Interval Temporal Logic, Satisfiability and Model Checking

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Part I: introduction & satisfiability checking

- interval temporal logics
- the logic of Allen's relations: Halpern and Shoham's modal logic of time intervals (HS)
- decidable fragments of HS
- undecidable fragments of HS

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Origins and application areas

- Philosophy and ontology of time, e.g., the choice between time instants and time intervals as the primary objects of a temporal ontology
- Linguistics: analysis of progressive tenses, semantics and processing of natural languages
- Artificial intelligence: temporal knowledge representation, systems for time planning and maintenance, theory of events
- Computer science: specification and design of hardware components, concurrent real-time processes, temporal databases, bioinformatics

Interval temporal logics and temporal ontologies

Interval temporal reasoning is subject to the same ontological dilemmas as the point-based temporal reasoning, viz., should the time structure be assumed:

- *linear or branching?*
- *discrete or dense?*
- *with or without beginning/end?*

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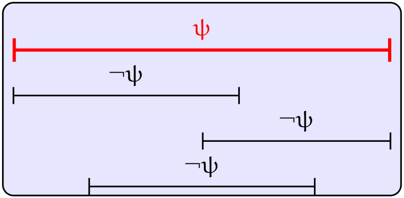
New dilemmas arise regarding the nature of the intervals:

- *How are points and intervals related? Which is the primary concept? Should an interval be identified with the set of points in it, or there is more into it?*
- *Can intervals be unbounded?*
- *Are intervals with coinciding endpoints admissible or not?*

The distinctive feature of interval temporal logics

Truth of formulae is defined over intervals (not points).



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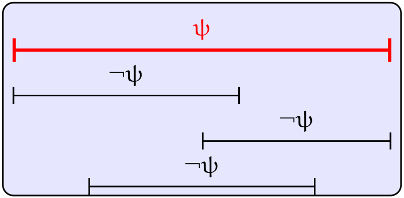

Interval temporal logics are very expressive (compared to point-based temporal logics)

In particular, formulas of interval logics express properties of pairs of time points rather than of single time points, and are evaluated as sets of such pairs, i.e., as binary relations

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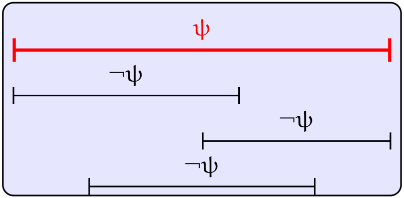

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It does not come as a surprise that, in general, there is not a reduction of the satisfiability/validity problem for interval temporal logics to the satisfiability/validity problem for monadic second-order logic, and therefore Rabin's theorem is not applicable here

Binary ordering relations over intervals

The thirteen binary ordering relations between two intervals on a linear order (those below and their inverses) form the set of *Allen's interval relations*:

current interval:

equals:

finished-by:

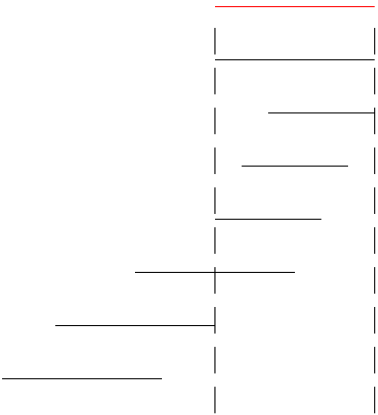
contains:

started-by:

overlapped-by:

met-by:

after:

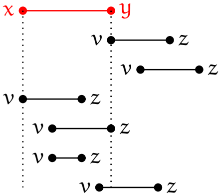


HS: the modal logic of Allen's interval relations

Allen's interval relations give rise to corresponding unary modalities over frames where intervals are primitive entities

- HS features a modality for any Allen ordering relation between pairs of intervals (except for equality)

Allen rel.	HS	Definition	Example
meets	$\langle A \rangle$	$[x, y]\mathcal{R}_A[v, z] \leftrightarrow y = v$	
before	$\langle L \rangle$	$[x, y]\mathcal{R}_L[v, z] \leftrightarrow y < v$	
started-by	$\langle B \rangle$	$[x, y]\mathcal{R}_B[v, z] \leftrightarrow x = v \wedge z < y$	
finished-by	$\langle E \rangle$	$[x, y]\mathcal{R}_E[v, z] \leftrightarrow y = z \wedge x < v$	
contains	$\langle D \rangle$	$[x, y]\mathcal{R}_D[v, z] \leftrightarrow x < v \wedge z < y$	
overlaps	$\langle O \rangle$	$[x, y]\mathcal{R}_O[v, z] \leftrightarrow x < v < y < z$	



All modalities can be expressed by means of $\langle A \rangle$, $\langle B \rangle$, $\langle E \rangle$, and their transposed modalities only

HS: the modal logic of Allen's interval relations

Halpern and Shoham's modal logic of time intervals HS (LICS 1986), interpreted over interval structures (not to be confused with Allen's Interval Algebra)

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Halpern and Shoham's modal logic of time intervals HS (LICS 1986), interpreted over interval structures (not to be confused with Allen's Interval Algebra)

The satisfiability/validity problem for HS is highly undecidable over all standard classes of linear orders. What about its fragments?

More than 4000 fragments of HS (over the class of all linear orders) can be identified by choosing a different subset of the set of basic modal operators. However, 1347 genuinely different ones exist only

D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco, *Expressiveness of the Interval Logics of Allen's Relations on the Class of all Linear Orders: Complete Classification*, IJCAI 2011

(Un)decidability of HS fragments: main parameters

Research agenda:

- search for maximal decidable HS fragments
- search for minimal undecidable HS fragments

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The large majority of HS fragments turns out be undecidable, but some meaningful exceptions exist

(Un)decidability of HS fragments depends on two factors:

- the set of interval modalities (in fact, one may also think of constraining the use of Boolean connectives)
- the class of interval structures (linear orders) over which the logic is interpreted

A real character: the logic D

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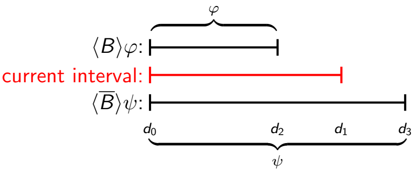
It is unknown, when D is interpreted over the class of all linear orders

An easy case: the logic $\mathrm{B}\overline{\mathrm{B}}$

Consider the logic $\mathrm{B}\overline{\mathrm{B}}$ of Allen's relations *begins* and *begun by*.



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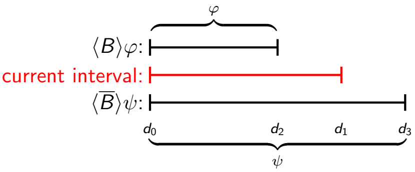

The decidability of $\mathrm{B\overline{B}}$ can be shown by embedding it into the propositional temporal logic of linear time LTL$[\mathrm{F},\mathrm{P}]$: formulas of $\mathrm{B\overline{B}}$ can be translated into formulas of LTL$[\mathrm{F},\mathrm{P}]$ by replacing $\langle \mathrm{B}\rangle$ with P (sometimes in the past) and $\langle \overline{\mathrm{B}}\rangle$ with F (sometimes in the future):

LTL$[\mathrm{F},\mathrm{P}]$ has the small (pseudo)model property and is decidable

The case of the logic $\mathrm{E\overline{E}}$ is similar

A well-behaved fragment: the logic $A\overline{A}$

Formulas of the logic $A\overline{A}$ of Allen's relations *meets* and *met by* are recursively defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle A \rangle \varphi \mid \langle \overline{A} \rangle \varphi \ ([A] = \neg\langle A \rangle \neg; \text{ same for } [\overline{A}])$$

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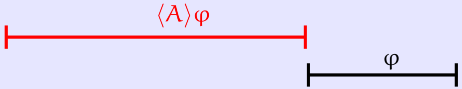

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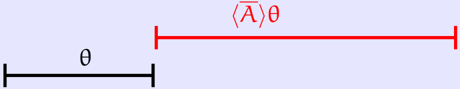

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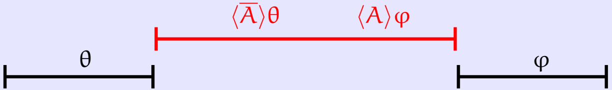
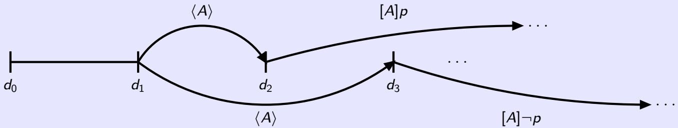


We cannot abstract way from any of the endpoints of intervals:

- contradictory formulas may hold over intervals with the same right endpoint and a different left endpoint

$\langle A\rangle[A]p \wedge \langle A\rangle[A]\neg p$ is satisfiable: for any $d > d_3$, p holds over $[d_2, d]$ and $\neg p$ holds over $[d_3, d]$



