WQO AND BQO THEORY
IN SUBSYSTEMS OF SECOND ORDER ARITHMETIC

ALBERTO MARCONE

Abstract. We consider the reverse mathematics of wqo and bqo theory. We survey the literature on the subject, which deals mainly with the more advanced results about wqos and bqos, and prove some new results about the elementary properties of these combinatorial structures. We state several open problems about the axiomatic strength of both elementary and advanced results.

A quasi-ordering (i.e. a reflexive and transitive binary relation) is a wqo (well quasi-ordering) if it contains no infinite descending chains and no infinite sets of pairwise incomparable elements. This concept is very natural, and has been introduced several times, as documented in [19]. The usual working definition of wqo is obtained from the one given above with an application of Ramsey's theorem: a quasi-ordering $\preceq$ on the set Q is wqo if for every sequence $\{x_n \mid n \in \mathbb{N}\}$ of elements of Q there exist $m < n$ such that $x_m \preceq x_n$. The notion of bqo (better quasi-ordering) is a strengthening of wqo which was introduced by Nash-Williams in the 1960's in a sequence of papers culminating in [30] and [31]. This notion has proved to be very useful in showing that specific quasi-orderings are indeed wqo. Moreover the property of being bqo is preserved by a much wider class of operations than those that preserve the property of being wqo, the general pattern being that when wqos are closed under a finitary operation, bqos are closed under its infinitary generalization (see e.g. Higman's and Nash-Williams' theorems in Section 2). [28] is a survey of wqo and bqo theory, while [38] (see [6] for a simplification in that approach) and [32] are alternative introductions to bqo theory. We postpone the precise (and rather technical) definition of bqo to Section 1.

Wqo and bqo theory represents an area of combinatorics which has always interested logicians. From the viewpoint of *reverse mathematics* ([42] is the basic reference on the subject) one of the reasons for this interest stems from the fact that these theories appear to use axioms that are within the realm of second order arithmetic, yet are much stronger than those necessary to develop other areas of ordinary mathematics (as defined in the introduction of [42]).

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In this paper we survey the known results about the provability of theorems of wqo and bqo theory within subsystems of second order arithmetic, state open problems and conjectures, and prove a few new results. The latter are mainly about the most elementary part of the theory, i.e. deal with the various equivalent definitions and with the basic closure properties of wqo and bqo.

We now list the subsystems of second order arithmetic we will use. Some of them are those usually appearing in reverse mathematics: RCA_0 is the weakest theory and consists of the basic algebraic axioms for the natural numbers together with the schemes of $\mathbf{\Sigma}^0_1$-induction and $\mathbf{\Delta}^0_1$-comprehension; WKL_0 is obtained by adding to RCA_0 the compactness principle embodied in König's lemma for binary trees; ACA_0 is stronger, and is obtained from RCA_0 by extending the comprehension scheme to arithmetical formulas; ATR_0 further extends ACA_0 by allowing definitions by arithmetical transfinite recursion; $\mathbf{\Pi}^1_1$-CA_0 is the strongest system and is obtained by allowing $\mathbf{\Pi}^1_1$ formulas in the comprehension scheme. We will also mention the stronger subsystem $\mathbf{\Pi}^1_2$-CA_0, where comprehension is extended to $\mathbf{\Pi}^1_2$ formulas: this is a very strong system, and no theorems of ordinary mathematics provable in second order arithmetic are known to require $\mathbf{\Pi}^1_2$-CA_0.

Other subsystems of second order arithmetic that are relevant to wqo theory are obtained by adding to RCA_0 certain combinatorial principles: we denote these subsystems with the abbreviation used for the combinatorial principle. RT^2_2 is Ramsey's theorem for pairs and two colors (i.e. the statement that for every $f : [\mathbb{N}]^2 \to \{0,1\}$ there exists $A \in [\mathbb{N}]^\omega$ which is homogeneous for f, i.e. such that f is constant on $[A]^2$). [1] contains what is currently known on RT^2_2 and other instances of Ramsey's theorem, including the well-known fact that RT^2_2 is properly stronger than RCA_0, Hirst's ([14]) result that WKL_0 does not prove RT^2_2, and Seetapun's theorem ([36]) implying that RT^2_2 is properly weaker than ACA_0 (this paper sparked the recent research on Ramsey's theorem for pairs). It is unknown whether RT^2_2 proves WKL_0. $\mathsf{RT}^1_{<\infty}$ is the infinite pigeonhole principle (i.e. the statement that for every $k \in \mathbb{N}$ and every $f : \mathbb{N} \to \{0,\dots,k-1\}$ there exists $A \in [\mathbb{N}]^\omega$ such that f is constant on A), which can be viewed as Ramsey's theorem for singletons. Hirst ([14]) proved that $\mathsf{RT}^1_{<\infty}$ is properly stronger than RCA_0, independent of WKL_0, and implied by RT^2_2. It can also be shown that $\mathsf{RT}^1_{<\infty}$ is properly weaker than RT^2_2. Notice that the finite pigeonhole principle (i.e. the statement that for every $k \in \mathbb{N}$ there is no one-to-one function from $\{0,\dots,k\}$ to $\{0,\dots,k-1\}$) is provable in RCA_0. For every fixed $k \in \mathbb{N}$, RCA_0 proves also the infinite pigeonhole principle for k.

The limitations of the expressive power of second-order arithmetic force us to consider only countable sets Q: this is not very restrictive because a quasi-ordering is wqo (resp. bqo) if and only if each of its restrictions to a countable subset of its domain is wqo (resp. bqo). In this paper Q will always denote a countable set; however we will consider also quasi-orderings defined on uncountable sets (such as $\mathcal{P}(Q)$, sets of infinite sequences of elements of Q, and the set of all countable linear orderings) and statements about these (with an appropriate quasi-ordering) being wqo or bqo are dealt with in a natural way (see Definition 1.12 below).

We now explain the organization of the paper. In Section 1 we establish our notation and terminology. Section 2 is a survey of the results from the literature about the axiomatic strength of theorems of wqo and bqo theory. In Section 3 we study whether weak systems suffices to establish that some very simple (e.g. finite) specific quasi-orderings are bqo. Section 4 explores the axiomatic strength of the equivalences between some of the various definitions of wqo and bqo. Section 5 deals with the provability of some of the elementary closure properties of wqo and bqo.

§1. Notation and terminology. Whenever we begin a definition, lemma or theorem with the name of a subsystem of second order arithmetic between parenthesis we mean that the definition is given, or the statement provable, within that subsystem.

First we define wqo within RCA_0:

Definition 1.1 (RCA_0). Let $\preceq$ be a quasi-ordering on Q. $\preceq$ is *wqo* if for every map $f : \mathbb{N} \to Q$ there exist $m < n$ such that $f(m) \preceq f(n)$.

Definition 1.2. Let A be an infinite subset of $\mathbb{N}$ and $f : A \to Q$. We say that f is a *good sequence* (with respect to $\preceq$) if there exist $m, n \in A$ such that $m < n$ and $f(m) \preceq f(n)$; f is *bad* otherwise.

The following characterization of wqo is immediate:

Lemma 1.3 (RCA_0). *Let $\preceq$ be a quasi-ordering on Q. The following are equivalent*:

i) *$\preceq$ is wqo*;
ii) *every sequence of elements of Q is good with respect to $\preceq$.*

To give the definition of bqo we need some terminology and notation for sequences and sets (here we follow [26]). All the definitions are given in RCA_0. If s is a finite sequence we denote by $\operatorname{lh} s$ its length and, for every $i < \operatorname{lh} s$, by $s(i)$ its $(i+1)$-th element. Then we write this sequence as $s = \langle s(0), \dots, s(\operatorname{lh} s - 1)\rangle$. If s and t are finite sequences we write $s \sqsubseteq t$ if s is an initial segment of t, i.e. if $\operatorname{lh} s \leq \operatorname{lh} t$ and $\forall i < \operatorname{lh} s\ s(i) = t(i)$. We write $s \subseteq t$ if the range of s is a subset of the range of t, i.e. if $\forall i < \operatorname{lh} s\ \exists j < \operatorname{lh} t$ $s(i) = t(j)$. $s \sqsubset t$ and $s \subset t$ have the obvious meanings. We write $s^\frown t$ for the concatenation of s and t, i.e. the sequence u such that $\operatorname{lh} u = \operatorname{lh} s + \operatorname{lh} t$, $u(i) = s(i)$ for every $i < \operatorname{lh} s$, and $u(\operatorname{lh} s + i) = t(i)$ for every $i < \operatorname{lh} t$. These notations are extended to infinite sequences (i.e. functions with domain $\mathbb{N}$) as well.

If $X \subseteq \mathbb{N}$ is infinite we denote by $[X]^{<\omega}$ the set of all finite subsets of X. We identify a subset of $\mathbb{N}$ with the unique sequence enumerating it in increasing order, so that we can use the notation introduced above. If $k \in \mathbb{N}$, $[X]^k$ is the subset of $[X]^{<\omega}$ consisting of the sets with exactly k elements. Similarly $[X]^\omega$ stands for the collection of all infinite subsets of X. Note that $[X]^\omega$ does not formally exist in second order arithmetic, and is only used in expressions of the form $Y \in [X]^\omega$; here again we identify Y with the unique sequence enumerating it in increasing order (notice that in RCA_0 an element of $[X]^\omega$ exists as a set if and only if it exists as an increasing sequence, so that this identification is harmless). For $X \in [\mathbb{N}]^\omega$ and $m \leq n$ let $X[m,n]$ be the finite set enumerated by $\langle X(m), \dots, X(n-1)\rangle$, while $X[m,\infty]$ is the infinite set $X \setminus \{X(0), \dots, X(m-1)\}$; we write $X[n]$ in place of $X[0,n]$. The same notation applies also to finite sets.

The notion of the base of a set of finite sets is basic for defining blocks and barriers and hence bqo. If $B \subseteq [\mathbb{N}]^{<\omega}$ then $\operatorname{base}(B)$ is the set

$$\{n \mid \exists s \in B\ \exists i < \operatorname{lh} s\ s(i) = n\}.$$

RCA_0 does not prove the existence of $\operatorname{base}(B)$ for arbitrary $B \subseteq [\mathbb{N}]^{<\omega}$; indeed we have

LEMMA 1.4 (RCA_0). *The following are equivalent*:

i) ACA_0;
ii) *for every set* $B \subseteq [\mathbb{N}]^{<\omega}$, $\operatorname{base}(B)$ *exists as a set*.

