has the property TD^κ;*
- *(c) there exists a compact abelian group K of weight κ such that $K \in TD^\kappa$;*
- *(d) there exists a compact abelian group of weight κ with a proper dense weakly κ-bounded subgroup.*

Proof. (a)⇒(b) Let $K = \prod_{i \in I} K_i$ be a w-divisible product with $|I| = \kappa$. As shown in Example 7.5(b) $B = \bigcup_{\lambda < \kappa} \Sigma_\lambda K$ is a weakly κ-bounded subgroup of K. For every $i \in I$ let c_i be a non-torsion element of K_i. Defining $C = \langle (c_i)_{i \in I} \rangle$ we have $B \cap C = \{0\}$ and so $r_0(K/B) \geq 1$. Apply Lemma 7.11 to conclude that $K \in TD^\kappa$.

(b)⇒(c) and (c)⇒(d) are obvious.

(d)⇒(a) Let H be a proper dense weakly κ-bounded subgroup of a compact abelian group K. Fix a point $x \in K \setminus H$ and assume for a contradiction that $\lambda = \mathrm{cf}(\kappa) < \kappa$, i.e.,

$$\kappa = \sup_{\alpha < \lambda} \kappa_\alpha, \text{ with } \kappa_\alpha < \kappa \text{ for all } \alpha < \lambda.$$

We can assume without loss of generality that K is a subgroup of $\mathbb{T}^\kappa$. Write $\mathbb{T}^\kappa = \prod_{\alpha < \lambda} T_\alpha$, where $T_\alpha \cong_{top} \mathbb{T}^{\kappa_\alpha}$ for each $\alpha < \lambda$. For $\alpha < \lambda$, let $N_\alpha = \prod_{\beta < \alpha} T_\beta$ and let $p_\alpha : \mathbb{T}^\kappa \to N_\alpha$ be the canonical projection. By Fact 2.6(a) $p_\alpha(H)$ is dense in $p_\alpha(K)$. Since $w(N_\alpha) = \kappa_\alpha < \kappa$, $p_\alpha(H)$ is also compact (so closed) in $p_\alpha(K)$. So $p_\alpha(H) = p_\alpha(K)$. Then there exists $h_\alpha \in H$ such that

$$p_\alpha(h_\alpha) = p_\alpha(x). \tag{7.1}$$

The set $A = \{h_\alpha : \alpha < \lambda\} \subseteq H$ has size $\leq \lambda < \kappa$. Hence the weak κ-boundedness of H implies that $C = \overline{A}^H$ is compact. Then C is closed in K. On the other hand, for every neighborhood U of 0 in $\mathbb{T}^\kappa$ there exists a projection p_α such that $\ker p_\alpha \subseteq U$. Now (7.1) yields $h_\alpha - x \in U$, so $A \cap (x + U) \neq \emptyset$. This proves that $x \in C \subseteq H$, a contradiction. □

The next example shows a w-divisible group K of non-regular weight having no proper dense weakly $w(K)$-bounded subgroup at all (so in particular $K \notin TD^{w(K)}$). Moreover K admits no continuous surjective homomorphism onto a w-divisible power $S^{w_d(K)}$.

Example 7.14. Let $\mathbb{P} = \{p_n : n \in \mathbb{N}_+\}$ be all primes written in increasing order. Then the group $K = \prod_{n=1}^{\infty} \mathbb{Z}(p_n)^{\aleph_n}$ is w-divisible of weight $\aleph_\omega$. Nevertheless, there exists no continuous surjective homomorphism of K onto a w-divisible power $S^{\aleph_\omega}$. Indeed if such a projection $K \to S^{\aleph_\omega}$ exists, by Remark 1.44(c) for $n \in \mathbb{N}_+$ the restriction $K_{p_n} = \mathbb{Z}(p_n)^{\aleph_n} \to S_{p_n}^{\aleph_\omega}$ is a continuous surjective homomorphism; this is not possible. From Theorem 7.13 it follows that $K \notin TD^{\aleph_\omega}$.

The subgroup $\bigcup_{\lambda<\aleph_\omega} \Sigma_\lambda K$ of K (defined in Example 7.5(b)) is not even ω-bounded. In fact it is not even countably compact, as it contains a sequence $(x_n)_{n\in\mathbb{N}_+}$ that converges to a point of $K \setminus \bigcup_{\lambda<\kappa} \Sigma_\lambda K$, although it is pseudocompact. In what follows we construct such a sequence. For every $n \in \mathbb{N}_+$ let $e_n \in \mathbb{Z}(p_n)^{\aleph_n}$ be the element $e_n = (e_{n,i})_{i<\aleph_n}$ with $e_{n,i} = 1_{\mathbb{Z}(p^n)}$ for every $i < \aleph_n$. Then for every $n \in \mathbb{N}_+$ let $x_n = (e_1, \dots, e_n, 0, \dots) \in \Sigma_{\aleph_n} K$. The sequence $(x_n)_{n\in\mathbb{N}_+}$ converges to the element $e = (e_n)_{n\in\mathbb{N}_+}$ which has $|\mathrm{supp}(e)| = \aleph_\omega$ and so $e \notin \bigcup_{\lambda<\aleph_\omega} \Sigma_\lambda K$.

The following theorem is Theorem B$^\kappa$ of the introduction and it is a complete generalization of Theorem B (indeed, to get Theorem B it suffices to take $\lambda = \omega$). Theorem 4.11 and Corollary 4.18 apply to prove it.

Theorem 7.15. *Let κ be an infinite cardinal and let K be a compact abelian group. The following conditions are equivalent:*

(a) K has a proper totally dense κ-pseudocompact subgroup;

(b) K is non-κ-singular;

(c) there exists a continuous surjective homomorphism of K onto S^{κ^+}, where S is a compact non-torsion abelian group;

(d) K has the property TD_κ.

Moreover, $K \in TD^{w_d(K)}$ if and only if $w_d(K)$ is an uncountable regular cardinal.

Proof. All conditions (a)–(e) imply that K has to be non-singular.

(a)⇒(b) Suppose that K is κ-singular. By Lemma 3.35 there exists a torsion $N \in \Lambda_\kappa(K)$ and Theorem 2.20 implies that $w(K/N) \leq \kappa$. Let H be a proper totally dense κ-pseudocompact subgroup of K. Since N is torsion and H is totally dense, $N \leq H$ by Lemma 1.47(b). By Corollary 2.14 H is G_κ-dense in K and so $K = N + H$ by Lemma 2.22(a). Therefore $N \leq H$ yields $K = H$, a contradiction.

(b)⇔(c) is Corollary 4.18.

(c)⇒(d) Assume there exists a continuous surjective homomorphism of K onto S^{κ^+}, where S is a compact non-torsion abelian group. By Remark 4.4 it is possible to suppose that S is metrizable. By Proposition 7.12 S^{κ^+} has the property TD_κ, hence also K has the property TD_κ thanks to Lemma 7.7.

(d)⇒(a) follows from Lemma 7.3.