The importance of the past in $A\overline{A}$

Unlike what happens with point-based linear temporal logic, $A\overline{A}$ is strictly more expressive than its future fragment A (proof technique: invariance of modal formulas with respect to bisimulation)

$A\overline{A}$ is able to separate $\mathbb{Q}$ and $\mathbb{R}$, while A is not

There is a log-space reduction from the satisfiability problem for $A\overline{A}$ over $\mathbb{Z}$ to its satisfiability problem over $\mathbb{N}$, that turns out to be much more involved than the corresponding reduction for point-based linear temporal logic

D. Della Monica, A. Montanari, and P. Sala, The importance of the past in interval temporal logics: the case of Propositional Neighborhood Logic, in A. Artikis et al. (Eds.), Sergot Festschrift, LNAI 7360, Springer, 2012

Expressive completeness of $A\overline{A}$ with respect to $\mathrm{FO}^2[<]$

Expressive completeness of $A\overline{A}$ with respect to the two-variable fragment of first-order logic for binary relational structures over various linearly-ordered domains $\mathrm{FO}^2[<]$

M. Otto, Two Variable First-order Logic Over Ordered Domains, Journal of Symbolic Logic, 2001

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Remark. The two-variable property is a sufficient condition for decidability, but it is not a necessary one (for instance, D is decidable over dense linear orders, but it does not satisfy the two-variable property - three variables are needed)

Decidability of $A\overline{A}$

As a by-product, decidability (in fact, NEXPTIME-completeness) of $A\overline{A}$ over the class of all linear orders, the class of well-orders, the class of finite linear orders, and the linear order on the natural numbers

D. Bresolin, V. Goranko, A. Montanari, and G. Sciavicco, Propositional Interval Neighborhood Logics: Expressiveness, Decidability, and Undecidable Extensions, Annals of Pure and Applied Logic, 2009

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This was not the end of the story ..

- It was/is far from being trivial to extract a decision procedure from Otto's proof
- Some meaningful cases were missing (dense linear orders, weakly discrete linear orders, real numbers)

Tableau-based decision procedures for $A\overline{A}$ - 1

An optimal tableau-based decision procedure for the future fragment of $A\overline{A}$ (the future modality $\langle A \rangle$ only) over the natural numbers

D. Bresolin and A. Montanari, A Tableau-based Decision Procedure for Right Propositional Neighborhood Logic, TABLEAUX 2005 (extended and revised version in *Journal of Automated Reasoning*, 2007)

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Later extended to full $A\overline{A}$ over the integers (it can be tailored to natural numbers and finite linear orders)

D. Bresolin, A. Montanari, and P. Sala, An Optimal Tableau-based Decision Algorithm for Propositional Neighborhood Logic, STACS 2007

Tableau-based decision procedures for $A\overline{A}$ - 2

Then, optimal tableau-based decision procedures for $A\overline{A}$ over all, dense, and weakly-discrete linear orders have been developed

D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco, Optimal Tableau Systems for Propositional Neighborhood Logic over All, Dense, and Discrete Linear Orders, TABLEAUX 2011

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D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco, Optimal Tableau Systems for Propositional Neighborhood Logic over All, Dense, and Discrete Linear Orders, TABLEAUX 2011

Finally, an optimal tableau-based decision procedure for $A\overline{A}$ over the reals has been devised

A. Montanari and P. Sala, An optimal tableau system for the logic of temporal neighborhood over the reals, TIME 2012

Maximal decidable fragments

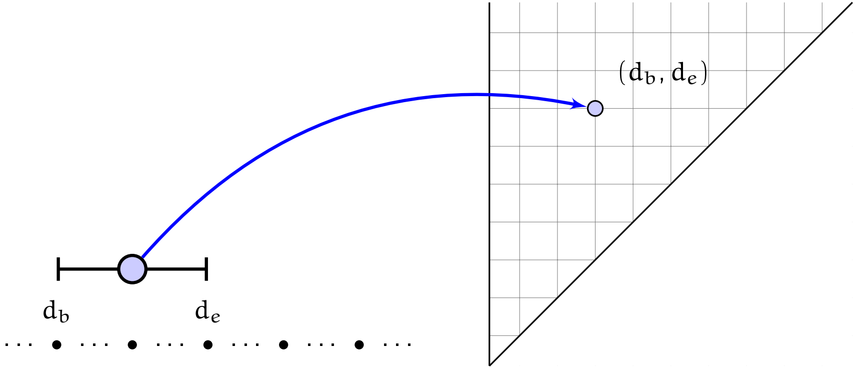
Issue: can we add other modalities from the HS repository to the logic of temporal neighborhood $A\overline{A}$ or to the logic of the subinterval relation D preserving decidability?

The search for maximal decidable fragments of HS benefitted from a natural geometrical interpretation of interval logics proposed by Venema

In the following, we restrict our attention to (the decidable extensions of) $A\overline{A}$

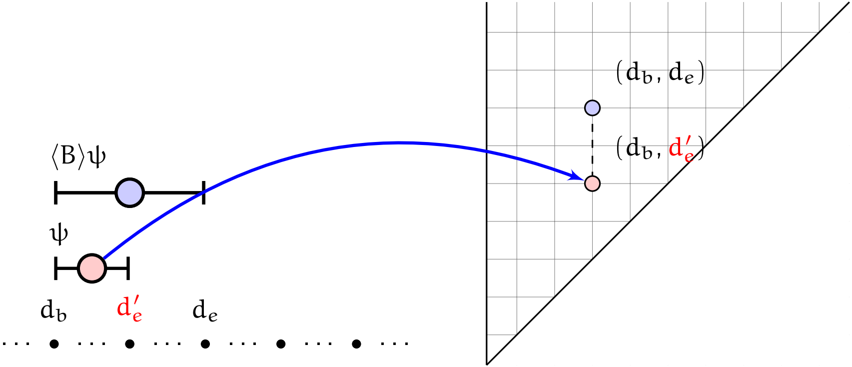
We illustrate the basic ingredients of such a geometrical interpretation and we summarize the main results

A geometrical account of interval logic: intervals



Every interval can be represented by a point in the second octant (in general, in the half plane $y \geqslant x$)

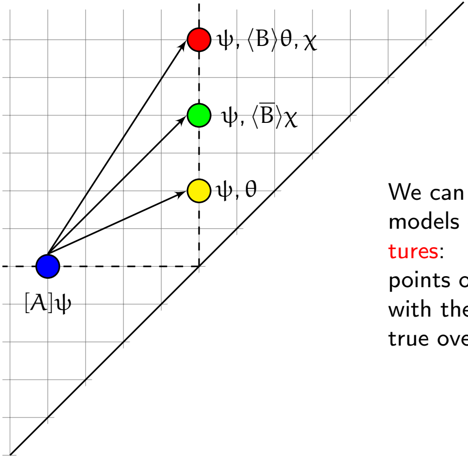
A geometrical account of interval logic: interval relations



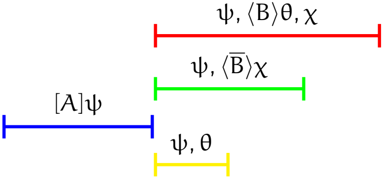
$d_b < d'_e < d_e$

Every interval relation has a spatial counterpart

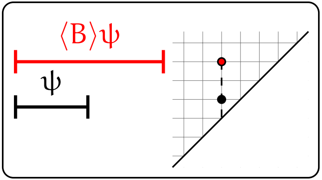
A geometrical account of interval logic: models

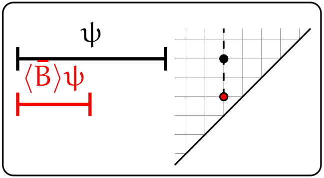


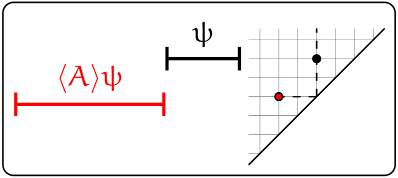
We can give a spatial interpretation to models of a formula φ as compass structures:
points of a compass structure are colored with the set of subformulas of φ that are true over the corresponding intervals