PROOF. It is obvious that i) implies ii). To prove the converse recall that ACA_0 is equivalent (over RCA_0) to the statement that the range of any function exists as a set. Given $f : \mathbb{N} \to \mathbb{N}$ let B be

$$\{\langle 2n, 2m+1\rangle \mid f(n) = m \wedge n \leq m\} \cup \{\langle 2m+1, 2n\rangle \mid f(n) = m \wedge n > m\}.$$

By ii) $\operatorname{base}(B)$ exists as a set. Then $\{m \mid 2m+1 \in \operatorname{base}(B)\}$ is a set which coincides with the range of f. $\dashv$

Lemma 1.4 does not affect the possibility of defining blocks and barriers within RCA_0: e.g. "$\operatorname{base}(B)$ is infinite" (which is condition (1) in the definition of block below) can be expressed by $\forall m\ \exists n > m\ \exists s \in B\ n \in s$. (For barriers see Lemma 1.6 below.)

DEFINITION 1.5 (RCA_0). A set $B \subseteq [\mathbb{N}]^{<\omega}$ is a *block* if:

(1) $\operatorname{base}(B)$ is infinite;
(2) $\forall X \in [\operatorname{base}(B)]^\omega\ \exists s \in B\ s \sqsubset X$;
(3) $\forall s, t \in B\ s \not\sqsubset t$.

B is a *barrier* if it satisfies (1), (2) and

(3') $\forall s, t \in B\ s \not\subset t$.

RCA_0 proves that if $X \in [\mathbb{N}]^\omega$ then $[X]^k$ is a barrier for every $k > 0$. Other, more complex, examples of barriers are $\{s \in [\mathbb{N}]^{<\omega} \mid \operatorname{lh} s = s(0)+1\}$ and $\{s \in [\mathbb{N}]^{<\omega} \mid \operatorname{lh} s = s(s(0))+1\}$.

It is immediate that every barrier is a block. Notice that if B is a block and $Y \in [\operatorname{base}(B)]^\omega$ then RCA_0 proves that there exists a unique block $B' \subseteq B$ such that $\operatorname{base}(B') = Y$, namely $B' = \{s \in B \mid s \subset Y\}$. Notice also that if B is a barrier then B' is also a barrier and we say that B' is a *subbarrier* of B.

Lemma 1.6 (RCA_0). *If B is a barrier then* $\operatorname{base}(B)$ *exists as a set and B is isomorphic to a barrier B' with* $\operatorname{base}(B') = \mathbb{N}$.

Proof. If B is a barrier the equivalences

$$\begin{aligned} n \in \operatorname{base}(B) &\iff \exists s \in B \; n \in s \\ &\iff \forall t \in B \; (t(0) > n \to \exists i < \operatorname{lh} t \;\; \langle n \rangle^\frown t[i] \in B) \end{aligned}$$

show that $\operatorname{base}(B)$ has a $\boldsymbol{\Delta}^0_1$ definition.

Since $\operatorname{base}(B)$ is a set it can be enumerated in increasing order and using this enumeration it is easy to define an isomorphic copy of B on $\mathbb{N}$. ⊣

Problem 1.7. Does RCA_0 prove that if B is a block then $\operatorname{base}(B)$ exists as a set?

Definition 1.8 (RCA_0). Let $s, t \in [\mathbb{N}]^{<\omega}$: we write $s \lhd t$ if there exists $u \in [\mathbb{N}]^{<\omega}$ such that $s \sqsubseteq u$ and $t \sqsubseteq u[1, \operatorname{lh} u]$.

Notice that $\langle 0,3,5\rangle \lhd \langle 3,5,7,8\rangle \lhd \langle 5,7,8,9\rangle$ and $\langle 0,3,5\rangle \not\lhd \langle 5,7,8,9\rangle$, so that $\lhd$ is not transitive.

Definition 1.9 (RCA_0). Let $\preceq$ be a quasi-ordering on Q, B be a block and $f : B \to Q$. We say that the map f is *good with respect to* $\preceq$ if there exist $s, t \in B$ such that $s \lhd t$ and $f(s) \preceq f(t)$. If f is not good then we say that it is *bad*. f is *perfect* if for every $s, t \in B$ such that $s \lhd t$ we have $f(s) \preceq f(t)$.

We can now give the definition of bqo:

Definition 1.10 (RCA_0). Let $\preceq$ be a quasi-ordering on Q. $\preceq$ is *bqo* if for every barrier B and every map $f : B \to Q$, f is good with respect to $\preceq$.

To understand the notion of bqo it may be helpful to make the following observations, which lead to the alternative approach to bqo theory developed in [38]. A block (in particular, a barrier) B represents an infinite partition of $[\operatorname{base}(B)]^\omega$ into clopen sets with respect to the usual topology of $[\operatorname{base}(B)]^\omega$. Thus any subset of a block represents a clopen subset of $[\operatorname{base}(B)]^\omega$ and any map $f : B \to Q$ represents a continuous map $F : [\operatorname{base}(B)]^\omega \to Q$ where Q has the discrete topology; f is good if for some $X \in [\operatorname{base}(B)]^\omega$ we have $F(X) \preceq F(X[1,\infty])$.

Lemma 1.11 (RCA_0). *Every bqo is wqo.*

Proof. $[\mathbb{N}]^1$ is a barrier, and it suffices to notice that $\langle n \rangle \lhd \langle m \rangle$ if and only if $n < m$. ⊣

We conclude this section by defining the notion of bqo in the case of an uncountable quasi-ordering (the definition of wqo is similar). In this case we need to give a definition schema, since a quasi-ordering on an uncountable structure is a formula satisfying the properties of reflexivity and transitivity. To be definite, let us assume that $\preceq^*_*$ is a quasi-ordering defined on $\mathcal{P}(Q)$.

Definition 1.12 (RCA_0). A sequence $\langle X_s : s \in B \rangle$ of elements of $\mathcal{P}(Q)$ indexed by a barrier B is *good* if there exist $s, t \in B$ such that $s \lhd t$ and $X_s \preceq^*_* X_t$. If every such sequence is good we say that $\preceq^*_*$ is *bqo*.

We will however use the notation $f : B \to \mathcal{P}(Q)$ to denote what formally is a sequence of elements indexed by B.

§2. Survey of known results. One of the main tools of wqo theory is the minimal bad sequence lemma (apparently isolated for the first time in [29]).

Definition 2.1 (RCA_0). Let $\preceq$ be a quasi-ordering on Q. A transitive binary relation $<'$ on Q is *compatible with* $\preceq$ if for every $q_0, q_1 \in Q$ we have that $q_0 <' q_1$ implies $q_0 \preceq q_1$. We write $q_0 \leq' q_1$ for $q_0 <' q_1 \vee q_0 = q_1$. In this situation, if $A, A' \in [\mathbb{N}]^\omega$, $f : A \to Q$, and $f' : A' \to Q$ we write $f \leq' f'$ if $A \subseteq A'$ and $\forall n \in A\ f(n) \leq' f'(n)$; we write $f <' f'$ if $f \leq' f'$ and $\exists n \in A\ f(n) <' f'(n)$. f is *minimal bad with respect to* $<'$ if it is bad with respect to $\preceq$ and there is no $f' <' f$ which is bad with respect to $\preceq$.

Statement 2.2 (minimal bad sequence lemma). Let $\preceq$ be a quasi-ordering on Q and $<'$ a well-founded relation which is compatible with $\preceq$: if $A \in [\mathbb{N}]^\omega$ and $f : A \to Q$ is bad with respect to $\preceq$ then there exists $f' : A' \to Q$ such that $f' \leq' f$ and f' is minimal bad with respect to $<'$.

The minimal bad sequence lemma has been analyzed by Simpson and Marcone:

Theorem 2.3 ([26]). *Within* RCA_0, $\mathbf{\Pi}^1_1$-CA_0 *is equivalent to the minimal bad sequence lemma.*

Two important theorems that can be proved using the minimal bad sequence lemma are Higman's theorem ([13]) and Kruskal's theorem ([18]). The latter establishes a conjecture of Vázsonyi from the 1930's which was popularized by Erdős.

Definition 2.4 (RCA_0). If $\preceq$ is a quasi-ordering on Q we define a quasi-ordering on $Q^{<\omega}$, the set of finite sequences of elements of Q, by setting $s \leq t$ if and only if there exists an embedding of s into t, i.e. a strictly increasing $f : \operatorname{lh} s \to \operatorname{lh} t$ such that $s(i) \preceq t(f(i))$ for every $i < \operatorname{lh} s$.

STATEMENT 2.5 (Higman's theorem). If $\preceq$ is wqo on Q then $\leq$ is wqo on $Q^{<\omega}$.

DEFINITION 2.6 (RCA_0). If $\mathcal{T}$ is the set of all finite trees define a quasi-ordering on $\mathcal{T}$ by setting $T_0 \preceq_{\mathcal{T}} T_1$ if and only if there exists a homeomorphic embedding of T_0 in T_1, i.e. an injective $f : T_0 \to T_1$ such that $f(s \wedge t) = f(s) \wedge f(t)$ for every $s, t \in T_0$ (here $\wedge$ denotes greatest lower bound).

If $\preceq$ is a quasi-ordering on Q define a quasi-ordering on $\mathcal{T}^Q$, the set of finite trees labelled with elements of Q, by setting $(T_0, \ell_0) \preceq_{\mathcal{T}^Q} (T_1, \ell_1)$ if and only if there exists a homeomorphic embedding f of T_0 in T_1 such that $\ell_0(s) \preceq \ell_1(f(s))$ for every $s \in T_0$.

STATEMENT 2.7 (Kruskal's theorem). $\preceq_{\mathcal{T}}$ is wqo on $\mathcal{T}$.

STATEMENT 2.8 (generalized Kruskal's theorem). If $\preceq$ is wqo on Q then $\preceq_{\mathcal{T}^Q}$ is wqo on $\mathcal{T}^Q$.

Theorem 2.3 implies that the usual proofs of both Higman's theorem and the generalized Kruskal's theorem can be carried out within $\mathbf{\Pi}^1_1$-CA_0. On the other hand, these theorems are $\mathbf{\Pi}^1_2$ statements and hence it follows from standard model theoretic considerations (see e.g. [26, corollary 1.10]) that neither of them implies $\mathbf{\Pi}^1_1$-CA_0 over ATR_0. The axiomatic status of these two theorems is however quite different. Higman's theorem is fairly weak:

THEOREM 2.9. *Within* RCA_0, ACA_0 *is equivalent to Higman's theorem.*

SKETCH OF THE PROOF. The proof of Higman's theorem within ACA_0 is based on the technique of "reification" of wqos by well-orderings ([16, 34], see also [17]), and follows from the results in Section 4 of [41] (see [5, theorem 3] for details). ACA_0 is used twice in this proof: first to show that every wqo admits a reification by a well-ordering and then, since RCA_0 proves that if Q has a reification of order type α then $Q^{<\omega}$ has a reification of order type $\omega^{\omega^{\alpha+1}}$ ([41, sublemma 4.8], which is Lemma 5.2 of [35]), to show that $\omega^{\omega^{\alpha+1}}$ is a well-ordering when α is a well-ordering ([12], see [15]). RCA_0 then shows that if a quasi-order admits a reification by a well-ordering then it is wqo.

