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Linear Temporal Logic

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Temporal Logics

Temporal logics are the de-facto standard languages for specifying properties of systems in *formal verification* and *artificial intelligence*.

- born in the ’50s as a tool for philosophical argumentation about time

Reference:

Arthur N. Prior (1957). *Time and Modality*. London: Oxford University Press

- the idea of its use in formal verification can be traced back to the ’70s

Reference:

Amir Pnueli (1977). “The temporal logic of programs”. In: *18th Annual Symposium on Foundations of Computer Science (sfcs 1977)*. IEEE, pp. 46–57. DOI: 10.1109/SFCS.1977.32

Modal Logic

Modal Logic extends classic propositional (Boolean) logic with the concepts of *necessity* and *possibility*.

- World = set of propositions that are supposed to be true in that world
- Worlds are connected with edges
  - directed graph with labels on the nodes: Kripke structure
- in Modal Logic, the truth of a formula depends on the *world* in which is interpreted (many-worlds interpretation) and on the worlds accessible from it.
- Necessity ($\Box$): is asking something to be true in *all* accessible states
- Possibility ($\Diamond$): is asking something to be true in *at least one* accessible state

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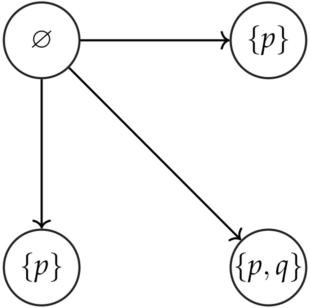


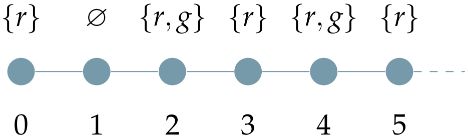
- $\mathcal{AP} = \{p, q\}$
- "$\Box p$" is true
- "$\Diamond q$" is true
- "$\Box q$" is false
- "$\Box p \vee \Box q$" is true

Linear Temporal Logic
LTL

Linear Temporal Logic (LTL, for short) is a (special case of) Modal Logic.

- World = State = set of proposition letters that are supposed to hold (*i.e.*, to be true) in that state
- Kripke Structure = (infinite) linear order of states = state sequence = word in a language
  - accessibility relation = temporal ordering
- Necessity ($\Box$) = Always in the future (G)
- Possibility ($\Diamond$) = Sometimes in the future (F)

$\mathcal{AP} := \{r, g\}$



- introduced by Pnueli in the '70s
- interpreted over *state sequences*
- it extends classical *propositional* logic
- temporal *modalities* are used to talk about how propositions change over time

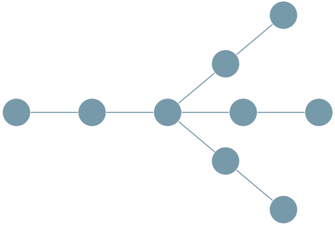
Reference:

Amir Pnueli (1977). "The temporal logic of programs". In: *18th Annual Symposium on Foundations of Computer Science (sfcs 1977)*. IEEE, pp. 46–57. DOI: 10.1109/SFCS.1977.32

There are many choices to be made for the representation of *time*.

Linear

Branching



There are many choices to be made for the representation of *time*.

Infinite

Finite

Representing time

There are many choices to be made for the representation of *time*.

Qualitative

Real-time

There are many choices to be made for the representation of *time*.

Discrete

Dense

Representing time

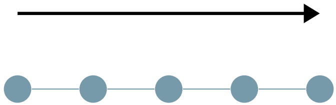
There are many choices to be made for the representation of *time*.