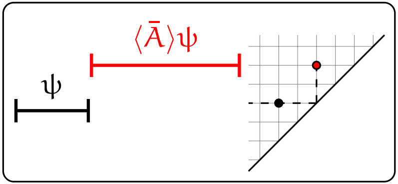


The maximal decidable fragment $AB\overline{BA}$



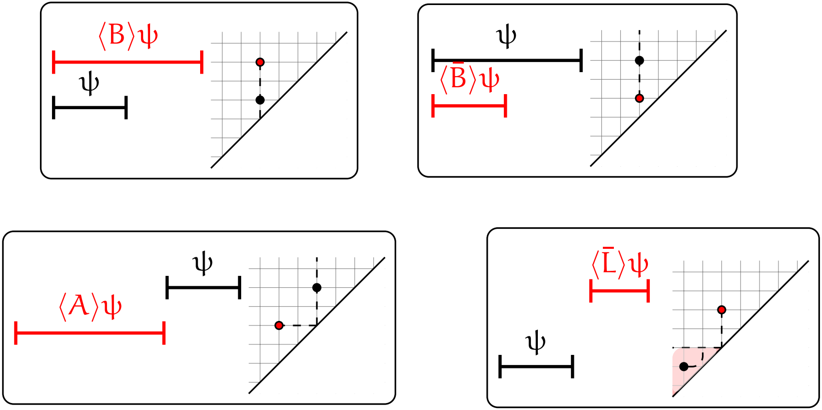






- $AB\overline{BA}$ is decidable, but NONPRIMITIVE RECURSIVE-hard over the class of finite linear orders, the rationals, and the class of all linear orders;
- it is undecidable over the natural numbers and the reals, and the class of all Dedekind-complete linear orders

The maximal decidable fragment $AB\overline{BL}$



- Replace $\langle\overline{A}\rangle$ by $\langle\overline{L}\rangle$: $AB\overline{BL}$ is EXSPACE-complete over the classes of all, dense, and (weakly) discrete linear orders

Maximal decidable fragments: references

Decidability of $A B \overline{B} \overline{A}$ over finite linear orders

A. Montanari, G. Puppis, and P. Sala, Maximal decidable fragments of Halpern and Shoham's modal logic of intervals, ICALP 2010

Decidability of $A B \overline{B} \overline{A}$ over the rationals and all linear orders

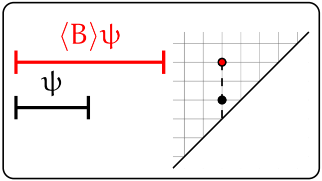
A. Montanari, G. Puppis, and P. Sala, Decidability of the interval temporal logic $A B \overline{B} \overline{A}$ over the rationals, MFCS 2014

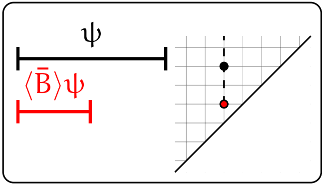
Decidability of $A B \overline{B} \overline{L}$ over all, dense, and discrete linear orders

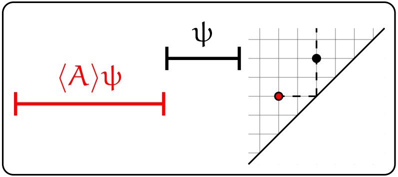
D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco, What's decidable about Halpern and Shoham's interval logic? The maximal fragment $A B \overline{B} \overline{L}$, LICS 2011

The case of the logic $AB\overline{B}$ (over finite linear orders and $\mathbb{N}$)

A. Montanari, G. Puppis, P. Sala, G. Sciavicco, Decidability of the interval temporal logic $AB\overline{B}$ over the natural numbers, STACS 2010







Why $A B \overline{B}$ is of particular interest?

Goal (statement): to recover standard (pointed-based) temporal logics as special cases of interval-based ones

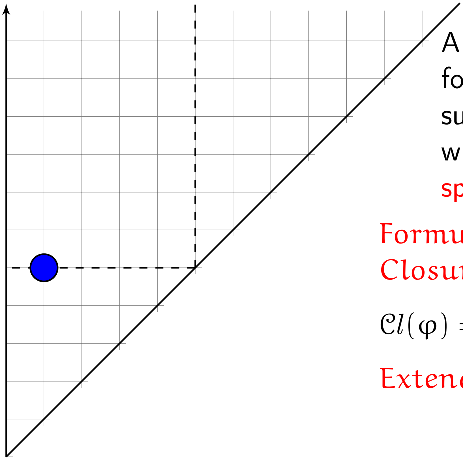
Let us consider propositional Linear Temporal Logic (LTL).
The until modality of LTL can be expressed in $A B \overline{B}$ (in fact, AB suffices)

$$\psi \; \mathcal{U} \; \varphi$$

can be encoded as

$$\langle A \rangle \mathrm{Unit}(\varphi) \; \vee \; \langle A \rangle (\langle A \rangle \mathrm{Unit}(\varphi) \; \wedge \; [B] \langle A \rangle \mathrm{Unit}(\psi)),$$

where $\mathrm{Unit}(\theta)$ is a shorthand for $[B]\bot \wedge \theta$



A color is the set of subformulas of the extended closure of the given formula φ which are true over the corresponding interval

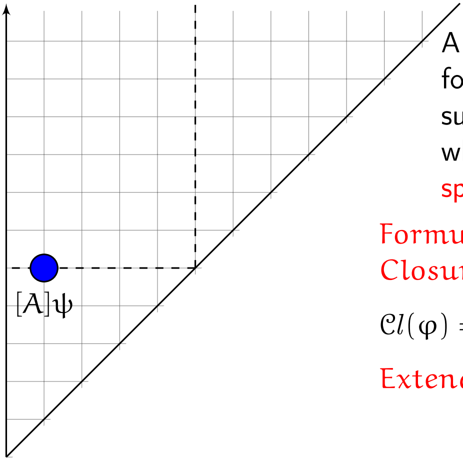
Formula : $\varphi = \langle A \rangle \langle B \rangle p$

Closure :

$$\mathcal{C}l(\varphi) = \left\{ \begin{array}{l} p, \neg p, \langle B \rangle p, [B]\neg p, \\ \langle A \rangle \langle B \rangle p, [A][B]\neg p \end{array} \right\}$$

Extended closure :

$$\mathcal{E}\mathcal{C}l(\psi) = \left\{ \begin{array}{l} p, \neg p, \langle B \rangle p, [B]\neg p, \\ \langle A \rangle p, [A]\neg p, \langle \overline{B} \rangle p, [\overline{B}]\neg p, \\ \langle A \rangle \langle B \rangle p, [A][B]\neg p, \\ \langle B \rangle \langle B \rangle p, [B][B]\neg p, \\ \langle \overline{B} \rangle \langle B \rangle p, [\overline{B}][B]\neg p, \ldots \end{array} \right\}$$



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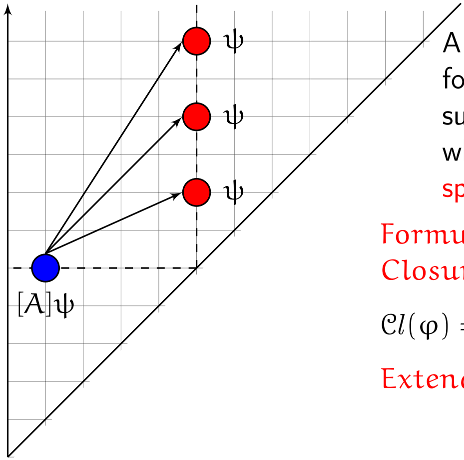

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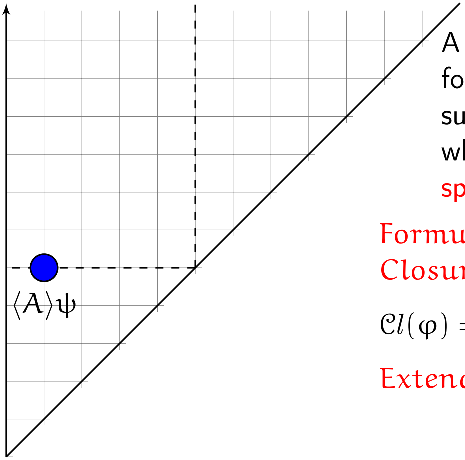

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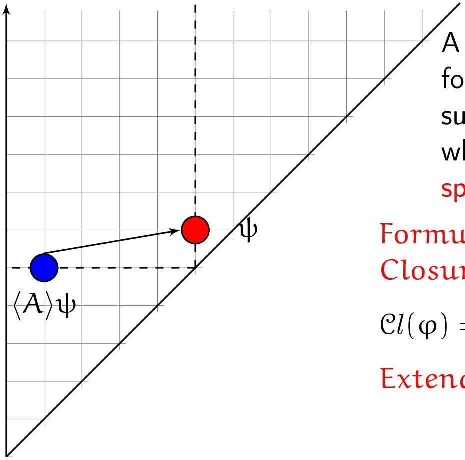

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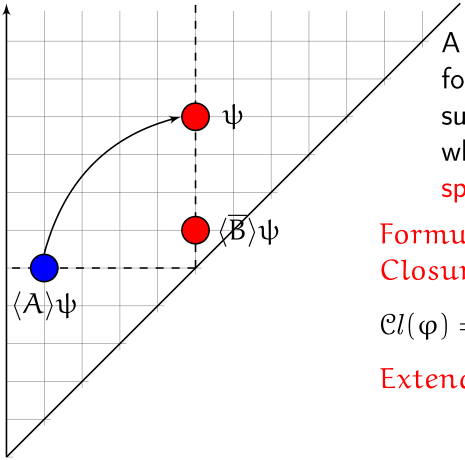