The reversal is proved in [5] (and implied, albeit within RT^2_2, by Lemma 5.10 below). ⊣

The following theorem of Friedman's shows that Kruskal's theorem is much stronger than Higman's theorem:

THEOREM 2.10 ([39]). *Kruskal's theorem* (*and, a fortiori, the generalized Kruskal's theorem*) *is not provable in* ATR_0.

SKETCH OF THE PROOF. The proof proceeds by constructing a bijection φ between $\mathcal{T}$ and a certain primitive recursive notation system for the ordinals less than Γ_0. Since ACA_0 proves that $T_0 \preceq_{\mathcal{T}} T_1$ implies $\varphi(T_0) \leq_o \varphi(T_1)$ (where $\leq_o$ is the ordering on the ordinal notation system), ACA_0 proves that

Kruskal's theorem implies the well-ordering of the system of ordinal notations. Since Γ_0 is the proof-theoretic ordinal of ATR_0, it follows that ATR_0 does not prove Kruskal's theorem. ⊣

Therefore from the viewpoint of reverse mathematics Kruskal's theorem and its generalization are intermediate between the theorems equivalent to ATR_0 and those equivalent to $\mathbf{\Pi}^1_1$-CA_0.

Friedman proved another strengthening of Kruskal's theorem (obtained by adding a "gap condition" on the homeomorphic embedding) which is known in the literature as the extended Kruskal's theorem. Friedman himself ([39]) showed, generalizing the technique of Theorem 2.10 to larger ordinals, that the extended Kruskal's theorem is not provable in $\mathbf{\Pi}^1_1$-CA_0, thus providing a rare example of a theorem of ordinary mathematics provable within second order arithmetic but not within $\mathbf{\Pi}^1_1$-CA_0.

An even more striking (being a problem studied well before its axiomatic analysis) instance of this phenomenon is provided by the graph minor theorem, proved by Robertson and Seymour in a long series of papers (see [44, section 5] for an overview):

Definition 2.11 (RCA_0). If $\mathcal{G}$ is the set of all finite directed graphs (allowing loops and multiple edges) define a quasi-ordering on $\mathcal{G}$ by setting $G_0 \preceq_m G_1$ if and only if G_0 is isomorphic to a minor of G_1.

Statement 2.12 (graph minor theorem). $\preceq_m$ is wqo on $\mathcal{G}$.

The proof of even special cases of the graph minor theorem (where $\preceq_m$ is restricted to some subset of $\mathcal{G}$) uses iterated applications of the minimal bad sequence lemma. This technique is not available in $\mathbf{\Pi}^1_1$-CA_0 and this is no accident, as the following theorem (again proved generalizing the technique of Theorem 2.10 to larger ordinals) shows:

Theorem 2.13 ([10]). *The graph minor theorem is not provable in* $\mathbf{\Pi}^1_1$-CA_0.

One may ask whether Higman's theorem can be extended to infinite sequences. The answer is negative, as the following example due to Rado ([33], see [28, p. 492]) shows: let $R = [\mathbb{N}]^2$ and define $\langle i, j\rangle \preceq_R \langle k, l\rangle$ if and only if either $i = k$ and $j \leq l$, or $j < k$. It is easy to see that $\preceq_R$ is wqo, while $f : \mathbb{N} \to R^\omega$ defined by $f(n) = \langle\langle n, n+1\rangle, \langle n, n+2\rangle, \dots\rangle$ shows that $\preceq_R$ is not wqo on R^ω. Moreover Rado's example is canonical, because if $\preceq$ is wqo on Q then $\leq$ is wqo on Q^ω if and only if Q does not contain any copy of R ([33]; see [21] for a simpler proof).

It is clear that Rado's example is not bqo: $[\mathbb{N}]^2$ is a barrier and the identity map is bad with respect to $\preceq_R$. Starting from Rado's example Nash-Williams developed the idea of bqo and proved one of the first theorems of the subject in [31] by generalizing Higman's theorem:

Definition 2.14 (RCA_0). If $\preceq$ is a quasi-ordering on Q extend the quasi-ordering $\leq$ defined on $Q^{<\omega}$ in Definition 2.4 to $\tilde{Q}$, the set of all countable sequences of elements of Q (i.e. the set of all functions from a countable well-ordering to Q).

Statement 2.15 (Nash-Williams' theorem). If $\preceq$ is bqo on Q then $\leq$ is bqo on $\tilde{Q}$.

Notice that $\tilde{Q}$ is uncountable, and hence in the preceding statement we are using Definition 1.12.

A weak version of Nash-Williams' theorem turns out to play a role in the axiomatic analysis:

Statement 2.16 (generalized Higman's theorem). If $\preceq$ is bqo on Q then $\leq$ is bqo on $Q^{<\omega}$.

The proofs of Nash-Williams' theorem use a generalization of the minimal bad sequence lemma known as the minimal bad array lemma (the maps of Definition 1.9 are sometimes called arrays) or the forerunning technique (this method was explicitly isolated and clarified in [22]).

Definition 2.17 (RCA_0). Let $\preceq$ be a quasi-ordering on Q and $<'$ be compatible with $\preceq$ in the sense of Definition 2.1. If B and B' are barriers, $f : B \to Q$, and $f' : B' \to Q$ we write $f \leq' f'$ if $\mathrm{base}(B) \subseteq \mathrm{base}(B')$, and for every $s \in B$ there exists $s' \in B'$ such that $s' \sqsubseteq s$ and $f(s) \leq' f'(s')$. We write $f <' f'$ if $f \leq' f'$ and for some $s \in B$, $s' \in B'$ with $s' \sqsubseteq s$ we have $f(s) <' f'(s')$. f is *minimal bad with respect to* $<'$ if it is bad with respect to $\preceq$ and there is no $f' <' f$ which is bad with respect to $\preceq$.

Statement 2.18 (minimal bad array lemma). Let $\preceq$ be a quasi-ordering on Q and $<'$ a well-founded relation which is compatible with $\preceq$. If B is a barrier and $f : B \to Q$ is bad with respect to $\preceq$ then there exist a barrier B' and $f' : B' \to Q$ such that $f' \leq' f$ and f' is minimal bad with respect to $<'$.

The proof of the minimal bad array lemma appears to use very strong set-existence axioms: a crude analysis shows that they can be carried out within $\mathbf{\Pi}^1_2$-CA_0.

Problem 2.19. What is the axiomatic strength of the minimal bad array lemma? We conjecture that it is not provable within $\mathbf{\Pi}^1_1$-CA_0.

To deal with Nash-Williams' theorem in a system within the usual scope of reverse mathematics the following milder generalization of the minimal bad sequence lemma is useful: this is actually the first version of the minimal bad array lemma proved for a specific quasi-ordering by Nash-Williams in [30], and was used in [24] to obtain a fine analysis of Nash-Williams' theorem.

Definition 2.20 (RCA_0). Let $\preceq$ be a quasi-ordering on Q and $<'$ be compatible with $\preceq$ in the sense of Definition 2.1. If B and B' are barriers, $f : B \to Q$, and $f' : B' \to Q$ we write $f \leq'_\ell f'$ if $B \subseteq B'$ and $\forall s \in B$ $f(s) \leq' f'(s)$. We write $f <'_\ell f'$ if $f \leq'_\ell f'$ and $\exists s \in B\ f(s) <' f'(s)$. f is *locally minimal bad with respect to* $<'$ if it is bad with respect to $\preceq$ and there is no $f' <'_\ell f$ which is bad with respect to $\preceq$.

Statement 2.21 (locally minimal bad array lemma). Let $\preceq$ be a quasi-ordering on Q and $<'$ a well-founded relation which is compatible with $\preceq$: if B is a barrier and $f : B \to Q$ is bad with respect to $\preceq$ then there exist a barrier B' and $f' : B' \to Q$ such that $f' \leq'_\ell f$ and f' is locally minimal bad with respect to $<'$.

Theorem 2.22 ([26]). *Within* RCA_0, $\mathbf{\Pi}^1_1$-CA_0 *is equivalent to the locally minimal bad array lemma.*

Sketch of the proof. The proof of the locally minimal bad array lemma within $\mathbf{\Pi}^1_1$-CA_0 is the generalization of the proof of the minimal bad sequence lemma.

For the other direction, notice that if $B \subseteq [\mathbb{N}]^1$ then, modulo the obvious identification of $[\mathbb{N}]^1$ and $\mathbb{N}$, a map $f : B \to Q$ is locally minimal bad if and only if it is minimal bad in the sense of Definition 2.1. Therefore the minimal bad sequence lemma is a particular instance of the locally minimal bad array lemma and, by Theorem 2.3, the locally minimal bad array lemma implies $\mathbf{\Pi}^1_1$-CA_0 within RCA_0. $\dashv$

The forward direction of Theorem 2.22 is one of the main tools in the proof of:

Theorem 2.23 ([26]). $\mathbf{\Pi}^1_1$-CA_0 *proves Nash-Williams' theorem.*

In the proof of Theorem 2.23 $\mathbf{\Pi}^1_1$ comprehension is used only in proving the locally minimal bad array lemma, which is in turn used only to establish the generalized Higman's theorem; since all the remaining arguments go through in ATR_0 we obtain:

Theorem 2.24 ([26]). *Within* ATR_0 *Nash-Williams' theorem and the generalized Higman's theorem are equivalent.*

In [26] the following conjecture was stated:

Conjecture 2.25. ATR_0 proves Nash-Williams' theorem.

In [5] Clote claimed to have proved Conjecture 2.25, but his proof is incorrect, as Clote himself has acknowledged (personal communication). Conjecture 2.25 is supported by the following theorem ([26]):

Theorem 2.26. ATR_0 *plus Nash-Williams' theorem does not imply* $\mathbf{\Pi}^1_1$-CA_0.

Sketch of the proof. By Theorem 2.24 it suffices to show that the generalized Higman's theorem does not imply $\mathbf{\Pi}^1_1$-CA_0 within ATR_0. Since "being bqo” is $\mathbf{\Pi}^1_2$ and no better ([24, 25]), the generalized Higman’s theorem is a $\mathbf{\Pi}^1_3$ statement and we cannot directly apply the model theoretic considerations mentioned before Theorem 2.9. However the proof of the generalized Higman’s theorem within $\mathbf{\Pi}^1_1$-CA_0 actually yields a slightly stronger result, which is a $\mathbf{\Pi}^1_2$ statement; this suffices to complete the proof. ⊣

A natural way of proving Conjecture 2.25 is extending the ordinal analysis used in the proof of the forward direction of Theorem 2.9 to the generalized Higman’s theorem (actually to the $\mathbf{\Pi}^1_2$ statement mentioned in the sketch of the proof of Theorem 2.26): this has eluded all attempts to date. Another, so far unfruitful as well, approach consists in establishing within ATR_0 a weaker version of the locally minimal bad array lemma which suffices to prove the generalized Higman’s theorem. In this version the well-foundedness of $<'$ is replaced by the stronger property that for each $q \in Q$ the set $\{q' \in Q \mid q' <' q\}$ is finite. (The proof of Theorem 2.3 does not show that this version of the locally minimal bad array lemma implies $\mathbf{\Pi}^1_1$-CA_0.)

