Interval Temporal Logics over Strongly Discrete Linear Orders: the Complete Picture

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Outline

Interval Temporal Logics: origin and motivations

Halpern-Shoham's modal logic HS

HS over strongly discrete linear orders

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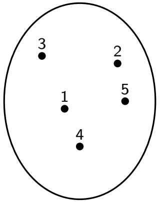
Temporal logics: origins and application fields

- Temporal logics play a major role in computer science
  - automated system verification
- Temporal logics are (multi-)modal logics



set of worlds
primitive temporal entity
time points/instants

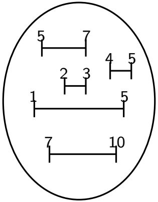
accessibility relations

$\longrightarrow$: next

$\longrightarrow^{\star}$: finally

A different approach: from points to intervals

- worlds are intervals (time period — pairs of points)



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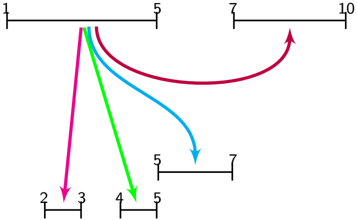

accessibility relations
all binary relations between pairs of intervals

Motivations

- properties intrinsically related to intervals (instead of points)
- points have no duration

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- true over a precise interval of time
- not true over all other intervals
  (starting/ending intervals, inner intervals, ecc.)

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- Zeno's flying arrow paradox ("if at each instant the flying arrow stands still, how is movement possible?")

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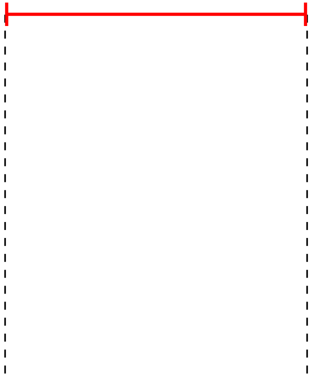
- Zeno's flying arrow paradox ("if at each instant the flying arrow stands still, how is movement possible?")
- The dividing instant dilemma ("if the light is on and it is turned off, what is its state at the instant between the two events?")

Binary interval relations on linear orders



J. F. Allen

Maintaining knowledge about temporal intervals.

Communications of the ACM, volume 26(11), pages 832-843, 1983.

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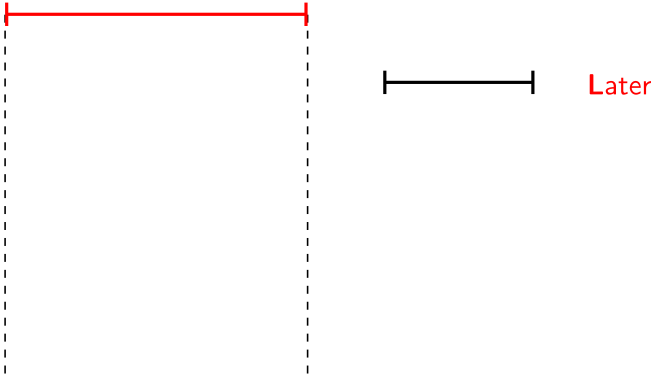

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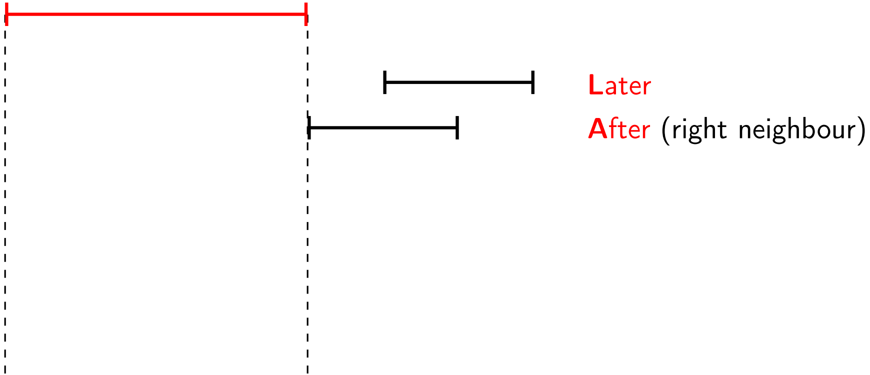