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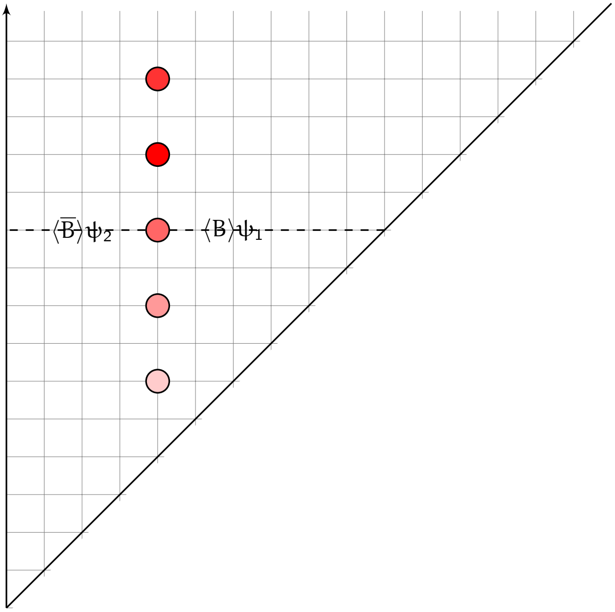
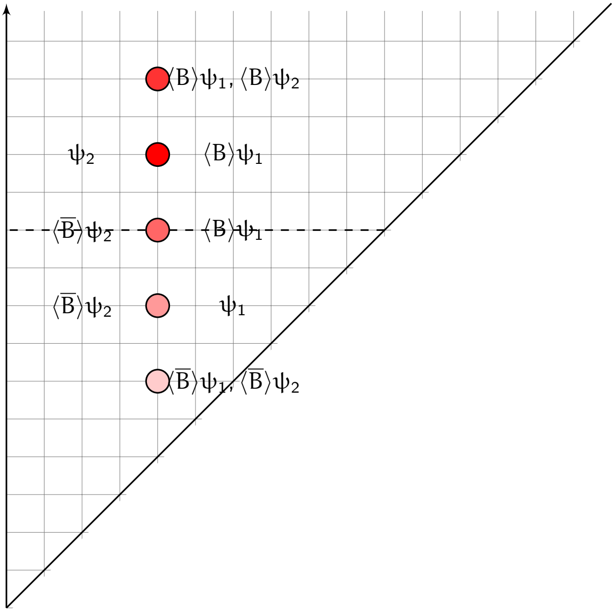
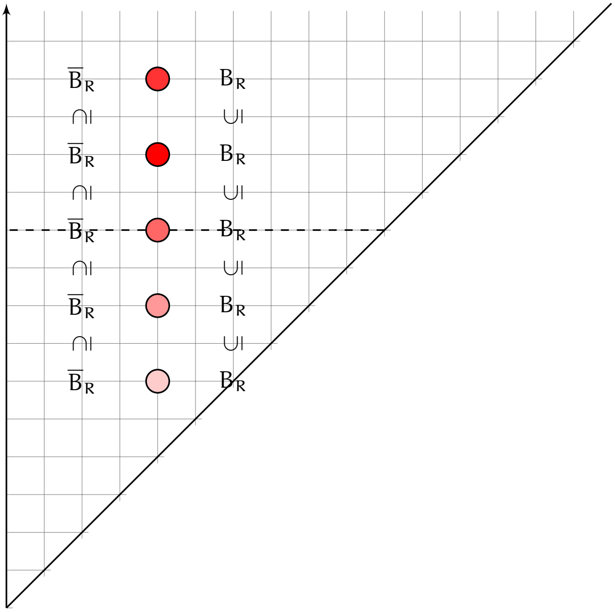
Extended closure :

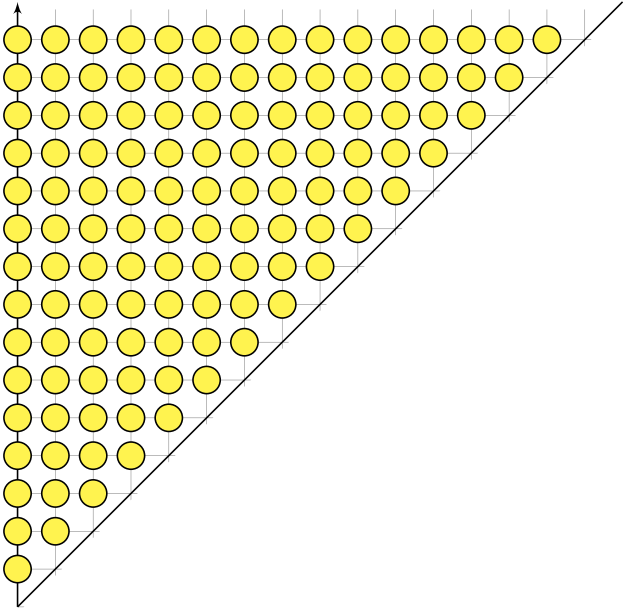
$$\mathcal{EC}l(\psi) = \left\{ \begin{array}{l} p, \neg p, \langle B \rangle p, [B] \neg p, \\ \langle A \rangle p, [A] \neg p, \langle \overline{B} \rangle p, [\overline{B}] \neg p, \\ \langle A \rangle \langle B \rangle p, [A][B] \neg p, \\ \langle B \rangle \langle B \rangle p, [B][B] \neg p, \\ \langle \overline{B} \rangle \langle B \rangle p, [\overline{B}][B] \neg p, \ldots \end{array} \right\}$$

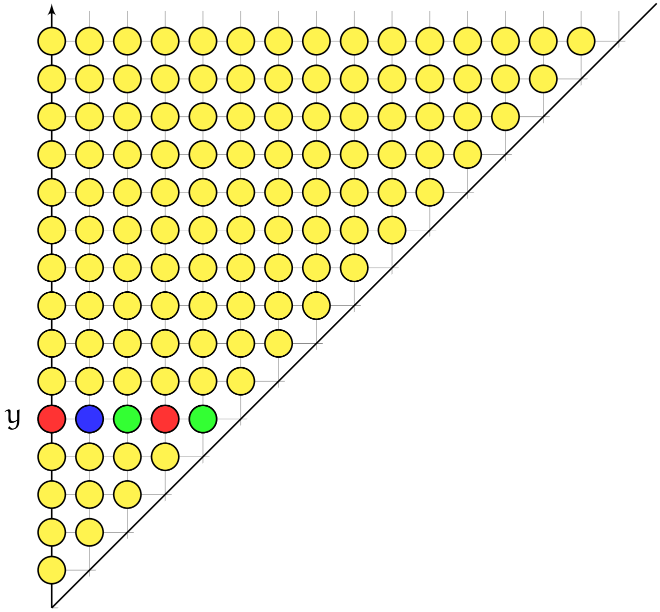


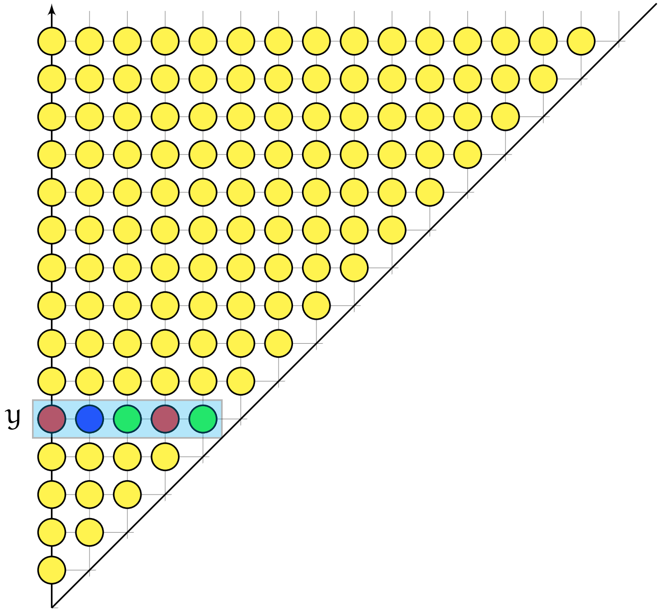
$\langle \overline{B} \rangle \psi_2$
$\langle B \rangle \psi_1$

