COLORING LINEAR ORDERS WITH RADO'S PARTIAL ORDER

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Abstract. Let $\preceq_{\mathsf{R}}$ be the preorder of embeddability between countable linear orders colored with elements of Rado's partial order (a standard example of a wqo which is not a bqo). We show that $\preceq_{\mathsf{R}}$ has fairly high complexity with respect to Borel reducibility (e.g. if P is a Borel preorder then $P \leq_B \preceq_{\mathsf{R}}$), although its exact classification remains open.

1. Introduction

This note is a contribution to the study of the relations of embeddability for countable colored linear orderings from the point of view of descriptive set theory. Fixing ω as the support of a countable linear ordering, we can consider the space LO of all linear orderings on ω. This is a standard Borel space (actually a Polish space, see [Kec95]). For our purposes, to color a linear order means assigning to each element of the support an element from a fixed countable set C. A colored linear ordering on ω is thus an element of $LO \times C^\omega$, which is also a standard Borel space.

If Q is a partial order on C we have a natural relation of embeddability $\preceq_Q$ on $LO \times C^\omega$ defined by letting $(\sqsubseteq, \varphi) \preceq_Q (\sqsubseteq', \varphi')$ if and only if there exists $g : \omega \to \omega$ such that:

(1) $\forall a, b \in \omega \ (a \sqsubseteq b \iff g(a) \sqsubseteq' g(b))$;
(2) $\forall a \in \omega \ \varphi(a) \, Q \, \varphi'(g(a))$.

Then $\preceq_Q$ is a $\mathbf{\Sigma}^1_1$ preorder which is not Borel and it can be studied in the framework of Borel reducibility.

Given binary relations P and P' on standard Borel spaces X and X' respectively, recall that $P \leq_B P'$ means that there exists a Borel function $f : X \to X'$ such that $x \, P \, y \iff f(x) \, P' \, f(y)$. A $\mathbf{\Sigma}^1_1$ preorder P' is called *$\mathbf{\Sigma}^1_1$-complete* if and only if $P \leq_B P'$ for any $\mathbf{\Sigma}^1_1$ preorder P.

For preorders of the form $\preceq_Q$ it is known that:

- if Q is a bqo then, by Laver's celebrated theorem ([Lav71]), $\preceq_Q$ is a bqo as well; in particular it is a wqo and the relation of equality on ω is not reducible to $\preceq_Q$; thus in this case $\preceq_Q$ is quite far from being a $\mathbf{\Sigma}^1_1$-complete preorder;

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- if Q is not a wqo, then $\preceq_Q$ is a $\boldsymbol{\Sigma}^1_1$-complete preorder (see [Cam05], a partial result in this direction is in [MR04]).

These results leave unanswered the question of the complexity of the relation $\preceq_Q$ when Q is a wqo but not a bqo.

If α is a countable ordinal then a preorder Q is ω^α-wqo if $\preceq_Q$ restricted to well-orders of order type strictly less than ω^α is a wqo. Therefore ω-wqo means wqo and a preorder is bqo if and only if it is ω^α-wqo for every α (details can be found e.g. in [Mar94]).

In this note we consider Rado's partial order R defined on the set $D = \{\, (n,m) \in \omega^2 \mid n < m \,\}$ by

$$(n,m)\,\mathsf{R}\,(n',m') \iff (n = n' \wedge m \leq m') \vee m < n'.$$

R is wqo but it is not ω^2-wqo. Moreover R embeds into every wqo which is not ω^2-wqo ([Rad54]). R is the best known example of a wqo which is not bqo.

We will need witnesses for the fact that R is not ω^2-wqo. Let $r_i \in D^\omega$ be defined by letting $r_i(n) = (i, i+1+n)$ for each $n \in \omega$. Note that if $i < i'$, then r_i and $r_{i'}$ are incomparable under $\preceq_{\mathsf{R}}$: indeed $(i',n)\,\mathsf{R}\,(i,m)$ does not hold for any n,m, while $(i,i')\,\mathsf{R}\,(i',m)$ does not hold for any m. Therefore, for $i \neq i'$, we can fix $\gamma_{ii'} \in \omega$ such that $r_i(\gamma_{ii'})\,\mathsf{R}\,r_{i'}(m)$ does not hold for any m.

In section 2 we show that for any Borel preorder P, the relation $P \leq_B \preceq_{\mathsf{R}}$ holds (by the above observation the result extends to all $\preceq_Q$ where Q is wqo but not ω^2-wqo). This shows that $\preceq_{\mathsf{R}}$ is indeed a quite complex preorder, though the question about its $\boldsymbol{\Sigma}^1_1$-completeness remains wide open. The proof is an application of Rosendal's construction of a cofinal family of Borel preorders ([Ros05]). This involves comparing $\preceq_{\mathsf{R}}$ with an ω_1-chain of Borel preorders on standard Borel spaces obtained by repeatedly applying a jump operation $P \mapsto P^{\mathrm{cf}}$.