Points

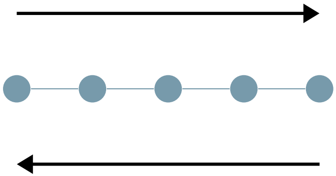
Intervals

Other parameters

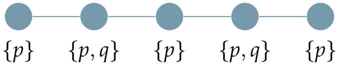
Pure Future



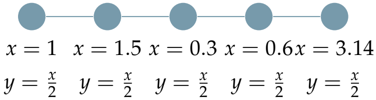
Past-Future



Propositional



First-Order



Our choice

We focus here on:

- *linear*-time
- *discrete*-time
- *qualitative*-time
- *infinite*-time
- *future only*
- *propositional*

Let $\mathcal{AP} := \{p, q, r, \ldots\}$ be a set of *atomic propositions*. The syntax of LTL is defined as follows:

$$\begin{aligned} \phi := p \mid \neg\phi \mid \phi \wedge \phi & \qquad \text{Boolean Modalities with } p \in \mathcal{AP} \\ \mid \mathsf{X}\phi \mid \phi \,\mathsf{U}\, \phi & \qquad \text{Future Temporal Modalities} \end{aligned}$$

- $\mathsf{X}\phi$ is the Next operator: *at the next time point (tomorrow), the formula ϕ holds*
- $\phi_1 \,\mathsf{U}\, \phi_2$ is the Until operator : *there exists a time point in the future where ϕ_2 is true, and ϕ_1 holds from now until (but not necessarily including) that point.*

Shortcuts:

- Eventually, $\mathsf{F}\phi$: *there exists a time point in the future where ϕ holds*. It is defined as $\mathsf{F}\phi \equiv \top \,\mathsf{U}\, \phi$.
- Globally, $\mathsf{G}\phi$: *for all time points in the future ϕ holds*. It is defined as $\mathsf{G}\phi \equiv \neg(\mathsf{F}\neg\phi)$.

Example:

Consider $\mathcal{AP} = \{p, q\}$ and the following formula:

$$\mathsf{GF}(p)$$

Which state sequences are *models* of the formula?

- $\{p\} \cdot \{q\} \cdot \{p\} \cdot (\{q\})^\omega$
- $(\{p, q\})^\omega$
- $(\{q\} \cdot \{q\} \cdot \{p\} \cdot \{q\})^\omega$
- $(\{p\})^* \cdot (\{q\})^\omega$

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Which state sequences are *models* of the formula?

- $\{p\} \cdot \{q\} \cdot \{p\} \cdot (\{q\})^\omega$ no
- $(\{p, q\})^\omega$ yes
- $(\{q\} \cdot \{q\} \cdot \{p\} \cdot \{q\})^\omega$ yes
- $(\{p\})^* \cdot (\{q\})^\omega$ no

Example:

Consider $\mathcal{AP} = \{p, q\}$ and the following formula:

$$\mathsf{FG}(q)$$

Which state sequences are *models* of the formula?

- $\{p\} \cdot \{q\} \cdot \{p\} \cdot (\{q\})^\omega$
- $(\{p, q\})^\omega$
- $(\{q\} \cdot \{q\} \cdot \{p\} \cdot \{q\})^\omega$
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- $(\{p, q\})^\omega$ yes
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- $(\{p\})^* \cdot (\{q\})^\omega$ yes

Example:

Let $\mathcal{AP} = \{r, g\}$. *Each request (r) is eventually followed by a grant (g).*

$$\mathsf{G}(r \rightarrow \mathsf{F}(g))$$

Which state sequences are *models* of the formula?

- $(\varnothing)^\omega$
- $\{r\} \cdot \{r\} \cdot \{r\} \cdot (\varnothing)^\omega$
- $\{r\} \cdot \{r\} \cdot \{r\} \cdot \{g\} \cdot (\varnothing)^\omega$
- $(\{r\} \cdot \varnothing \cdot \varnothing \cdot \{g\})^\omega$

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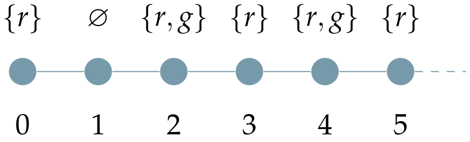
Which state sequences are *models* of the formula?