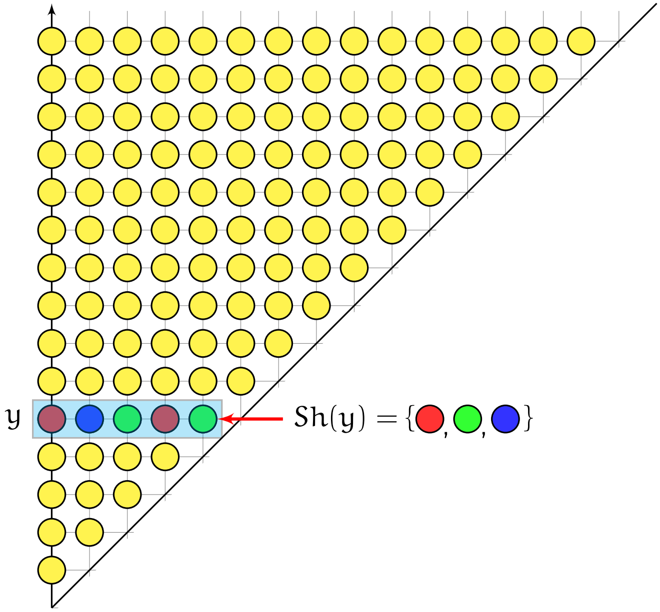

⟨B⟩ψ1, ⟨B⟩ψ2
ψ2
⟨B⟩ψ1
⟨B̄⟩ψ2
⟨B⟩ψ1
⟨B̄⟩ψ2
ψ1
⟨B̄⟩ψ1, ⟨B̄⟩ψ2

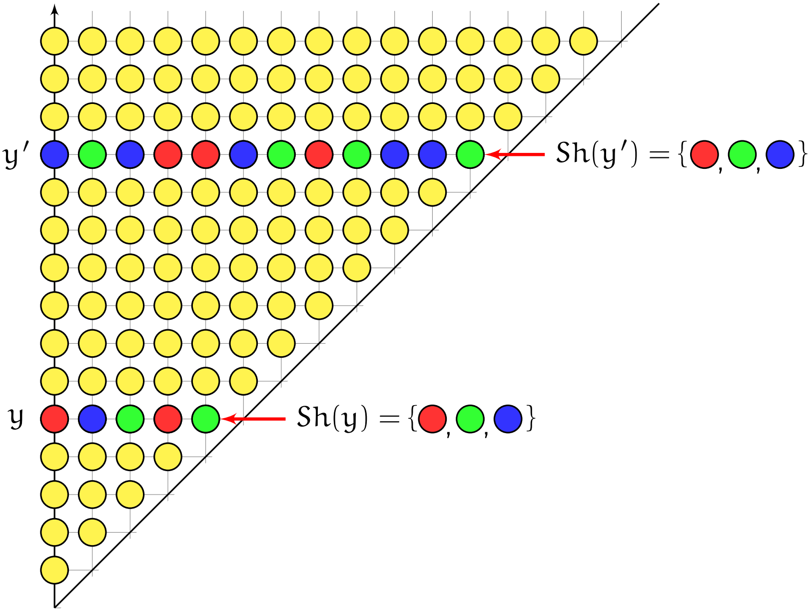

$\overline{B}_R$ ⊆ B_R
$\overline{B}_R$ ⊆ B_R
$\overline{B}_R$ ⊆ B_R
$\overline{B}_R$ ⊆ B_R
$\overline{B}_R$ ⊆ B_R



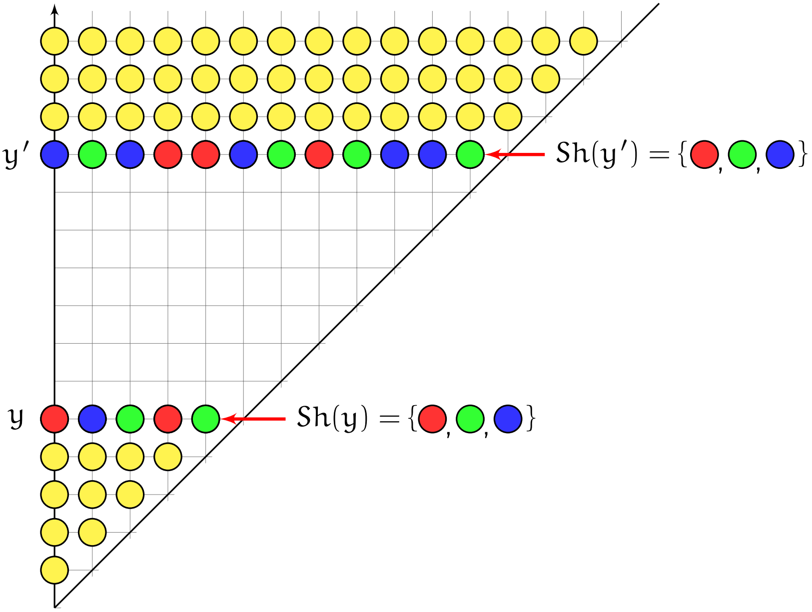

y


𝔜


y
Sh(y) = { , , }



y′
y
Sh(y′) = {, , }
Sh(y) = {, , }


y'
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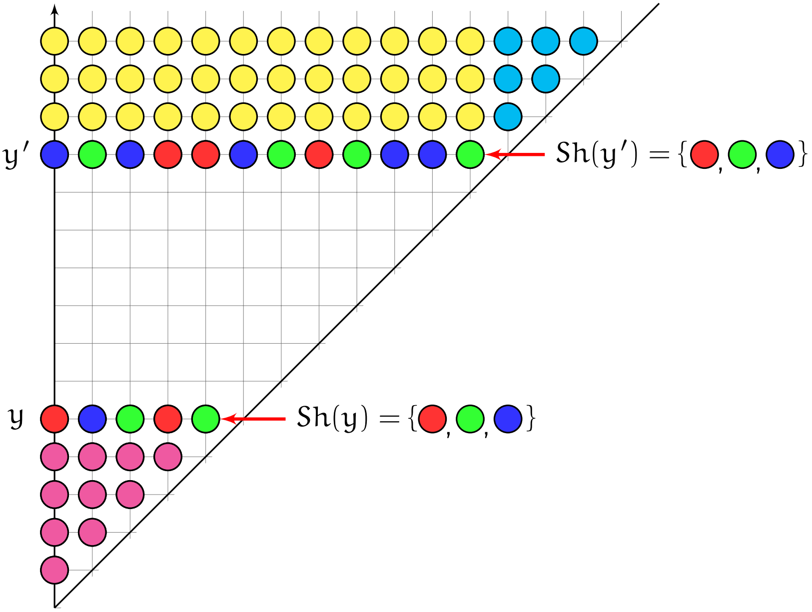
y
Sh(y) = {, , }



y′
Sh(y′) = {, , }
y
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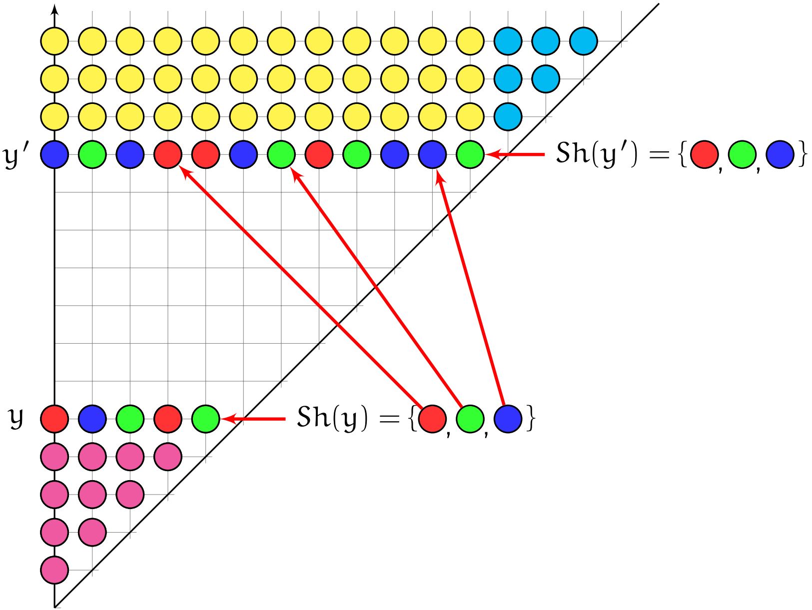
y′
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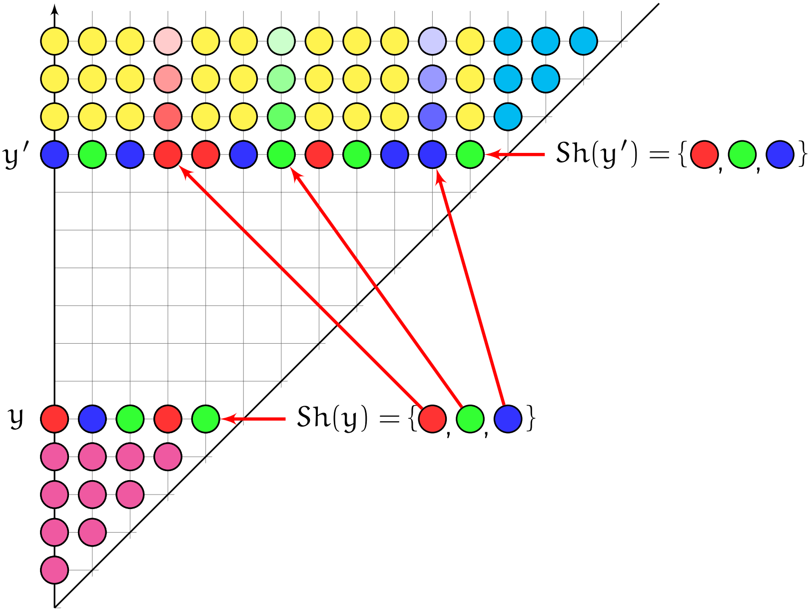

