Interval Temporal Logics: Back to the Future

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Road map

- interval temporal logics
- the logic of Allen's relations: Halpern and Shoham's modal logic of time intervals (HS)
- decidable fragments of HS
- undecidable fragments of HS
- latest developments
- research directions

Origins and application areas

- Philosophy and ontology of time, e.g., the choice between time instants and time intervals as the primary objects of a temporal ontology
- Linguistics: analysis of progressive tenses, semantics and processing of natural languages
- Artificial intelligence: temporal knowledge representation, systems for time planning and maintenance, theory of events
- Computer science: temporal databases, specification and design of hardware components, concurrent real-time processes, bioinformatics

Interval temporal logics and temporal ontologies

Interval temporal reasoning is subject to the same ontological dilemmas as the point-based temporal reasoning, viz., should the time structure be assumed:

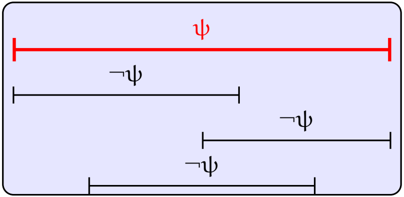
- *linear or branching?*
- *discrete or dense?*
- *with or without beginning/end?*

New dilemmas arise regarding the nature of the intervals:

- *How are points and intervals related? Which is the primary concept? Should an interval be identified with the set of points in it, or there is more into it?*
- *Can intervals be unbounded?*
- *Are intervals with coinciding endpoints admissible or not?*

The distinctive features of interval temporal logics

Truth of formulae is defined over intervals (not points).



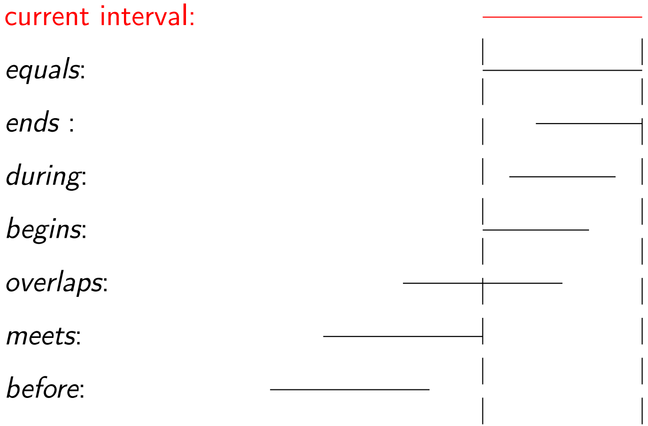
Interval temporal logics are very expressive (compared to point-based temporal logics)

In particular, formulas of interval logics express properties of pairs of time points rather than of single time points, and are evaluated as sets of such pairs, i.e., as binary relations

Thus, in general there is no reduction of the satisfiability/validity in interval logics to monadic second-order logic, and therefore Rabin's theorem is not applicable here

Binary ordering relations over intervals

The thirteen binary ordering relations between two intervals on a linear order (those below and their inverses) form the set of *Allen's interval relations*:



HS: the modal logic of Allen's interval relations

Allen's interval relations give rise to corresponding unary modalities over frames where intervals are primitive entities:
Halpern and Shoham's modal logic of time intervals HS (LICS 1986), interpreted over interval structures (not to be confused with Allen's Interval Algebra)

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The satisfiability/validity problem for HS is highly undecidable over all standard classes of linear orders. What about its fragments?

More than 4000 fragments of HS (over the class of all linear orders) can be identified by choosing a different subset of the set of basic modal operators. However, 1347 genuinely different ones exist only

D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco, *Expressiveness of the Interval Logics of Allen's Relations on the Class of all Linear Orders: Complete Classification*, IJCAI 2011

(Un)decidability of HS fragments: main parameters

Research agenda:

- search for maximal decidable HS fragments
- search for minimal undecidable HS fragments

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(Un)decidability of HS fragments depends on two factors:

- the set of interval modalities
- the class of interval structures (linear orders) over which the logic is interpreted

A real character: the logic D

The logic D of the subinterval relation (Allen's relation *during*) is quite interesting from the point of view of (un)decidability

The satisfiability problem for D, interpreted over the class of dense linear orders, is PSPACE-complete