One of the most famous achievements of bqo theory is Laver’s proof ([20]) of Fraïssé’s conjecture ([7]). Laver actually proved a stronger result and we keep the two statements distinct.

Definition 2.27 (RCA_0). If $\mathcal{L}$ is the set of countable linear orderings define a quasi-ordering on $\mathcal{L}$ by setting $L_0 \preceq_{\mathcal{L}} L_1$ if and only if there exists an order-preserving embedding of L_0 in L_1, i.e. an injective $f : L_0 \to L_1$ such that $x <_{L_0} y$ implies $f(x) <_{L_1} f(y)$ for every $x, y \in L_0$.

If $\preceq$ is a quasi-ordering on Q define a quasi-ordering on $\mathcal{L}^Q$, the set of countable linear orderings labelled with elements of Q, by setting $(L_0, \ell_0) \preceq_{\mathcal{L}^Q} (L_1, \ell_1)$ if and only if there exists an order-preserving embedding f of L_0 in L_1 such that $\ell_0(x) \preceq \ell_1(f(x))$ for every $x \in L_0$.

Statement 2.28 (Fraïssé’s conjecture). $\preceq_{\mathcal{L}}$ is wqo on $\mathcal{L}$.

Statement 2.29 (Laver’s theorem). If $\preceq$ is bqo on Q then $\preceq_{\mathcal{L}^Q}$ is bqo on $\mathcal{L}^Q$.

Notice that (using the bqo with a single element and Lemma 1.11) Laver’s theorem easily implies Fraïssé’s conjecture. It is also immediate that Laver’s theorem implies Nash-Williams’ theorem.

The known proofs of Fraïssé’s conjecture actually establish Laver’s theorem and in particular that $\preceq_{\mathcal{L}}$ is bqo on $\mathcal{L}$. These proofs use the minimal bad array lemma and can be carried out in $\mathbf{\Pi}^1_2$-CA_0 (using the results of [4] for the analysis of linear orderings). Since Fraïssé’s conjecture is a $\mathbf{\Pi}^1_2$ statement the usual considerations yield:

Theorem 2.30. ATR_0 *plus Fraïssé’s conjecture does not imply* $\mathbf{\Pi}^1_1$-CA_0.

Therefore the following conjecture is plausible:

Conjecture 2.31 ([42]). ATR_0 proves Fraïssé’s conjecture.

An interesting step towards establishing Conjecture 2.31 would be the analogous of Theorem 2.23, i.e. a proof of Fraïssé's conjecture within $\mathbf{\Pi}^1_1$-CA_0.

To refute Conjecture 2.31 one could assign ordinals to linear orderings in a way similar to the proof of Theorem 2.10; however this does not appear to be feasible, since linear orderings lack the structure which is present in trees.

So far we have only considered upper bounds (i.e. provability results or conjectures) for the main theorems of bqo theory. An important lower bound is due to Shore ([37]):

THEOREM 2.32 (RCA_0). *The statement "every infinite sequence of countable well-orderings contains two distinct elements which are comparable with respect to* $\preceq_{\mathcal{L}}$*" implies* ATR_0.

The statement contained in Theorem 2.32 implies both Nash-Williams' theorem and Fraïssé's conjecture, and hence each of Nash-Williams' theorem, Fraïssé's conjecture and Laver's theorem implies ATR_0 within RCA_0. Therefore Conjectures 2.25 and 2.31 are actually conjectures about equivalences with ATR_0.

Other theorems of wqo and bqo theory could be investigated within subsystems of second order arithmetic: some of them are listed in [42, pp. 407–410] and [11], while a more recent result is the main theorem of [43].

§3. Specific quasi-orderings. In the following discussion of finite quasi-orderings we use standard set-theoretic notation and identify $p \in \mathbb{N}$ with the set $\{0, 1, \dots, p-1\}$. It is obvious that if a map from a barrier to a set is good with respect to the quasi-ordering which makes the elements pairwise incomparable then it is good with respect to any quasi-ordering on that set. Thus we always consider p equipped with the quasi-ordering which makes the elements pairwise incomparable.

RCA_0 easily proves that all well-orderings and all finite quasi-orderings are wqo (indeed for the latter fact the finite pigeonhole principle suffices). The situation with the stronger property of bqo is more delicate. In fact, the straightforward proof that 2 is bqo uses the so-called barrier theorem, which by Theorem 4.9 below is equivalent to ATR_0. On the other hand, a reversal to a theory T weaker (not necessarily properly) than ATR_0 of a statement of the form "if Q is bqo then $\Phi(Q)$ is bqo" (see e.g. Problem 5.7 and Conjecture 5.18) is likely to require the construction of a quasi-ordering Q which is proved to be bqo in a theory properly weaker than T. Thus it appears to be worthwhile to find out which specific quasi-orderings can be shown to be bqo in theories properly weaker than ATR_0.

LEMMA 3.1 (RCA_0). *Every well-ordering is a bqo.*

PROOF. Let Q be well-ordered by $\preceq$, B be a barrier and $f : B \to Q$. For every i let $s_i \in B$ be such that $s_i \sqsubset \text{base}(B)[i, \infty]$; then $s_i \lhd s_{i+1}$ for every i.

Since Q is a well-ordering there exists i such that $f(s_i) \preceq f(s_{i+1})$, and hence f is good. $\dashv$

ATR_0 proves the clopen Ramsey's theorem for two colors, and hence for any $p \in \mathbb{N}$, it proves that p is bqo. When $p = 2$ this result can be improved:

Lemma 3.2 (RCA_0). *2 is bqo.*

Proof. We reason within RCA_0. Let B be a barrier. Pick $s \in B$, $n \in \operatorname{base}(B)$, and $t \in B$ such that, if $k = \operatorname{lh} s$, we have $s(k-1) < n < t(0)$.

Since B is a barrier there exist $s_0 = s, \dots, s_k, s_{k+1} = t \in B$ such that $s_i \sqsubset s[i,k]^\frown \langle n \rangle ^\frown t$ for $i < k$ and $s_k \sqsubset \langle n \rangle ^\frown t$. Clearly $s_i \lhd s_{i+1}$ for every $i \le k$. Similarly there exist $s'_0 = s, \dots, s'_k = t \in B$ such that for $i < k$ we have $s'_i \sqsubset s[i,k]^\frown t$. Therefore $s'_i \lhd s'_{i+1}$ for every $i < k$.

Towards a contradiction suppose that $f : B \to 2$ is bad, so that whenever $u \lhd v$ we have $f(u) \neq f(v)$ and hence $f(u) = 1 - f(v)$. If k is even we have

$$f(s_0) = f(s_k) \neq f(s_{k+1}) \quad \text{so that } f(t) \neq f(s) \quad \text{and}$$
$$f(s'_0) = f(s'_k) \quad \text{so that } f(t) = f(s).$$

A similar contradiction is reached if k is odd. $\dashv$

Theorem 5.11 below shows that if a theory T containing RCA_0 proves that 3 is bqo then for any $p \in \mathbb{N}$, T proves that p is bqo. Therefore the following problem is rather important:

Problem 3.3. Does any subsystem properly weaker than ATR_0 prove that 3 is bqo?

In attempting to answer affirmatively the question of Problem 3.3 we will now prove some partial results. Recall that every $[\mathbb{N}]^k$ is a barrier.

Let us write $\mathcal{E}^*(k;p)$ for an exponential stack of $k-1$ 2's with p placed on top. The recursive definition, which can be given in RCA_0, is $\mathcal{E}^*(1;p) = p$, $\mathcal{E}^*(k+1;p) = 2^{\mathcal{E}^*(k;p)}$. Easy inductions within RCA_0 prove that $\mathcal{E}^*(k+1;p) = \mathcal{E}^*(k;2^p)$ and $p < p' \to \mathcal{E}^*(k;p) < \mathcal{E}^*(k;p')$ hold. The statement of the next lemma is essentially due to Friedman ([8]), although our notation is slightly different from his.

Lemma 3.4 (RCA_0). *Let $k, p > 0$. For every function $f : [\mathcal{E}^*(k;p+1)]^k \to p$ there exist $x_0 < x_1 < \dots < x_k < \mathcal{E}^*(k;p+1)$ such that $f(x_0, \dots, x_{k-1}) = f(x_1, \dots, x_k)$.*

Proof. The proof is by $\mathbf{\Pi}^0_1$-induction ([42, corollary II.3.10]) on k. When $k = 1$ the statement is just an application of the finite pigeonhole principle, since $\mathcal{E}^*(1;p+1) = p+1$.

For the induction step let us fix a bijection $\pi : \mathcal{P}(p) \to 2^p$ between the subsets of p and 2^p. Given a function $f : [\mathcal{E}^*(k+1;p+1)]^{k+1} \to p$ define $g : [\mathcal{E}^*(k+1;p+1)]^k \to 2^p$ by

$$g(x_0, \dots x_{k-1}) = \pi(\{ f(x_0, \dots, x_{k-1}, y) \mid x_{k-1} < y < \mathcal{E}^*(k+1;p+1) \}).$$

Since $\mathcal{E}^*(k+1; p+1) = \mathcal{E}^*(k; 2^{p+1}) \geq \mathcal{E}^*(k; 2^p + 1)$ the induction hypothesis implies the existence of $x_0 < x_1 < \cdots < x_k < \mathcal{E}^*(k+1; p+1)$ such that $g(x_0, \ldots, x_{k-1}) = g(x_1, \ldots, x_k)$. Since $f(x_0, \ldots, x_k) \in \pi^{-1}(g(x_0, \ldots, x_{k-1})) = \pi^{-1}(g(x_1, \ldots, x_k))$ there exists x_{k+1} such that $x_k < x_{k+1} < \mathcal{E}^*(k+1; p+1)$ and $f(x_0, \ldots, x_k) = f(x_1, \ldots, x_k, x_{k+1})$. ⊣

Lemma 3.4 immediately implies the following:

Theorem 3.5 (RCA_0). *Let $k, p > 0$. Every map $f : [\mathbb{N}]^k \to p$ is good.*

Using the terminology of [24, 27], Theorem 3.7 says that RCA_0 suffices to prove that any finite quasi-ordering is ω^k-wqo for every finite k. The next theorem deals with the provability (using the same terminology) of "3 is ω^ω-wqo".