Rosendal ([Ros05]) showed that if P is a Borel preorder satisfying a simple combinatorial condition, then $P <_B P^{\mathrm{cf}}$. In section 3 we compare $\preceq_{\mathsf{R}}$ with its jump $(\preceq_{\mathsf{R}})^{\mathrm{cf}}$ and show that $\preceq_{\mathsf{R}} \equiv_B (\preceq_{\mathsf{R}})^{\mathrm{cf}}$ (that is $\preceq_{\mathsf{R}} \leq_B (\preceq_{\mathsf{R}})^{\mathrm{cf}} \leq_B \preceq_{\mathsf{R}}$), giving further evidence to the fact that $\preceq_{\mathsf{R}}$ has high descriptive complexity.

In contrast with the latter property of $\preceq_{\mathsf{R}}$, in section 4 we prove the existence of a downward closed class of non-Borel preorders P such that $P <_B P^{\mathrm{cf}}$.

In the sequel, we will freely use the operations of sum of colored linear orders, and of multiplication of a colored linear order L by a linear order L', indicated by $L \cdot L'$. Standard coding techniques allow to represent the result as an element of $LO \times C^\omega$ in a Borel way.

2. $\preceq_{\mathsf{R}}$ is above all Borel preorders

Definition 1. If P is a preorder on a standard Borel space X define P^{cf} on X^ω by setting

$$\vec{x}\, P^{\mathrm{cf}}\, \vec{y} \iff \forall n \in \omega\ \exists m \in \omega\ x_n\, P\, y_m.$$

Remark 2. Notice that $P_0 \leq_B P_1$ implies ${P_0}^{\mathrm{cf}} \leq_B {P_1}^{\mathrm{cf}}$. Moreover $P \leq_B P^{\mathrm{cf}}$, as witnessed by the map $x \mapsto \vec{y}$ where $y_n = x$ for every n.

A (colored) linear order L is *right indecomposable* if it is embeddable in any of its final segments. For example each r_i defined above is right indecomposable. An ordinal is right indecomposable if and only if it is additively indecomposable: in this case we adhere to standard terminology and say that the ordinal is indecomposable.

Theorem 3. *For any Borel preorder P on some standard Borel space, the relation $P \leq_B \preceq_{\mathsf{R}}$ holds.*

Proof. Following Rosendal ([Ros05]) define preorders P_ξ with domain X_ξ, for $\xi \in \omega_1$ as follows:

- $X_0 = \omega$ and P_0 is equality;
- given P_ξ let $X_{\xi+1} = {X_\xi}^\omega$ and $P_{\xi+1} = {P_\xi}^{\mathrm{cf}}$;
- for ξ limit let $X_\xi = \prod_{\beta<\xi} X_\beta$ and $\vec{x}\, P_\xi\, \vec{y} \iff \forall \beta < \xi\ x_\beta\, P_\beta\, y_\beta$.

Rosendal proved that this transfinite sequence is $\leq_B$-cofinal among Borel preorders. Therefore it suffices to show that $\forall \xi < \omega_1\ P_\xi \leq_B \preceq_{\mathsf{R}}$.

We prove by induction on ξ that there exist an indecomposable countable ordinal α_ξ and a Borel reduction f_ξ of P_ξ to $\preceq_{\mathsf{R}}$ whose values are colored well-orders of order type α_ξ which are right indecomposable.

Basis step. Let $\alpha_0 = \omega$ and $f_0(i) = r_i$.

Successor step. Let $\xi \in \omega_1$ and assume f_ξ, α_ξ satisfy the induction hypothesis. Set $\alpha_{\xi+1} = \alpha_\xi \cdot \omega$. Given $\vec{x} = (x_0, x_1, x_2, \dots) \in X_{\xi+1}$, define

$$f_{\xi+1}(\vec{x}) = \sum_k f_\xi(x_{n_k})$$

where $(n_k)_{k\in\omega}$ is an enumeration of ω where each natural number occurs infinitely often. Each occurrence of $f_\xi(x_i)$ in this definition will be called a *segment* of $f_{\xi+1}(\vec{x})$. Notice that $f_{\xi+1}(\vec{x})$ is a right indecomposable colored well-order of length $\alpha_{\xi+1}$.