I. Shapirovsky, On PSPACE-decidability in Transitive Modal Logic, Advances in Modal Logic 2005

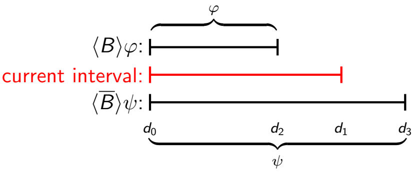
It is undecidable, when D is interpreted over the classes of finite and discrete linear orders

J. Marcinkowski and J. Michaliszyn, *The Ultimate Undecidability Result for the Halpern-Shoham Logic*, LICS 2011

It is unknown, when D is interpreted over the class of all linear orders

An easy case: the logic $B\overline{B}$

Consider the fragment $B\overline{B}$.



The decidability of $B\overline{B}$ can be shown by embedding it into the propositional temporal logic of linear time LTL$[F, P]$: formulas of $B\overline{B}$ can be translated into formulas of LTL$[F, P]$ by replacing $\langle B \rangle$ with P (sometimes in the past) and $\langle \overline{B} \rangle$ with F (sometimes in the future):

LTL$[F, P]$ has the small (pseudo)model property and is decidable

The case of $E\overline{E}$ is similar

A well-behaved fragment: the logic $A\overline{A}$

Formulas of the logic $A\overline{A}$ of Allen's relations *meets* and *met by* are recursively defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle A \rangle\varphi \mid \langle \overline{A} \rangle\varphi \ ([A] = \neg\langle A \rangle\neg; \text{ same for } [\overline{A}])$$

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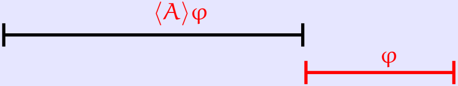

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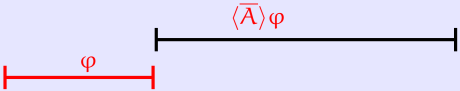

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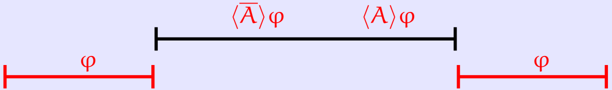
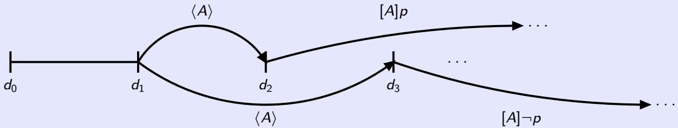


We cannot abstract way from any of the endpoints of intervals:

- contradictory formulas may hold over intervals with the same right endpoint and a different left endpoint

$\langle A \rangle [A] p \wedge \langle A \rangle [A] \neg p$ is satisfiable: for any $d > d_3$, p holds over $[d_2, d]$ and $\neg p$ holds over $[d_3, d]$



Expressive completeness of $A\overline{A}$ with respect to $\mathrm{FO}^2[<]$

Expressive completeness of $A\overline{A}$ with respect to the two-variable fragment of first-order logic for binary relational structures over various linearly-ordered domains $\mathrm{FO}^2[<]$

M. Otto, Two Variable First-order Logic Over Ordered Domains, Journal of Symbolic Logic, 2001

Remark. The two-variable property is a sufficient condition for decidability, but it is not a necessary one (for instance, D is decidable over dense linear orders, but it does not satisfy the two-variable property - three variables are needed)

Decidability of $A\overline{A}$

As a by-product, decidability (in fact, NEXPTIME-completeness) of $A\overline{A}$ over all linear orders, well-orders, finite linear orders, and the linear order on the natural numbers

D. Bresolin, V. Goranko, A. Montanari, and G. Sciavicco, Propositional Interval Neighborhood Logics: Expressiveness, Decidability, and Undecidable Extensions, Annals of Pure and Applied Logic, 2009

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Tableau-based decision procedures for $A\overline{A}$