Let $\kappa = w_d(K) > \omega$. Assume that it is regular. By Theorem 4.11 there exists a continuous surjective homomorphism of K onto a w-divisible product C of weight κ. By Theorem 7.13, C has the property TD^κ, hence also K has the property TD^κ thanks to Lemma 7.7. To end the proof assume $K \in TD^\kappa$. By Lemma 3.23 the subgroup $H = m_d(K)K$ of K is such that $w(H) = w_d(H) = \kappa$; in particular H is w-divisible and $H \in TD^\kappa$ by Remark 7.8. Apply again Theorem 7.13 to conclude that κ is regular. □

All conditions (a)–(e) imply that $\kappa < w(K)$. In case $\kappa \geq w(K)$ they are all false. The equivalence of (b) and (d) in Theorem 7.15 implies

$$w_d(K) = \sup\{\kappa : K \in TD^\kappa\},$$

but leaves open the question of when $K \in TD^{w_d(K)}$ holds true. This motivates the final part of the theorem that settles completely this issue. Consequently, for an infinite cardinal κ the property TD^κ coincides for compact abelian groups with the conjunction of all TD_λ for $\lambda < \kappa$ precisely when κ is regular. Analogously, the equivalence of (c) and (d) yields:

$$\begin{aligned} w_d(K) = \sup\{\kappa :\ &\text{there exists a compact non-torsion abelian group } S \\ &\text{and a surjective continuous homomorphism } K \to S^\kappa\}. \end{aligned}$$

7.2 Essential dense κ-pseudocompact subgroups

As an immediate corollary of Theorem 6.2 we obtain the following result in case the dense κ-pseudocompact subgroup is required to be also either r_κ- or s_κ-extremal.

Corollary 7.16. *Let κ be an infinite cardinal. A proper dense κ-pseudocompact subgroup of a topological abelian group cannot be either s_κ- or r_κ-extremal.*

Proof. If D is a dense either s_κ- or r_κ-extremal κ-pseudocompact subgroup of G, then D is compact by Theorem 6.2. Hence D is closed in G and so $D = G$, since D is also dense in G. □

This proves that the concept of r_κ-extremal κ-pseudocompact group is "dual" to that of minimal κ-pseudocompact group, in the sense that a κ-pseudocompact group is minimal if there exists no strictly coarser κ-pseudocompact group topology on G. So Corollary 7.16 suggests the following problem:

Problem 7.17. *For an infinite cardinal κ, when does a κ-pseudocompact abelian group admit a proper dense minimal κ-pseudocompact subgroup?*

In Theorem 7.21 we completely solve Problem 7.17 in the compact case, i.e., in view of Theorem 1.50, we solve the following:

Problem 7.18. *For an infinite cardinal κ, determine when a compact abelian group admits some proper essential dense κ-pseudocompact subgroup.*

It is immediately possible to prove the following proposition, which gives a necessary condition for a κ-pseudocompact abelian group to have proper essential dense κ-pseudocompact subgroups.

Proposition 7.19. *Let κ be an infinite cardinal and let G be a κ-pseudocompact abelian group. If G is super-κ-singular, then G has no proper essential dense κ-pseudocompact subgroup.*

Proof. Let H be an essential dense κ-pseudocompact subgroup of G. Since H is essential in G, it follows that $H \supseteq \mathrm{Soc}(G)$ by Lemma 1.47(a) and so $H \supseteq N \in \Lambda_\kappa(G)$. By Corollary 2.14 H is G_κ-dense in G. Therefore Lemma 2.22(a) implies that $H + N = G$ and so $H = G$. □

The next example shows that there exist compact abelian groups that have proper essential dense κ-pseudocompact subgroups, although they have no proper totally dense subgroup (because they are torsion).

Example 7.20. Let κ be an infinite cardinal and let $p \in \mathbb{P}$. Consider the compact p-group $K = \mathbb{Z}(p^m)^{\kappa^+}$ with $m \in \mathbb{N}, m > 1$. Then $K[p] + \Sigma_\kappa K$ is a proper essential dense κ-pseudocompact subgroup of K.

In fact $K[p]$ is essential in K and $\Sigma_\kappa K$ is G_κ-dense in K. Moreover $K[p] + \Sigma_\kappa K$ is proper in K. Indeed fix an element $x \in \mathbb{Z}(p^m) \setminus \mathbb{Z}(p)$ and define $\mathbf{x} = (x_i)_{i<\kappa^+} \in K$ by $x_i = x$ for every $i < \kappa^+$. Since $\Sigma_\kappa K$ intersects trivially the diagonal subgroup ΔK of K and $x \notin \mathbb{Z}(p)$, it follows that $\mathbf{x} \in K \setminus (K[p] + \Sigma_\kappa K)$. Since $K[p] + \Sigma_\kappa K$ is G_κ-dense in K, it is dense κ-pseudocompact by Corollary 2.14.

In view of Theorem 7.15, to solve Problem 7.18 we can consider the case when K is a κ-singular compact abelian group of weight $> \kappa$. The following theorem, which is Theorem C$^\kappa$ of the introduction, solves Problem 7.18 and gives necessary and sufficient conditions for K to admit a proper essential dense κ-pseudocompact subgroup. We give it in negative form to have a clearer statement.

Theorem 7.21. *Let κ be an infinite cardinal and let K be a compact abelian group. Then the following conditions are equivalent:*

(a) K admits no proper essential dense κ-pseudocompact subgroup;

(b) K is κ-singular and $w(pT_p(K)) \leq \kappa$ for every $p \in \mathbb{P}$;

(c) K is super-κ-singular.

Proof. (a)⇒(b) Suppose that K has no proper essential dense κ-pseudocompact subgroup. In particular K has no proper totally dense κ-pseudocompact subgroup; so by Theorem 7.15 K is κ-singular. Lemma 3.39 yields that $w(c(K)) \leq \kappa$ and Lemma 3.37 implies that $L = K/c(K)$ is κ-singular. Since L is totally disconnected, $L \cong_{top} \prod_{p\in\mathbb{P}} L_p$

by Remark 1.44(b), where each L_p is κ-singular by Lemma 3.37. If there exists $p \in \mathbb{P}$ such that $w(pT_p(K)) > \kappa$, then $w(pL_p) > \kappa$ by Remark 1.44(d).

In particular L_p is a κ-singular compact $\mathbb{Z}_p$-module of weight $> \kappa$. Thus there exists $m \in \mathbb{N}_+$ such that $w(p^m L_p) \leq \kappa$ and $w(p^{m-1}L_p) > \kappa$. By Lemma 1.46 there exist a compact $\mathbb{Z}_p$-module L'_p of weight $\leq \kappa$ and a compact bounded torsion abelian group $B \cong_{top} \prod_{n=1}^m \mathbb{Z}(p^n)^{\alpha_n}$ such that $L_p \cong_{top} L'_p \times B$ and $\alpha_m > \kappa$. Since $w(pL_p) > \kappa$, we have $m > 1$ and consequently there exists a proper essential dense κ-pseudocompact subgroup of $\mathbb{Z}(p^m)^{\alpha_m}$ as Example 7.20 shows.

The composition of the projections

$$K \to L,\ L \to L_p,\ L_p \cong_{top} L'_p \times B \to B \text{ and } B \cong_{top} \prod_{n=1}^m \mathbb{Z}(p^n)^{\alpha_n} \to \mathbb{Z}(p^m)^{\alpha_m}$$

is a continuous surjective homomorphism $K \to \mathbb{Z}(p^m)^{\alpha_m}$. So apply Lemmas 1.22, 1.49(a) and 2.23(c) to find a proper essential dense κ-pseudocompact subgroup of K.

(b)$\Rightarrow$(c) By Lemma 1.43 there exists a totally disconnected closed subgroup N of K such that $K = N + c(K)$. Since K is κ-singular, $w(c(K)) \leq \kappa$ by Lemma 3.39 and so $w(K/N) \leq \kappa$ because $K/N \cong_{top} c(K)/(N \cap c(K))$ also by Theorem 1.28. Hence $N \in \Lambda_\kappa(K)$ by Theorem 2.20. According to Lemma 3.54 it suffices to prove that N is super-κ-singular.