y'
$Sh(y') = \{ , , \}$
y
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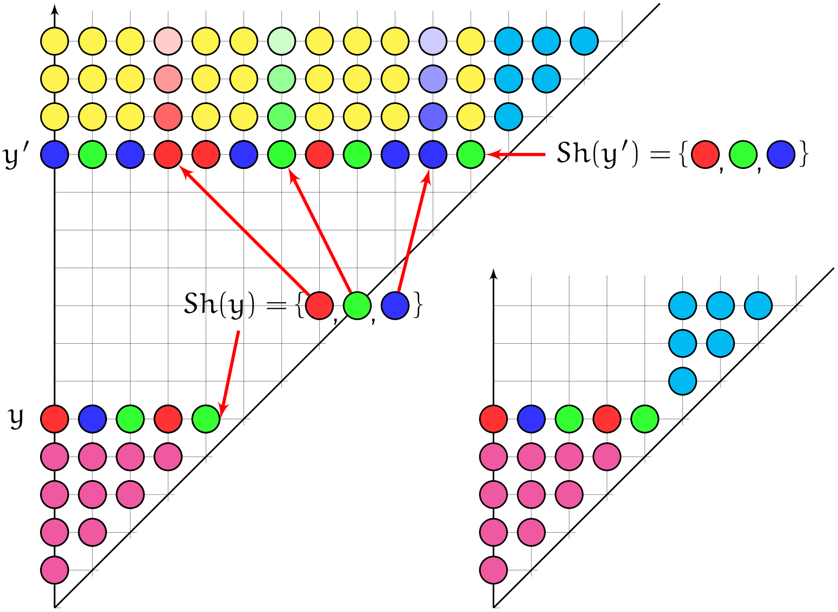
y′
y
Sh(y′) = {, , }
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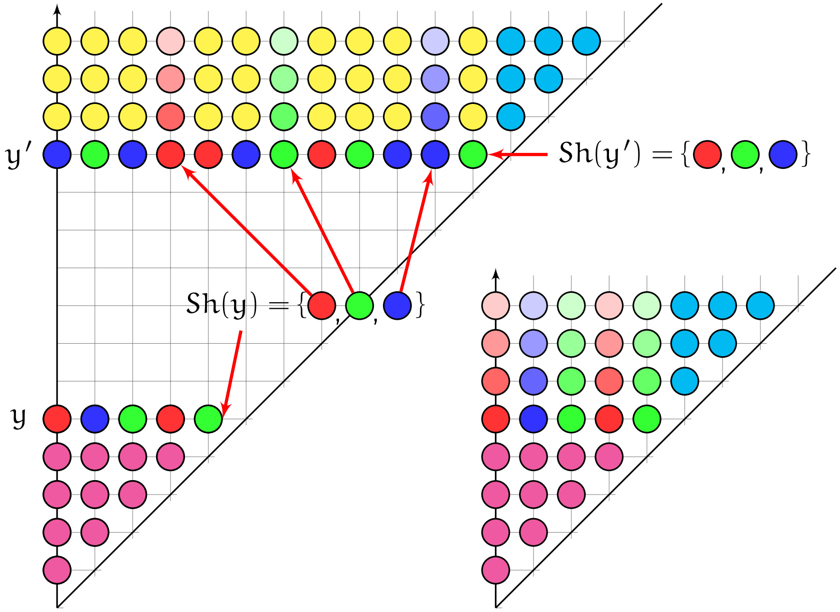
y′
y
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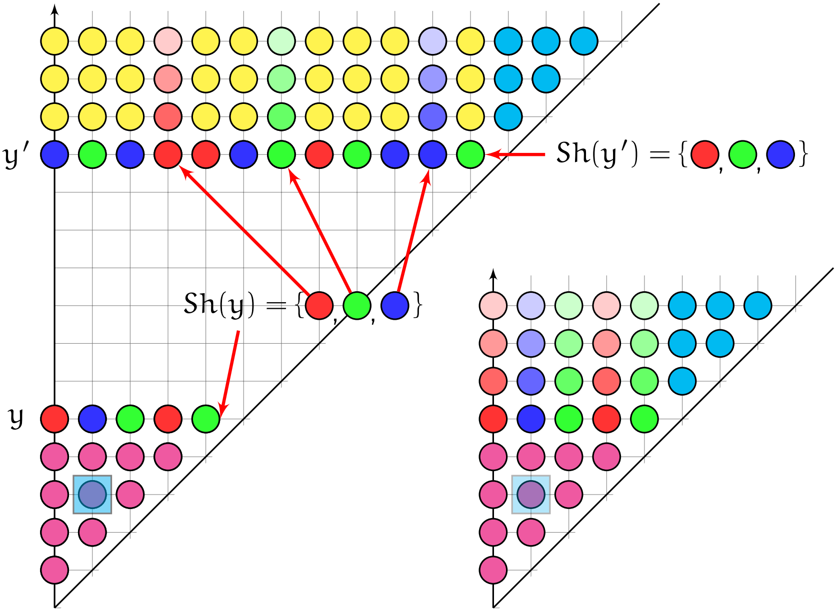
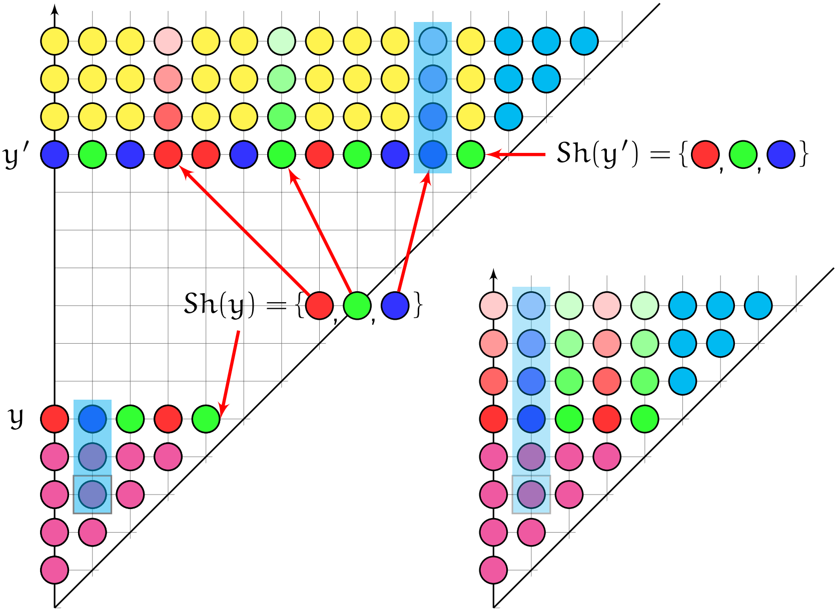
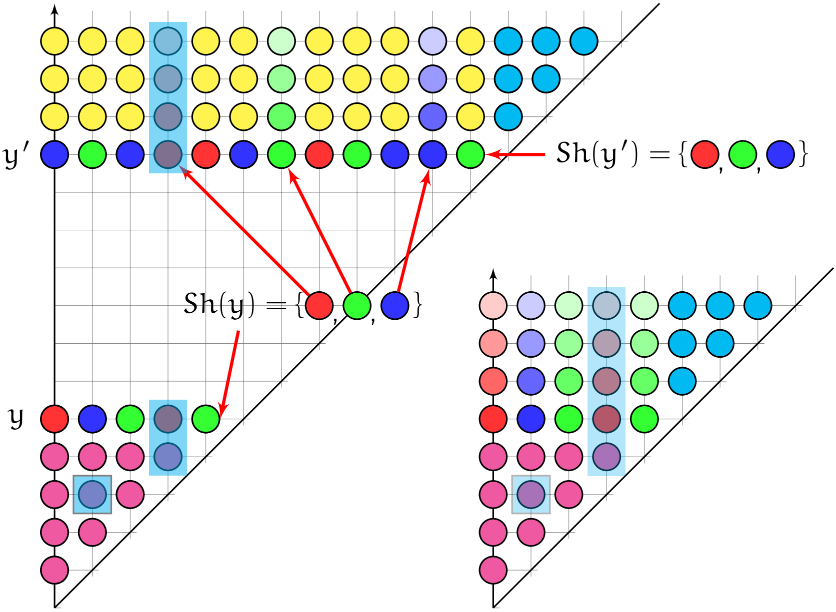


y'
$Sh(y') = \{$, , $\}$
$Sh(y) = \{$, , $\}$
y


y′
y
Sh(y′) = {, , }
Sh(y) = {, , }



y'
Sh(y') = { , , }
Sh(y) = { , , }
y

Paths to undecidability - 1

Undecidability results for HS fragments have been obtained by means of reductions from several undecidable problems:

- reduction from the non-halting problem for Turing machines (e.g., HS over all meaningful classes of linear orders, and BE over dense linear orders – that of BE over all linear orders immediately follows)

J. Halpern and Y. Shoham, A propositional modal logic of time intervals, Journal of the ACM, 1991

K. Lodaya, Sharpening the Undecidability of Interval Temporal Logic, ASIAN 2000

Paths to undecidability - 2

- reductions from several variants of the tiling problem, like the octant tiling problem and the finite tiling problem (O, $\overline{\text{O}}$, AD, $\overline{\text{A}}$D, A$\overline{\text{D}}$, $\overline{\text{AD}}$, BE, $\overline{\text{B}}$E, B$\overline{\text{E}}$, and $\overline{\text{BE}}$ over any class of linear orders that contains, for each $n > 0$, at least one linear order with length greater than n)

D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco, The dark side of Interval Temporal Logics: marking the undecidability border, Annals of Mathematics and Artificial Intelligence, 2014

- reduction from the halting problem for two-counter automata (D over finite and discrete linear orders)

J. Marcinkowski and J. Michaliszyn, The Ultimate Undecidability Result for the Halpern-Shoham Logic, LICS 2011

The case of the logic O (over discrete linear orders)

Regularities and (wrong) conjectures: are there necessary and sufficient conditions for the decidability of the satisfiability problem for HS fragments?

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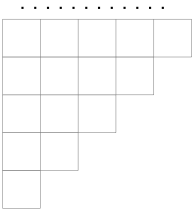
  Counterexample: O over discrete linear orders

In the following, we focus on the logic O over discrete linear orders

Proof overview ⏩ skip

Reduction from the Octant Tiling Problem

The Octant Tiling Problem is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile the octant $\mathcal{O} = \{(i, j) : i, j \in \mathbb{N} \wedge 0 \leqslant i \leqslant j\}$ respecting the color constraints



Proof overview ⏩ skip

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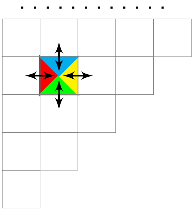

Proof overview ⏩ skip

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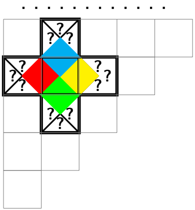

Proof overview skip

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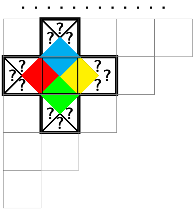


by König's Lemma

$\mathbb{N} \times \mathbb{N} \to \mathcal{O}$

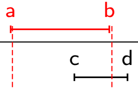
Proof overview

The logic O over discrete linear orders

We build a formula $\phi_{\mathcal{T}} \in \mathsf{O}$ such that

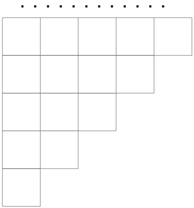
$$\phi_{\mathcal{T}} \text{ is satisfiable} \quad \Leftrightarrow \quad \mathcal{T} \text{ can tile the octant}$$

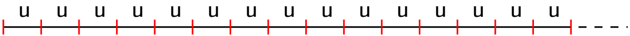
(over discrete linear orders)

Op.	Semantics	
$\langle \mathsf{O} \rangle$	$M, [a,b] \Vdash \langle \mathsf{O} \rangle \phi \Leftrightarrow \exists c, d (a < c < b < d . M, [c,d] \Vdash \phi)$	

Encoding the Octant

Encoding the Octant





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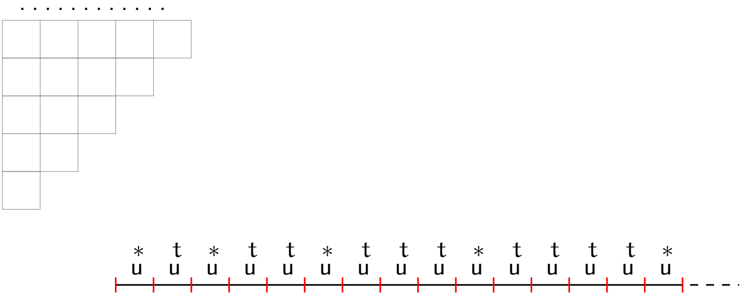

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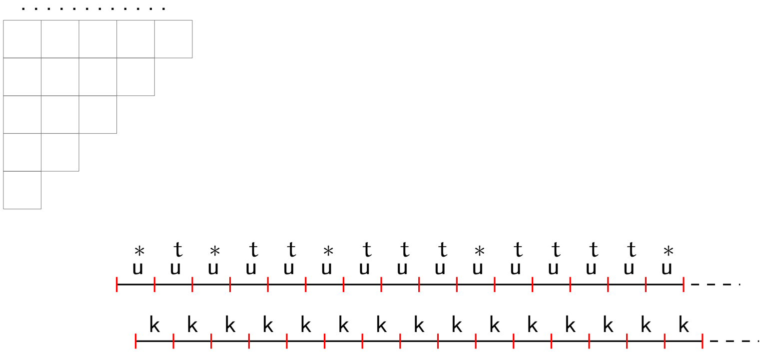

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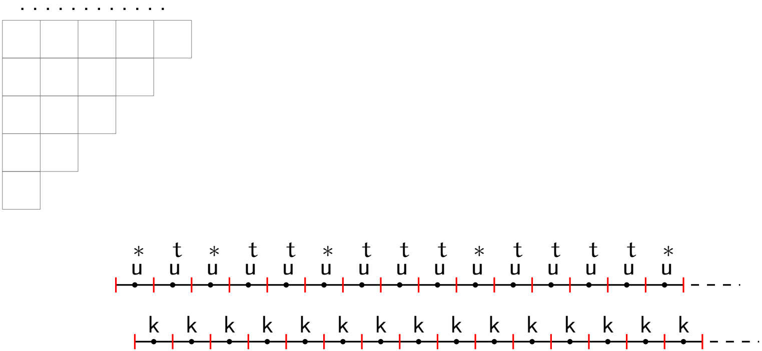
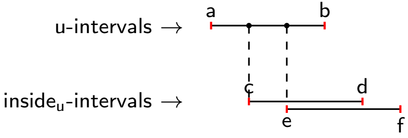
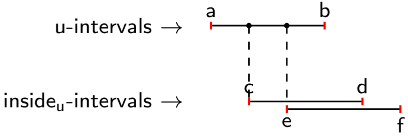


Encoding the Octant (u- and k-intervals of length 2)

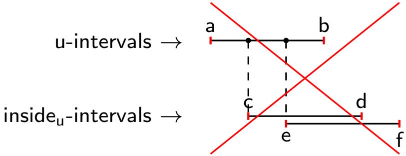


Encoding the Octant (u- and k-intervals of length 2)



inside$_u$-intervals **cannot overlap** inside$_u$-intervals starting inside the same u-interval

Encoding the Octant (u- and k-intervals of length 2)



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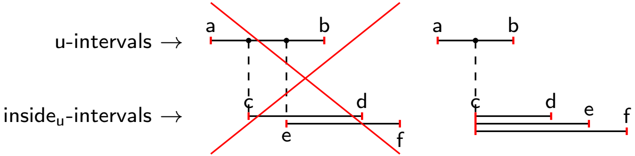
Encoding the Octant (u- and k-intervals of length 2)



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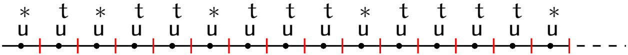
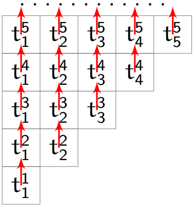
Encoding the Above-Neighbour Relation

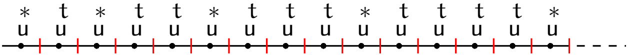
. . . .

t_1^5	t_2^5	t_3^5	t_4^5	t_5^5
t_1^4	t_2^4	t_3^4	t_4^4	
t_1^3	t_2^3	t_3^3		
t_1^2	t_2^2			
t_1^1				



Encoding the Above-Neighbour Relation





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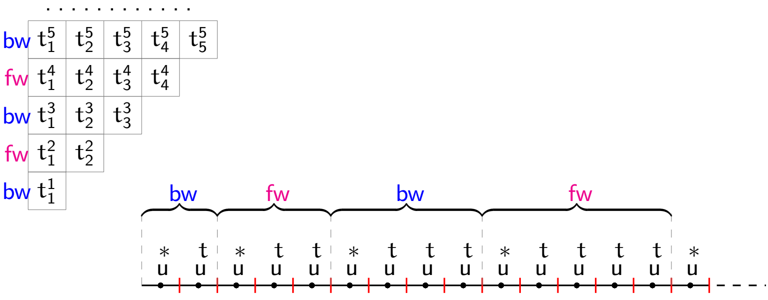

