Verification of infinite state systems

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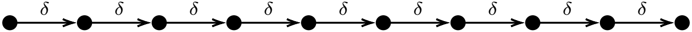
Go

In this part

We present basic results and techniques related to the decidability of *MSO-theories* of infinite transition systems.

- Automaton-based approaches
  - decidability of the MSO-theory of a finite (discrete) line
  - **Büchi's Theorem**
    (decidability of the MSO-theory of the semi-infinite line)
  - **Rabin's Theorem**
    (decidability of the MSO-theory of the infinite binary tree)
- Transformational approaches
  - **interpretations**, **inverse mappings**, **markings**
  - **unfoldings**
  - **Caucal hierarchy**

Consider a **finite line** $\mathcal{L}_n = \big(\{1, ..., n\}, \delta\big)$



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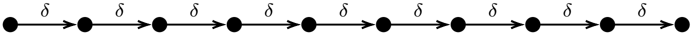

We expand $\mathcal{L}_n$ by **unary predicates** $P_1, ..., P_m \subseteq \{1, ..., n\}$
and we obtain a colored line $\mathcal{L}_{n,\bar{P}} = \big(\{1, ..., n\}, \delta, P_1, ..., P_m\big)$

The automaton-based approach – The finite line

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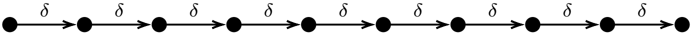

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We encode $\mathcal{L}_{n,\bar{P}}$ by a **finite word** $w_{n,\bar{P}}$ over $\mathbb{B}^m = \{0, 1\}^m$ such that

$$|w_{n,\bar{P}}| = n$$

$$w_{n,\bar{P}}[i] = (b_1, \ldots, b_m) \qquad \text{where } b_j = \begin{cases} 1 & \text{if } i \in P_j \\ 0 & \text{if } i \notin P_j \end{cases}$$

Example

The colored line $\mathcal{L}_{10,\bar{P}} = (\{1, \ldots, 10\}, \delta, P_{even}, P_{prime})$

is encoded by $w_{10,\bar{P}} = \binom{0}{0}\binom{1}{1}\binom{0}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{0}\binom{1}{0}$.

W.l.o.g. we can think of each *FO-variable* x in an MSO-formula ψ as an *MSO-variable* X *which stands for the singleton* $\{x\}$.

$\Rightarrow$ we can get rid of FO-variables
(by taking $\delta(X_i, X_j)$, with X_i, X_j singletons, and $X_i \subseteq X_j$ as atomic formulas)

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We evaluate an MSO-formula ψ with free variables $X_1, ..., X_m$ over the finite line expanded by any tuple of unary predicates $P_1, ..., P_m$:

$$\mathcal{L}_{n,\bar{P}} \overset{?}{\models} \psi[P_1/X_1, ..., P_m/X_m]$$

The above problem can be reduced to the (decidable) acceptance problem for finite state automata.

Theorem (Acceptance problem)

For any MSO-formula ψ with free variables $X_1, \ldots, X_m$, one can compute a **finite state automaton** $\mathcal{A}_\psi$ *over* $\mathbb{B}^m$ *such that, for every n-length colored line* $\mathcal{L}_{n,\bar{P}}$

$$\mathcal{L}_{n,\bar{P}} \models \psi[P_1/X_1, \ldots, P_m/X_m] \quad \textit{iff} \quad w_{n,\bar{P}} \in \mathscr{L}(\mathcal{A}_\psi)$$

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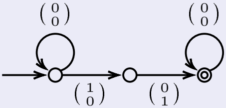
Note: if ψ contains no free variables ($m = 0$), then $\mathcal{A}_\psi$ is an **input-free** automaton.

Corollary

The MSO-theory of a finite line $\mathcal{L}_n = (\{1, \ldots, n\}, \delta)$ *is reducible to the (decidable)* **acceptance problem** *for finite state automata.*

Proof of the theorem

By *induction on the structure* of the formula ψ:

- if $\psi = \delta(X_1, X_2)$, then $\mathcal{A}_\psi :=$ 

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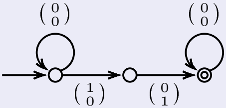
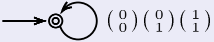
- if $\psi = \delta(X_1, X_2)$, then $\mathcal{A}_\psi :=$ 

- if $\psi = X_1 \subseteq X_2$, then $\mathcal{A}_\psi :=$ 

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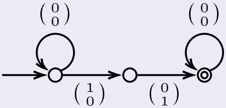
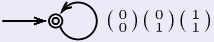
- if $\psi = \delta(X_1, X_2)$, then $\mathcal{A}_\psi :=$ 

- if $\psi = X_1 \subseteq X_2$, then $\mathcal{A}_\psi :=$ 

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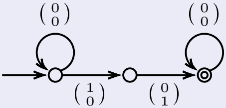
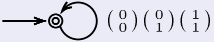
- if $\psi = \delta(X_1, X_2)$, then $\mathcal{A}_\psi :=$ 

- if $\psi = X_1 \subseteq X_2$, then $\mathcal{A}_\psi :=$ 

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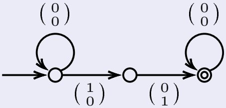
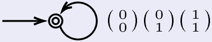
- if $\psi = \delta(X_1, X_2)$, then $\mathcal{A}_\psi :=$ 