Definition 3.6 (RCA_0). Let $k > 0$. A barrier B is an (ω^ω, k)-*barrier* if

$$\forall s \in B\ (s(0) = \min(\text{base}(B)) \to \text{lh}\, s \leq k).$$

The barrier $\{s \in [\mathbb{N}]^{<\omega} \mid \text{lh}\, s = s(1) + 1\}$ is not (ω^ω, k) for any k, but all barriers of order type $\leq \omega^\omega$ (in the sense of [24, 27]) are (ω^ω, k) for some k.

Theorem 3.7. *Let $\varphi(k)$ be the statement "if B is an (ω^ω, k)-barrier then every $f : B \to 3$ is good".*

1) RCA_0 *proves* $\varphi(1)$;
2) RCA_0 *proves* $\varphi(2)$;
3) RT^2_2 *proves* $\varphi(3)$;
4) *for every* $k \in \mathbb{N}$, ACA_0 *proves* $\varphi(k)$;
5) ACA_0 *plus "for every X and n, the nth Turing jump of X exists" (this system is usually called* ACA_0'*) proves* $\forall k\ \varphi(k)$.

Proof. By Lemma 1.6 we may assume throughout the proof that the barriers we deal with have base $\mathbb{N}$.

To prove 1) let B be an $(\omega^\omega, 1)$-barrier and observe that $\langle 0 \rangle \in B$ and $\langle 0 \rangle \lhd s$ for any other $s \in B$. If $f : B \to 3$ is bad we may assume that $f(\langle 0 \rangle) = 2$: thus for all $s \in B$ with $s \neq \langle 0 \rangle$ we must have $f(s) = 0$ or $f(s) = 1$. Since $B' = B \setminus \{\langle 0 \rangle\}$ is a subbarrier of B, we have a bad map $f \restriction B' : B' \to 2$, contradicting Lemma 3.2.

The proofs of 2), 3), 4) and 5) differ only in the use of different versions of Ramsey's theorem. Fix $k > 1$, an (ω^ω, k)-barrier B and let $f : B \to 3$ be bad. Define $g : [\mathbb{N} \setminus \{0\}]^{k-1} \to 3$ by setting $g(s) = f(\langle 0 \rangle ^\frown s')$ where $s' \sqsubseteq s$ is such that $\langle 0 \rangle ^\frown s' \in B$ (such an s' exists because B is an (ω^ω, k)-barrier). We apply Ramsey's theorem to g: RCA_0 proves the infinite pigeonhole principle for three colors which suffices when $k = 2$; RT^2_2 proves Ramsey's theorem for pairs and three colors which is used when $k = 3$; for any $k \in \mathbb{N}$, ACA_0 proves Ramsey's theorem for $(k-1)$-tuples, while ACA_0' proves that for every k Ramsey's theorem for $(k-1)$-tuples holds ([11]). In any case there exists $Y \in [\mathbb{N} \setminus \{0\}]^\omega$ such that $g \restriction [Y]^{k-1}$ is constant, say equal to 2. Let B' be the subbarrier of B with $\mathrm{base}(B') = Y$: we claim that the bad map $f \restriction B'$ has range 2, contradicting again Lemma 3.2.

To prove the claim let $s \in B'$: we have $s(0) > 0$ and hence (by condition (3') in the definition of barrier) there exists $s' \sqsubseteq s$, such that $t = \langle 0 \rangle^\frown s' \in B$. Then $f(t) = 2$ by the homogeneity of Y with respect to g; since $t \lhd s$ we have $f(s) \in \{0, 1\}$, as claimed. ⊣

REMARK 3.8. The proof of 5) of Theorem 3.7 can be generalized arguing by induction and obtaining for every $p \in \mathbb{N}$ a proof within ACA_0' of the following statement: "if B is a barrier and there exists $h : \mathrm{base}(B) \to \mathbb{N}$ such that $\forall s \in B\ \forall n \in \mathrm{base}(B)\ (s(0) = n \to \mathrm{lh}\, s \leq h(n))$, then every $f : B \to p$ is good".

REMARK 3.9. Theorems 3.5 and 3.7 should not be viewed as steps toward an inductive proof of "3 is bqo" in a system weaker than ATR_0: in fact it is known that ATR_0 is necessary for any comparability between ordinals to hold (Theorem 2.32 is the strongest result along these lines, but see also [9] and [42, theorem V.6.8]). These theorems are best understood as limitations on the sort of statement that can lead to a negative answer to the question of Problem 3.3. Notice however that the statement of Theorem 3.5 is $\mathbf{\Pi}^1_1$ and hence, by model theoretic arguments similar to the ones used in Section 2 with respect to $\mathbf{\Pi}^1_1$-CA_0 (using the results of Section VIII.4 of [42]), cannot imply ATR_0 over, say, the system of $\mathbf{\Sigma}^1_1$ dependent choice. On the other hand each of the statements appearing in Theorem 3.7 is $\mathbf{\Pi}^1_2$ and could have implied ATR_0.

§4. Equivalent definitions of wqo and bqo. The definitions of wqo and bqo of Section 1 are equivalent to many other statements about quasi-orderings (in [28] Milner lists seven equivalent characterizations of wqo). In this section we investigate which axioms are needed to prove some of these equivalences.

One of the alternative definitions of wqo is a strengthening requiring that every sequence of elements of Q contains an increasing subsequence.

LEMMA 4.1 (RT^2_2). *Let $\preceq$ be wqo on Q. For every $f : \mathbb{N} \to Q$ there exists $A \in [\mathbb{N}]^\omega$ such that for every $n, m \in A$ with $n < m$, $f(n) \preceq f(m)$.*

PROOF. Define $h : [\mathbb{N}]^2 \to \{0, 1\}$ by $h(\langle n, m \rangle) = 0$ if and only if $f(n) \preceq f(m)$ (since $\langle n, m \rangle$ represents a set we have $n < m$). By RT^2_2 there exists a set A which is homogeneous for h and, since Q is wqo, we must have $h \restriction [A]^2 = \{0\}$. ⊣

LEMMA 4.2 (RCA_0). *The statement of Lemma* 4.1 *implies* $\mathsf{RT}^1_{<\infty}$.

PROOF. Let $f : \mathbb{N} \to \{0, \ldots, k-1\}$. Since k is wqo, by the statement of Lemma 4.1 there exists an infinite set on which f is constant, i.e. we have $\mathsf{RT}^1_{<\infty}$. ⊣

CONJECTURE 4.3. Within RCA_0, the statement of Lemma 4.1 is equivalent to RT^2_2.

We now consider the intuitive definition of wqo we gave at the beginning of the paper:

LEMMA 4.4 (RCA_0). *Let $\preceq$ be a wqo on Q. With respect to $\preceq$ there are no infinite descending chains and no infinite sets of pairwise incomparable elements in Q.*

PROOF. If $f : \mathbb{N} \to Q$ is an infinite descending chain or a one-to-one enumeration of an infinite set of pairwise incomparable elements then for all $m < n$ we have $f(m) \npreceq f(n)$, contradicting the definition of wqo. ⊣

LEMMA 4.5 (RT^2_2). *Let $\preceq$ be a quasi-ordering on Q without infinite descending chains and infinite sets of pairwise incomparable elements. Then for every $f : \mathbb{N} \to Q$ there exists $A \in [\mathbb{N}]^\omega$ such that for every $n, m \in A$ with $n < m$, $f(n) \preceq f(m)$, and hence Q is wqo.*

PROOF. Let $f : \mathbb{N} \to Q$: define $g : [\mathbb{N}]^2 \to \{0, 1, 2\}$ by setting

$$g(\langle n, m\rangle) = \begin{cases} 0 & \text{if } f(n) \preceq f(m), \\ 1 & \text{if } f(m) \prec f(n), \\ 2 & \text{otherwise.} \end{cases}$$

By RT^2_2 there exists an infinite set $A \subseteq \mathbb{N}$ which is homogeneous for g. Since Q is has no infinite descending sequence, g cannot have value 1 on A. If g has value 2 on A then $f \restriction A$ is one-to-one: RCA_0 proves that the range of a one-to-one function always contains an infinite set and hence there exists an infinite set of pairwise incomparable elements in Q, contradicting our hypothesis. Thus g has value 0 on A and $f(m) \preceq f(n)$ for all $m, n \in A$ with $m < n$. ⊣

Lemma 4.5 was first noticed by Simpson (personal communication).

CONJECTURE 4.6. Within RCA_0, the statement "every quasi-ordering without infinite descending chains and infinite sets of pairwise incomparable elements is wqo" is equivalent to RT^2_2.

Another notion which is equivalent to wqo is the finite basis property:

DEFINITION 4.7. A quasi-ordering $\preceq$ on Q has the *finite basis property* if for every $X \subseteq Q$ there exists a finite set $Y \subseteq X$ such that $\forall x \in X\ \exists y \in Y\ y \preceq x$.

The following lemma is essentially Lemma 3.2 of [41], although in that paper Simpson considers the finite basis property for sequences rather than sets.

LEMMA 4.8 (RCA_0). *A quasi-ordering is wqo if and only if it has the finite basis property.*

PROOF. Let Q be wqo with respect to $\preceq$ and $X \subseteq Q$. If X is finite take $Y = X$, otherwise let $f : \mathbb{N} \to Q$ be an enumeration of X. Let $A = \{n \mid \forall m < n\ f(m) \npreceq f(n)\}$. Since $f(m) \npreceq f(n)$ for every $m, n \in A$ with $m < n$, if A is infinite then $f \restriction A$ is a bad sequence, contradicting our hypothesis.

Hence $Y = \{f(n) \mid n \in A\} \subseteq X$ is finite. For every $x \in X$ let n be least such that $f(n) \preceq x$; then $n \in A$ and therefore $\forall x \in X\, \exists y \in Y\; y \preceq x$.

Let Q have the finite base property with respect to $\preceq$ and let $f : \mathbb{N} \to Q$. Notice that if f is not one-to-one then any pair of numbers $m < n$ for which $f(m) = f(n)$ witnesses that f is good. Therefore, let us suppose that f is one-to-one. Define a strictly increasing $g : \mathbb{N} \to \mathbb{N}$ by setting $g(0) = 0$ and letting $g(i+1)$ be the least $n > g(i)$ such that $f(n) > f(g(i))$ (here the elements of Q are compared according to the usual order of $\mathbb{N}$). Since $f \circ g$ is strictly increasing (with respect to the usual order of $\mathbb{N}$) the set $X = \{x \in Q \mid \exists i\; x = f(g(i))\}$ exists within RCA_0. By the finite basis property there exists a finite $Y \subseteq X$ such that $\forall x \in X\, \exists y \in Y\; y \preceq x$. Let

$$j = \max\{i \mid f(g(i)) \in Y\} + 1.$$

Since $f(g(j)) \in X$ there exists $y \in Y$ such that $y \preceq f(g(j))$. By definition of j we have $y = f(g(i))$ for some $i < j$ and, since $g(i) < g(j)$, f is good. $\dashv$

We now turn to alternative characterizations of bqo. The equivalence between i) and ii) in the next theorem was conjectured by Clote in [3]. iii) is the analogue for bqo of the statement of Lemma 4.1.