Suppose $\vec{x}\, P_{\xi+1}\, \vec{y}$. For each $n \in \omega$ there exists $m \in \omega$ such that $f_\xi(x_n) \preceq_{\mathsf{R}} f_\xi(y_m)$. Since each $f_\xi(y_i)$ occurs infinitely often in $f_{\xi+1}(\vec{y})$, it follows that $f_{\xi+1}(\vec{x}) \preceq_{\mathsf{R}} f_{\xi+1}(\vec{y})$.

Conversely, suppose g embeds $f_{\xi+1}(\vec{x})$ into $f_{\xi+1}(\vec{y})$. Since no segment of $f_{\xi+1}(\vec{x})$ can be mapped by g cofinally into $f_{\xi+1}(\vec{y})$, a final segment of each $f_\xi(x_i)$ must be embedded by g into some $f_\xi(y_j)$. Since by induction hypothesis $f_\xi(x_i)$ is right indecomposable, we have $f_\xi(x_i) \preceq_{\mathsf{R}} f_\xi(y_j)$ which implies $x_i\, P_\xi\, y_j$. Thus $\vec{x}\, P_{\xi+1}\, \vec{y}$.

Limit step. Let ξ be a limit ordinal. Suppose that, for each $\beta < \xi$, α_β and f_β satisfy the induction hypothesis. Let ρ be an indecomposable countable ordinal larger than each α_β. Let $\{\beta_n\}_{n\in\omega}$ be an enumeration of ξ with each element occurring infinitely often. Given $\vec{x} \in X_\xi = \prod_{\beta<\xi} X_\beta$, define

$$F_n(\vec{x}) = f_{\beta_n}(x_{\beta_n}) + r_{n+1}(\gamma_{n+1,n}) + r_n \cdot \rho$$

(where $r_{n+1}(\gamma_{n+1,n})$ denotes the colored linear order with a single element colored by $r_{n+1}(\gamma_{n+1,n})$),

$$F(\vec{x}) = \sum_n F_n(\vec{x}),$$

and finally

$$f_\xi(\vec{x}) = F(\vec{x}) \cdot \omega.$$

So each $f_\xi(\vec{x})$ is a right indecomposable colored well-order of length $\alpha_\xi = \rho \cdot \omega^2$, which is an indecomposable countable ordinal.

Suppose $\vec{x}\, P_\xi\, \vec{y}$. Then $F_n(\vec{x}) \preceq_{\mathsf{R}} F_n(\vec{y})$ for every n, so $F(\vec{x}) \preceq_{\mathsf{R}} F(\vec{y})$ and finally $f_\xi(\vec{x}) \preceq_{\mathsf{R}} f_\xi(\vec{y})$.

Conversely, let g witness $f_\xi(\vec{x}) \preceq_{\mathsf{R}} f_\xi(\vec{y})$. Then a final segment of the first occurrence Φ of $F(\vec{x})$ in the definition of $f_\xi(\vec{x})$ is embedded by g into some occurrence Ψ of $F(\vec{y})$ in $f_\xi(\vec{y})$. Let $i \in \omega$ be least such that the occurrence of $F_i(\vec{x})$ in Φ is embedded by g into Ψ. So for each $j \geq i$ a final segment of the occurrence of $r_j \cdot \rho$ in Φ must be sent by g cofinally into the corresponding occurrence in Ψ. As, for $j > i$, the occurrence of $r_{j+1}(\gamma_{j+1,j})$ in $F_j(\vec{x})$ just preceding the occurrence of $r_j \cdot \rho$ cannot be sent by g to an element of the occurrence of $r_j \cdot \rho$ in $F_j(\vec{y})$, it follows that g embeds $f_{\beta_j}(x_{\beta_j})$ into $f_{\beta_j}(y_{\beta_j})$, witnessing $x_{\beta_j}\, P_{\beta_j}\, y_{\beta_j}$. Since each $\beta \in \xi$ occurs as β_j for some $j > i$, it follows that $\vec{x}\, P_\xi\, \vec{y}$. ☐

Corollary 4. *If Q is a countable wqo but not an ω^2-wqo then, for all Borel preorders P on standard Borel spaces, $P \leq_B \preceq_Q$.*

Proof. By Rado's Theorem R embeds into Q and from this it follows easily that $\preceq_{\mathsf{R}} \leq_B \preceq_Q$. ☐

3. A closure property for $\preceq_{\mathsf{R}}$

The goal of this section is proving the following result:

Theorem 5. $(\preceq_{\mathsf{R}})^{\mathrm{cf}} \equiv_B \preceq_{\mathsf{R}}$.

The proof of Theorem 5 uses the following definition and a couple of lemmas.