- if $\psi = X_1 \subseteq X_2$, then $\mathcal{A}_\psi :=$ 

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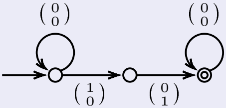
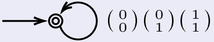
- if $\psi = \delta(X_1, X_2)$, then $\mathcal{A}_\psi :=$ 

- if $\psi = X_1 \subseteq X_2$, then $\mathcal{A}_\psi :=$ 

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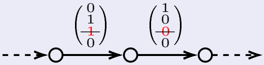
- if $\psi = \neg\varphi$, then $\mathcal{A}_\psi$ is the complement automaton of $\mathcal{A}_\varphi$
- if $\psi = \exists\, X_i.\ \varphi(X_1, \ldots, X_i, \ldots, X_m)$, then $\mathcal{A}_\psi$ is obtained from $\mathcal{A}_\varphi$ by removing the i-th component of each input symbol



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What about the **semi-infinite line** $\mathcal{L}_\omega = (\mathbb{N}, \delta)$?

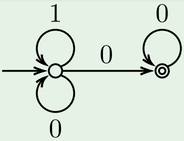
Basic ingredients

- We need to use **infinite words**, rather than finite ones, to encode expansions of $\mathcal{L}_\omega$ by unary predicates
- We need to introduce a suitable class of automata working on infinite words: **Büchi automata**!

Definition (**Büchi automaton**)

It is a non-deterministic finite state automaton that accepts an infinite word w iff there is a run ρ on w such that $\mathcal{I}nf(\rho) \cap F \neq \emptyset$ (‘*ρ contains at least one final state that occurs infinitely often*’).

Example



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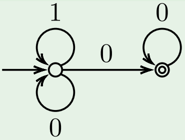

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Lemma (Büchi ’62)

Büchi automata are effectively closed under union, intersection, complementation, and projection.

Theorem (Büchi '62)

For any MSO-formula ψ *with free variables* $X_1, \ldots, X_m$, *one can compute a Büchi automaton* $\mathcal{A}_\psi$ *over* $\mathbb{B}^m$ *such that, for every tuple of unary predicates* $P_1, \ldots, P_m \subseteq \mathbb{N}$

$$\mathcal{L}_{\omega,\bar{P}} \models \psi[P_1/X_1, \ldots, P_m/X_m] \quad \textit{iff} \quad w_{\omega,\bar{P}} \in \mathscr{L}(\mathcal{A}_\psi)$$

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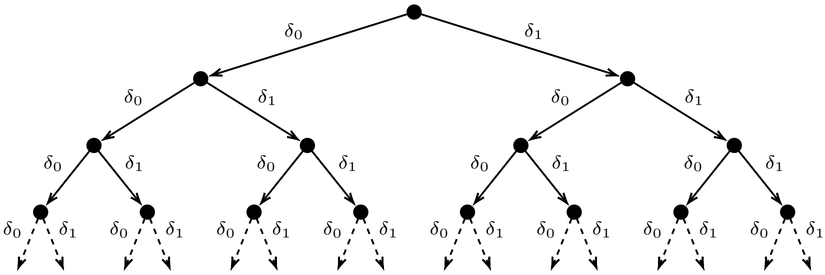
Note: if ψ contains no free variables ($m = 0$)
then $\mathcal{A}_\psi$ is input free and it recognizes
either the empty language or a singleton.

Corollary

The MSO-theory of the semi-infinite line $\mathcal{L} = (\mathbb{N}, \delta)$ is reducible to the (decidable) **emptiness problem** *for Büchi automata.*

What about MSO-theories of branching structures, in particular, of the **infinite binary tree** $\mathcal{T}_2 = (\mathbb{B}^*, \delta_0, \delta_1)$?

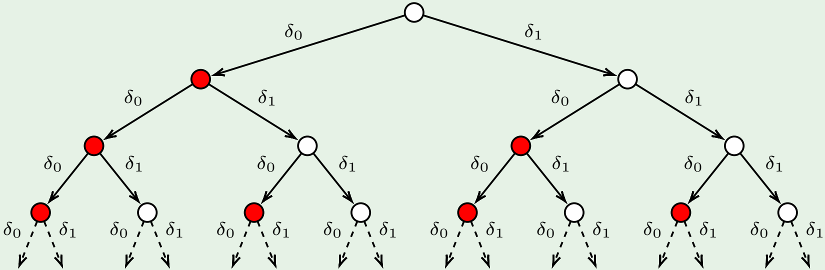


In analogy to the previous cases, we shall describe an automaton-based method to decide the MSO-theory of $\mathcal{T}_2$.

Now, expansions of the infinite binary tree $\mathcal{T}_2$ with unary predicates $P_1, \ldots, P_m \subseteq \mathbb{B}^*$ are encoded by $\mathbb{B}^m$-colored trees.

Example

The expanded tree $\left(\mathbb{B}^*, \delta_0, \delta_1, P\right)$, where $P = \{\textit{left successors}\}$, is encoded by the colored tree $\mathcal{T}_{2,P}$



We need a suitable class of automata running on *colored trees*, rather than words: **Rabin tree automata**!

Definition (Rabin tree automaton)

A **Rabin tree automaton** is a tuple $\mathcal{A} = \big(Q, C, \Delta, q_0, \{(G_1, F_1), \ldots, (G_k, F_k)\}\big)$, where:

- Q is a finite set of states
- C is a finite set of vertex colors (e.g., $\mathbb{B}^m$)
- $\Delta \subseteq Q \times C \times Q \times Q$ is a transition relation
- $q_0 \in Q$ is the initial state
- for all $1 \leq i \leq k$, (G_i, F_i) is an **accepting pair**, with $G_i, F_i \subseteq Q$.