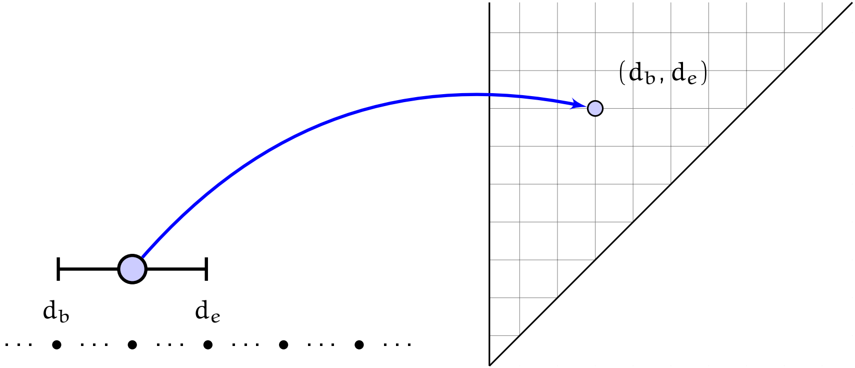
Maximal decidable fragments

Issue: can we add other modalities from the HS repository to the logic of temporal neighborhood $\mathrm{A}\overline{\mathrm{A}}$ or to the logic of the subinterval relation D preserving decidability?

The search for maximal decidable fragments of HS benefitted from a natural geometrical interpretation of interval logics proposed by Venema

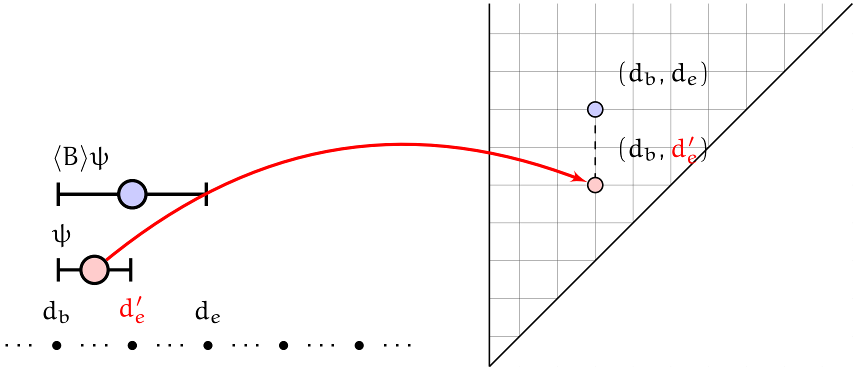
We illustrate the basic ingredients of such a geometrical interpretation, and we summarize the main results

A geometrical account of interval logic: intervals



Every interval can be represented by a point in the second octant (in general, in the half plane $y \geqslant x$)

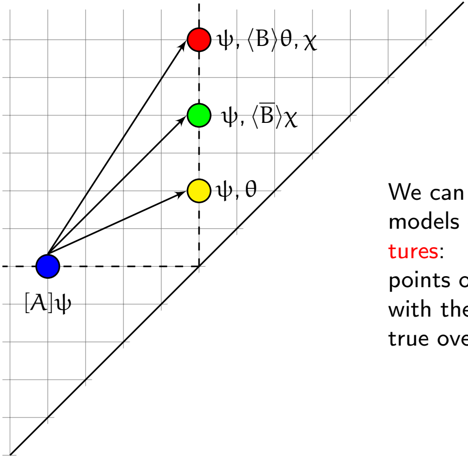
A geometrical account of interval logic: interval relations



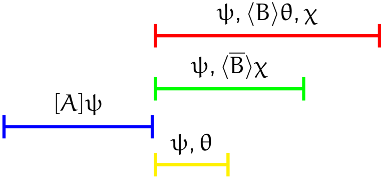
$$d_b < d'_e < d_e$$

Every interval relation has a spatial counterpart

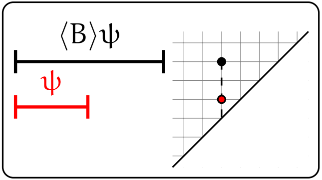
A geometrical account of interval logic: models

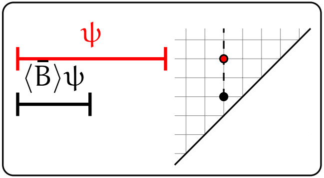


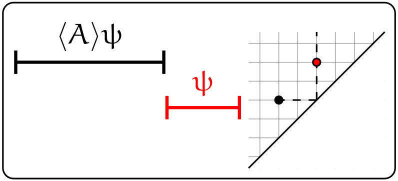
We can give a spatial interpretation to models of a formula φ as compass structures:
points of a compass structure are colored with the set of subformulas of φ that are true over the corresponding intervals

