Verification of infinite state systems

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Go

In this part

We present other two relevant classes of transition systems:

- Rational graphs
  described by (unrestricted) **word transducers**
- Automatic graphs
  described by **left-synchronized word transducers**

We study the decidability of their FO-theories and we provide alternative representations of graphs in both classes.

Definition (Word transducer)

A **word transducer** is a tuple $\mathcal{A} = (Q, \Gamma, \Delta, I, F)$, where

- Q is a finite set of states
- Γ is a finite alphabet
- $\Delta \subseteq Q \times \Gamma^* \times \Gamma^* \times Q$
- $I \subseteq Q$ is the set of initial states
- $F \subseteq Q$ is the set of final states.

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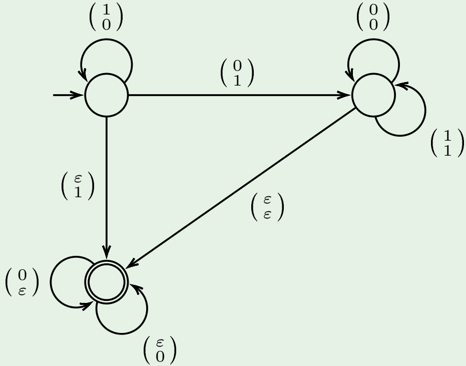
Word transducers are used as acceptors of pairs of finite words over Γ: $(u, v) \in \mathscr{L}(\mathcal{A})$ if there are $u_1, \dots, u_n$, $v_1, \dots, v_n$, $s_0, \dots, s_n$ s.t.

- $u = u_1 \cdot \ldots \cdot u_n$ and $v = v_1 \cdot \ldots \cdot v_n$
- $s_0 \in I$ and $s_n \in F$
- $(s_{i-1}, u_i, v_i, s_i) \in \Delta$ for all $1 \leq i \leq n$

$$\text{(shortly, } I \xrightarrow[\mathcal{A}]{(u,v)} F\text{)}.$$

Example

A word transducer recognizing the set of all reversed binary expansions of pairs of numbers of the form $(n, n+1)$, with $n \in \mathbb{N}$ (e.g., $(111, 0001)$)



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Definition (Rational graph)

Given an alphabet Γ, a finite set A of edge labels,
a finite state automaton $\mathcal{A}_{dom}$ over Γ, and
a tuple $(\mathcal{A}_a)_{a \in A}$ of word transducers over Γ,
the generated graph, called **rational graph**, is of the form

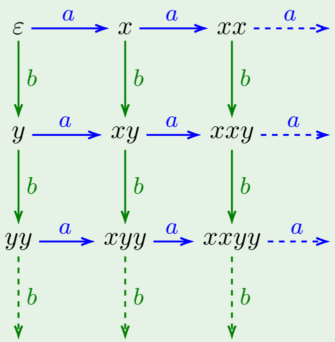
$$\mathcal{T} = (S, (\delta_a)_{a \in A})$$

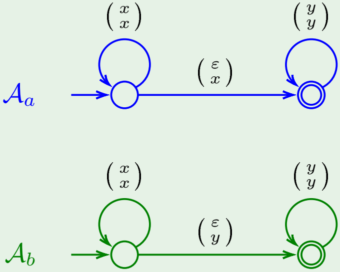
- $S := \mathscr{L}(\mathcal{A}_{dom})$ *(vertices := words accepted by $\mathcal{A}_{dom}$)*
- $\delta_a := \mathscr{L}(\mathcal{A}_a)$ *(edges := pairs accepted by transducers)*

Example

The infinite grid is a *rational graph*:

- vertices are words over $\{x, y\}$ of the form $x^i y^j$
- a-labeled edges connect a word $x^i y^j$ to $x^{i+1} y^j$
  b-labeled edges connect a word $x^i y^j$ to $x^i y^{j+1}$





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Actually, things go bad even for FO logic:

Theorem

The model checking problem for FO logic over rational graphs is undecidable.

Proof

We reduce the Post's Correspondence Problem (**PCP**) to the model checking problem for FO logic over rational graphs.