How does a Rabin tree automaton $\mathcal{A}$ accept a colored tree?

Definition (Successful run)

A **successful run** of $\mathcal{A}$ on an infinite binary C-colored tree $\mathcal{T}$ is an infinite binary Q-colored tree $\mathcal{R}$ such that:

- $\mathcal{R}(\varepsilon) = q_0$
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- for every vertex v, $\big(\mathcal{R}(v), \mathcal{T}(v), \mathcal{R}(v \cdot 0), \mathcal{R}(v \cdot 1)\big) \in \Delta$
  '*if* $\mathcal{A}$ *lies at* v *with color* $c = \mathcal{T}(v)$ *and state* $q = \mathcal{R}(v)$, $\mathcal{A}$ *can associate the states* $q' = \mathcal{R}(v \cdot 0), q'' = \mathcal{R}(v \cdot 1)$ *with the two successors of* v *iff* (q, c, q', q'') *is a valid transition*'

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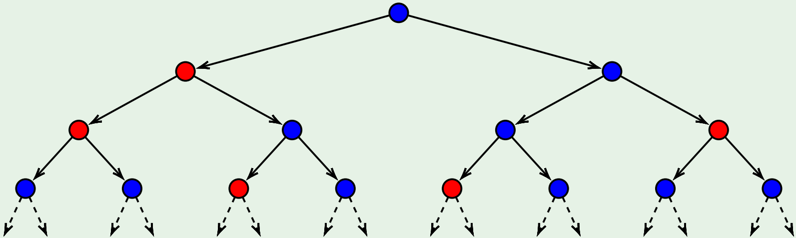
- for every infinite path π, there is $1 \leq i \leq k$ such that $\mathcal{I}nf(\mathcal{R}|\pi) \cap G_i \neq \emptyset$ and $\mathcal{I}nf(\mathcal{R}|\pi) \cap F_i = \emptyset$
  '*at least one state of* G_i *occurs infinitely often in* $\mathcal{R}$ *along* π' and '*all states of* F_i *occur only finitely often in* $\mathcal{R}$ *along* π'

Example

Consider the {*red*, *blue*}-colored tree



and the Rabin tree automaton having

- two states, r and b, that keep track of which color was seen last
- transitions (r, red, r, r), (b, red, r, r),
  $(r, blue, b, b)$, $(b, blue, b, b)$
- a single accepting pair (G_1, F_1), with $G_1 = \{b\}$, $F_1 = \{r\}$

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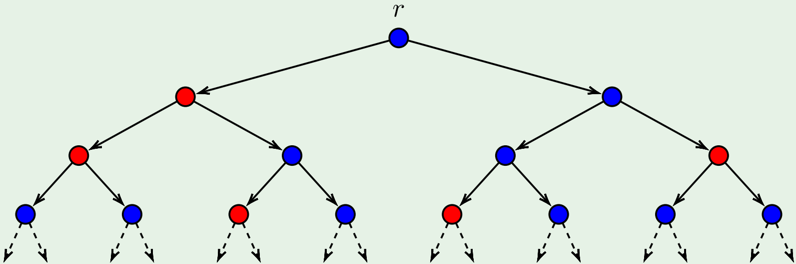

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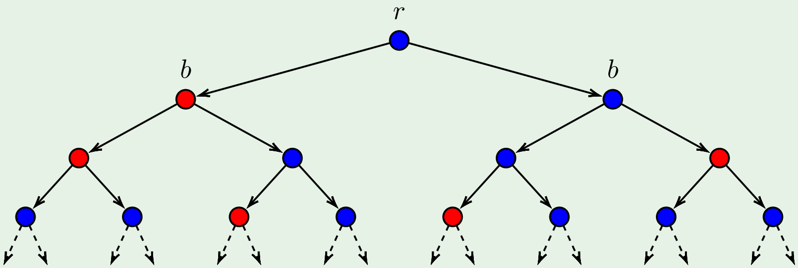

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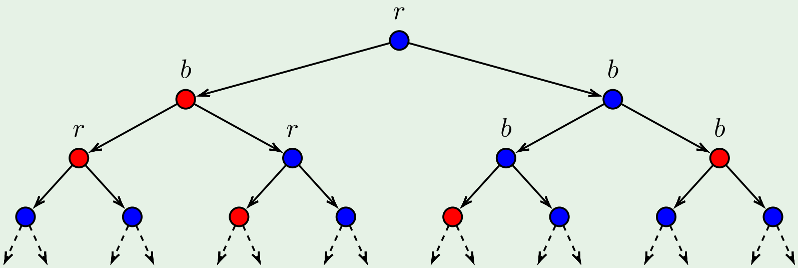

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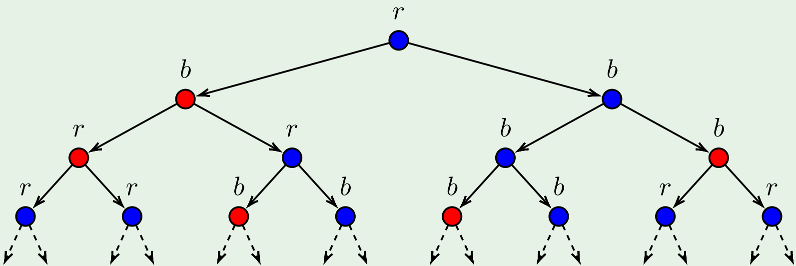