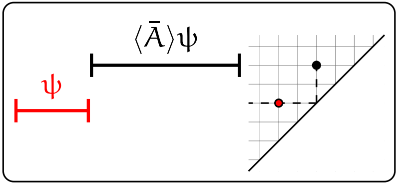


The maximal decidable fragment $AB\overline{BA}$



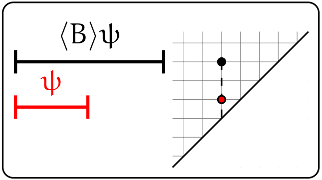


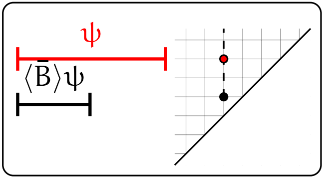


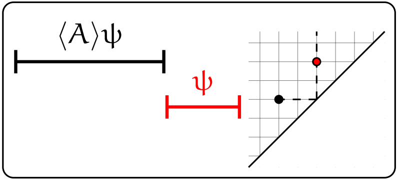


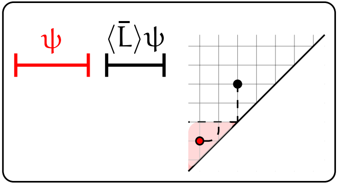
$AB\overline{BA}$ is NONPRIMITIVE RECURSIVE-hard over finite linear orders, the rationals, and the class of all linear orders; undecidable over the natural numbers and the reals, and the class of all Dedekind-complete linear orders

The maximal decidable fragment $AB\overline{B}\overline{L}$









Replace $\langle \overline{A} \rangle$ by $\langle \overline{L} \rangle$: $AB\overline{B}\overline{L}$ is EXSPACE-complete over the classes of all, dense, and (weakly) discrete linear orders

Maximal decidable fragments: references

Decidability of $AB\overline{B}\overline{A}$ over finite linear orders

A. Montanari, G. Puppis, and P. Sala, Maximal decidable fragments of Halpern and Shoham's modal logic of intervals, ICALP 2010

Decidability of $AB\overline{B}\overline{A}$ over the rationals and all linear orders

A. Montanari, G. Puppis, and P. Sala, Decidability of the interval temporal logic $AB\overline{B}\overline{A}$, over the rationals, MFCS 2014

Decidability of $AB\overline{B}\overline{L}$ over all, dense, and discrete linear orders

D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco, What's decidable about Halpern and Shoham's interval logic? The maximal fragment $AB\overline{B}\overline{L}$, LICS 2011

Paths to undecidability - 1

Undecidability results for HS fragments have been obtained by means of reductions from several undecidable problems:

- reduction from the non-halting problem for Turing machines (e.g., HS over all meaningful classes of linear orders, and BE over dense linear orders – that of BE over all linear orders immediately follows)

J. Halpern and Y. Shoham, A propositional modal logic of time intervals, Journal of the ACM, 1991

K. Lodaya, Sharpening the Undecidability of Interval Temporal Logic, ASIAN 2000

Paths to undecidability - 2

- reductions from several variants of the tiling problem, like the octant tiling problem and the finite tiling problem (O, $\overline{\mathsf{O}}$, AD, $\overline{\mathsf{A}}\mathsf{D}$, $\mathsf{A}\overline{\mathsf{D}}$, $\overline{\mathsf{AD}}$, BE, $\overline{\mathsf{B}}\mathsf{E}$, $\mathsf{B}\overline{\mathsf{E}}$, and $\overline{\mathsf{BE}}$ over any class of linear orders that contains, for each $n > 0$, at least one linear order with length greater than n)

D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco, The dark side of Interval Temporal Logics: marking the undecidability border, Annals of Mathematics and Artificial Intelligence, 2014

- reduction from the halting problem for two-counter automata (D over finite and discrete linear orders)

J. Marcinkowski and J. Michaliszyn, The Ultimate Undecidability Result for the Halpern-Shoham Logic, LICS 2011

Latest developments

In their standard formulation, model checking algorithms describe systems as (finite) labelled state-transition graphs (Kripke structures) and make use of point-based, linear or branching temporal logics to constrain the way in which the truth value of the state-labelling proposition letters changes along the paths of the Kripke structure $\mathcal{K}$.

