An optimal tableau system for the logic of temporal neighborhood over the reals

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TIME 2012

Leicester, UK, September 12–14, 2012

Outline of the talk

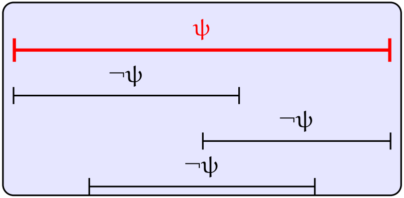
- an introduction to interval temporal logics
- the logic $A\overline{A}$ of temporal neighborhood
- decidability (NEXPTIME-completeness) of the satisfiability problem for $A\overline{A}$ over the reals
- an optimal tableau system for $A\overline{A}$ over the reals
- conclusions

Interval temporal logics: areas of interest

- Philosophy and ontology of time
- Linguistics (quoting Kamp and Reyle, "truth, as it pertains to language in the way we use it, relates sentences not to instants but to temporal intervals")
- Artificial intelligence: temporal knowledge representation, systems for time planning and maintenance, theory of events
- Computer science: specification and design of hardware components, concurrent real-time processes, temporal databases, bioinformatics

The distinctive features of interval temporal logics

Truth of formulae is defined over intervals (not points).



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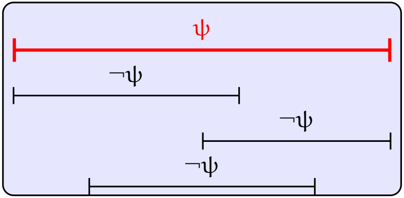

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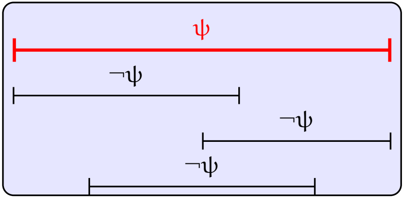

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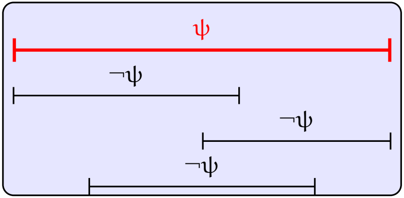


Interval temporal logics are very expressive (compared to point-based temporal logics):

- formulas of interval logics express properties of pairs of time points rather than of single time points, and are evaluated as sets of such pairs (binary relations)

- there is no reduction of satisfiability/validity in interval logics to those in monadic second-order logic (Rabin's theorem is not applicable)

Binary ordering relations over intervals

The thirteen binary ordering relations between two intervals on a linear order (those below and their inverses) form the set of *Allen's interval relations*:

current interval:

equals:

ends :

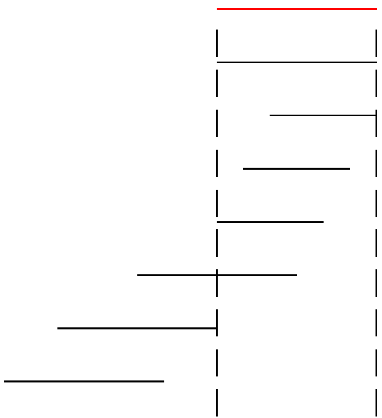
during:

begins:

overlaps:

meets:

before:



HS: the modal logic of Allen's interval relations

Allen's interval relations give rise to corresponding unary modalities over frames where intervals are primitive entities:
Halpern and Shoham's modal logic of time intervals HS, interpreted over interval structures (not to be confused with Allen's Interval Algebra)

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The satisfiability/validity problem for HS is highly undecidable over all standard classes of linear orders.

(Un)decidability of HS fragments: main parameters

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(Un)decidability of HS fragments depends on two factors:

- the set of interval modalities;
- the class of interval structures (linear orders) over which the logic is interpreted.