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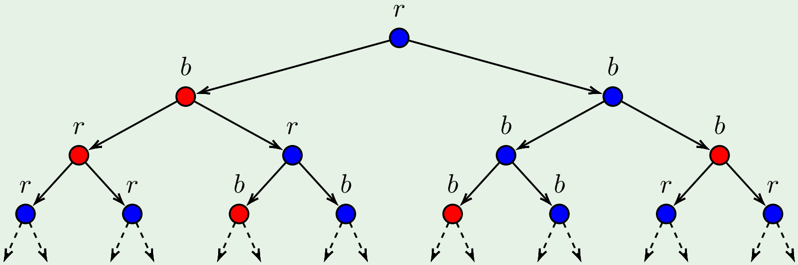

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- a single accepting pair (G_1, F_1), with $G_1 = \{b\}$, $F_1 = \{r\}$

$\Rightarrow$ $\mathcal{A}$ accepts those trees whose paths encompass *only finitely many red-colored vertices*

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$$(\mathbb{B}^*, \delta_0, \delta_1, \bar{P}) \vDash \psi[P_1/X_1, \ldots, P_m/X_m] \quad \textit{iff} \quad \mathcal{T}_{2,\bar{P}} \in \mathscr{L}(\mathcal{A}_\psi)$$

Corollary

The MSO-theory of the infinite binary tree $(\mathbb{B}^*, \delta_0, \delta_1)$ is reducible to the (decidable) **emptiness problem** *for Rabin tree automata.*

Summing up, we have the following decidability results:

MSO-theory	**Model**	**Automata**
S1S	finite line	finite state automata
S1S	semi-infinite line	Büchi automata
S2S	infinite tree	Rabin tree automata

Summing up, we have the following decidability results:

MSO-theory	**Model**	**Automata**
S1S	finite line	finite state automata
S1S	semi-infinite line	Büchi automata
S2S	infinite tree	Rabin tree automata

What next?
to find infinite transition systems *in between* the infinite tree and the infinite grid that enjoy a decidable MSO-theory.

Basic ingredients of the transformational approach:

1. We start from a structure $\mathcal{T}$ that enjoys a decidable MSO-theory (e.g., the infinite binary tree)
2. We apply to $\mathcal{T}$ a suitable transformation that preserves the decidability of MSO-theories (e.g., interpretation), thus obtaining a new (decidable) structure $\mathcal{T}'$
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A noticeable class of transformations that preserve decidability of MSO-theories is the class of **MSO-compatible transformations**.

Definition (MSO-compatible transformation)

A transformation t for transition systems is said to be **MSO-compatible** if for any transition system $\mathcal{T}$ and any MSO-sentence ψ over $t(\mathcal{T})$, one can compute an MSO-sentence $\overleftarrow{\psi}$ over $\mathcal{T}$ (which depends on ψ only) such that

$$t(\mathcal{T}) \models \psi \quad \text{iff} \quad \mathcal{T} \models \overleftarrow{\psi}$$

Intuitively, MSO-compatibility allows one to map a property about $t(\mathcal{T})$ into a corresponding property about $\mathcal{T}$

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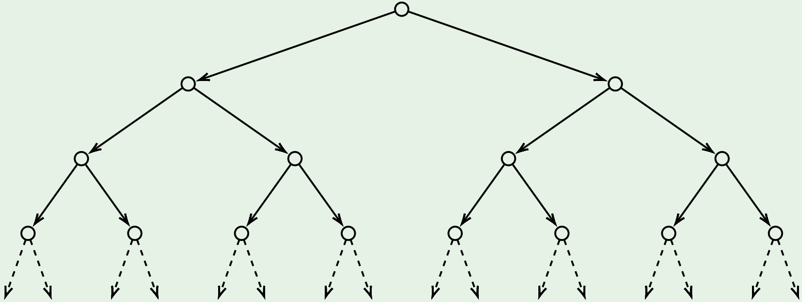
The first transformation we consider is the MSO-interpretation.

Example (MSO-interpretation)

Consider the infinite binary tree $\mathcal{T}_2$.



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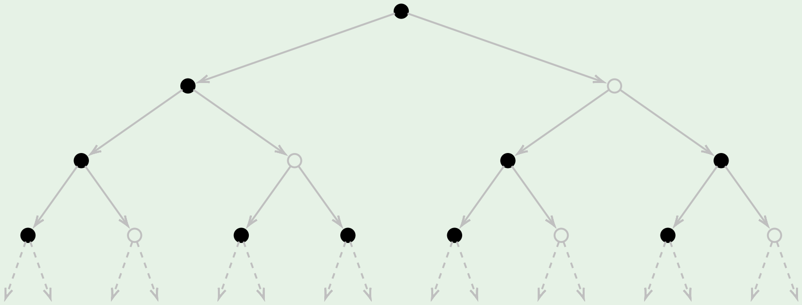

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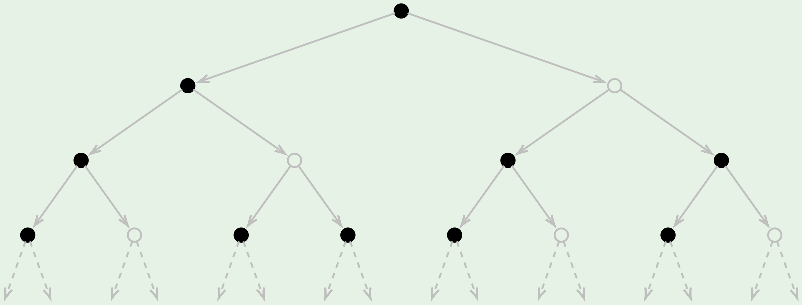