Theorem 4.9 (RCA_0). *The following are equivalent*:

i) ATR_0;

ii) *the "barrier theorem": if B is a barrier and $f : B \to \{0,1\}$ then there exists a subbarrier $B' \subseteq B$ such that $f \restriction B'$ is constant*;

iii) *if Q is bqo with respect to $\preceq$ then for every barrier B and every map $f : B \to Q$ there exists a subbarrier $B' \subseteq B$ such that $f \restriction B'$ is perfect.*

Proof. i) implies ii) follows easily from the fact that ATR_0 proves the clopen Ramsey's theorem (see [26, theorem 2.13] for details).

For ii) implies iii) see the proof of Lemma 2.15 in [26]: the proof goes through in RCA_0 except where ii) is used.

To prove that iii) implies ii) view $\{0,1\} = 2$ as in Section 3. By Lemma 3.2 this is a bqo and a perfect map $f : B' \to 2$ must be constant, so that ii) is a special case of iii).

To prove that ii) implies i) first of all notice that the proof of Lemma V.9.5 of [42] (showing that the clopen Ramsey's theorem implies ACA_0) works verbatim assuming the barrier theorem. Thus we can work within ACA_0. We will work out some details of the proof (due to Jockusch) of Theorem V.9.6 of [42]; this theorem states that the clopen Ramsey's theorem implies ATR_0 over ACA_0. We will actually show that the same proof works assuming the (apparently weaker) barrier theorem.

Arguing as in that proof we suppose $\{T^i_k \mid k \in \mathbb{N}\}$ are, for $i = 0, 1$, sequences of trees with the property that for every k at least one of T^0_k and T^1_k has no path: our goal is to prove the existence of a set Z such that if T^1_k has a path then $k \in Z$, while if T^0_k has a path then $k \notin Z$.

As in [42], for a tree T and $X \in [\mathbb{N}]^\omega$, X *majorizes* T means that T has a path g such that $g(n) \leq X(n)$ for every n. Similarly, $s \in [\mathbb{N}]^{<\omega}$ *majorizes* T if there exists $\tau \in T$ with $\operatorname{lh} \tau = \operatorname{lh} s$ such that $\tau(n) \leq s(n)$ for every $n < \operatorname{lh} s$. Using König's lemma (which is provable within ACA_0) it is easy to show that for every $W \in [\mathbb{N}]^\omega$ and every k there exists m such that $W[m]$ does not majorize at least one of T^0_k and T^1_k.

For any $W \in [\mathbb{N}]^\omega$ let m_W be the least m such that $W[1,m]$ does not majorize at least one of T^0_k and T^1_k for every $k \leq W(0)$. Now let n_W be the least n such that $W[m_W, m_W + n]$ does not majorize at least one of T^0_k and T^1_k for every $k \leq W(0)$.

At this stage in [42] one defines a clopen subset $P \subseteq [\mathbb{N}]^\omega$ by declaring that $W \in P$ if and only if for all $k \leq W(0)$ we have

$$W[1, m_W] \text{ majorizes } T^1_k \longleftrightarrow W[m_W, m_W + n_W] \text{ majorizes } T^1_k.$$

This suffices for applying the clopen Ramsey's theorem, but to use the barrier theorem we need to go deeper into the details and show that this clopen set can be coded by a barrier.

Let $B = \{W[m_W + n_W] \mid W \in [\mathbb{N}]^\omega\}$. This definition is perspicuous but not arithmetical. To check that B can be defined within ACA_0 (actually RCA_0 suffices here) notice that $s \in B$ if and only if there exist $m_s, n_s < \operatorname{lh} s$ such that

a) $\operatorname{lh} s = m_s + n_s$;
b) $s[1, m_s]$ does not majorize at least one of T^0_k and T^1_k and $s[m_s, m_s + n_s]$ does not majorize at least one of T^0_k and T^1_k for all $k \leq s(0)$;
c) for all $m' < m_s$ there exists $k \leq s(0)$ such that $s[1, m']$ majorizes both T^0_k and T^1_k;
d) for all $n' < n_s$ there exists $k \leq s(0)$ such that $s[m_s, m_s + n']$ majorizes both T^0_k and T^1_k.

For each $s \in B$ let us fix such m_s, n_s.

Claim 1. *B is a barrier.*

Proof of claim. Conditions (1) and (2) of the definition of barrier are immediate, so we are left with checking (3'). Notice that if $s, t \in [\mathbb{N}]^{<\omega}$ are such that $s \subset t$ then $\operatorname{lh} s < \operatorname{lh} t$ and $s(i) \geq t(i)$ for every $i < \operatorname{lh} s$; therefore if $t[\operatorname{lh} s]$ majorizes a tree so does s.

Suppose now, towards a contradiction, that $s, t \in B$ are such that $s \subset t$. Then $s(0) \geq t(0)$ and $s[1, \operatorname{lh} s] \subset t[1, \operatorname{lh} t]$: the latter condition implies that either $s[1, m_s] \subset t[1, m_t]$ or $s[m_s, m_s + n_s] \subset t[m_t, m_t + n_t]$. In the first case $m_s < m_t$ and $s[1, m_s]$ does not majorize at least one of T^0_k and T^1_k for all $k \leq s(0)$. Since $t(0) \leq s(0)$ the above observation implies that $t[1, m_s]$ does not majorize at least one of T^0_k and T^1_k for all $k \leq t(0)$, contradicting the minimality of m_t. The second case contradicts analogously the minimality of n_t. ⊣

Let $f : B \to \{0, 1\}$ be defined so that $f(s) = 1$ if and only if for all $k \leq s(0)$ we have

$$s[1, m_s] \text{ majorizes } T^1_k \longleftrightarrow s[m_s, m_s + n_s] \text{ majorizes } T^1_k.$$

Then the clopen P defined in [42] is exactly

$$\{W \in [\mathbb{N}]^\omega \mid \exists s \in B \ (s \sqsubset W \wedge f(s) = 1)\}.$$

The barrier theorem implies that there exists a subbarrier B' such that $f \restriction B'$ is constant. If we let $U = \text{base}(B')$ then either $[U]^\omega \subseteq P$ or $[U]^\omega \cap P = \emptyset$ and we can follow the proof in [42], showing that the second alternative is impossible and defining the set Z we are seeking. ⊣

§5. Closure of wqo and bqo under elementary operations.

We start the study of elementary operations on quasi-orderings by considering two quasi-orderings that can be defined on the power set $\mathcal{P}(Q)$ of a quasi-ordering Q. (For a thorough study of these quasi-orderings from the viewpoint of the fine analysis of bqos see [27].)

Definition 5.1 (RCA_0). Let $\preceq$ be a quasi-ordering on Q. If $X, Y \subseteq \mathcal{P}(Q)$ let

$$X \preceq^{\exists}_{\forall} Y \iff \forall x \in X\, \exists y \in Y\ x \preceq y \quad \text{and}$$
$$X \preceq^{\forall}_{\exists} Y \iff \forall y \in Y\, \exists x \in X\ x \preceq y.$$

$\equiv^{\exists}_{\forall}$ and $\equiv^{\forall}_{\exists}$ will denote the equivalence relations induced by the two quasi-orderings.

We will also consider $\preceq^{\exists}_{\forall}$ and $\preceq^{\forall}_{\exists}$ restricted to $\mathcal{P}_{\mathrm{f}}(Q)$, the set of all finite subsets of Q.

If $\preceq$ is bqo on Q then $\preceq^{\exists}_{\forall}$ and $\preceq^{\forall}_{\exists}$ are both bqo on $\mathcal{P}(Q)$, and if $\preceq$ is wqo on Q then $\preceq^{\exists}_{\forall}$ is wqo on $\mathcal{P}_{\mathrm{f}}(Q)$ (as we will show). On the other hand Rado's example described in Section 2 shows that $\preceq$ wqo on Q does not imply $\preceq^{\forall}_{\exists}$ wqo on $\mathcal{P}_{\mathrm{f}}(Q)$ (consider the bad sequence defined by $f(n) = \{\langle m, n\rangle \mid m < n\}$).

The following well-known construction will be useful in our study of $\preceq^{\exists}_{\forall}$ and $\preceq^{\forall}_{\exists}$.

Definition 5.2 (RCA_0). If B is a block let $B^2 = \{s \cup t \mid s, t \in B \wedge s \lhd t\}$.

The main properties of B^2 are provable within RCA_0 and are collected in the following lemma:

Lemma 5.3 (RCA_0). *Let B be a block.*

a) *B^2 is a block;*
b) *for every $t \in B^2$ there exist unique $\pi_0(t), \pi_1(t) \in B$ such that $\pi_0(t) \lhd \pi_1(t)$ and $t = \pi_0(t) \cup \pi_1(t)$;*
c) *if $t, t' \in B^2$ and $t \lhd t'$ then $\pi_1(t) = \pi_0(t')$;*
d) *if B is a barrier then B^2 is a barrier.*

Theorem 5.4. *Let $\preceq$ be a quasi-ordering on Q and let $\preceq_*^*$ be either $\preceq_\forall^\exists$ or $\preceq_\exists^\forall$.*

1) ACA_0 *proves that if Q is bqo with respect to $\preceq$ then $\mathcal{P}(Q)$ is bqo with respect to $\preceq_*^*$;*
2) RCA_0 *proves that if Q is bqo with respect to $\preceq$ then $\mathcal{P}_\mathrm{f}(Q)$ is bqo with respect to $\preceq_*^*$.*

Proof. We first deal with both statements when $\preceq_*^*$ is $\preceq_\forall^\exists$. Assume that $\mathcal{P}(Q)$ is not bqo with respect to $\preceq_\forall^\exists$ and let $f : B \to \mathcal{P}(Q)$ be bad, where B is a barrier. We define a bad $g : B^2 \to Q$. To this end if $t \in B^2$ let $X_t = \{q \in f(\pi_0(t)) \mid \forall y \in f(\pi_1(t))\ q \not\preceq y\}$: in general arithmetical comprehension is needed to prove the existence of X_t, but if the range of f consists of finite sets RCA_0 suffices. $X_t \neq \emptyset$ because $f(\pi_0(t)) \not\preceq_\forall^\exists f(\pi_1(t))$, which is a consequence of the badness of f. Now let $g(t)$ be the minimum (with respect to the usual ordering of $\mathbb{N}$) of X_t. g is bad with respect to $\preceq$: indeed if $t, t' \in B^2$ are such that $t \lhd t'$ we have $\pi_1(t) = \pi_0(t')$ and hence $g(t) \not\preceq g(t')$. Thus Q is not bqo.