To check interval properties of computations, one needs to collect information about states into computation stretches. This amounts to interpret each finite path of $\mathcal{K}$ as an interval, and to suitably define its labelling on the basis of the labelling of the states that compose it (interval representation of $\mathcal{K}$).

Warning: since $\mathcal{K}$ has loops, the number of its tracks is infinite, and thus the number of corresponding intervals is infinite.

Interval temporal logics can then be used to express and to check interval properties.

Latest developments (cont'd)

Molinari et al. showed that, given a finite Kripke structure $\mathcal{K}$ and a bound k on the structural complexity of HS formulas (that is, on the nesting of E and B modalities), it is possible to obtain a finite interval representation for $\mathcal{K}$, which is equivalent to the original one with respect to satisfiability of HS formulas with structural complexity less than or equal to k.
By exploiting such a representation, they proved that the model checking problem for (full) HS is decidable (the given algorithm has a non-elementary upper bound).
Moreover, they showed that the problem for the fragment $A\overline{A}BE$, and thus for full HS, is PSPACE-hard (EXPSPACE-hard if a suitable succinct encoding of formulas is exploited).

A. Molinari, A. Montanari, A. Murano, G. Perelli G., and A. Peron, Checking Interval Properties of Computations, Acta Informatica 2016 (extended version of TIME 2014)

Latest developments (cont'd)

Later, Molinari et al. devised an EXPSPACE model checking algorithm for the fragments $A\overline{A}B\overline{BE}$ and $A\overline{A}E\overline{EB}$, that needs to consider only a subset of relatively short tracks: for any given bound k on the complexity of formulas, they define an equivalence relation over tracks of finite index and show that model checking can be restricted to track representatives of bounded length.

A. Molinari, A. Montanari, A. Peron, A Model Checking Procedure for Interval Temporal Logics based on Track Representatives, CSL 2015

- Related work: Lomuscio and Michaliszyn addressed the model checking problem for some fragments of HS extended with epistemic modalities.

Latest developments (cont'd)

- Aceto et al. extended the expressiveness classification result for the family of HS fragments to the classes of dense, finite, and discrete linear orders

L. Aceto, D. Della Monica, A. Ingolfsdottir, A. Montanari, G. Sciavicco, A Complete Classification of the Expressiveness of Interval Logics of Allen's Relations: The General and the Dense Cases, Acta Informatica 2016 (extended version of IJCAI 2011 and TIME 2013)

L. Aceto, D. Della Monica, A. Ingolfsdottir, A. Montanari, G. Sciavicco, On the expressiveness of the interval logic of Allen's relations over finite and discrete linear orders, JELIA 2014

The only missing cases are those of the relations *overlaps* and *overlapped by* over finite and discrete linear orders.

Latest developments (cont'd)

- Montanari et al. studied the effects of the addition of one or more equivalence relations to (Metric) $A\overline{A}$ (since $A\overline{A}$ is expressively complete with respect to $\mathrm{FO}^2[<]$, the results obtained for the former can be immediately transferred to the latter)

  They first showed that finite satisfiability for $A\overline{A}$ extended with an equivalence relation $\sim$ is still NEXPTIME-complete. Then, they proved that finite satisfiability for Metric $A\overline{A}$ can be reduced to the decidable 0-0 reachability problem for vector addition systems and vice versa (EXPSPACE-hardness immediately follows)

A. Montanari, M. Pazzaglia, P. Sala, Metric Propositional Neighborhood Logic with an Equivalence Relation, Acta Informatica 2016 (extended version of TIME 2014)

Latest developments (cont'd)

- Then, they proved that AB extended with an equivalence relation is decidable (non-primitive recursive) on the class of finite linear orders and undecidable over the natural numbers.

  Finally, they showed that the addition of two or more equivalence relations makes finite satisfiability for AB undecidable

A. Montanari, M. Pazzaglia, P. Sala, Adding one or more equivalence relations to the interval temporal logic $AB\overline{B}$, Theoretical Computer Science 2016 (extended version of LICS 2013 and ICTCS 2014)

Latest developments (cont'd)

- Montanari and Sala established a link between interval temporal logics and classes of extended regular and ω-regular languages.