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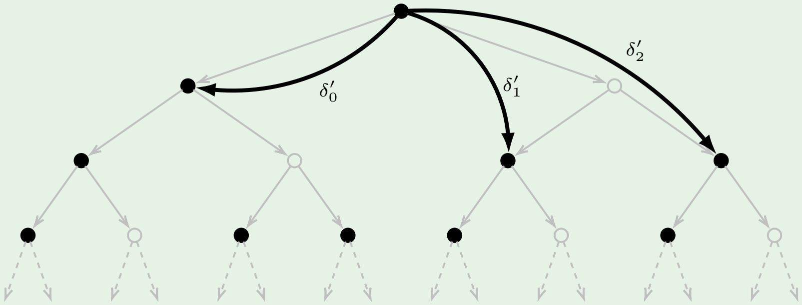

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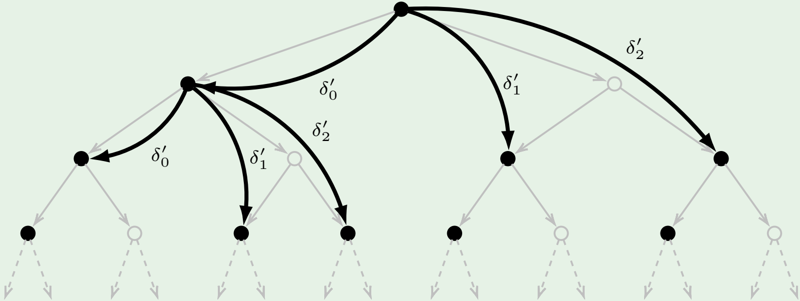

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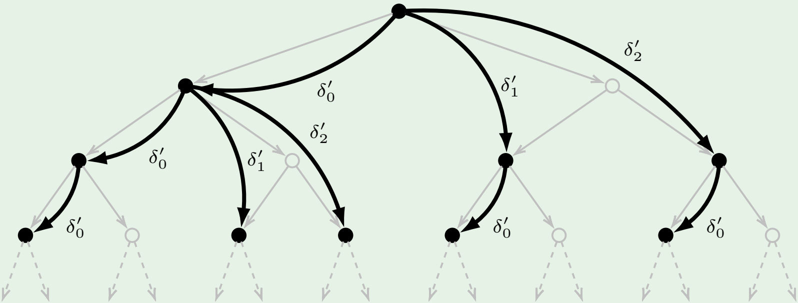

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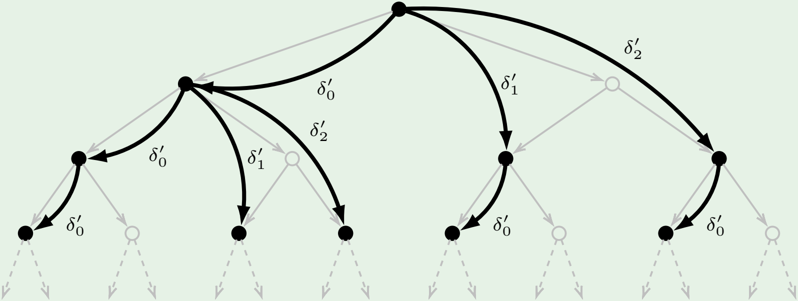

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  $\psi_{\delta'_0}(x, y) := \delta_0(x, y) \qquad \psi_{\delta'_{i+1}}(x, y) := (\delta_1 \circ \delta_i)(x, y)$

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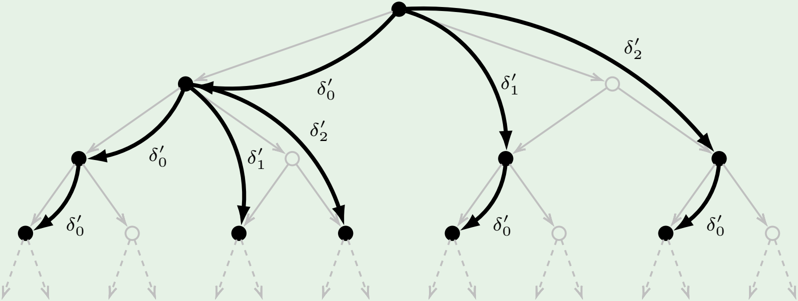

Any MSO-formula ϕ over $\mathcal{T}_3$ can be mapped into a corresponding formula $\overleftarrow{\phi}$ over $\mathcal{T}_2$.

For instance, the formula $\phi = \forall\, x.\ \exists\, y.\ \delta_2(x, y)$ becomes
$\overleftarrow{\phi} = \forall\, x.\ (\psi_{dom}(x)\ \rightarrow\ \exists\, y.\ (\psi_{dom}(y)\ \wedge\ \psi_{\delta'_2}(x, y)))$

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An **MSO-interpretation** is a tuple of MSO-formulas

$$\underbrace{\psi_{dom}(x)}_{\text{domain formula}} \quad \underbrace{\psi_{b_1}(x,y) \quad \dots \quad \psi_{b_k}(x,y)}_{\text{edge formulas}} \quad \underbrace{\psi_{d_1}(x) \quad \dots \quad \psi_{d_m}(x)}_{\text{color formulas}}$$

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Theorem

MSO-interpretations are MSO-compatible.