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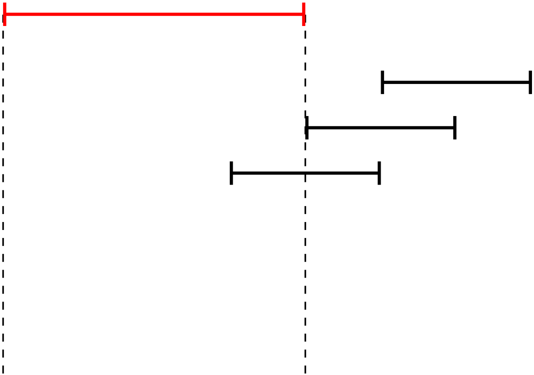

Later

After (right neighbour)

Overlaps (to right)

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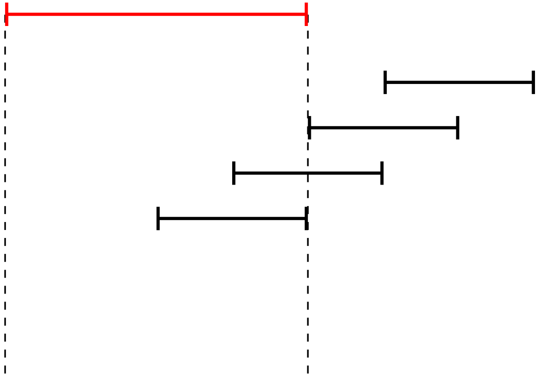

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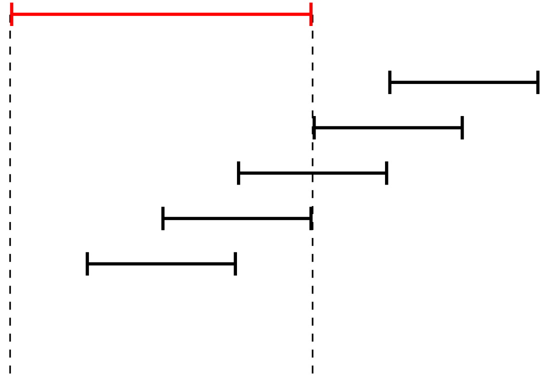

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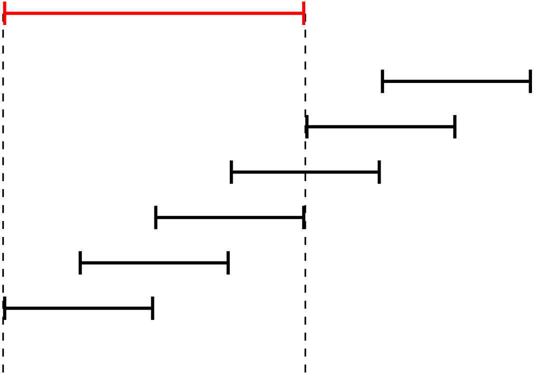

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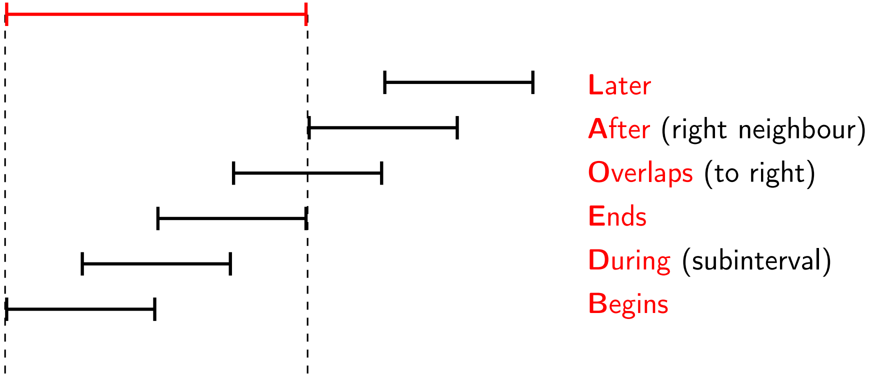

6 relations + their inverses + equality = 13 Allen's relations.

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Halpern-Shoham's modal logic of interval relations

interval relations give rise to modal operators

HS logic

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interval relations give rise to modal operators → HS logic

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Syntax:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle X \rangle \varphi$$

$$\langle X \rangle \in \{\langle A \rangle, \langle L \rangle, \langle B \rangle, \langle E \rangle, \langle D \rangle, \langle O \rangle, \langle \overline{A} \rangle, \langle \overline{L} \rangle, \langle \overline{B} \rangle, \langle \overline{E} \rangle, \langle \overline{D} \rangle, \langle \overline{O} \rangle\}$$

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Models:

$$\mathbf{M} = \langle \mathbb{I}(\mathbb{D}), V \rangle$$

$$V : \mathbb{I}(\mathbb{D}) \mapsto 2^{\mathcal{AP}}$$

$\mathcal{AP}$ atomic propositions (over intervals)

Formal semantics of HS

$\langle B \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle B \rangle \phi$ iff there exists d_2 such that $d_0 \leqslant d_2 < d_1$ and $\mathbf{M}, [d_0, d_2] \Vdash \phi$.