Since N is totally disconnected and compact, it follows that $N \cong_{top} \prod_{p\in\mathbb{P}} N_p$ by Remark 1.44(b). By Remark 1.44(a) $N_p \leq T_p(K)$ and so $w(pN_p) \leq \kappa$ for every $p \in \mathbb{P}$. Moreover N is κ-singular. According to Lemma 3.42 $\mathbb{P} = P_{s,\kappa}(N)$ and $\mathbb{P} \setminus P_{m,\kappa}(N)$ is finite. This means that

$$N \cong_{top} M \times \prod_{\mathbb{P}\setminus P_{m,\kappa}(N)} N_p,$$

where $M = \prod_{p\in P_{m,\kappa}(N)} N_p$ has weight $\leq \kappa$ and $\prod_{\mathbb{P}\setminus P_{m,\kappa}(N)} N_p$ is a finite product; each N_p with $p \in \mathbb{P} \setminus P_{m,\kappa}(N)$ is a κ-singular compact $\mathbb{Z}_p$-module of weight $> \kappa$. Since $w(pN_p) \leq \kappa$ for every $p \in \mathbb{P}$, for $p \in \mathbb{P} \setminus P_{m,\kappa}(N)$ Lemma 1.46 with $m = 1$ implies $N_p \cong_{top} N'_p \times B_p$, where N'_p is a compact $\mathbb{Z}_p$-module of weight $\leq \kappa$ and $B_p = \mathbb{Z}(p)^{\alpha_p}$ with $\alpha_p > \kappa$. Therefore

$$N \cong_{top} M \times \prod_{p\in\mathbb{P}\setminus P_{m,\kappa}(N)} N'_p \times \prod_{p\in\mathbb{P}\setminus P_{m,\kappa}(N)} B_p.$$

Let B be a subgroup of N such that

$$B \cong_{top} \prod_{p\in P_{m,\kappa}(N)} \{0_{N_p}\} \times \prod_{p\in\mathbb{P}\setminus P_{m,\kappa}(N)} \{0_{N'_p}\} \times \prod_{p\in\mathbb{P}\setminus P_{m,\kappa}(N)} B_p.$$

Then $\mathrm{Soc}(N) \supseteq B$ as $\mathbb{P} \setminus P_{m,\kappa}(N)$ is finite. Since $w(N/B) \leq \kappa$, $B \in \Lambda_\kappa(N)$ by Theorem 2.20. This means that N is super-κ-singular by Lemma 3.53.

(c)$\Rightarrow$(a) follows from Proposition 7.19. $\square$

7.3 Either totally dense or essential dense subgroups

In this section we consider and solve the counterparts of Problems 7.1 and 7.18 omitting the condition κ-pseudocompact for the subgroups. Indeed we characterize compact abelian groups admitting some proper totally dense subgroup and compact abelian groups admitting some proper essential dense subgroup.

As a corollary of Lemma 7.11, we obtain a very simple characterization of compact abelian groups having proper totally dense subgroups:

Theorem 7.22. *Let K be a compact abelian group. Then K admits some proper totally dense subgroup if and only if K is non-torsion.*

Proof. Assume that K is non-torsion, that is $K \neq t(K)$ and so $r_0(K/t(K)) \geq 1$. Apply Lemma 7.11 with $B = t(K)$ to find a proper totally dense subgroup of K.

If K is torsion, then K cannot have any proper totally dense subgroup, because every totally dense subgroup contains $t(K) = K$ by Lemma 1.47(b). □

The group in the following example is a (metrizable) compact abelian group that has a proper essential dense subgroup but no proper dense pseudocompact subgroup.

Example 7.23. Let $p \in \mathbb{P}$ and $K = \mathbb{Z}(p^m)^\omega$ with $m \in \mathbb{N}$. Then $K[p]$ is a proper essential dense subgroup of K if and only if $m > 1$.

The following result characterizes compact abelian groups admitting some proper essential dense subgroup, in terms similar to those of Theorem 7.21.

Theorem 7.24. *Let K be a compact abelian group. Then the following conditions are equivalent:*

(a) K has no proper essential dense subgroup;

(b) K is torsion and $pT_p(K)$ is finite for every $p \in \mathbb{P}$;

(c) $\mathrm{Soc}(K)$ is open.

Proof. (a)⇒(b) Suppose that K has no proper essential dense subgroup, in particular K has no proper totally dense subgroup. Then Theorem 7.22 implies that K is torsion. By Fact 1.36 K is bounded torsion and so $K \cong_{top} K_{p_1} \times \ldots \times K_{p_n}$ for some $n \in \mathbb{N}_+$ and $p_1, \ldots, p_n \in \mathbb{P}$ (in particular K is totally disconnected and Remark 1.44(b) can be applied). This means that $K_p = \{0\}$ for all $p \in \mathbb{P} \setminus \{p_1, \ldots, p_n\}$.

Since K has no proper essential dense subgroup, in particular K has no proper essential dense pseudocompact subgroup and so pK_p is metrizable for every $p \in \mathbb{P}$ by Theorem 7.21 with $\kappa = \omega$. According to Lemmas 1.22 and 1.49(a) K_p cannot have any proper essential dense subgroup for every $p \in \mathbb{P}$. Let us prove that pK_p is finite for every $p \in \mathbb{P}$. Suppose that pK_p is infinite for some $p \in \mathbb{P}$ (then $p \in \{p_1, \ldots, p_n\}$). Since K is bounded torsion, K_p is a compact bounded p-torsion abelian group. Consequently

$$K_p \cong_{top} \mathbb{Z}(p)^{\alpha_1} \times \ldots \times \mathbb{Z}(p^s)^{\alpha_s}$$

for some cardinals $\alpha_1, \ldots, \alpha_s$, where $s \in \mathbb{N}_+$. Since pK_p is infinite and metrizable, there exists $m \in \mathbb{N}$, with $1 < m \leq s$ such that $\alpha_m = \omega$. By Example 7.23 $\mathbb{Z}(p^m)^{\alpha_m}$ has a proper essential dense subgroup. Consider the composition of the projections $K \to K_p$ and $K_p \to \mathbb{Z}(p^m)^{\alpha_m}$ and apply Lemmas 1.22 and 1.49(a) to find a proper essential dense subgroup of K, a contradiction.

(b)$\Rightarrow$(c) Since K is torsion, it follows that K is bounded torsion by Fact 1.36. In particular it is totally disconnected, and we can apply Remark 1.44(b). We have

$$K \cong_{top} K_{p_1} \times \ldots \times K_{p_n},$$

where $n \in \mathbb{N}_+$ and $p_1, \ldots, p_n \in \mathbb{P}$. Since $K_p/K_p[p] \cong_{top} pK_p$ is finite by hypothesis for every $p \in \mathbb{P}$, the subgroup $\mathrm{Soc}(K) = K_{p_1}[p_1] \oplus \ldots \oplus K_{p_n}[p_n]$ has finite index in K. Since $\mathrm{Soc}(K)$ is closed, it is also open in K.

(c)$\Rightarrow$(a) Let H be an essential dense subgroup of K. Then $H \supseteq \mathrm{Soc}(K)$ by Lemma 1.47(a); since $\mathrm{Soc}(K)$ is open in K so is H. This implies that H is closed in K. Since H is also dense in K, we conclude that $H = K$. $\square$

Corollary 7.25. *Let K be a compact abelian group, and let H be an open essential subgroup of K. Then K has no proper essential dense subgroup if and only if H has no proper essential dense subgroup.*

7.4 Small essential pseudocompact subgroups

In [15] Comfort and Robertson studied the compact groups that admit small strongly totally dense pseudocompact subgroups and proved the following theorem.