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Proof (sketch)

Rewrite a given MSO-sentence ψ over $\mathcal{T}'$ into a corresponding MSO-sentence $\overleftarrow{\psi}$ over $\mathcal{T}$:

- if $\psi = \delta'_{b_i}(x, y)$, then $\overleftarrow{\psi} := \psi_{b_i}(x, y)$
- if $\psi = P_{d_i}(x)$, then $\overleftarrow{\psi} := \psi_{d_i}(x)$
- if $\psi = \exists\, x.\ \varphi(x)$, then $\overleftarrow{\psi} := \exists\, x.\ (\psi_{dom}(x)\ \wedge\ \overleftarrow{\varphi}(x))$
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Corollary

The infinite ternary tree $\mathcal{T}_3$ has a decidable MSO-theory.

Most MSO-formulas *with two free variables* can be conveniently written as **regular** (path) **expressions**:

- let A and C be disjoint sets of edge labels and vertex colors
- for each label $a \in A$, we introduce an **inverse label** $\bar{a}$ denoting a-labeled edges traversed in backward direction
- we describe paths traversing edges in both directions by words over the alphabet $A \cup \bar{A} \cup C$

Example

The set of paths on an A-labeled C-colored transition system that

- start from a vertex with color c
- traverse a sequence of edges labeled with a
- reach a vertex colored with c'
- and finally traverse a edge labeled with a' in backward direction

is described by the regular expression $c \cdot a^* \cdot c' \cdot \bar{a}'$

Fact

Regular path expressions are **shorthands**
of (a subset of) MSO-formulas with two free variables.

For instance:

- the expression a
  abbreviates $\psi(x,y) := \delta_a(x,y)$
- the expression $a \cdot a'$
  abbreviates $\psi(x,y) := \exists\, z.\ \delta_a(x,z) \ \wedge\ \delta_{a'}(z,y)$
- the expression $a + \overline{a}'$
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- the expression a^*
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Note: the converse is not true in the general case (e.g., $\psi(x,y) := \nexists\, z.\ (\delta_0(z,x) \ \vee\ \delta_1(z,y)) \ \wedge\ \nexists\, z.\ (\delta_0(y,z) \ \vee\ \delta_1(y,z))$).

However, regular path expressions suffice for most cases.

For the usual MSO-interpretations, we can replace every edge formula $\psi_b(x, y)$ with a regular path expression, namely, a regular language over $A \cup \bar{A} \cup C$.

Definition (Inverse rational mapping)

A **rational mapping** is a function $h : B \rightarrow \mathscr{P}((A \cup \bar{A} \cup C)^*)$ such that $\forall b \in B$, $h(b)$ is a regular language over $A \cup \bar{A} \cup C$.

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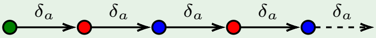
The 'inverse' h^{-1} of h (**inverse rational mapping**) can be applied to an A-labeled transition system $\mathcal{T}$ to produce the B-labeled transition system $h^{-1}(\mathcal{T})$ such that:

- $h^{-1}(\mathcal{T})$ has the same vertices of $\mathcal{T}$
- (u, v) is a b-labeled edge of $h^{-1}(\mathcal{T})$ iff $\mathcal{T}$ contains a w-**marked path** from u to v, for some $w \in h(b)$.

Example (Inverse rational mapping)

Colored semi-infinite line $\mathcal{L}_\omega = \left(\mathbb{N}, \delta_a, P_{even}, P_{odd}, P_0\right)$



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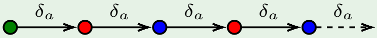
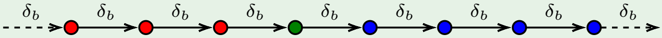


Colored bi-infinite line $\mathcal{L}_{-\omega,\omega} = (\mathbb{Z}, \delta_b, P_{pos}, P_{neg}, P_0)$



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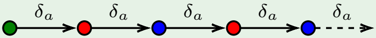

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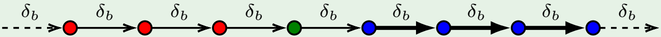

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- between *blue*-colored vertices

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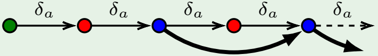

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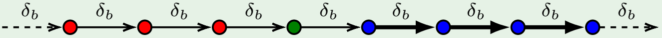

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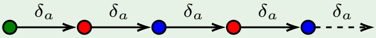

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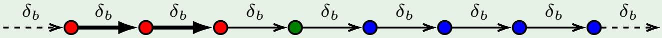

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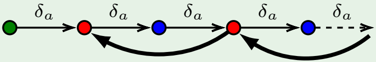

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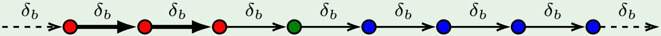

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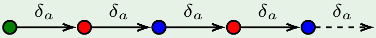

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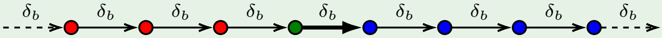

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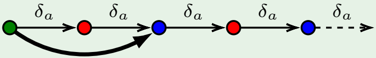

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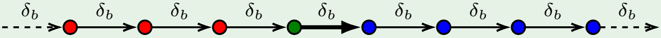

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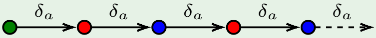

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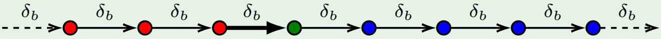