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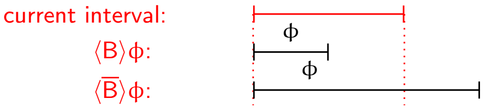

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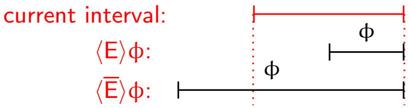

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$\langle A \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle A \rangle \phi$ iff there exists d_2 such that $d_1 < d_2$ and $\mathbf{M}, [d_1, d_2] \Vdash \phi$.

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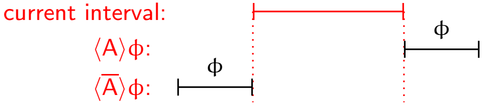

Formal semantics of HS - contd'

$\langle L \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle L \rangle \phi$ iff there exists d_2, d_3 such that $d_1 < d_2 < d_3$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

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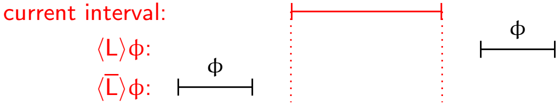

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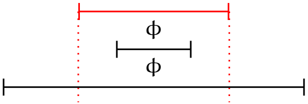
current interval:
$\langle D \rangle \phi$:
$\langle \overline{D} \rangle \phi$:



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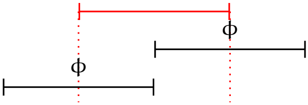
current interval:

$\langle O \rangle \phi$:

$\langle \overline{O} \rangle \phi$:



(Un)decidability of HS fragments: main parameters

Research agenda:

- search for maximal decidable HS fragments;
- search for minimal undecidable HS fragments.

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(Un)decidability of HS fragments depends on two factors:

- the set of interval modalities;
- the class of interval structures (linear orders) over which the logic is interpreted.

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Strong discreteness

Strongly discrete linear orders

There is a finite number of points between any pairs of points over the linear order

Example

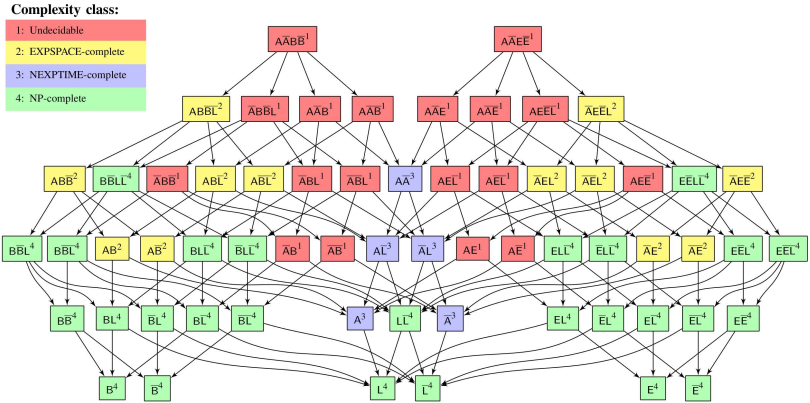
- natural numbers $\mathbb{N}$
- integers $\mathbb{Z}$

Counterexample

- rational numbers $\mathbb{Q}$
- $\mathbb{Z} + \mathbb{Z}$

From now on we will talk about *strongly discrete* linear orders

The complete picture



Main contributions of the paper

To prove that the diagram is correct (assuming strong discreteness)

Relative expressive power

$$\langle L \rangle p \equiv \langle A \rangle \langle A \rangle p$$
$$\langle \overline{L} \rangle p \equiv \langle \overline{A} \rangle \langle \overline{A} \rangle p$$

Lemma

The above set of inter-definabilities is sound and complete within the fragment $\mathsf{A\overline{A}B\overline{B}}$

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Theorem Invariance of modal formulae wrt bisimulations