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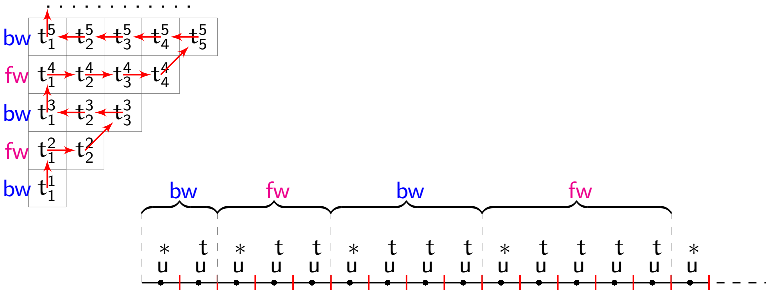

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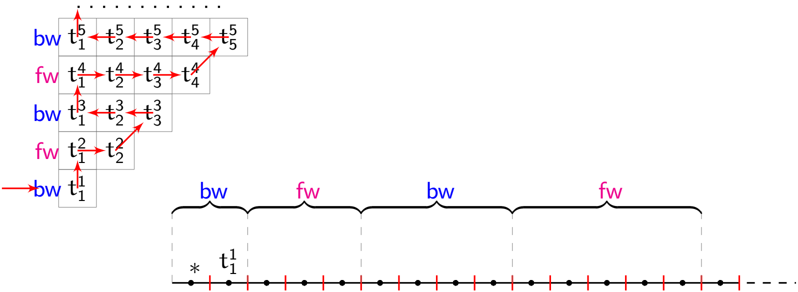

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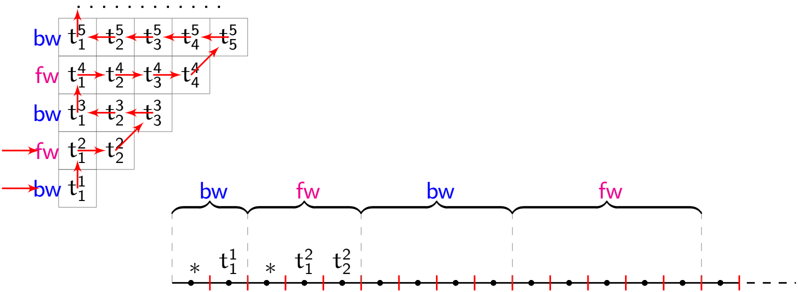

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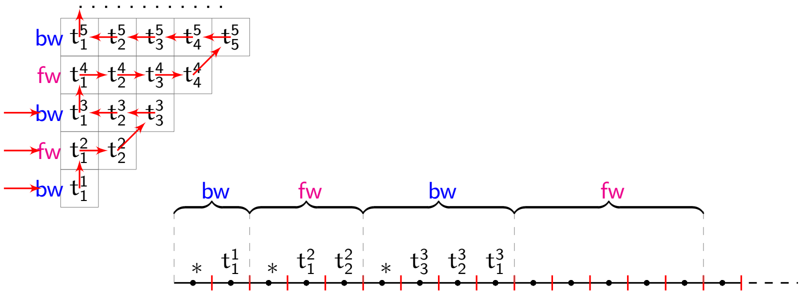

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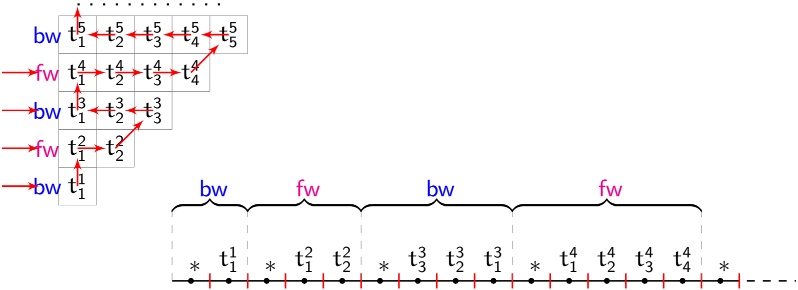

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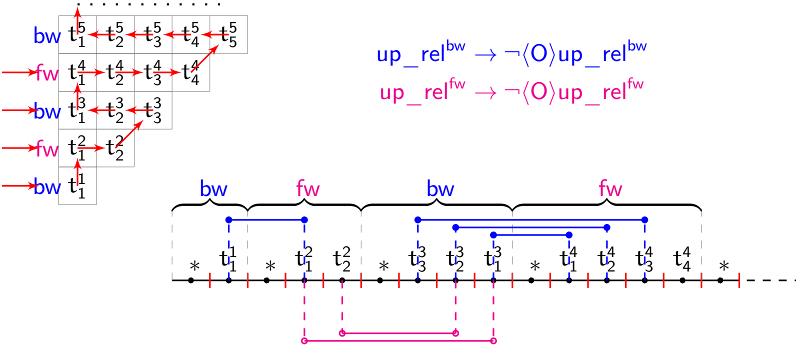


$$up\_rel^{bw} \rightarrow \neg\langle O\rangle up\_rel^{bw}$$

$$up\_rel^{fw} \rightarrow \neg\langle O\rangle up\_rel^{fw}$$

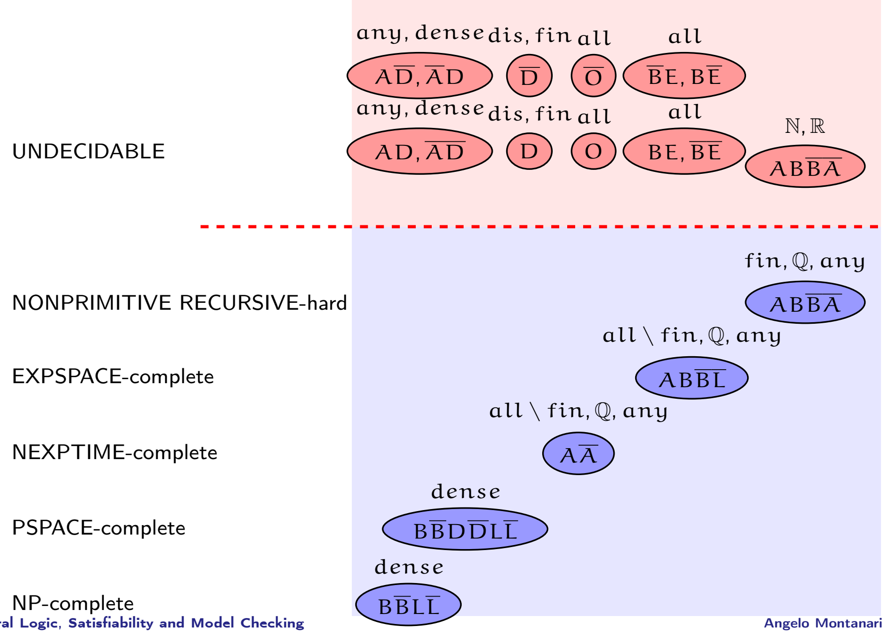
Theorem

Theorem [Undecidability of the logic O (resp., $\overline{\mathrm{O}}$) over discrete linear orders]

The satisfiability problem for the HS fragment O (resp., $\overline{\mathrm{O}}$) is undecidable over any class of discrete linear orders that contains at least one linear order with an infinite ascending (resp., descending) sequence

D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco, Undecidability of the Logic of Overlap Relation over Discrete Linear Orderings, M4M 2009

The (almost) complete picture



(Maximal) decidable fragments: additional references

NP-completeness of $\mathrm{B}\overline{\mathrm{B}}\mathrm{L}\overline{\mathrm{L}}$ over dense linear orders

D. Bresolin, D. Della Monica, A. Montanari, P. Sala, and G. Sciavicco, On the Complexity of Fragments of the Modal Logic of Allen's Relations over Dense Structures, LATA 2015

PSPACE-completeness of $\mathrm{B}\overline{\mathrm{B}}\mathrm{D}\overline{\mathrm{D}}\mathrm{L}\overline{\mathrm{L}}$ over dense linear orders

A. Montanari, G. Puppis, and P. Sala, A decidable weakening of Compass Logic based on cone-shaped cardinal directions. Logical Methods in Computer Science, 2015

Current research agenda (an excerpt)

- To obtain a complete classification of the family of HS fragments with respect to decidability/undecidability of their satisfiability problem and with respect to their relative expressive power
- To extend the study of metric variants of interval logics (we already did it for $A\overline{A}$ over $\mathbb{N}$, and finite linear orders) to other HS fragments and over other metrizable linear orders, notably that of $\mathbb{Q}$

D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, G. Sciavicco, Metric Propositional Neighborhood Logics on Natural Numbers, Software and Systems Modeling 2013

D. Bresolin, A. Montanari, G. Sciavicco, P. Sala, Optimal decision procedures for MPNL over finite structures, the natural numbers, and the integers, Theoretical Computer Science 2013