Theorem 7.26. [15, Theorem 6.2] *ZFC cannot decide whether there exists a compact group K with small strongly totally dense pseudocompact subgroups.*

In the proof of this theorem a totally disconnected compact abelian group admitting some small totally dense pseudocompact subgroup is produced. Consequently, as stated in [15, Theorem 6.3], it is possible to deduce the following result, which is the counterpart of Theorem 7.26 in the abelian case.

Theorem 7.27. *ZFC cannot decide whether there exists a totally disconnected compact abelian group with small totally dense pseudocompact subgroups.*

In this section, in Theorems 7.45 and 7.46 we prove the counterpart of this theorem for essential dense pseudocompact and essential pseudocompact subgroups.

7.4.1 CR-cardinals

First we define the following concept, because cardinals with these properties were useful to prove Theorem 7.26.

Definition 7.28. *A cardinal κ is a* CR-cardinal *if* $\mathrm{cf}(\kappa) = \omega$ *and* $\log 2^\kappa = \kappa$.

By the definition every CR-cardinal is a limit cardinal. So without loss of generality we can consider limit cardinals. A first example of CR-cardinal is ω.

In the following claim we give some properties of CR-cardinals.

Claim 7.29. *Let κ be a cardinal.*

(a) If $\mathrm{cf}(\kappa) = \omega$, then $\log 2^\kappa = \kappa$ if and only if $2^{<\kappa} < 2^\kappa$. In particular κ is a CR-cardinal if and only if $\mathrm{cf}(\kappa) = \omega$ and $2^{<\kappa} < 2^\kappa$.

(b) If κ is a CR-cardinal such that $\kappa^+ = 2^\kappa$, then κ is a strong limit.

(c) If κ is a strong limit, then it is a CR-cardinal if and only if $\mathrm{cf}(\kappa) = \omega$.

(d) Under GCH, κ is a CR-cardinal if and only if $\mathrm{cf}(\kappa) = \omega$.

Proof. (a) Suppose that $\log 2^\kappa = \kappa$. Let $\{\kappa_n\}_{n\in\mathbb{N}}$ be an increasing sequence of cardinals $< \kappa$ such that $\sup_{n\in\mathbb{N}} \kappa_n = \kappa$. Then $2^{\kappa_n} < 2^\kappa$ for every $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $2^{\kappa_n} = 2^{<\kappa}$, then $2^{<\kappa} < 2^\kappa$. If $2^{\kappa_n} < 2^\kappa$ for every $n \in \mathbb{N}$, then $\mathrm{cf}(2^{<\kappa}) = \omega$, because $2^{<\kappa} = \sup_{n\in\mathbb{N}} 2^{\kappa_n}$. Since $\mathrm{cf}(2^\kappa) > \kappa \geq \omega$ by Fact 1.1, it follows that $2^{<\kappa} < 2^\kappa$.

The condition $2^{<\kappa} < 2^\kappa$ implies $2^\lambda < 2^\kappa$ for every $\lambda < \kappa$ and this is equivalent to $\log 2^\kappa = \kappa$.

(b) Let $\{\kappa_n\}_{n\in\mathbb{N}}$ be an increasing sequence of cardinals $< \kappa$ such that $\sup_{n\in\mathbb{N}} \kappa_n = \kappa$. Since $\log 2^\kappa = \kappa$, $2^{\kappa_n} < 2^\kappa$ for every $n \in \mathbb{N}$. By hypothesis $2^\kappa = \kappa^+$. Then $2^{\kappa_n} < \kappa$ for every $n \in \mathbb{N}$ by Claim 1.2. This implies that κ is a strong limit.

(c) Since κ is a strong limit if and only if $\log \kappa = \kappa$ by Claim 1.3, it follows that $\log 2^\kappa = \kappa$. So this condition of the definition of CR-cardinal is satisfied. Then κ is a CR-cardinal if and only if it has countable cofinality.

(d) Since $2^\kappa = \kappa^+$ by GCH, by (c) κ is a CR-cardinal if and only if $\mathrm{cf}(\kappa) = \omega$. □

Remark 7.30. The problem of the existence of a small essential dense pseudocompact subgroup of a compact abelian group could be divided into two parts: the existence of a small essential (dense) subgroup and the existence of a small dense pseudocompact subgroup. Indeed, having these subgroups, it suffices to take their sum to find a small essential dense pseudocompact subgroup. We have considered the second part in Section 5.1; in particular Corollary 5.10 shows that

> *under GCH no compact abelian group K such that $w(K)$ is a CR-cardinal admits some small dense pseudocompact subgroup.*

7.4.2 Measuring essential subgroups

The following cardinal invariant is introduced in analogy with $ED(-)$ and $TD(-)$ (see Definition 1.53):

Definition 7.31. *For a topological group G let*

$$E(G) = \min\{|H| : H \leq G \text{ essential}\}.$$

We immediately give an example. The argument is due to Prodanov [59].

Example 7.32. For $p \in \mathbb{P}$, $E(\mathbb{Z}_p^2) = \mathfrak{c}$.

In fact, $E(\mathbb{Z}_p^2) \leq |\mathbb{Z}_p^2| = \mathfrak{c}$. Moreover there exist $\mathfrak{c}$ many subgroups of $\mathbb{Z}_p^2$ topologically isomorphic to $\mathbb{Z}_p$ and pairwise with trivial intersection: for every $\xi \in \mathbb{Z}_p \setminus \{0\}$, let $N_\xi = \overline{\langle(1,\xi)\rangle} \cong_{top} \mathbb{Z}_p$. If $\xi_1, \xi_2 \in \mathbb{Z}_p$, then $N_{\xi_1} \cap N_{\xi_2} = \{0\}$. These N_ξ are exactly $|\mathbb{Z}_p| = \mathfrak{c}$ many. Since an essential subgroup non-trivially intersects each non-trivial closed subgroup, it follows that $E(\mathbb{Z}_p^2) \geq \mathfrak{c}$ and so $E(\mathbb{Z}_p^2) = \mathfrak{c}$.

Lemma 7.33. *Let G be a topological abelian group. Then:*

(a) $E(G) \leq ED(G) \leq TD(G)$;

(b) $ED(G) = d(G) \cdot E(G)$.

Proof. (a) is obvious.

(b) It is clear that $ED(G) \geq d(G) \cdot E(G)$. Let D be a dense subgroup of G of cardinality $d(G)$ and let E be an essential subgroup of G of cardinality $E(G)$. Then $D + E$ has cardinality $d(G) \cdot E(G)$. Moreover $D + E$ is an essential dense subgroup of G and so $|D + E| \geq ED(G)$. It follows that $d(G) \cdot E(G) \geq ED(G)$ and hence $ED(G) = d(G) \cdot E(G)$. □

The next lemma in particular shows that $E(-)$ is monotone under taking closed subgroups.

Lemma 7.34. *Let K be a compact abelian group and N a closed subgroup of K. Then:*

(a) $E(N) \leq E(K)$;

(b) if $|N| = |K|$ *then* $E(K) < |K|$ *yields* $E(N) < |N|$;

(c) $E(K_p) \leq E(K)$ *for every* $p \in \mathbb{P}$.

Proof. (a) If H is essential in K, then $H \cap N$ is essential in N.