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A **PCP-instance** is a tuple $(u_1, \ldots, u_k, v_1, \ldots, v_k)$ of words.
A PCP-instance is **positive** iff there are indices $i_1, \ldots, i_n$ s.t.

$$w = u_{i_1} \cdot \ldots \cdot u_{i_n} \quad = \quad v_{i_1} \cdot \ldots \cdot v_{i_n} = w'$$

(checking if a given PCP-instance $(\bar{u}, \bar{v})$ is positive
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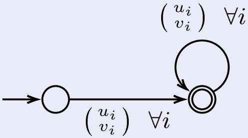
Given $(\bar{u}, \bar{v})$, we define the transducer $\mathcal{A}_{\bar{u},\bar{v}}$:



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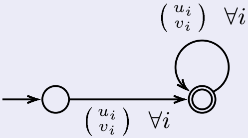

The rational graph generated by $\mathcal{A}_{\bar{u},\bar{v}}$ satisfies
$\psi := \exists\, x.\ \delta(x, x)$ iff $(\bar{u}, \bar{v})$ is a positive PCP-instance.

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$\Rightarrow$ let us consider restricted forms of word transducers:

Definition (Synchronized word transducer)

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Note: if $\mathcal{A}$ is a synchronized transducer and $(u, v) \in \mathscr{L}(\mathcal{A})$, then $|u| = |v|$

$\Rightarrow$ *every connected component in a graph generated by a synchronized transducer is finite.*

A more relaxed form of synchronization can be introduced by using a **padding symbol** $\#$ to fill up the words and achieve equal length:

Definition (Left-synchronized word transducer)

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We say that the pair $(u, v) \in \Gamma^* \times \Gamma^*$ is accepted by $\mathcal{A}$ iff either $(u \cdot \#^{|v|-|u|}, v) \in \mathscr{L}(\mathcal{A})$ or $(u, v \cdot \#^{|u|-|v|}) \in \mathscr{L}(\mathcal{A})$.

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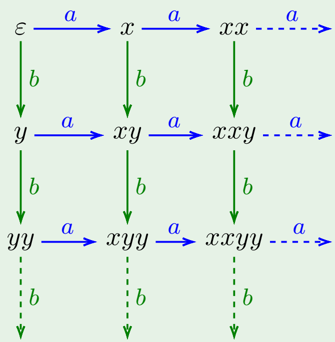
Definition (Automatic graph)

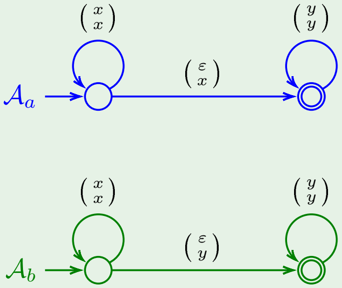
An **automatic graph** is a graph generated by a finite state automaton and a tuple of left-synchronized word transducers.

Example

The infinite grid is actually an *automatic graph*.

We simply need to put the transducers
$\mathcal{A}_a$ and $\mathcal{A}_b$ in a left-synchronized form:





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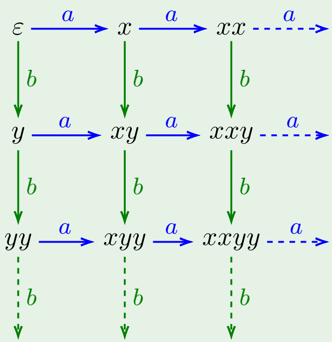

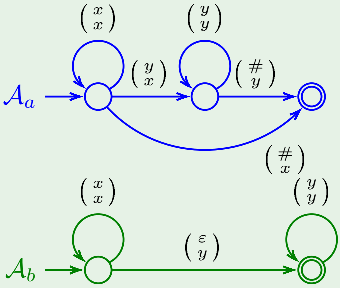

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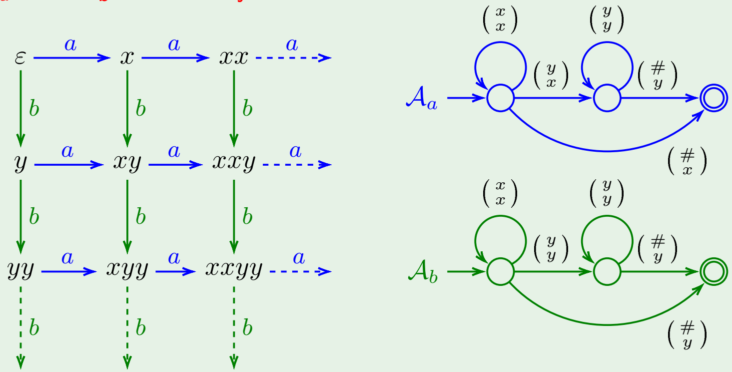

