The completeness theorem for modal logic based on strictly ordered $A$-spaces

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Ershov’s topological spaces

Let $\langle X, \tau \rangle$ be a topological $T_0$-space ($\tau$ is a topology on $X$).

The order $\leq$ on $X$ is defined, related to the topology $\tau$, as follows [1]:

$$\forall x, y \in X \ (x \leq y \iff \forall U \in \tau \ (x \in U \implies y \in U)).$$

One more relation is introduced, namely the approximation relation $\prec$ on elements of $X$ as follows:

$$\forall x, y \in X \ (x \prec y \iff \exists U \in \tau \ (y \in U \forall z \in U (x \leq z))).$$

An equivalent definition may be given as follows

$$x \prec y \iff y \in \text{Int}\{z \mid x \leq z\}.$$  

A topological $T_0$-space $X$ is called an $\alpha$-space if the following condition holds:

$$\forall U \in \tau \forall x \in U \exists y \in U (y \prec x).$$

Let $\langle X, \tau \rangle$ be an $\alpha$-space. A set $X_0 \subseteq X$ is called a base subset of $X$ if the following condition holds:

$$\forall U \in \tau \forall x \in U \exists x_0 \in U \cap X_0 (x_0 \prec x).$$

An $\alpha$-space $X$ is called an $A$-space, if there exists a base subset $X_0 \subseteq X$ such that $\langle X_0, \leq \rangle$ — partial upper semilattice.

(Let $X_0 \subseteq X$, then $\langle X_0, \leq \rangle$ is called a partial upper semilattice if for any $x, y \in X_0$, a consistency of $x$ and $y$ implies the existence of the least upper bound $x \cup y \in X_0$.)

An $\alpha$-space $X$ is called $f$-space if the set of finitary elements $F(X) = \{x \mid x \in X, x \prec x\} = \emptyset$ is a base subset of $X$ and $\langle F(X), \leq \rangle$ is a partial upper semilattice.

An $f$-space with a least element is called an $f_0$-space.
Before axiomatizations of modal and temporal logics associated with strictly ordered $f$-spaces were obtained [2], [3].

A calculus $L_f$ (in modal language with modality $\Diamond$ and a constant $\beta$) and calculy $L^*_f, L^*_f_0$ (in temporal language with modalities $\Diamond, \Diamond^*$, and a constant $\beta$) were introduced.

**Completeness theorem**

(a) The calculus $L_f$ is complete with respect to the class of all strictly ordered $f$-frames and with respect to the class of all strictly ordered $f_0$-frames.

(b) The calculus $L^*_f$ is complete with respect to the class of all strictly ordered $f$-frames.

(c) The calculus $L^*_f_0$ is complete with respect to the class of all strictly ordered $f_0$-frames.

We prove the following theorem:

**Theorem** The quadruple $⟨X, X_0, \leq, \prec⟩$, where $\leq, \prec$ are binary relations on $X$ and $X_0 \subseteq X$, is defined by a linearly ordered $A$-space if and only if the following conditions are possessed:

**I**

1) $X_0 \subseteq X$;
2) $\leq$ is a linear order on $X$;
3) $x \prec y \implies x \leq y$;
4) $x \prec y \leq z \implies x \prec z$;
5) $x \leq y \prec z \implies x \prec z$.

**II**

1) $x_0 \in X_0, x_0 \prec x \implies \exists x'_0 \in X_0 (x_0 \prec x'_0 \prec x)$;
2) $x \prec y \implies \exists x_0 \in X_0 (x \leq x_0 \text{ and } x_0 \prec y)$;
3) $x \prec y \implies \exists x_0 \in X_0 (x \prec x_0 \text{ and } x_0 \prec y)$, where $x < y \iff x \leq y$ and $x \neq y$;
4) If $x$ is a least element of $X$, then $x \in X_0$ and $x \prec x$. 
Semantics

Frames \( \langle X, X_0, <, \prec \rangle \) are considered being similar to Kripke’s frames. If the quadruple \( \langle X, X_0, \leq, \prec \rangle \) is a linearly ordered \( A \)-space with the base subset \( X_0 \) then the frame \( \langle X, X_0, <, \prec \rangle \), where for any \( x, y \in X \) \( x < y \iff x \leq y \) and \( x \neq y \), is called a strictly ordered \( A \)-frame.

The modalities \( \Box_<, \Box_\prec \) associated with the relations \( < \) and \( \prec \) respectively are introduced. The set \( X_0 \) is represented as a constant \( \beta \):

\[
x \models \beta \iff x \in X_0,
\]

\[
x \models \Box_R A \iff \forall y (xRy \implies y \models A), \text{ where } R \in \{\leq, \prec\}.
\]
The calculus $L\alpha$

We define a calculus $L\alpha$ by adding the following axioms to the minimal modal calculus $K$:

(I)
1. $\Box(A_1 \rightarrow A_2) \lor \Box(A_2 \& \Box A_1 \rightarrow A_1)$;
2. $\Box A \rightarrow \Box \Box A$;
3. $\Box A \& A \rightarrow \Box A$;
4. $\Box A \rightarrow \Box \Box A$;
5. $\Box A \rightarrow \Box \Box A$.

(II)
1. $\Diamond A \& \beta \rightarrow \Diamond (\beta \& (\beta \& A))$;
2. $\Diamond A \rightarrow (\Diamond (\beta \& \Diamond A) \lor \beta)$;
3. $\Diamond A \rightarrow (\Diamond (\beta \& \Diamond A))$.

Completeness theorem The calculus $L\alpha$ is complete with respect to the class of all strictly ordered $A$-frames.

Finite axiomatizable The calculus $L\alpha$ has the finite model property.

Since the calculus $L\alpha$ has the finite model property then it is decidable.
References


3. V. F. Murzina, Temporal logics which are complete with respect to strictly linearly ordered f-models, Vestnik NSU, vol.3, 1, 2003, 61–82 (in russian)