(b) Suppose that $E(N) = |N|$. Since $|N| = |K|$ by hypothesis and $E(K) \geq E(N)$ by (a), it follows that $E(K) = |K|$.

(c) follows from Corollary 1.52. □

Remark 7.35. If K is a compact abelian group with a finite essential subgroup H, then K is finite.

Proof. We want to reduce to the totally disconnected case. Since H is finite, it follows that there exists $q \in \mathbb{P}$ such that $r_q(H) = 0$. Therefore $H \cap K_q = H_q = \{0\}$ and so $K_q = \{0\}$ by the essentiality of H in K. By Remark 1.44(f) K is totally disconnected. Then $K \cong_{top} \prod_{p\in\mathbb{P}} K_p$, where each K_p is closed, by Remark 1.44(b). Since H is finite essential, $K_p = \{0\}$ for every $p \in \mathbb{P} \setminus P$, where P is a finite subset of $\mathbb{P}$. Let $p \in P$. Then K_p cannot contain any topologically isomorphic copy of $\mathbb{Z}_p$, because it is torsion free. Consequently $K_p \cong_{top} \prod_{n=1}^\infty \mathbb{Z}(p^n)^{\alpha_n}$ for some cardinals α_n, with $n \in \mathbb{N}_+$, by Remark

1.17. Since $H \cap K_p$ is a finite essential subgroup of K_p by Lemma 7.34, $H \cap K_p \supseteq \mathrm{Soc}(K_p) \cong \prod_{n=1}^{\infty} \mathbb{Z}(p)^{\alpha_n}$ implies that $\alpha_n = 0$ for all but finitely many $n \in \mathbb{N}_+$ and each α_n is finite. Therefore K_p is finite and so K is finite, because P is finite. □

Remark 7.36. Let K be a compact abelian group of weight ω. If $E(K) < |K|$ then $E(K) = TD(K) = \omega$ [60]. So

(a) either $E(K) = ED(K) = TD(K) = |K|$,

(b) or $E(K) = ED(K) = TD(K) = w(K)$.

Proposition 7.37. *Let $p \in \mathbb{P}$ and let M be a topological $\mathbb{Z}_p$-module such that $M \notin \mathcal{P}$ and $M[p]$ is compact. Then*

$$E(M) = |M| = \mathfrak{c} \cdot \mathrm{rank}_{\mathbb{Z}_p}(M) \cdot r_p(M).$$

Proof. First we prove that

$$E(M) \geq \mathfrak{c}.$$

Indeed, since $M \notin \mathcal{P}$, by Corollary 1.58(a) either $\mathrm{rank}_{\mathbb{Z}_p}(M) \geq 2$ or $r_p(M)$ is infinite. If $\mathrm{rank}_{\mathbb{Z}_p}(M) \geq 2$, then M has a subgroup isomorphic to $\mathbb{Z}_p^2$ and so $E(M) \geq E(\mathbb{Z}_p^2) = \mathfrak{c}$ by Lemma 7.34(a) and Example 7.32. If $r_p(M)$ is infinite, then $r_p(M) \geq \mathfrak{c}$, because $M[p]$ is compact and so $|M[p]| \geq \mathfrak{c}$ by Theorem 1.37. Moreover $E(M) \geq r_p(M)$ in view of Lemma 1.47(a) and so $E(M) \geq \mathfrak{c}$. In both cases $E(M) \geq \mathfrak{c}$.

Now we show that

$$E(M) \geq \mathrm{rank}_{\mathbb{Z}_p}(M).$$

Let H be an essential subgroup of M. Let S be a maximal $\mathbb{Z}_p$-independent subset of M. For every $x \in S$ consider $\overline{\langle x \rangle} = \langle x \rangle_{\mathbb{Z}_p} \cong_{top} \mathbb{Z}_p$. Then

$$\mathcal{F} = \left\{ \overline{\langle x \rangle} : x \in S \right\}$$

is a family of closed subgroups of M topologically isomorphic to $\mathbb{Z}_p$ and pairwise with trivial intersection. Since H is essential in M, H non-trivially intersects each of these subgroups in $\mathcal{F}$. Consequently $|H| \geq |\mathcal{F}| = |S| = \mathrm{rank}_{\mathbb{Z}_p}(M)$. This proves that $E(M) \geq \mathrm{rank}_{\mathbb{Z}_p}(M)$.

Moreover $E(M) \geq r_p(M)$ by Lemma 1.47(a). Therefore $E(M) \geq \mathfrak{c} \cdot \mathrm{rank}_{\mathbb{Z}_p}(M) \cdot r_p(M)$. By Corollary 1.58(c) $E(M) \leq |M| = \mathfrak{c} \cdot \mathrm{rank}_{\mathbb{Z}_p}(M) \cdot r_p(M)$ and so $E(M) = |M| = \mathfrak{c} \cdot \mathrm{rank}_{\mathbb{Z}_p}(M) \cdot r_p(M)$. □

Proposition 7.37 can be proved in the general case of a compact abelian group K, thanks to the fact that the essentiality of a subgroup can be verified "locally", namely in K_p for every $p \in \mathbb{P}$ by Corollary 1.52. We do this in Theorem 7.39.

Remark 7.38. Let K be a compact abelian group such that $K \notin \mathcal{P}$. By Remark 1.59 $\pi_{\mathcal{P}}(K)$ is not empty. Moreover

$$\sup_{p \in \mathbb{P}} E(K_p) = \sup_{p \in \mathbb{P} \setminus \pi_{\mathcal{P}}(K)} E(K_p) \cdot \sup_{p \in \pi_{\mathcal{P}}(K)} E(K_p).$$

By Corollary 1.58(b), if $p \in \mathbb{P} \setminus \pi_{\mathcal{P}}(K)$, then $|K_p| \leq \mathfrak{c}$ and in particular $E(K_p) \leq |K_p| \leq \mathfrak{c}$. Consequently

$$\sup_{p\in\mathbb{P}\setminus\pi_{\mathcal{P}}(K)} E(K_p) \leq \sup_{p\in\mathbb{P}\setminus\pi_{\mathcal{P}}(K)} |K_p| \leq \mathfrak{c}.$$

If $p \in \pi_{\mathcal{P}}(K)$ then $|K_p| = \mathfrak{c} \cdot \mathrm{rank}_{\mathbb{Z}_p}(K_p) \cdot r_p(K_p)$ by Corollary 1.58(c). By Proposition 7.37 $E(K_p) = |K_p| \geq \mathfrak{c}$ for every $p \in \mathbb{P}$. Consequently

$$\sup_{p\in\mathbb{P}} E(K_p) = \sup_{p\in\pi_{\mathcal{P}}(K)} E(K_p) = \sup_{p\in\pi_{\mathcal{P}}(K)} |K_p| = \sup_{p\in\mathbb{P}} |K_p|. \tag{7.2}$$

Theorem 7.39. *Let K be a compact abelian group such that $K \notin \mathcal{P}$. Then*

$$E(K) = ED(K) = TD(K) = \sup_{p\in\mathbb{P}} |K_p|.$$

Proof. By Lemma 7.34(c) $E(K) \geq E(K_p)$ for every $p \in \mathbb{P}$. Then $E(K) \geq \sup_{p\in\mathbb{P}} E(K_p)$ and $\sup_{p\in\mathbb{P}} E(K_p) = \sup_{p\in\mathbb{P}} |K_p|$ by (7.2) of Remark 7.38. Then

$$E(K) \geq \sup_{p\in\mathbb{P}} |K_p|.$$

Since the subgroup $wtd(K) = \bigoplus_{p\in\mathbb{P}} K_p$ is totally dense in K [65], so

$$TD(K) \leq |wtd(K)| = \sup_{p\in\mathbb{P}} |K_p|.$$

Consequently $E(K) \geq TD(K)$. By Lemma 7.33(a) $E(K) = ED(K) = TD(K)$. □

A consequence of this theorem and Remark 7.38 is that for a compact abelian group $K \notin \mathcal{P}$

$$E(K) = ED(K) = TD(K) = \sup_{p\in\pi_{\mathcal{P}}(K)} \mathfrak{c} \cdot \rho_p(K) \cdot r_p(K).$$

This confirms the idea that the cardinal invariant $E(-)$ is strictly related to the algebraic structure of the group. Indeed we see that it can be computed in terms of two purely algebraic cardinal invariants, unlike the behavior of $m(-)$, that is the other cardinal invariant involved in this section. In fact we have explained in Section 5.1 that the value of $m(-)$ depends only on the weight and so it has a more topological nature.