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Fortunately, we achieve decidability when considering FO logic:

Theorem (Büchi '60, Hodgson '76, Khoussainov and Nerode '94)

The model checking problem for FO logic over automatic graphs is decidable.

Proof

Let $\mathcal{T} = \big(S, (\delta_a)_{a\in A}\big)$ be an automatic graph generated by a finite state automaton $\mathcal{A}_{dom}$ and by tuple of left-synchronized transducers $(\mathcal{A}_a)_{a\in A}$.

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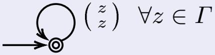
Let us prove that for any given FO-formula $\psi(x_1, \ldots, x_k)$ there is a left-synchronized transducer $\mathcal{A}_\psi$ such that, for all tuples $(u_1, \ldots, u_k) \in S^k$

$$\mathcal{T} \vDash \psi[u_1/x_1, \ldots, u_k/x_k] \quad \text{iff} \quad (u_1, \ldots, u_k) \in \mathscr{L}(\mathcal{A}_\psi)$$

Warning: here a transducer defines a relation $r \subseteq S^k$
($\Rightarrow$ it accepts tuples of vertices rather than vertex colorings)

Proof (continued)

The proof goes by induction on the structure of $\psi(x_1, \ldots, x_k)$:

- if ψ is $x_1 = x_2$, then $\mathcal{A}_\psi :=$ 

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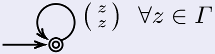
- if ψ is $x_1 = x_2$, then $\mathcal{A}_\psi :=$ 

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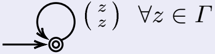
- if ψ is $x_1 = x_2$, then $\mathcal{A}_\psi :=$ 

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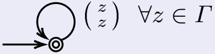
- if ψ is $x_1 = x_2$, then $\mathcal{A}_\psi :=$ 

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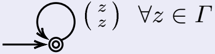
- if ψ is $x_1 = x_2$, then $\mathcal{A}_\psi :=$ 

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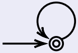
- if ψ is $x_1 = x_2$, then $\mathcal{A}_\psi :=$ 

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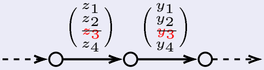
- if ψ is $\exists\, x_i .\ \varphi(x_1, \ldots, x_i, \ldots, x_m)$, then $\mathcal{A}_\psi$ is obtained from $\mathcal{A}_\varphi$ by first intersecting with $\mathcal{A}_{dom}$ and then removing the i-th symbol in each transition



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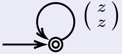
- if ψ is $x_1 = x_2$, then $\mathcal{A}_\psi :=$ 

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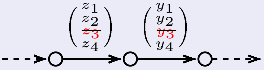

$\Rightarrow$ The model checking problem for FO logic is reduced to the emptiness problem for word transducers.

Proposition

The reachability problem over automatic graphs is undecidable.

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Proof

Consider a generic Turing machine M where

- configurations are encoded by words $a_1 \dots a_{m-1} q a_m \dots a_n$
- transitions are of the following forms

$$\begin{array}{c} a_1 \dots a_{m-1} q a_m \dots a_n \\ \downarrow \\ a_1 \dots a_{m-1} q' a'_m \dots a_m \end{array} \qquad \begin{array}{c} a_1 \dots a_{m-1} q a_m \dots a_n \\ \downarrow \\ a_1 \dots a_{m-2} q' a_{m-1} a'_m \dots a_m \end{array} \qquad \begin{array}{c} a_1 \dots a_{m-1} q a_m \dots a_n \\ \downarrow \\ a_1 \dots a_{m-1} a'_m q' a_{m+1} \dots a_m \end{array}$$

Note that

- the words that encode a valid configuration of M can be recognized by a suitable *finite state automaton*
- the transition relations can be recognized by a suitable *left-synchronized word transducer*.