The proofs when $\preceq_*^*$ is $\preceq_\exists^\forall$ are similar. Given $f : B \to \mathcal{P}(Q)$ bad with respect to $\preceq_\exists^\forall$, for every $t \in B^2$ let $Y_t = \{q \in f(\pi_1(t)) \mid \forall x \in f(\pi_0(t))\ x \not\preceq q\}$. The existence of Y_t requires ACA_0, but RCA_0 suffices if the range of f is contained in $\mathcal{P}_\mathrm{f}(Q)$. $Y_t \neq \emptyset$ because f is bad and we can define $g : B^2 \to Q$ by letting $g(t)$ be the minimum of Y_t. Arguing as above we can prove that g is bad with respect to $\preceq$. ⊣

For $\preceq_\exists^\forall$ Theorem 5.4.1 can be improved:

Lemma 5.5 (RCA_0). *If Q is wqo then for every $X \in \mathcal{P}(Q)$ there exists $Y \in \mathcal{P}_\mathrm{f}(Q)$ such that $Y \equiv_\exists^\forall X$. Moreover this can be proved uniformly, i.e. if $\{X_k \mid k \in \mathbb{N}\}$ is a sequence of elements of $\mathcal{P}(Q)$ there exists a sequence $\{Y_k \mid k \in \mathbb{N}\}$ of elements of $\mathcal{P}_\mathrm{f}(Q)$ such that $Y_k \equiv_\exists^\forall X_k$ for every k.*

Proof. The first statement follows immediately from Lemma 4.8: indeed if $Y \subseteq X$ then $X \preceq_\exists^\forall Y$, and $\forall x \in X\ \exists y \in Y\ y \preceq x$ means $X \preceq_\exists^\forall Y$. Moreover it is clear that the proof of that lemma is uniform, since each X_k can be enumerated in increasing order. ⊣

Lemma 5.5 and Theorem 5.4.2 yield:

Theorem 5.6 (RCA_0). *If Q is bqo then $\mathcal{P}(Q)$ is bqo with respect to $\preceq_\exists^\forall$.*

Theorem 5.6 shows that an (admittedly fairly weak) infinitary closure property of bqo can be established within a weak subsystem of second order arithmetic. This contrasts sharply with the results mentioned in Section 2, and in particular with Theorem 2.32 and its consequences.

An interesting phenomenon can be noticed by considering the two finitary operations that associate to a quasi-ordering Q respectively $\mathcal{P}_\mathrm{f}(Q)$ quasi-ordered by $\preceq_\exists^\forall$ and $Q^{<\omega}$ quasi-ordered by embeddability. As noticed above the former operation does not preserve the notion of wqo, while Higman's theorem states that the latter does: this state of affairs suggests that the former operation is more complex than the latter. The former operation does preserve the notion of bqo and by Theorem 5.4.2 this is provable in RCA_0. The generalized Higman's theorem shows that the latter operation also preserves the notion of bqo. By Theorem 2.9 Higman's theorem is equivalent to ACA_0 and by Theorem 2.32 the generalized Higman's theorem implies ATR_0. Thus from an axiomatic viewpoint the latter operation appears to be much more complex than the former.

PROBLEM 5.7. What is the axiomatic strength of "if Q is bqo then $\mathcal{P}(Q)$ is bqo with respect to $\preceq^{\exists}_{\forall}$"?

To complete the study of $\preceq^{\exists}_{\forall}$ we will use the product quasi-ordering:

DEFINITION 5.8 (RCA_0). If Q_1 and Q_2 are quasi-ordered by $\preceq_1$ and $\preceq_2$ respectively, then $Q_1 \times Q_2$ is quasi-ordered by the *product quasi-ordering* defined by

$$(x_1, x_2) \preceq_\times (y_1, y_2) \iff x_1 \preceq_1 y_1 \land x_2 \preceq_2 y_2.$$

LEMMA 5.9 (RT^2_2). *If Q_1 and Q_2 are wqo with respect to $\preceq_1$ and $\preceq_2$ then $Q_1 \times Q_2$ is wqo with respect to the product quasi-ordering.*

PROOF. Let $f : \mathbb{N} \to Q_1 \times Q_2$ and suppose $f(n) = (f_1(n), f_2(n))$ for every n. Since Q_1 is wqo with respect to $\preceq_1$ by Lemma 4.1 there exists $A \in [\mathbb{N}]^\omega$ such that for every $n, m \in A$ with $n < m$, $f_1(n) \preceq f_1(m)$. Since Q_2 is wqo with respect to $\preceq_1$ the sequence $f_2 \restriction A$ is good and there exists $m, n \in A$ with $m < n$ such that $f_2(m) \preceq_2 f_2(n)$. Therefore $f(m) \preceq_\times f(n)$. ⊣

THEOREM 5.10 (RT^2_2). *The following are equivalent*:

i) ACA_0;

ii) *if Q is wqo then $\mathcal{P}_{\mathrm{f}}(Q)$ is wqo with respect to $\preceq^{\exists}_{\forall}$*;

PROOF. i) implies ii) follows easily from Theorem 2.9, because Higman's theorem implies ii).

To prove that ii) implies i) we will use the fact that ACA_0 is equivalent over RCA_0 (and a fortiori over RT^2_2) to the statement "for every well-ordering L the linear ordering 2^L is a well-ordering" ([12, p. 299], see [15, theorem 2.6] for a direct proof). Here, if $\preceq$ is a linear ordering on L, 2^L consists of all finite sequences $\langle \ell_0, \dots, \ell_{k-1} \rangle$ such that $\ell_i \in L$ and $\ell_{i+1} \prec \ell_i$, ordered lexicographically with respect to $\preceq$.

We reason within RT^2_2. Let L be well ordered by $\preceq$; in particular L is wqo. Denote by ω the quasi-ordering consisting of $\mathbb{N}$ with the usual ordering (it is obviously wqo). Define $Q = L \times \omega$ with the product ordering $\preceq_\times$. By Lemma 5.9, RT^2_2 proves that Q is wqo, and hence ii) implies that $\mathcal{P}_{\mathrm{f}}(Q)$ is wqo with respect to $\preceq^{\exists}_{\forall}$.

Now let $g : \mathbb{N} \to 2^L$ be such that $g(n) = \langle \ell^n_0, \dots, \ell^n_{k(n)-1} \rangle$. We need to show that g is not strictly descending. Define $f : \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(Q)$ by $f(n) = \{(\ell^n_i, i) \mid i < k(n)\}$. There exist $m < n$ such that $f(m) \preceq^{\exists}_{\forall} f(n)$. Therefore

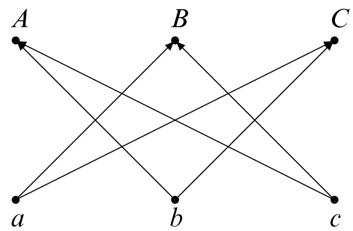

FIGURE 1

for each $i < k(m)$ there exists $j < k(n)$ such that $\ell_i^m \preceq \ell_j^n$ and $i \leq j$: this implies that $\ell_i^m \preceq \ell_i^n$. Therefore $\langle \ell_0^m, \dots, \ell_{k(m)-1}^m \rangle$ precedes $\langle \ell_0^n, \dots, \ell_{k(n)-1}^n \rangle$ in the lexicographic ordering. ⊣

In [5, theorem 3] Clote claims to prove within RCA_0 the implication ii) $\Longrightarrow$ i) of Theorem 5.10, but his proof works only for a finer quasi-ordering on $\mathcal{P}(Q)$. In his quasi-ordering, X precedes Y if there exists a *one-to-one* map $h : X \to Y$ such that $x \preceq h(x)$ for every $x \in X$.

As an application of the results about the quasi-orderings on $\mathcal{P}(Q)$, we can prove (using the terminology of Section 3):

THEOREM 5.11. *Let* T *be a subsystem of second order arithmetic containing* RCA_0 *and suppose that* T *proves that* 3 *is bqo. Then for any* $p > 0$ *"p is bqo" is a theorem of* T.

PROOF. The proof is by induction on p, the base case being provided by the hypothesis (since $p = 1$ is trivial and $p = 2$ follows from Lemma 3.2).

The first inductive step (leading from $p = 3$ to $p = 4$) is nontrivial. We argue within T and notice that $\mathcal{P}(3)$ quasi-ordered by $\preceq_{\exists}^{\forall}$ contains the quasi-ordering Q_1 represented by the connected graph of Fig. 1 (where e.g. $A = \{0\}$ and $a = \{1, 2\}$). By Theorem 5.6 (and the hypothesis) Q_1 is bqo. Applying again Theorem 5.6 we have that $\mathcal{P}(Q_1)$ is bqo with respect to $\preceq_{\exists}^{\forall}$. The elements $\{A, B, C\}$, $\{A, B, c\}$, $\{A, b, C\}$, and $\{a, B, C\}$ of $\mathcal{P}(Q_1)$ are pairwise incomparable with respect to $\preceq_{\exists}^{\forall}$, and hence it follows that 4 is bqo.

The following inductive steps are obtained by observing that when $p \geq 4$ there are sets (e.g. $[p]^2$) of more than p elements of $\mathcal{P}(p)$ which are pairwise incomparable with respect to $\preceq_{\exists}^{\forall}$, so that Theorem 5.6 yields the desired conclusion. ⊣

We will now show that even basic closure properties of bqo need fairly strong set-existence axioms. One such operation is product (Definition 5.8), while others are sum and disjoint union:

DEFINITION 5.12 (RCA_0). Let Q_1 and Q_2 be quasi-ordered by $\preceq_1$ and $\preceq_2$ respectively. We may assume that $Q_1 \cap Q_2 = \emptyset$ (or replace each Q_i by its isomorphic copy on $Q_i \times \{i\}$). The set $Q_1 \cup Q_2$ is denoted either by $Q_1 + Q_2$

or by $Q_1 \uplus Q_2$ and quasi-ordered respectively by the *sum quasi-ordering* and the *disjoint union quasi-ordering* defined by

$$x \preceq_+ y \iff (x \in Q_1 \wedge y \in Q_2) \vee \exists i < 2\,(x, y \in Q_i \wedge x \preceq_i y);$$
$$x \preceq_\uplus y \iff \exists i < 2\,(x, y \in Q_i \wedge x \preceq_i y).$$

The following lemma is an easy consequence of the infinite pigeonhole principle for two colors:

Lemma 5.13 (RCA_0). *If Q_1 and Q_2 are wqo then $Q_1 + Q_2$ and $Q_1 \uplus Q_2$ are wqo with respect to the sum and disjoint union quasi-orderings.*

RCA_0 proves also that the sum preserves bqo:

Lemma 5.14 (RCA_0). *If Q_1 and Q_2 are bqo then $Q_1 + Q_2$ is bqo with respect to the sum quasi-ordering.*

Proof. Let B be a barrier and suppose towards a contradiction that $f : B \to Q_1 + Q_2$ is bad. If $f(s) \in Q_2$ for all $s \in B$ then $f : B \to Q_2$ is bad, and this contradicts Q_2 bqo.