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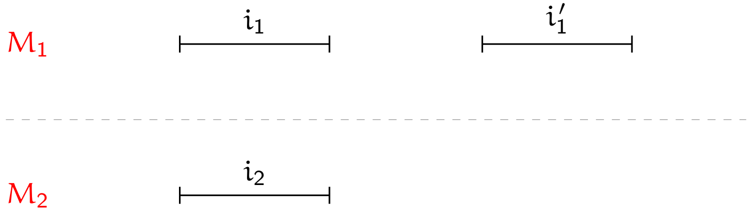

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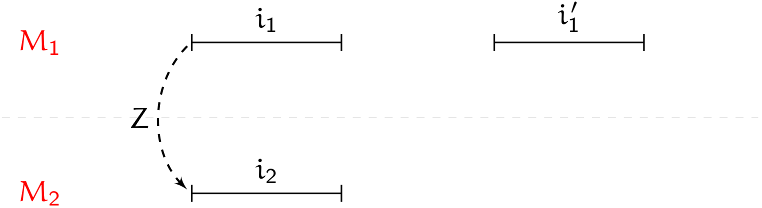

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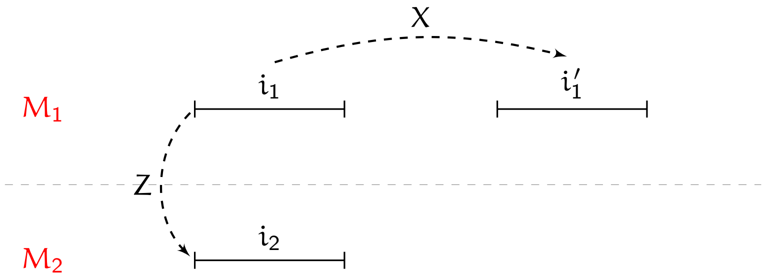
$$(i_1, i_2) \in Z$$
$$(i_1, i_1') \in X$$



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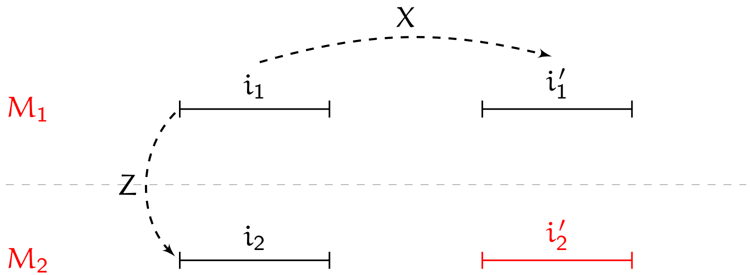
$$\left.\begin{array}{l}(i_1, i_2) \in Z \\ (i_1, i_1') \in X\end{array}\right\} \Rightarrow \exists i_2' \text{ s.t.}$$



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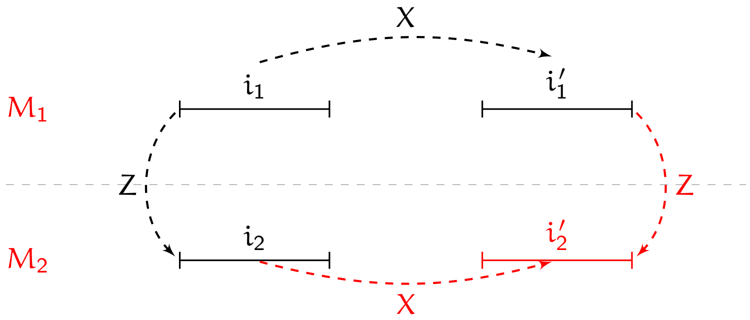
$$(i_1, i_2) \in Z \Rightarrow (p \text{ is true over } i_1 \Leftrightarrow p \text{ is true over } i_2)$$

2. the bisimulation relation is "preserved" by modal operators, i.e., for every modal operator $\langle X \rangle$:

$$\left.\begin{array}{l}(i_1, i_2) \in Z \\ (i_1, i_1') \in X\end{array}\right\} \Rightarrow \exists i_2' \text{ s.t. } \left\{\begin{array}{l}(i_1', i_2') \in Z \\ (i_2, i_2') \in X\end{array}\right.$$



How to use bisimulations to disprove definability

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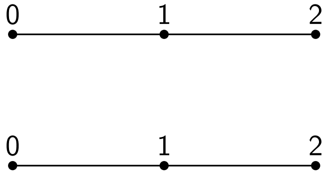

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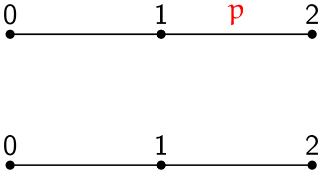