$\Rightarrow$ the transition graph of any Turing machine is automatic.

The previous definitions and results can be easily generalized to **relational structures** having relations of arbitrary arities:

Definition (Automatic structure)

An **automatic structure** is a relational structure $\mathcal{R} = \big(S, r_1, \ldots, r_m\big)$, where

- $S \subseteq \Gamma^*$ is a regular language of finite words
- $r_i \subseteq (\Gamma^*)_i^k$ is a k_i-ary relation over finite words recognized by a left-synchronized word transducer working with k_i-tuples of letters.

Theorem

The model checking problem for FO logic over automatic structures is decidable.

Example

Building on previous ideas, one can show that the **Presburger arithmetic** (i.e., the FO-theory of $(\mathbb{N}, +)$) is decidable.

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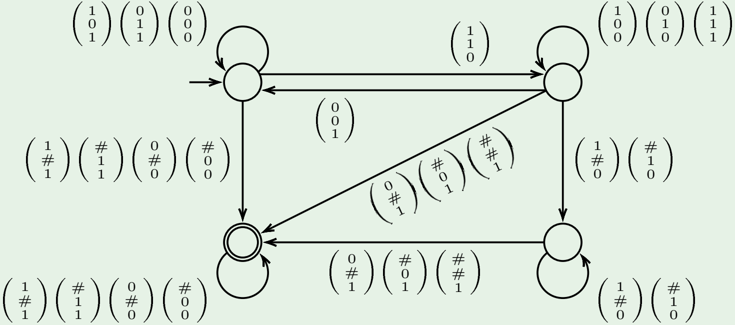
It is sufficient to represent

- the natural numbers by their reversed binary expansions
- the *ternary* relation $+$ by the left-synchronized transducer



Alternative characterizations of automatic/rational graphs/structures have been given in the literature:

Theorem (Morvan '00)

A relational graph $\mathcal{T}$ is rational iff it can be obtained from the infinite binary tree via an inverse linear mapping and a rational restriction.

(an inverse mapping h^{-1} is **linear** *if, for every label b, $h(b)$ is a linear context-free language, namely, generated by a grammar with rules of the form $z \rightarrow \varepsilon$ and $z \rightarrow \bar{u} \cdot z' \cdot v$).*

$\Rightarrow$ Special forms of inverse linear mappings characterize the automatic graphs.

Theorem (Blumensath '99)

A relational structure $\mathcal{S}$ is automatic iff it can be obtained from $(\mathbb{B}^*, \delta_0, \delta_1, \sqsubseteq, L)$ via a FO-interpretation endowed with a congruence $\sim$ that defines the vertices of $\mathcal{S}$ as $\sim$-classes.

(the structure $(\mathbb{B}^*, \delta_0, \delta_1, \sqsubseteq, L)$ is the infinite binary tree expanded with the **ancestor relation** $\sqsubseteq$ *and the* **equi-level relation** L*).*

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Theorem (Elgot and Rabin '66, Rubin '04)

A relational structure $\mathcal{S}$ is automatic iff it can be obtained from $\mathcal{L} := (\mathbb{N}, \delta)$ via a **WMSO-to-FO-interpretation**.

(a WMSO-to-FO-interpretation is an interpretation where free variables are instantiated by finite sets
$\Rightarrow$ the vertices of $\mathcal{S}$ are subsets of the domain of $\mathcal{L}$).

Richer (automatic-like) structures can be defined using:

- transducers over infinite words
  $\Rightarrow$ this leads to ω-**automatic structures** (e.g., $(\mathbb{R}, +)$)
- transducers over finite trees
  $\Rightarrow$ this leads to **tree-automatic structures**
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- WMSO-to-FO-interpretations over the binary tree
  $\Rightarrow$ this leads to tree-automatic structures as well
- WMSO-to-FO-interpretations over Caucal graphs
  $\Rightarrow$ this leads to a (strictly increasing)
  hierarchy of automatic structures

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