Hence we may assume that $f(s) \in Q_1$ for some $s \in B$. Fix such s and let

$$B' = \{t \in B \mid t(0) > s(\operatorname{lh} s - 1)\};$$

B' is a subbarrier of B. If $t \in B'$, arguing as in the proof of Lemma 3.2, we can find $s_0 = s, s_1, \dots, s_{\operatorname{lh} s} = t \in B$ such that $s_i \lhd s_{i+1}$ for every $i < \operatorname{lh} s$. Since f is bad, by induction we can prove that $f(s_i) \in Q_1$ for every $i \leq \operatorname{lh} s$, and in particular $f(t) \in Q_1$. Hence $f \restriction B' : B' \to Q_1$ is bad, and this contradicts Q_1 bqo. $\dashv$

Remark 5.15. Combining Lemmas 3.2 and 5.14 we obtain that for every finite quasi-ordering Q with the property that every element is incomparable to at most one other element, RCA_0 proves that Q is bqo.

The closure of bqos with respect to other operations require more than RCA_0:

Lemma 5.16 (RCA_0). *The following are equivalent*:

i) *if Q_1 and Q_2 are bqo then $Q_1 \times Q_2$ is bqo with respect to the product quasi-ordering*;
ii) *if Q_1 and Q_2 are bqo then $Q_1 \uplus Q_2$ is bqo with respect to the disjoint union quasi-ordering*;
iii) *if Q is bqo with respect to the quasi-orderings $\preceq_1$ and $\preceq_2$ then Q is bqo with respect to the quasi-ordering $\preceq_1 \cap \preceq_2$.*

Proof. To prove that i) implies ii) assume Q_1 and Q_2 are disjoint bqos and for $i = 0, 1$ let Q_i' be the quasi-ordering consisting of a single element $m_i \notin Q_i$ (which is obviously a bqo). By Lemma 5.14, $Q_i + Q_i'$ is bqo.

Given $f : B \to Q_1 \cup Q_2$, where B is a barrier, define $g : B \to (Q_1 + Q_1') \times (Q_2 + Q_2')$ as follows:

$$g(s) = \begin{cases} (f(s), m_2) & \text{if } f(s) \in Q_1, \\ (m_1, f(s)) & \text{if } f(s) \in Q_2. \end{cases}$$

By i) $(Q_1 + Q_1') \times (Q_2 + Q_2')$ is bqo with respect to the product quasi-ordering and hence there exist $s, t \in B$ such that $s \lhd t$ and $g(s) \preceq_\times g(t)$. If $f(s) \in Q_1$ and $f(t) \in Q_2$ then $m_2 \preceq_+ f(t)$, which is impossible, and similarly we rule out $f(s) \in Q_2$ and $f(t) \in Q_1$. Therefore $f(s), f(t) \in Q_i$ for some i, and clearly $f(s) \preceq_i f(t)$: thus $f(s) \preceq_\cup f(t)$, and f is good.

To prove that ii) implies i) assume $Q_1 \cap Q_2 = \emptyset$, let B be a barrier and $f : B \to Q_1 \times Q_2$. Let $f_1 : B \to Q_1$ and $f_2 : B \to Q_2$, be such that $f(s) = (f_1(s), f_2(s))$ for every $s \in B$. Define $g : B \to \mathcal{P}_\mathrm{f}(Q_1 \cup Q_2)$ by $g(s) = \{f_1(s), f_2(s)\}$. By ii) $Q_1 \cup Q_2$ is bqo with respect to the disjoint union quasi-ordering and hence, by Theorem 5.4.2, $\mathcal{P}_\mathrm{f}(Q_1 \cup Q_2)$ is bqo under either of the quasi-orderings $\preceq_\forall^\exists$ and $\preceq_\exists^\forall$ induced by $\preceq_\cup$. Hence there exist $s, t \in B$ such that $s \lhd t$ and $g(s) \preceq_\exists^\forall g(t)$. It is immediate that $f_1(s) \preceq_1 f_1(t)$ and $f_2(s) \preceq_2 f_2(t)$ hold, so that $f(s) \preceq_\times f(t)$. Therefore f is good.

To prove that i) implies iii) let B be a barrier and $f : B \to Q$. Define $g : B \to Q \times Q$ by $g(s) = (f(s), f(s))$. By i) the cartesian product of $(Q, \preceq_1)$ and $(Q, \preceq_2)$ is bqo and there exist $s, t \in B$ with $s \lhd t$ and $g(s) \preceq_\times g(t)$. This means that $f(s) \preceq_1 f(t)$ and $f(s) \preceq_2 f(t)$, so that f is good with respect to $\preceq_1 \cap \preceq_2$.

To prove that iii) implies i), on the set $Q_1 \times Q_2$ define two quasi-orderings by setting $(x, y) \preceq_1^* (x', y')$ if and only if $x \preceq_1 x'$, and $(x, y) \preceq_2^* (x', y')$ if and only if $y \preceq_2 y'$. It is easy to show that $Q_1 \times Q_2$ is bqo with respect to both $\preceq_1^*$ and $\preceq_2^*$. Since the product quasi-ordering of $\preceq_1$ and $\preceq_2$ coincides with $\preceq_1^* \cap \preceq_2^*$, we are done. ⊣

Lemma 5.17 (RCA_0). *Each of the equivalent clauses of Lemma* 5.16 *implies* ACA_0.

Proof. Obviously it suffices to show that i) implies ACA_0. RCA_0 proves that every well-ordering is bqo (Lemma 3.1), and that Q bqo implies $\mathcal{P}_\mathrm{f}(Q)$ bqo, and hence wqo, with respect to $\preceq_\forall^\exists$ (Theorem 5.4.2 and Lemma 1.11). Using these facts, the proof of ii) implies i) in Theorem 5.10 can be translated into a proof within RCA_0 that the closure of bqos under cartesian products implies ACA_0. ⊣

Actually the proof of Lemma 5.17 shows that the statement "the product of a well-ordering and ω is bqo" already implies ACA_0. The obvious proof of any of i), ii), and iii) of Lemma 5.16 uses the barrier theorem, and hence each of these statements is provable in ATR_0.

CONJECTURE 5.18. The statements contained in Lemma 5.16 are equivalent to ATR_0.

REMARK 5.19. Notice that RCA_0 plus either i) or ii) of Lemma 5.16 implies that 3 is bqo. Therefore, either directly or by Theorem 5.11, for any p this theory proves "p is bqo".

On the other hand, if T contains RCA_0 and does not prove "3 is bqo" then T does not prove the statements of Lemma 5.16. Therefore if the answer to Problem 3.3 is negative then Conjecture 5.18 holds.

What about the corresponding question for wqo? In Lemma 5.13 we noticed that wqos are closed both with respect to sums and disjoint unions. On the other hand the proof of the equivalence between i) and iii) of Lemma 5.16 translates to the wqo case and yields:

LEMMA 5.20 (RCA_0). *The following are equivalent*:

i) *if Q_1 and Q_2 are wqo then $Q_1 \times Q_2$ is wqo with respect to the product quasi-ordering*;
ii) *if Q is wqo with respect to the quasi-orderings $\preceq_1$ and $\preceq_2$ then Q is wqo with respect to the quasi-ordering $\preceq_1 \cap \preceq_2$.*

By Lemma 5.9, i) and ii) of Lemma 5.20 are provable in RT^2_2.

It may be worthwhile to see why the proof of ii) implies i) of Lemma 5.16 does not translate to the wqo case: if we use $\preceq^{\forall}_{\exists}$ then it is false that $Q_1 \cup Q_2$ wqo implies $\mathcal{P}_{\mathrm{f}}(Q_1 \cup Q_2)$ wqo, while if we use $\preceq^{\exists}_{\forall}$ then the proof requires ACA_0 by Theorem 5.10.

PROBLEM 5.21. What is the axiomatic strength of i) and ii) of Lemma 5.20?

REMARK 5.22. It follows from [16] that if Q_1 and Q_2 are wqo and admit reifications of order type resp. α_1 and α_2 then $Q_1 \times Q_2$ admits a reification of order type $\alpha_1 \otimes \alpha_2$ (where $\otimes$ denotes natural product of ordinals). Since RCA_0 proves that the product of two well-orderings is a well-ordering one could hope to prove i) of Lemma 5.20 within RCA_0, by using the technique described in the sketch of the proof of Theorem 2.9. However this is not straightforward since it is not clear that RCA_0 proves that each wqo admits a reification. Moreover the natural product of ordinals is based on Cantor's normal form theorem, which is equivalent to ATR_0 ([15]).

Special instances of the closure of wqos under product have been studied by Simpson ([41]):

THEOREM 5.23. RCA_0 *proves each of the following*:

1. *The product of two copies of ω is wqo with respect to the product quasi-ordering.*

2. *The following are equivalent*:
   i) ω^ω *is well-ordered*;
   ii) *for every* $k \in \mathbb{N}$ *the product of* k *copies of* ω *is wqo with respect to the* (*obvious generalization of the*) *product quasi-ordering.*

Since ω^ω is the proof theoretic ordinal of RCA_0, it follows that RCA_0 does not prove the statement of ii) above.

Note added July 31, 2003. Some of the open problems listed in this paper have been addressed in the paper *Reverse mathematics and the equivalence of definitions for well and better quasi-orders* [2] by Peter Cholak, Alberto Marcone and Reed Solomon, to appear in *The Journal of Symbolic Logic*. This paper studies in detail the strength of the implications between different definitions of wqo and bqo.

In particular, Problem 1.7 is answered affirmatively, some results relevant to Conjectures 4.3 and 4.6 (whose truth depends in part on the as yet quite unclear relationship between a consequence of RT^2_2 and full RT^2_2) are obtained, and Problem 5.21 is partially answered by showing that WKL_0 does not suffice to prove the statements under consideration.

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DIPARTIMENTO DI MATEMATICA E INFORMATICA
UNIVERSITÀ DI UDINE
VIALE DELLE SCIENZE 206
33100 UDINE, ITALY
E-mail: marcone@dimi.uniud.it