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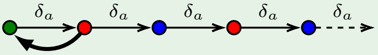
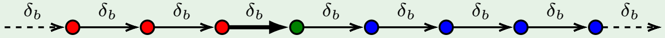
- between *red*-colored vertices: $red \cdot \bar{a} \cdot \bar{a}$
- from the *green*-colored vertex to the *blue*-colored vertex: $0 \cdot a \cdot a$
- from the *red*-colored vertex to the *green*-colored one

Example (Inverse rational mapping)

Colored semi-infinite line $\mathcal{L}_\omega = (\mathbb{N}, \delta_a, P_{even}, P_{odd}, P_0)$



Colored bi-infinite line $\mathcal{L}_{-\omega,\omega} = (\mathbb{Z}, \delta_b, P_{pos}, P_{neg}, P_0)$



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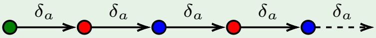

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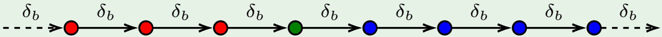

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$\Rightarrow \quad h(b) = blue \cdot a \cdot a + red \cdot \bar{a} \cdot \bar{a} + 0 \cdot a \cdot a + \bar{a} \cdot 0$

Similarly, color formulas can be replaced with rational markings:

Definition (Rational marking)

A **rational marking** is a function $k : D \rightarrow \mathscr{P}((A \cup \bar{A} \cup C)^*)$ such that $\forall d \in D$, $k(d)$ is a regular language over $A \cup \bar{A} \cup C$.

It induces a **recoloring** of the rooted transition system $\mathcal{T}$ as follows:

- for each $d \in D$, the color d is assigned
  to all vertices v of $\mathcal{T}$ such that there is
  a w-marked path from the root to v, for some $w \in k(d)$.

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Example (Rational marking)

The $\{pos, neg, 0\}$-coloring of the bi-infinite line is encoded in the rooted semi-infinite line $\mathcal{L}_\omega = (\mathbb{N}, \delta_a, P_0)$ via the rational marking k such that

$$k(pos) = 0 \cdot a \cdot a \cdot (a \cdot a)^* \quad k(neg) = 0 \cdot a \cdot (a \cdot a)^* \quad k(0) = 0$$

Finally, domain formulas can be replaced with rational restrictions:

Definition (Rational restriction)

A **rational restriction** is specified by a regular language L over $A \cup \bar{A} \cup C$.

It induces a **restriction** $\mathcal{T}|_L$ of the rooted transition system $\mathcal{T}$ as follows:

- for each vertex v of $\mathcal{T}$, v belongs to $\mathcal{T}|_L$ iff there is a w-marked path from the root to v, for some $w \in L$.

Another useful transformation is the unfolding:

Definition (Unfolding)

The **unfolding** of a rooted transition system $\mathcal{T}$ is the tree $\mathcal{U}nf(\mathcal{T})$ such that:

- the vertices of $\mathcal{U}nf(\mathcal{T})$ are all and only the finite paths in $\mathcal{T}$ originating from the root
- the edges of $\mathcal{U}nf(\mathcal{T})$ are given by the path-extension relation, namely, if π is path in $\mathcal{T}$ from the root and π' is the extension of π with an a-labeled edge, then (π, π') is an a-labeled edge in $\mathcal{U}nf(\mathcal{T})$
- the color of a vertex in $\mathcal{U}nf(\mathcal{T})$ is the color of the target vertex of the corresponding path in $\mathcal{T}$

Example (unfoldings)

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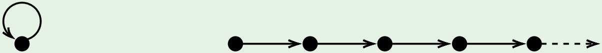

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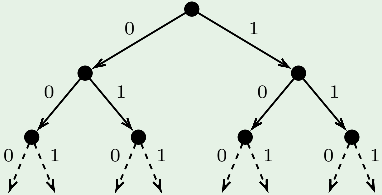

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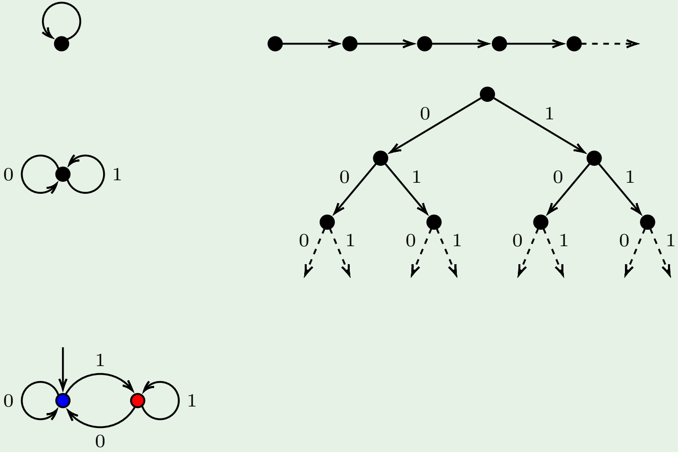

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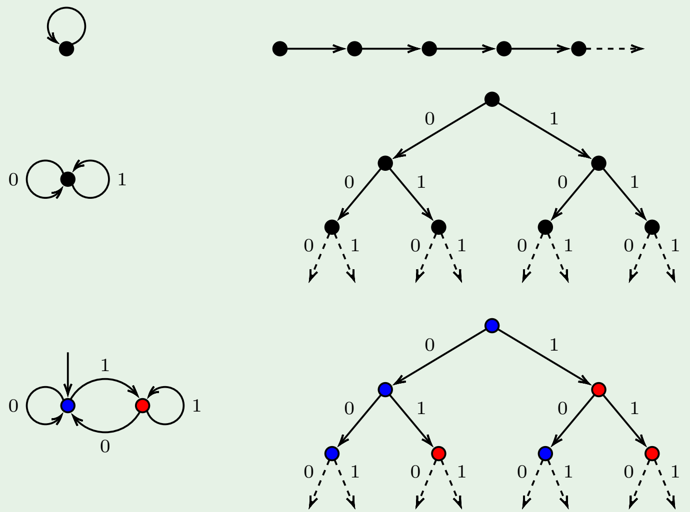


Theorem (Semenov-Muchnik '84 – proved by Walukiewicz '96)

The unfolding operation is MSO-compatible.