- $(\varnothing)^\omega$ — yes
- $\{r\} \cdot \{r\} \cdot \{r\} \cdot (\varnothing)^\omega$ — no
- $\{r\} \cdot \{r\} \cdot \{r\} \cdot \{g\} \cdot (\varnothing)^\omega$ — yes
- $(\{r\} \cdot \varnothing \cdot \varnothing \cdot \{g\})^\omega$ — yes

Linear Temporal Logic
LTL Semantics

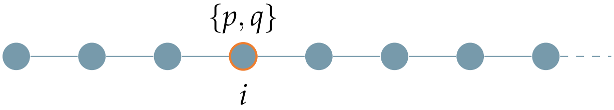
- Given a set of atomic propositions $\mathcal{AP}$, any LTL formula defined over $\mathcal{AP}$ is interpreted over *infinite words* $\sigma \in (2^{\mathcal{AP}})^{\omega}$.
- Let $\sigma = \langle \sigma_0, \sigma_1, \ldots \rangle$. For each $i \geq 0$, $\sigma_i \subseteq \mathcal{AP}$ is called a state contains the atomic propositions that are supposed to hold in that state.
- In this context, sequences in $(2^{\mathcal{AP}})^{\omega}$ are also called state sequences or traces.

$\mathcal{AP} := \{r, g\}$



We say that σ satisfies at position i the LTL formula ϕ, written $\sigma, i \models \phi$, iff:

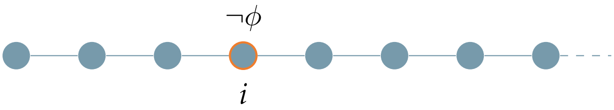
- $\sigma, i \models p$ iff $p \in \sigma_i$



p holds at position i

We say that σ satisfies at position i the LTL formula ϕ, written $\sigma, i \models \phi$, iff:

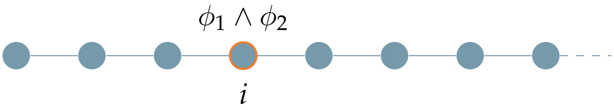
- $\sigma, i \models \neg\phi$ iff $\sigma, i \not\models \phi$



ϕ does not hold at position i

We say that σ satisfies at position i the LTL formula ϕ, written $\sigma, i \models \phi$, iff:

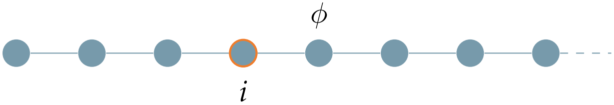
- $\sigma, i \models \phi_1 \land \phi_2$ iff $\sigma, i \models \phi_1$ and $\sigma, i \models \phi_2$



ϕ_1 and ϕ_2 hold at position i

We say that σ satisfies at position i the LTL formula ϕ, written $\sigma, i \models \phi$, iff:

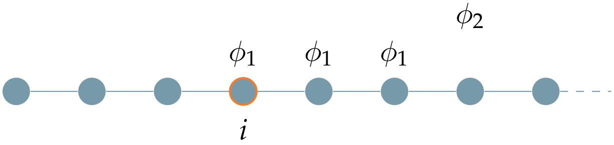
- $\sigma, i \models \mathsf{X}\phi$ iff $\sigma, i+1 \models \phi$



ϕ holds at the *next* position of i

We say that σ satisfies at position i the LTL formula ϕ, written $\sigma, i \models \phi$, iff:

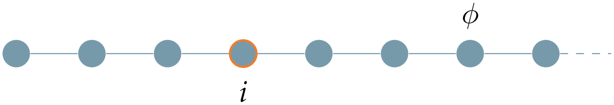
- $\sigma, i \models \phi_1 \mathsf{U} \phi_2$ iff $\exists j \geq i \,.\, \sigma, j \models \phi_2$ and $\forall i \leq k < j \,.\, \sigma, k \models \phi_1$



ϕ_1 holds *until* ϕ_2 holds

Shortcuts:

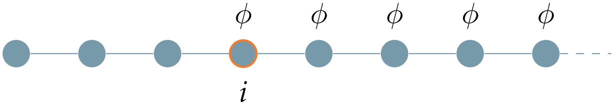
- (*eventually*) $\mathsf{F}\phi \equiv \top \;\mathsf{U}\; \phi$



ϕ will *eventually* hold

Shortcuts:

- (*globally*) $\mathsf{G}\phi \equiv \neg\mathsf{F}\neg\phi$



ϕ holds *always*

- We say that σ *satisfies* ϕ (written $\sigma \models \phi$) iff $\sigma, 0 \models \phi$. In this case, we say that σ is a *model* of ϕ.
- For any LTL formula ϕ, we define *the language of ϕ* as:

$$\mathcal{L}(\phi) = \{\sigma \in (2^{\mathcal{AP}})^\omega \mid \sigma \models \phi\}$$

- We say that ϕ is satisfiable iff $\mathcal{L}(\phi) \neq \varnothing$.
- We say that ϕ is valid iff $\mathcal{L}(\phi) = (2^{\mathcal{AP}})^\omega$.

Example:

Consider $\mathcal{AP} = \{p, q\}$ and the following formula:

$$\mathsf{F}(p \wedge \mathsf{X}q)$$

Which state sequences are *models* of the formula?

- $(\varnothing)^\omega$
- $(\{q\})^\omega$
- $(\varnothing)^* \cdot \{p\} \cdot \varnothing \cdot \{q\} \cdot (\varnothing)^\omega$
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Consider $\mathcal{AP} = \{p, q\}$ and the following formulas:

$$\mathsf{F}(p) \wedge \mathsf{F}(q) \qquad \mathsf{F}(p \wedge \mathsf{F}q) \qquad \mathsf{F}(p \wedge q)$$

Which state sequences are *models* of the formula?

- $(\varnothing)^\omega$
- $(\varnothing)^* \cdot \{p\} \cdot \varnothing \cdot \{q\} \cdot (\varnothing)^\omega$
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	$\mathsf{F}(p) \wedge \mathsf{F}(q)$	$\mathsf{F}(p \wedge \mathsf{F}q)$	$\mathsf{F}(p \wedge q)$
$(\varnothing)^\omega$	no	no	no
$(\varnothing)^* \cdot \{p\} \cdot \varnothing \cdot \{q\} \cdot (\varnothing)^\omega$	yes	yes	no
$(\varnothing)^* \cdot \{q\} \cdot \varnothing \cdot \{p\} \cdot (\varnothing)^\omega$	yes	no	no
$(\varnothing)^* \cdot \{p, q\} \cdot (\varnothing)^\omega$	yes	yes	yes

Example:

Consider $\mathcal{AP} = \{p, q\}$. What is the language of the following formula?

$$p \,\mathsf{U}\, (\mathsf{G}q)$$

Write an equivalent ω-regular expression.

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$$\mathcal{L}(p \, \mathsf{U} \, (\mathsf{G} q)) = (\{p\} \cup \{p, q\})^* \cdot (\{q\} \cup \{p, q\})^\omega$$

Example:

Consider $\mathcal{AP} = \{p, q\}$.
Is the formula $\mathsf{FX}p$ equivalent to $\mathsf{XF}p$?

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Exercise:

Consider $\mathcal{AP} = \{p, q\}$.
Write the formula $(\mathsf{G}p)\ \mathsf{U}\ q$ without using the *Until* operator, that is, using only F, G, and Boolean modalities.

- A property that can be expressed in LTL: *p holds in all and only even positions/states* $\{0, 2, 4, 6, \ldots\}$

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It is not possible that a transition is enabled infinitely many times but taken only finitely many times.

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$$\neg(\mathsf{GF}(en) \wedge \mathsf{FG}(\neg tk)) \qquad \equiv \qquad \mathsf{GF}(en) \rightarrow \mathsf{GF}(tk)$$

This is very different from $\mathsf{GF}(en \rightarrow tk)$ and from $\mathsf{G}(en \rightarrow \mathsf{F}(tk))$.

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This is very different from $\mathsf{GF}(en \rightarrow tk)$ and from $\mathsf{G}(en \rightarrow \mathsf{F}(tk))$.

Justice:

Consider $\mathcal{AP} = \{en, tk\}$.
It is never the case that a transition is always enabled but never taken.