Corollary 7.40. *If K is a compact abelian group such that $K \notin \mathcal{P}$, then $E(K) \geq w(K)$.*

Proof. By Theorem 7.39 $E(K) = ED(K)$ and $ED(K) \geq w(K)$ by Fact 1.57. □

The next theorem motivates Definition 7.28.

Theorem 7.41. *For a non-metrizable compact abelian group K the following conditions are equivalent:*

(a) K has a small totally dense subgroup ($TD(K) < |K|$);

(b) K has a small essential dense subgroup ($ED(K) < |K|$);

(c) K has a small essential subgroup ($E(K) < |K|$);

(d) $w(K)$ is a CR-cardinal.

Proof. By Theorem 7.39 (a), (b) and (c) are equivalent.

(a)⇒(d) Suppose that $TD(K) < |K|$. By Theorem 1.61 this implies that $w(K) > w(c(K))$ and $w(K) > w((K/c(K))_p)$ for every $p \in \mathbb{P}$. Then $w(K)$ is a limit and $TD(K) = 2^{<w(K)}$ by Theorem 1.61. Hence $2^{<w(K)} < 2^{w(K)}$ and Claim 7.29(a) implies that $w(K)$ is a CR-cardinal.

(d)⇒(a) Assume that $w(K)$ is a CR-cardinal. By Claim 7.29(a) $w(K)$ has countable cofinality and $2^{<w(K)} < 2^{w(K)} = |K|$. In particular $w(K)$ is a limit. By Theorem 1.61 $TD(K) = 2^{<w(K)}$ and so $TD(K) < 2^{w(K)} = |K|$. □

In the following corollary it is stressed the direct relation between the concept of being a CR-cardinal and the existence of small subgroups of compact abelian groups, which are either totally dense, or essential dense, or essential.

Corollary 7.42. *Let K be a compact abelian group K with $w(K) = \kappa$, where κ is a cardinal which is not a strong limit. Then the following conditions are equivalent:*

(a) K has a small totally dense pseudocompact subgroup H;

(b) K has a small essential dense pseudocompact subgroup H;

(c) K has a small essential pseudocompact subgroup H;

(d) κ is a CR-cardinal.

If the above conditions hold, H can be chosen of size any κ with $2^{<\kappa} \leq \kappa < 2^{\kappa}$.

Proof. (a)⇒(b)⇒(c) is obvious.

(c)⇒(d) Since $E(K) < |K|$, it follows that κ is a CR-cardinal by Theorem 7.41.

(d)⇒(a) By Theorem 7.41 $TD(K) < |K|$. Since κ is not a strong limit, it follows that $m(\kappa) < 2^{\kappa}$ by Corollary 5.10 together with Claim 7.29(a). Hence $m(\kappa) < 2^{\kappa} = |K|$. Therefore there exist a small totally dense subgroup H of K and a small dense pseudocompact subgroup D of K. Consequently $H + D$ is a small totally dense pseudocompact subgroup of K. □

7.4.3 Independence results

The next proposition is inspired by [15, Theorem 6.2, Theorem 5.8 and Theorem 2.7(d)]. We construct an example of a totally disconnected compact abelian group K such that $TD(K) < |K|$ and $m(w(K)) < |K|$ and which has the minimal weight it can have: indeed $w(K)$ has to be uncountable and with countable cofinality and so $\aleph_\omega$ is the smallest cardinal $w(K)$ can be.

Proposition 7.43. *It is consistent with ZFC that there exists a totally disconnected compact abelian group of weight $\aleph_\omega$ with a small totally dense pseudocompact subgroup.*

Proof. We construct a totally disconnected compact abelian group with $w(K) = \aleph_\omega$, such that $TD(K) < |K|$ and $m(\aleph_\omega) < 2^{\aleph_\omega} = |K|$. To this end we make use of Easton's theorem [41] to provide a model of ZFC with $2^{\aleph_n} = \aleph_{\omega+n}$ for every $n \in \mathbb{N}_+$. Hence $2^{<\aleph_\omega} = \aleph_{\omega+\omega}$. Since $\mathrm{cf}(\aleph_{\omega+\omega}) = \omega$, and $\mathrm{cf}(2^{\aleph_\omega}) > \omega$ by Fact 1.1, it follows that $2^{<\aleph_\omega} < 2^{\aleph_\omega}$. Therefore $\aleph_\omega$ is a CR-cardinal by Claim 7.29(a). Let $\mathbb{P} = \{p_n : n \in \mathbb{N}_+\}$ and $K = \prod_{n=1}^\infty \mathbb{Z}(p_n)^{\aleph_n}$. In particular K is a totally disconnected compact abelian group such that $w(K) = \sup_{n\in\mathbb{N}_+} \aleph_n = \aleph_\omega$. By Theorem 7.41 $TD(K) < |K|$.

Being $2^{\aleph_1} = \aleph_{\omega+1} > \aleph_\omega$, then $\log \aleph_\omega \le \aleph_1$. Since ${\aleph_1}^\omega \le 2^{\aleph_1} \le 2^{<\aleph_\omega}$, it follows that $2^{<\aleph_\omega} < \aleph_\omega$ implies $(\log \aleph_\omega)^\omega < 2^{\aleph_\omega}$. This implies $m(\aleph_\omega) < |K|$, because $m(\aleph_\omega) \le (\log \aleph_\omega)^\omega$ by Fact 5.5(a). Therefore there exist small subgroups H_1, H_2 of K such that H_1 is totally dense in K and H_2 is dense pseudocompact in K. Hence $H = H_1 + H_2$ is a small totally dense pseudocompact subgroup of K. It is pseudocompact because H_2 is G_δ-dense in K by Corollary 2.14 with $\kappa = \omega$; then H is G_δ-dense in K as well and so H is dense pseudocompact in K by Corollary 2.14 with $\kappa = \omega$. □

The next proposition follows from Theorem 7.41 and together with the previous proposition is the proof of Theorem 7.45.