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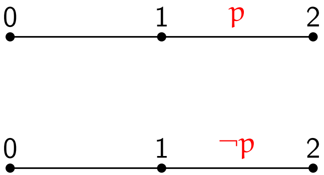

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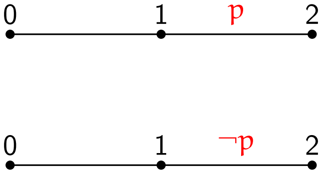

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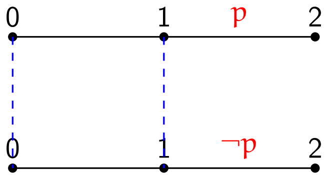

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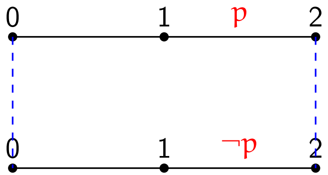

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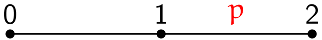

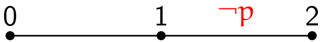

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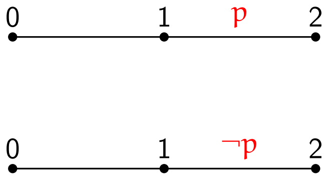

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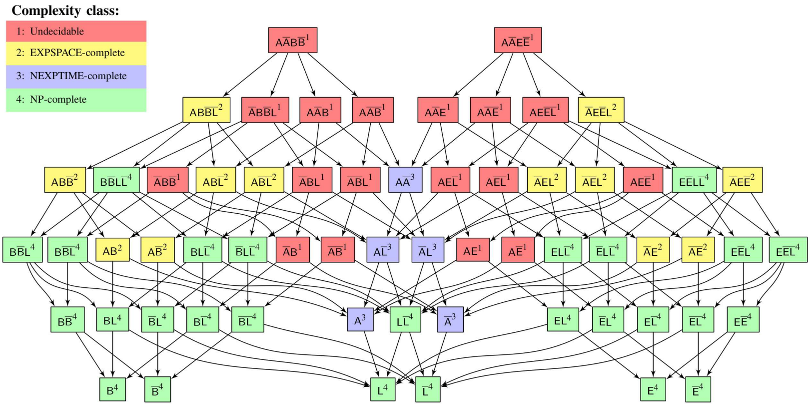

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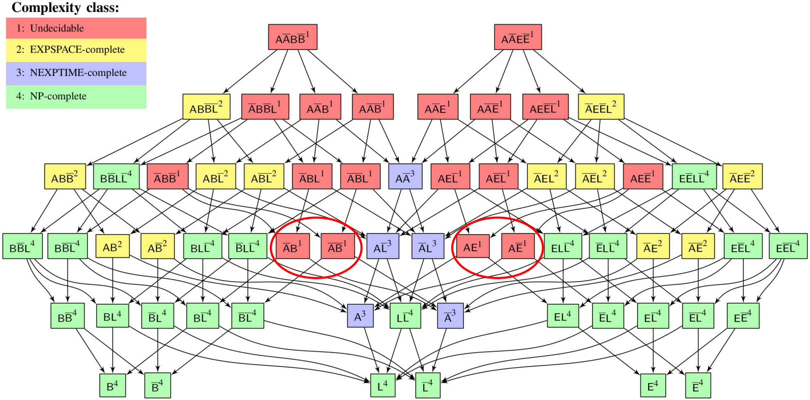

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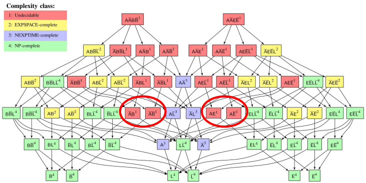

Reduction from the non-emptiness problem for incrementing counter automata over ω-words

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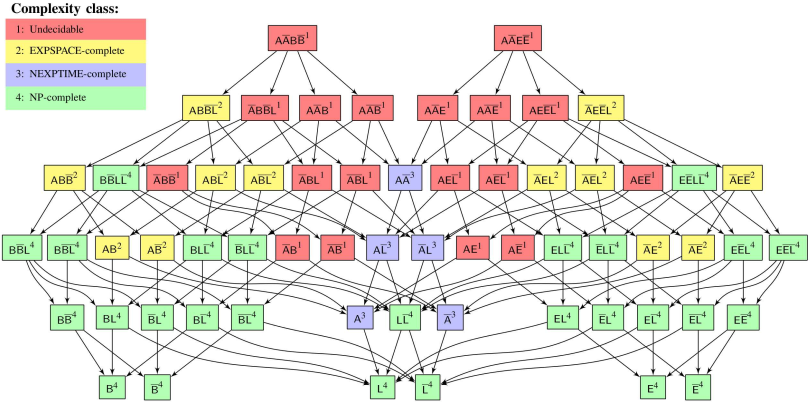
Complexity of decidable fragments