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Justice:

Consider $\mathcal{AP} = \{en, tk\}$.
It is never the case that a transition is always enabled but never taken.

$$\neg\mathsf{F}(\mathsf{G}(en) \wedge \mathsf{G}(\neg tk))$$

This is equivalent to $\mathsf{G}(\mathsf{G}(en) \rightarrow \mathsf{F}(tk))$.

It is possible to define a *strict* version of the until as follows:

- $\sigma, i \models \phi_1 \mathsf{U}^s \phi_2$ iff $\exists j > i \,.\, \sigma, j \models \phi_2$ and $\forall i < k < j \,.\, \sigma, k \models \phi_1$

How can be encode formulas of type $\mathsf{X}\phi$ with only the strict version of the until?

Therefore, if we adopt the *strict* version, then it is possible to define LTL with the only temporal operator being the *until*.

- ... but encoding the standard until with the strict until requires more space:

$$\phi_1 \mathsf{U} \phi_2 \equiv \phi_2 \vee (\phi_1 \wedge \phi_1 \mathsf{U}^s \phi_2)$$

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How can be encode formulas of type $\mathsf{X}\phi$ with only the strict version of the until?

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Definition (Negation Normal Form)

We define the $\mathtt{nnf}(\cdot) : \mathsf{LTL} \to \mathsf{LTL}$ (*Negation Normal Form*) function as follows:

- $\mathtt{nnf}(p) = p$
- $\mathtt{nnf}(\phi_1 \land \phi_2) = \mathtt{nnf}(\phi_1) \land \mathtt{nnf}(\phi_2)$
- $\mathtt{nnf}(\phi_1 \lor \phi_2) = \mathtt{nnf}(\phi_1) \lor \mathtt{nnf}(\phi_2)$
- $\mathtt{nnf}(\mathsf{X}\phi) = \mathsf{X}(\mathtt{nnf}(\phi))$
- $\mathtt{nnf}(\phi_1 \mathsf{U} \phi_2) = (\mathtt{nnf}(\phi_1)) \mathsf{U} (\mathtt{nnf}(\phi_2))$
- $\mathtt{nnf}(\phi_1 \mathsf{R} \phi_2) = (\mathtt{nnf}(\phi_1)) \mathsf{R} (\mathtt{nnf}(\phi_2))$

For any $\phi \in \mathsf{LTL}$, the formula $\mathtt{nnf}(\phi)$ has *negation only applied to atomic propositions*.

Definition (Negation Normal Form)

We define the $\mathtt{nnf}(\cdot) : \mathsf{LTL} \rightarrow \mathsf{LTL}$ (*Negation Normal Form*) function as follows:

- $\mathtt{nnf}(\neg p) = \neg p$
- $\mathtt{nnf}(\neg\neg\phi) = \mathtt{nnf}(\phi)$
- $\mathtt{nnf}(\neg(\phi_1 \wedge \phi_2)) = \mathtt{nnf}(\neg\phi_1) \vee \mathtt{nnf}(\neg\phi_2)$
- $\mathtt{nnf}(\neg(\phi_1 \vee \phi_2)) = \mathtt{nnf}(\neg\phi_1) \wedge \mathtt{nnf}(\neg\phi_2)$
- $\mathtt{nnf}(\neg \mathsf{X}\phi) = \mathsf{X}(\mathtt{nnf}(\neg\phi))$
- $\mathtt{nnf}(\neg(\phi_1 \mathsf{U} \phi_2)) = (\mathtt{nnf}(\neg\phi_1)) \mathsf{R} (\mathtt{nnf}(\neg\phi_2))$
- $\mathtt{nnf}(\neg(\phi_1 \mathsf{R} \phi_2)) = (\mathtt{nnf}(\neg\phi_1)) \mathsf{U} (\mathtt{nnf}(\neg\phi_2))$

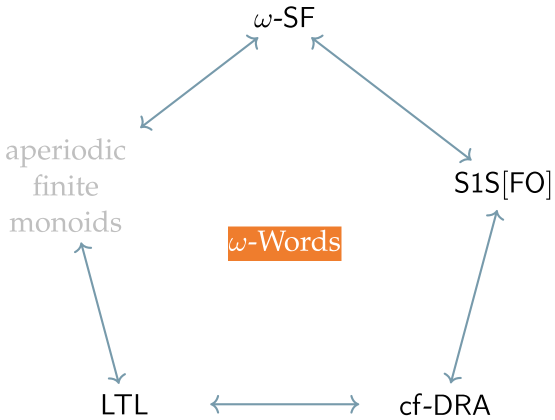
For any $\phi \in \mathsf{LTL}$, the formula $\mathtt{nnf}(\phi)$ has *negation only applied to atomic propositions*.