$\Rightarrow$ Since finite transition systems enjoy decidable MSO-theories, Muchnik's Theorem subsumes Büchi's and Rabin's theorems (in fact, the proof is strongly based on Rabin's Theorem...)

We know that MSO-interpretation and unfolding preserve the decidability of MSO-theories of transition systems.

We can start from finite (hence decidable) graphs and iterate MSO-interpretation and unfolding:

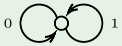
$$\begin{aligned} \mathit{Graph}_0 &:= \{\text{finite rooted graphs}\} \\ \mathit{Tree}_{n+1} &:= \{\text{trees obtained by unfolding graphs in } \mathit{Graph}_n\} \\ \mathit{Graph}_n &:= \{\text{rooted graphs obtained via} \\ &\qquad \text{interpretation from trees in } \mathit{Tree}_n\} \end{aligned}$$

(e.g., $\mathit{Tree}_1 = \{\text{regular trees}\}$)

$\Rightarrow$ a **hierarchy** of graphs with decidable MSO-theories arises (this is commonly known as **Caucal's hierarchy**)

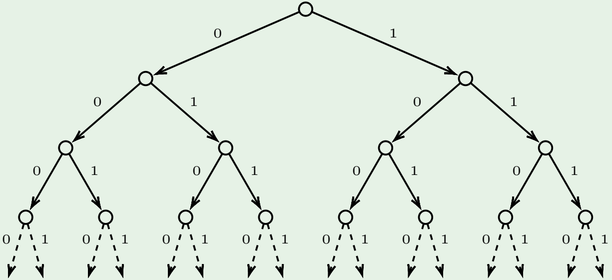
Example

We start from the finite graph



Example

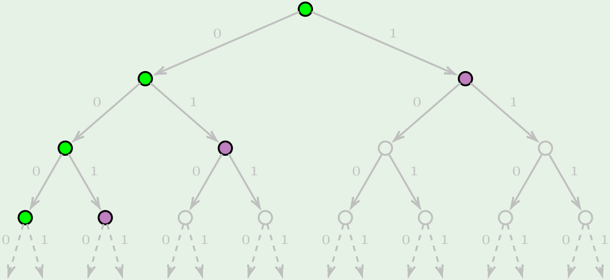
We unfold it, obtaining the infinite binary tree ...



Example

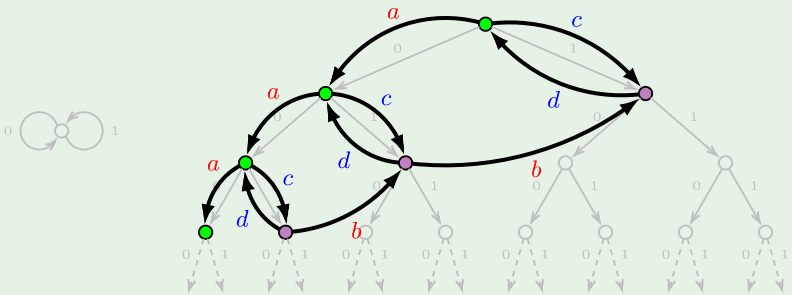
... we apply the rational marking

$k(A) = 0^*$, $k(B) = 0^*1$



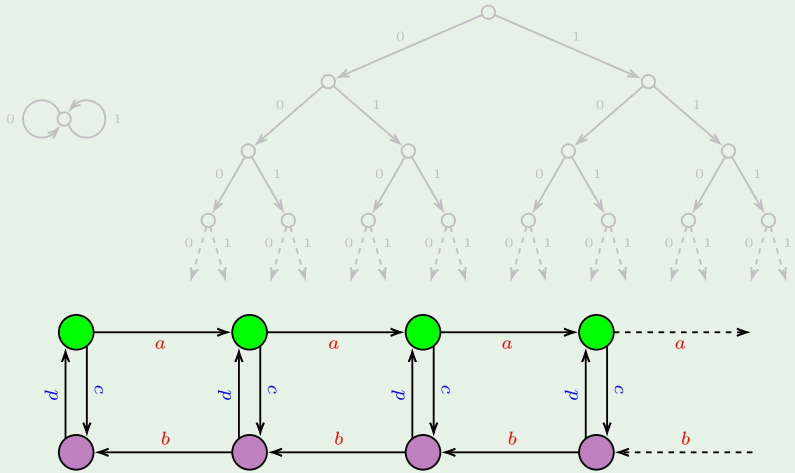
Example

... the inverse rational mapping
$h(a) = 0,\ h(b) = \bar{1}\bar{0}1,\ h(c) = 1,\ h(d) = \bar{1}$



Example

... and finally the rational restriction $L = 0^* + 0^*1$, obtaining the following transition system (do you remember it?)



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