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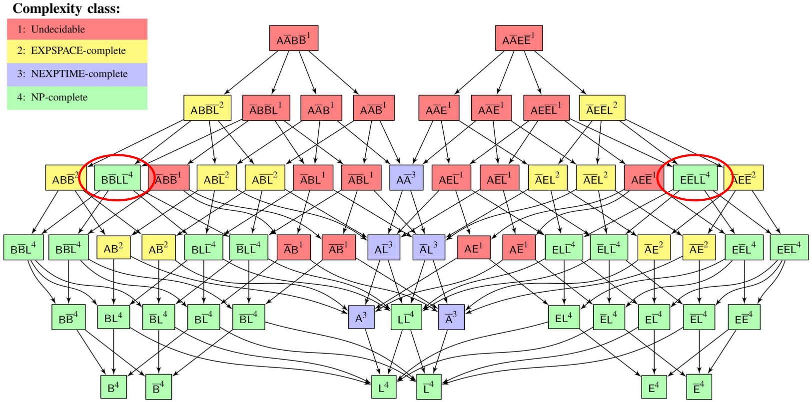

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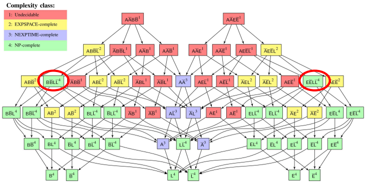

Other complexity results (membership and hardness)

- P. Sala, Decidability of Interval Temporal Logic, *PhD Thesis*, 2010
- Bresolin et al., Interval Temporal Logics over Finite Linear Orders: the Complete Picture, *ECAI*, 2012

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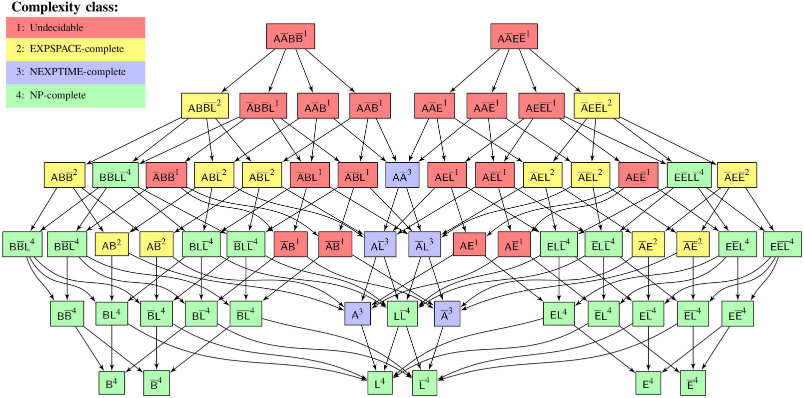
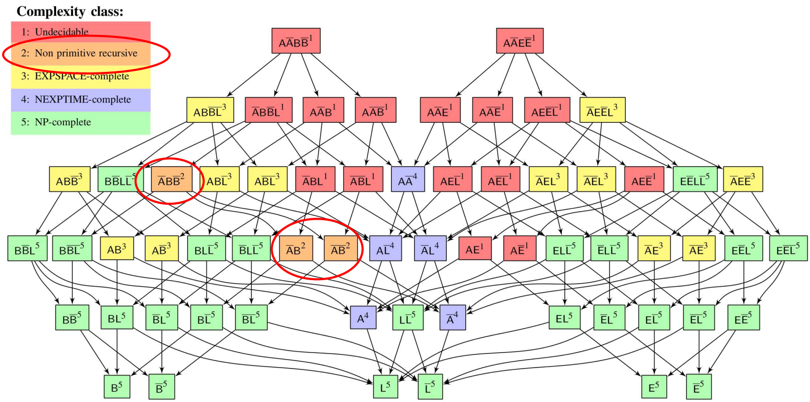
Strongly discrete linear orders: recall the picture for



The complete picture over natural numbers



Conclusions and future work

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