Theorem (Kamp's Theorem over ω-words)

- *For each* LTL *formula* ϕ, *there exists an* S1S[FO] *formula* ψ *such that* $\mathcal{L}(\phi) = \mathcal{L}(\psi)$.
- *For each* S1S[FO] *formula* ψ, *there exists an* LTL *formula* ϕ *such that* $\mathcal{L}(\psi) = \mathcal{L}(\phi)$.

Reference:

Johan Anthony Wilem Kamp (1968). ***Tense logic and the theory of linear order.***
University of California, Los Angeles



ω-SF
aperiodic finite monoids
S1S[FO]
ω-Words
LTL
cf-DRA

The syntax of LTL+P is defined as follows:

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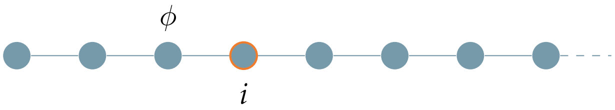
- $\mathsf{Y}\phi$ is the Yesterday operator: *the previous time point exists and it satisfies the formula* ϕ.
- $\phi_1 \,\mathsf{S}\, \phi_2$ is the Since operator: *there exists a time point in the past where* ϕ_2 *is true, and* ϕ_1 *holds since (and excluding) that point up to now.*

Shortcuts:

- Once, $\mathsf{O}\phi$: *there exists a time point in the past where* ϕ *holds.* $\mathsf{O}\phi \equiv \top \,\mathsf{S}\, \phi$.
- Historically, $\mathsf{H}\phi$: *for all time points in the past* ϕ *holds.* $\mathsf{H}\phi \equiv \neg(\mathsf{O}\neg\phi)$.

We say that σ satisfies at position i the LTL formula ϕ, written $\sigma, i \models \phi$, iff:

- $\sigma, i \models \mathsf{Y}\phi$ iff $i > 0$ and $\sigma, i-1 \models \phi$

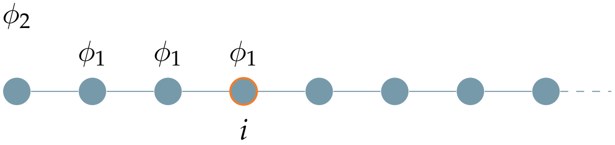


position i has a predecessor and ϕ holds at the *previous* position of i

Note: $\sigma, 0 \models \mathsf{Y}\phi$ is always false.

We say that σ satisfies at position i the LTL formula ϕ, written $\sigma, i \models \phi$, iff:

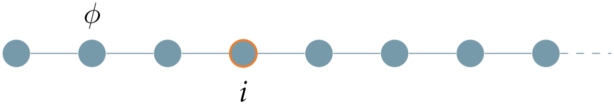
- $\sigma, i \models \phi_1 \mathsf{S}\, \phi_2$ iff $\exists j \leq i \,.\, \sigma, j \models \phi_2$ and $\forall j < k \leq i \,.\, \sigma, k \models \phi_1$



ϕ_1 holds *since* ϕ_2 held

Shortcuts:

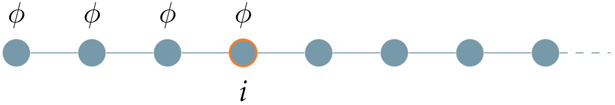
- (*once*) $\mathsf{O}\phi \equiv \top \mathsf{S}\, \phi$



ϕ *once* held

Shortcuts:

- (*historically*) $\mathsf{H}\phi \equiv \neg\mathsf{O}\neg\phi$

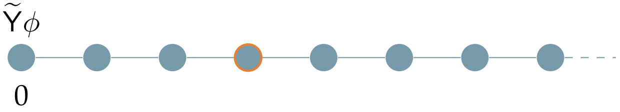


ϕ holds *always in the past*

Linear Temporal Logic
LTL Shortcuts

Shortcuts:

- (*weak yesterday*) $\widetilde{\mathsf{Y}}\phi \equiv \neg\mathsf{Y}\neg\phi$



ϕ holds at the *previous* position of i, *if any*

Note: $\sigma, i \models \widetilde{\mathsf{Y}}\bot$ is true iff $i = 0$.