Proposition 7.44. *Under GCH no compact abelian group has a small essential pseudocompact subgroup.*

Proof. Let H be a small essential subgroup of a compact abelian group K with $w(K) = \kappa$. By Corollary 7.40 $E(K) \ge w(K)$ and so $\kappa \le |H| < 2^\kappa = \kappa^+$. By Theorem 7.41 κ is a CR-cardinal and then κ is a strong limit of countable cofinality. Consequently H cannot be pseudocompact by Theorem 1.37. □

Theorem 7.45. *ZFC cannot decide whether there exists a totally disconnected compact abelian group of weight $\aleph_\omega$ with small essential dense pseudocompact subgroups.*

Proof. According to Proposition 7.43 it is consistent with ZFC that there exists a totally disconnected compact abelian group of weight $\aleph_\omega$ with a small essential dense pseudocompact subgroup. Moreover under GCH there exists no compact abelian group with small essential pseudocompact subgroups by Proposition 7.44. □

The following is Theorem F of the introduction. It is similar to Theorem 7.45 and its proof is based on the same results. But Theorem 7.45 could be proved also as a consequence of the fact that $ED(K) = TD(K) \ge w(K)$ for compact abelian groups K (see Facts 1.54 and 1.57), while the difficulty in Theorem 7.46 is to consider essential pseudocompact subgroups which are non-necessarily dense. This more complicated part is contained in Theorem 7.39, that is in proving that $E(K) = ED(K) = TD(K)$ for non-metrizable compact abelian groups.

Theorem 7.46. *ZFC cannot decide whether there exists a compact abelian group with small essential pseudocompact subgroups.*

Proof. According to Proposition 7.43 it is consistent with ZFC that there exists a compact abelian group with a small essential pseudocompact subgroup. Moreover under GCH there exists no compact abelian group with small essential pseudocompact subgroups by Proposition 7.44. □

7.5 Open problems

In this section we collect all open problems related to the previous sections of this chapter.

In Section 7.1 we solve Problem 7.17 in the compact case, but it remains open in general. It is open also its counterpart for essential subgroups. We collect both in the following:

Problem 7.47. *Let κ be an infinite cardinal.*

(a) Describe the κ-pseudocompact abelian groups that admit proper dense minimal κ-pseudocompact subgroups.

(a) Describe the κ-pseudocompact abelian groups that admit proper essential dense κ-pseudocompact subgroups.*

This problem suggests to consider its counterpart for total minimality and total density. In fact, in Section 7.2 we give a characterization of compact abelian groups admitting some proper totally dense κ-pseudocompact subgroup, but it is possible to consider the same question for κ-pseudocompact abelian groups. As in Problem 7.47 we have two parts:

Problem 7.48. *Let κ be an infinite cardinal.*

(b) Describe the κ-pseudocompact abelian groups that admit proper dense totally minimal κ-pseudocompact subgroups.

(b) Describe the κ-pseudocompact abelian groups that admit proper totally dense κ-pseudocompact subgroups.*

The groups considered in (a) are necessarily minimal, and similarly the groups considered in (b) are necessarily totally minimal. For compact abelian groups (a) coincides with (a*) and (b) coincides with (b*), and we have a complete description given respectively by Theorems 7.15 and 7.21.

In analogy with Problems 7.47 and 7.48, Theorems 7.22 and 7.24 suggest the following:

Problem 7.49. *Let κ be an infinite cardinal.*

(c) Describe the κ-pseudocompact abelian groups that admit proper dense minimal subgroups.

(c*) *Describe the κ-pseudocompact abelian groups that admit proper essential dense subgroups.*

(d) *Describe the κ-pseudocompact abelian groups that admit proper totally minimal subgroups.*

(d*) *Describe the κ-pseudocompact abelian groups that admit proper totally dense subgroups.*

A problem related to Theorem 7.46 about small essential pseudocompact subgroups of compact abelian groups is to generalize this theorem for non-necessarily abelian groups:

Problem 7.50. *Can ZFC decide whether there exists a compact group admitting some small essential pseudocompact subgroups?*

The difficulty in doing this is to generalize Proposition 7.44. We think that the answer to this question is negative, since we have proved that it is negative in the abelian case in Theorem 7.45. Moreover the answer in negative for totally dense subgroups as shown by Theorem 7.26. It could be convenient to consider first the case when the small essential pseudocompact subgroups are requested to be also dense. We study this problem in [28].

Another question related to this topic is the following.

Problem 7.51. *For κ an infinite cardinal, is it possible to prove the counterpart of Theorems 7.45 and 7.46 replacing pseudocompact with κ-pseudocompact?*

To answer this question it is sufficient to prove the counterpart of Proposition 7.43 for κ-pseudocompact subgroups. The same question can be posed for Theorems 7.26 and 7.27.

Bibliography

[1] S. Berhanu, W.W. Comfort and J.D. Reid, *Counting subgroups and topological group topologies*, Pacific J. Math. **116** (1985) no. 2, 217–241.

[2] A. J. Berner, *$\beta(X)$ can be Frechet*, Proc. Amer. Math. Soc. **80** (1980), 367–373.

[3] G. M. Bergman and A. G. Waterman, *Connected fields of arbitrary characteristic*, J. Math. Kyoto Univ. **5** (1966), 177–184.

[4] E. Boschi and D. Dikranjan, *Essential subgroups of topological groups*, Comm. Algebra **28** (1996) no. 10, 2325–2339.

[5] J. Braconnier, *Sur les groupes topologiques localement compacts* (French), J. Math. Pures Appl. (9) **27** (1948), 1–85.

[6] F. S. Cater, P. Erdös and F. Galvin, *On the density of λ-box products*, Gen. Topol. Appl. **9** (1978), 307–312.

[7] W. W. Comfort, *Tampering with pseudocompact groups*, Topology Proc. **28** (2004) no. 2, 401–424.

[8] W. Comfort, *Dense subgroups of compact groups*, Open Problems in Topology II, E. Pearl ed., Elsevier Science (2007), 377–388.

[9] W. W. Comfort, *Pseudocompact groups: progress and problems*, Topology and its Appl. **155** (2008), 172–179.

[10] W. W. Comfort and D. Dikranjan, *Essential density and total density in topological groups*, J. Group Theory **5** (2002), 325–350.

[11] W. W. Comfort and J. Galindo, *Pseudocompact topological group refinements of maximal weight*, Proc. Amer. Math. Soc. **131** (2003), 1311–1320.

[12] W. W. Comfort and J. Galindo, *Extremal pseudocompact topological groups*, J. Pure Appl. Algebra **197** (2005) no. 1–3, 59–81.

[13] W. W. Comfort, H. Gladdines, and J. van Mill, *Proper pseudocompact subgroups of pseudocompact Abelian groups*, Papers on general topology and applications, Ann. New York Acad. Sci. **728** (1994), 237–247.

[14] W. W. Comfort and L. C. Robertson, *Proper pseudocompact extensions of compact abelian group topologies*, Proc. Amer. Math. Soc. **86** (1982), 173–178.

[15] W. W. Comfort and L. C. Robertson, *Cardinality constraints for pseudocompact and for totally dense subgroups of compact topological groups*, Pacific J. Math. **119** (1985), 265–285.

[16] W. W. Comfort and L. C. Robertson, *Extremal phenomena in certain classes of totally bounded groups*, Dissertationes Math. (Rozprawy Mat.) **272** (1988), 48 pp.

[17] W. W. Comfort and K. A. Ross, *Topologies induced by groups of characters*, Fundamenta Math. **55** (1964), 283–291.

[18] W. W. Comfort and K. A. Ross, *Pseudocompactness and uniform continuity in topological groups*, Pacific J. Math. **16** (1966), 483–496.

[19] W. W. Comfort and V. Saks, *Countably compact groups and finest totally bounded topologies*, Pacific J. Math. **49** (1973), 33–44.

[20] W. W. Comfort and T. Soundararajan, *Pseudocompact group topologies and totally dense subgroups*, Pacific J. Math. **100** (1982), 61–84.

[21] W. W. Comfort and J. van Mill, *Concerning connected, pseudocompact abelian groups*, Topology and its Appl. **33** (1989), 21–45.