A well-behaved fragment: the logic $A\overline{A}$

Formulas of the logic $A\overline{A}$ of Allen's relations *meets* and *met by* are recursively defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle A\rangle\varphi \mid \langle\overline{A}\rangle\varphi \ ([A] = \neg\langle A\rangle\neg; \text{ same for } [\overline{A}])$$

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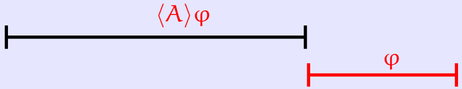

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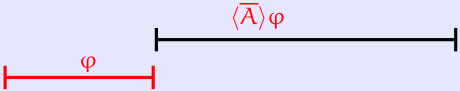

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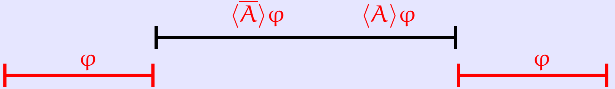
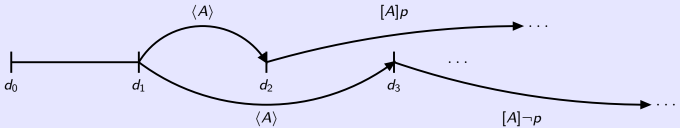


We cannot abstract way from any of the endpoints of intervals:

- contradictory formulas may hold over intervals with the same right endpoint and a different left endpoint

$\langle A\rangle[A]p \wedge \langle A\rangle[A]\neg p$ is satisfiable:



For any $d > d_3$, p holds over $[d_2, d]$ and $\neg p$ holds over $[d_3, d]$.

Expressive completeness of $A\overline{A}$ with respect to $\mathrm{FO}^2[<]$

Expressive completeness of $A\overline{A}$ with respect to the two-variable fragment of first-order logic for binary relational structures over various linearly-ordered domains $\mathrm{FO}^2[<]$

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Remark. The two-variable property is a sufficient condition for decidability, but it is not a necessary one (for instance, the logic D of the subinterval relation is decidable over dense linear orders, but it does not satisfy the two-variable property - three variables are needed)

Decidability of $A\overline{A}$

As a by-product, decidability (in fact, NEXPTIME-completeness) of $A\overline{A}$ over all linear orders, well-orders, finite linear orders, and the linear order on the natural numbers

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This is not the end of the story ..

- It is far from being trivial to extract a decision procedure from Otto's proof
- some meaningful cases are not dealt with by Otto's proof (dense linear orders, weakly discrete linear orders, ..)

Tableau-based decision procedures for $A\overline{A}$

Tableau-based decision procedures have been developed for:

- the future fragment of $A\overline{A}$ (it features the future modality $\langle A \rangle$ only) over the natural numbers;
- full $A\overline{A}$ over the integers (it can be tailored to natural numbers and the class of finite linear orders) and the rationals;
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$A\overline{A}$ is expressive enough to "separate" $\mathbb{Q}$ and $\mathbb{R}$ (this is not the case with A), but, unfortunately, there is no way to reduce the satisfiability problem for $A\overline{A}$ over $\mathbb{R}$ to that over $\mathbb{Q}$.

$A\overline{A}$ over $\mathbb{Q}$ and $\mathbb{R}$

Proposition 1. For any $A\overline{A}$-formula φ, if φ is satisfiable over $\mathbb{R}$, then it is also satisfiable over $\mathbb{Q}$.

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Key observation: it is always possible to replace every $\mathrm{d} \in \mathbb{R} \setminus \mathbb{Q}$ by a suitable $\mathrm{d}' \in \mathbb{Q}$ without affecting the truth value of an $A\overline{A}$ (sub)formula.

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Proposition 2. There exist $A\overline{A}$-formulas which are satisfiable over $\mathbb{Q}$, but not over $\mathbb{R}$.

The differentiating formula

Let θ be the $A\overline{A}$-formula:

$$p \wedge \langle A \rangle \langle A \rangle q \wedge [G]((p \rightarrow \langle A \rangle p) \wedge (q \rightarrow \langle \overline{A} \rangle q) \wedge$$
$$(p \rightarrow [A]([\overline{A}]p \wedge [\overline{A}][\overline{A}]p)) \wedge (q \rightarrow [\overline{A}]([A]q \wedge [A][A]q)) \wedge$$
$$\neg(p \wedge q) \wedge (\neg p \wedge \neg q \rightarrow \langle \overline{A} \rangle p \wedge \langle A \rangle q)),$$

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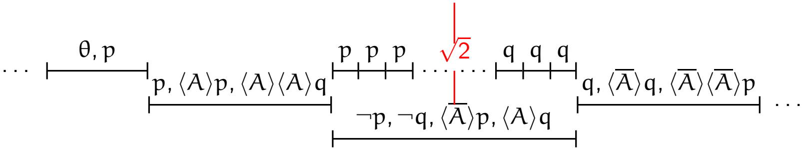

Main Result

Theorem The satisfiability problem for PNL over $\mathbb{R}$ is decidable.