Theorem

LTL+P *is expressively equivalent to* LTL.

Reference:

Dov M. Gabbay et al. (1980). "On the Temporal Analysis of Fairness". In: *Conference Record of the Seventh Annual ACM Symposium on Principles of Programming Languages, Las Vegas, Nevada, USA, January 1980.* Ed. by Paul W. Abrahams, Richard J. Lipton, and Stephen R. Bourne. ACM Press, pp. 163–173. URL: https://doi.org/10.1145/567446.567462

Theorem

LTL+P *can be exponentially more succinct than* LTL.

Reference:

Nicolas Markey (2003). "Temporal logic with past is exponentially more succinct". In: *Bull. EATCS* 79, pp. 122–128

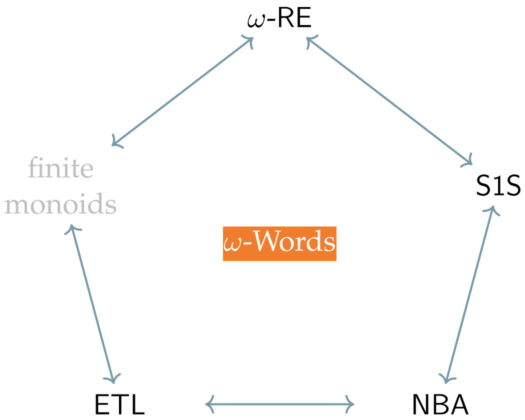
Extended Linear Temporal Logic

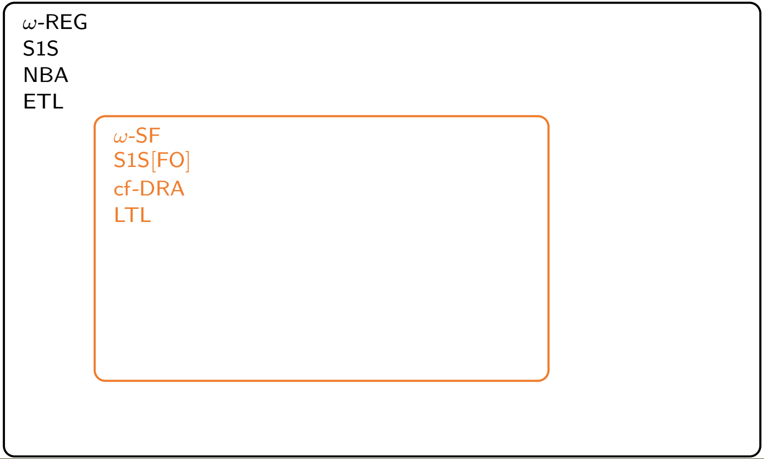
We have seen that LTL captures *star-free* ω-regular languages.
In order to capture all ω-regular languages, one can consider *Extended Linear Temporal Logic* (ETL, for short).

ETL = LTL + operators corresponding to *right-linear grammars*

Reference:

Pierre Wolper (1983). "Temporal logic can be more expressive". In: *Information and control* 56.1-2, pp. 72–99. DOI: 10.1016/S0019-9958(83)80051-5


ω-RE
finite
monoids
S1S
ω-Words
ETL
NBA


ω-REG
S1S
NBA
ETL
ω-SF
S1S[FO]
cf-DRA
LTL

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Pierre Wolper (1983). "Temporal logic can be more expressive". In: *Information and control* 56.1-2, pp. 72–99. DOI: 10.1016/S0019-9958(83)80051-5.