[22] W. W. Comfort and J. van Mill, *Extremal pseudocompact abelian groups are compact metrizable*, Proc. Amer. Math. Soc. **135** (2007), 4039–4044.

[23] W.W. Comfort and J. van Mill, *On the supremum of the pseudocompact group topologies*, Topology and its Appl. **155** (2008), 213–224.

[24] D. Dikranjan, *Recent advances in minimal topological groups*, Topology and its Appl. **85** (1998), 53–91.

[25] D. Dikranjan, *Countably compact groups satisfying the open mapping theorem*, Topology and its Appl. **98** (1999), 81–129.

[26] D. Dikranjan and A. Giordano Bruno, *Pseudocompact totally dense subgroups*, Proc. Amer. Math. Soc. **136** (2008) no. 3, 1093–1103.

[27] D. Dikranjan and A. Giordano Bruno, *w-Divisible groups*, Topology and its Appl. **155** (2008) 252–272.

[28] D. Dikranjan and A. Giordano Bruno, *Small essential pseudocompact subgroups of compact groups*, preprint.

[29] D. Dikranjan, A. Giordano Bruno and C. Milan, *Weakly metrizable pseudocompact groups*, Appl. Gen. Topol. **7** (2006) no. 1, 1–39.

[30] D. Dikranjan and I. Prodanov, *Totally minimal groups*, Ann. Univ. Sofia Fac. Math. Méc. **69** (1974/75), 5–11.

[31] D. Dikranjan, Iv. Prodanov and L. Stoyanov, *Topological Groups: Characters, Dualities and Minimal Group Topologies*, Pure and Applied Mathematics, vol. **130**, Marcel Dekker Inc., New York-Basel (1989).

[32] D. Dikranjan and D. Shakhmatov, *Compact-like totally dense subgroups of compact groups*, Proc. Amer. Math. Soc. **114** (1992) no. 4, 1119–1129.

[33] D. Dikranjan and D. Shakhmatov, *Pseudocompact and countably compact abelian groups: cartesian products and minimality*, Trans. Amer. Math. Soc. **335** (1993), 775–790.

[34] D. Dikranjan and D. Shakhmatov, *Algebraic structure of pseudocompact groups*, Mem. Amer. Math. Soc. **133** (1998), 83 pages.

[35] D. Dikranjan and D. Shakhmatov, *Forcing hereditarily separable compact-like group topologies on abelian group*, Topology and its Appl. **151** (2005) no. 1–3, 2–54.

[36] D. Dikranjan and D. Shakhmatov, *Selected topics from the structure theory of topological groups*, Open Problems in Topology II, E. Pearl ed., Elsevier Science (2007), 389–406.

[37] D. Dikranjan and D. Shakhmatov, *Algebraic structure of pseudocompact abelian groups*, preprint.

[38] D. Dikranjan and M. Tkačenko, *Sequentially complete groups: dimension and minimality*, J. Pure Appl. Algebra **157** (2001) no. 2–3, 215–239.

[39] D. Dikranjan and A. Tonolo, *On the minimality of powers of minimal ω-bounded abelian groups*, Proc. Amer. Math. Soc. **127** (1999) no. 10, 3101–3110.

[40] R. Engelking, *General Topology*, Heldermann Verlag, Berlin (1989).

[41] W. B. Easton, *Powers of regular cardinals*, Annals of Math. Logic **1** (1970), 139–178.

[42] L. Fuchs, *Infinite abelian groups*, vol. I, Academic Press New York and London (1973).

[43] J. Galindo, *Dense pseudocompact subgroups and finer pseudocompact group topologies*, Sci. Math. Jpn. **55** (2001), 627–640.

[44] S. García-Ferreira, M. Sanchis, A. Tamariz-Mascarúa, *On C_α-compact subsets*, Topology and its Appl. **77** (1997), 139–160.

[45] A. Giordano Bruno, *Dense minimal pseudocompact subgroups of compact abelian groups*, to appear in Topology and its Appl.

[46] A. Giordano Bruno, *Extremal κ-pseudocompact groups*, to appear in Forum Math.

[47] M. Gitik and S. Shelah, *On densities of box products*, Topology Appl. **88** (1998) no. 3, 219–237.

[48] G. Grant, *Topological groups which satisfy an open mapping theorem*, Pacific J. Math. **68** (1977), 411–423.

[49] E. Hewitt, *Rings of real-valued continuous functions I*, Trans. Amer. Math. Soc. **64** (1948), 45–99.

[50] E. Hewitt and K. Ross, *Abstract harmonic analysis I*, Springer-Verlag, Berlin-Heidelberg-New York (1963).

[51] E. Hewitt and K. Ross, *Abstract harmonic analysis II*, Springer-Verlag, Berlin-Heidelberg-New York (1970).

[52] K. H. Hofmann and S. A. Morris, *The structure of compact groups*, De Gruyter Studies in Mathematics **25**, Walter de Gruyter, Berlin-New York (1998).

[53] S. Kakutani, *On cardinal numbers related with a compact abelian group*, Proc. Imperial Acad. Tokyo **19** (1943), 366–372.

[54] J. F. Kennison, *m-pseudocompactness*, Trans. Amer. Math. Soc. **104** (1962), 436–442.

[55] J. König, *Zum Kontinuumproblem*, Mathematische Annalen **60** (1904), 177–180.

[56] L. Y. Kulikov, *On the theory of abelian groups of arbitrary cardinality*, Math. Sb. **16** (1945), 129–162.

[57] K. Kunen, *Set theory. An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics **102**, North-Holland Publishing Co., Amsterdam-New York (1980).

[58] L. S. Pontryagin, *Topological groups*, Gordon and Breach, New York (1966).

[59] I. Prodanov, *Precompact minimal group topologies and p-adic numbers*, Annuaire Univ. Sofia Fac. Math. Méc. **66** (1971/72), 249–266.

[60] I. Prodanov, *Minimal topologies on countable abelian groups*, Annuaire Univ. Sofia Fac. Math. Méc. **70** (1975/76), 107–118.

[61] L. Ribes and P. Zalesskii, *Profinite Groups*, Springer, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 40 (2000).

[62] L. Salce, *Struttura dei p-gruppi abeliani*, Quaderni dell'Unione Matematica Italiana **18**, Pitagora Editrice (1980).

[63] T. Soundararajan, *Totally dense subgroups of topological groups*, General Topology and its Relations to Modern Analysis and Algebra III, Proc. (1968) Kanpur Topological Conference, pp. 299–300. Academia Press, Prague (1971).

[64] R. M. Stephenson, Jr., *Minimal topological groups*, Math. Ann. **192** (1971), 193–195.

[65] L. Stoyanov, *A property of precompact minimal abelian groups*, Ann. Univ. Sofia Fac. Math. Méc. **70** (1975/76), 253–260.

[66] T. Szele, *On the basic subgroups of abelian p-groups*, Acta Math. Acad. Sci. Hungar **5** (1954), 129–141.

[67] E. van Douwen, *The weight of a pseudocompact (homogeneous) space whose cardinality has countable cofinality*, Proc. Amer. Math. Soc. **80** (1980), 678–682.

[68] N. T. Varopulos, *A theorem on the continuity of homomorphisms of locally compact groups*, Proc. Cambridge Phil. Soc. **60** (1964), 449–463.

[69] A. Weil, *Sur les espaces à structure uniforme et sur la topologie générale*, Publ. Math. Univ. Strasbourg, Hermann & Cie., Paris (1937).

Index

List of Symbols