  They give a characterization of regular (resp., ω-regular) languages in the logic $\mathrm{AB\overline{B}}$ of Allen's relations *meets*, *begun by*, and *begins* over finite linear orders (resp., $\mathbb{N}$). Then, they lift such a correspondence to $\omega\mathrm{B}$-regular languages (they allow one to constrain the distance between consecutive occurrences of a symbol to be bounded) by substituting $\mathrm{AB\overline{B}\overline{A}}$ for $\mathrm{AB\overline{B}}$.

A. Montanari, P. Sala, Interval logics and $\omega\mathrm{B}$-regular languages, LATA 2013

- Finally, they showed that the addition of an equivalence relation $\sim$ to $\mathrm{AB\overline{B}}$ makes the resulting logic expressive enough to define ωS-regular languages (strongly unbounded ω-regular languages).

A. Montanari, P. Sala, Adding an equivalence relation to the interval logic $\mathrm{AB\overline{B}}$: complexity and expressiveness, LICS 2013

Latest developments (cont'd)

- Montanari and Sala formally stated the synthesis problem for HS extended with an equivalence relation $\sim$.

  They proved that the synthesis problem for $\mathrm{AB\overline{B}} \sim$ over finite linear orders is decidable (non-primitive recursive hard), while that for$\mathrm{AB\overline{B}\overline{A}}$ turns out to be undecidable.

  Moreover, they showed that if one replaces finite linear orders by natural numbers, then the problem becomes undecidable even for $\mathrm{AB\overline{B}}$

A. Montanari, P. Sala, Interval-based Synthesis, GandALF 2014

Current research agenda

- To obtain a complete classification of the family of HS fragments with respect to decidability/undecidability of their satisfiability problem and with respect to their relative expressive power

- To extend the study of metric variants of interval logics (we already did it for $A\overline{A}$ over natural numbers, integers, and finite linear orders) to other HS fragments / other metrizable linear orders

D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, G. Sciavicco, Metric Propositional Neighborhood Logics on Natural Numbers, Software and Systems Modeling 2013

D. Bresolin, A. Montanari, G. Sciavicco, P. Sala, Optimal decision procedures for MPNL over finite structures, the natural numbers, and the integers, Theoretical Computer Science 2013

Current research agenda (cont'd)

- To complete the classification of the family of HS fragments w.r. to the complexity of their model checking problem (and to cope with more general semantics, relaxing the homogeneity assumption)

> A. Molinari, A. Montanari, A. Peron, Complexity of ITL model checking: some well-behaved fragments of the interval logic HS, TIME 2015
> A. Molinari, A. Montanari, A. Peron, P. Sala, Model checking well-behaved fragments of HS: the (almost) final picture, KR 2016

- To explore possible connections between interval temporal logics and description logics

> A. Artale, D. Bresolin, A. Montanari, V. Ryzhikov, G. Sciavicco, DL-Lite and Interval Temporal Logics: a Marriage Proposal, ECAI 2014

Mid-term research agenda

- Systematic application of game-theoretic techniques in interval-based synthesis
- Quest for automaton-based techniques for proving decidability of interval temporal logics
- Identification and development of major applications of interval temporal logics. Besides system specification, verification, and synthesis, planning and plan validation (to represent and to reason about actions/events with duration, accomplishments, and interval constraints), temporal databases (to deal with temporal aggregation), workflow systems (to cope with additional temporal constraints), and natural language processing (to model features like progressive tenses)

Long-term research agenda

- To show how point-based temporal logics can be recovered as special cases of interval temporal logics

  As an example, the until modality of Linear Temporal Logic can be expressed in the interval logic AB (interpreted over linear orders):

$$\psi \, \mathcal{U} \, \varphi$$

  can be encoded as

$$\langle A \rangle \big([B] \bot \wedge \varphi \big) \vee \langle A \rangle \big(\langle A \rangle ([B] \bot \wedge \varphi) \wedge [B] (\langle A \rangle ([B] \bot \wedge \psi)) \big)$$

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