The proof of the theorem consists of two fundamental lemmas.

Lemma 1. Let φ be a PNL formula and $\mathbf{L} = \langle \mathbb{I}(\mathbb{R}), \mathcal{L} \rangle$ be a fulfilling Labelled Interval Structure (LIS) that satisfies it. Then, there exists an $\mathbb{R}$-pseudo-model $\mathbf{L}_{\mathbb{R}} = \langle \langle \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle, \mathcal{F}_{inf}, \mathcal{F}_{sup} \rangle$ for φ with $|D| \leqslant \left(\frac{2^{2 \cdot |\varphi|} + 3 \cdot 2^{|\varphi|} - 2}{2} \right) \cdot (2 \cdot |\varphi| + 1) + 2 \cdot |\varphi| \cdot 2^{3 \cdot |\varphi| + 1}$.

Lemma 2. Let φ be a PNL formula and $\mathbf{L}_{\mathbb{R}} = \langle \langle \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle, \mathcal{F}_{inf}, \mathcal{F}_{sup} \rangle$ be an $\mathbb{R}$-pseudo-model for it. Then, there exists a fulfilling LIS $\mathbf{L}$ over $\mathbb{R}$ that satisfies φ.

Labelled Interval Structures and $\mathbb{R}$-pseudo-models

Some fundamental notions:

φ-labeled interval structures (LIS): candidate models for φ that guarantee the truth of formulas devoid of temporal operators (local formulas) and satisfy universal temporal conditions imposed by $[A]$ and $[\overline{A}]$ operators

Fulfilling LIS: a LIS that guarantees the satisfaction of existential temporal conditions imposed by $\langle A \rangle$ and $\langle \overline{A} \rangle$ operators

$\mathbb{R}$-pseudo-model: a finite LIS (not necessarily a fulfilling one) that satisfies suitable structural conditions

Proof of Lemma 2

The proof is organized in two phases:

1) Building a fulfilling LIS $\mathbf{L}'$ over $\mathbb{Q}$
2) Turning $\mathbf{L}'$ into a fulfilling LIS $\mathbf{L}$ over $\mathbb{R}$

The first phase produces a candidate fulfilling LIS over $\mathbb{Q}$ as the 'limit' of the repeated application of the following three steps:

1.1) Step 1 forces the 'limit' LIS to be fulfilling
1.2) Step 2 forces infinite bounded chains of requests in the 'limit' LIS to accumulate only on rational numbers
1.3) Step 3 forces the 'limit' LIS to be dense by simply adding a point in between any pair of consecutive points

A Tableau System for $A\overline{A}$ over $\mathbb{R}$

A tableau for an $A\overline{A}$ formula φ is a special *decorated tree* $\mathcal{T}$ that features both expansion nodes and accumulation nodes.

Such a tree is built by applying the following expansion rules:

- $\langle A \rangle$-rule and $\overline{A}$-rule;
- Fill-in rule;
- Dense rule;
- Inf-rule and Sup-rule;
- Inf-chain rule and Sup-chain rule.

A suitable blocking condition is given

Theorem. Let φ be a $A\overline{A}$ formula. If $\mathcal{T}$ is a final tableau for φ that features one blocked branch, then φ is satisfiable over $\mathbb{R}$ and, conversely, if φ is satisfiable over $\mathbb{R}$, then there exists a final tableau for φ with at least one blocked branch.

Conclusions

We proved the decidability (NEXPTIME-completeness) of the satisfiability problem for $A\overline{A}$ over the reals, and we developed an optimal tableau system for $A\overline{A}$ over the reals

Remark. Unsatisfiability of formulae like θ over $\mathbb{R}$ can be interpreted as a plus of $\mathbb{R}$-models: structural properties of $\mathbb{R}$ exclude pathological models like the above-described $\mathbb{Q}$-model satisfying θ, and thus $\mathbb{R}$-models can be viewed as the most appropriate models for practical applications where density is an essential ingredient of the temporal domain

Future work is concerned with the implementation of the tableau-based decision procedure