Definition 6. If Q is a partial order on a countable set C of colors, define $\preceq_Q^*$ on $LO \times C^\omega$ by $L \preceq_Q^* L'$ if and only if $L \cdot \omega \preceq_Q L' \cdot \omega$.

Remark 7. For any Q the map $L \mapsto L \cdot \omega$ witnesses that $\preceq_Q^* \leq_B \preceq_Q$.

Moreover it is easy to check that $L \preceq_Q^* L'$ is equivalent to the existence of $k \in \omega$ such that $L \preceq_Q L' \cdot k$.

Lemma 8. *If Q is a partial order on a countable set then we have $(\preceq_Q^*)^{\mathrm{cf}} \leq_B \preceq_Q$.*

Proof. Fix a sequence $(n_k)_{k\in\omega}$ enumerating each natural number infinitely many times and use the map $\vec{L} \mapsto \sum_{k\in\omega}(L_{n_k} \cdot \omega)$. $\square$

Lemma 9. *Let P be any preorder on $LO \times D^\omega$ such that $\preceq_{\mathsf{R}} \subseteq P \subseteq \preceq_{\mathsf{R}}^*$. Then $\preceq_{\mathsf{R}} \leq_B P$. In particular $\preceq_{\mathsf{R}} \leq_B \preceq_{\mathsf{R}}^*$ and thus (by Remark 7) $\preceq_{\mathsf{R}} \equiv_B \preceq_{\mathsf{R}}^*$.*

Proof. Given $L \in LO \times D^\omega$ let $L^{(i)}$ be the colored linear order obtained from L by replacing each color (n, m) with $(2(i+1)n+1, 2(i+1)m+1)$ (the underlying linear order is unchanged). Notice that $L^{(i)} \preceq_{\mathsf{R}} M^{(i)}$ if and only if $L \preceq_{\mathsf{R}} M$.

To show that $\preceq_{\mathsf{R}} \leq_B P$ we use the map $L \mapsto F(L)$ where

$$F(L) = \sum_{i\in\omega}(L^{(i)} + r_{2i} \cdot \mathbb{Q}).$$

It is immediate that if $L \preceq_{\mathsf{R}} M$ then $F(L) \preceq_{\mathsf{R}} F(M)$ and hence $F(L)\, P\, F(M)$.

Now assume that $F(L)\, P\, F(M)$, which implies $F(L) \preceq_{\mathsf{R}}^* F(M)$, so that for some $k \in \omega$ we have $F(L) \preceq_{\mathsf{R}} F(M) \cdot k$ via some g. We need to show that $L \preceq_{\mathsf{R}} M$. There are two cases:

- First suppose that for some i, j we have $r_{2i} \cdot \mathbb{Q} \preceq_{\mathsf{R}} M^{(j)}$ via a function $p = p(n, q)$ (the domain of $r_{2i} \cdot \mathbb{Q}$ is $\omega \times \mathbb{Q}$, and (n, q) has color $(2i, 2i+n+1)$). In this case we claim that $N \preceq_{\mathsf{R}} M^{(j)}$ for any D-colored countable linear ordering N.

  To prove the claim fix N and an order preserving map $h : N \to \mathbb{Q}$ (here we are looking at N as a linear order). For any $a \in N$ let $(\varphi(a), \psi(a))$ the label assigned by N to a. Define $f : N \to M^{(j)}$ order preserving by $f(a) = p(\psi(a), h(a))$. Since $M^{(j)}$ does not use any label of the form $(2i, k)$, the first component of the label of $f(a)$ is greater than $2i+\psi(a)+1 > \psi(a)$. Therefore f witnesses $N \preceq_{\mathsf{R}} M^{(j)}$ and the claim is proved.

  Then for any $N \in LO \times D^\omega$ we have $N^{(j)} \preceq_{\mathsf{R}} M^{(j)}$, and hence $N \preceq_{\mathsf{R}} M$. In particular $L \preceq_{\mathsf{R}} M$.
- Now assume that $r_{2i} \cdot \mathbb{Q} \not\preceq_R M^{(j)}$ for all i and j. As g maps $F(L)$ into $F(M) \cdot k$, for some ℓ, g maps

  $$\sum_{i\geq\ell}(L^{(i)} + r_{2i} \cdot \mathbb{Q})$$

  into a single copy of $F(M)$. Since $r_{2i} \cdot \mathbb{Q} \not\preceq_R r_{2j} \cdot \mathbb{Q}$ when $i \neq j$ (because $r_{2i}(\gamma_{2i,2j})$ is a color used by $r_{2i} \cdot \mathbb{Q}$), g maps each $r_{2i} \cdot \mathbb{Q}$ for $i \geq \ell$ into the copy of $r_{2i} \cdot \mathbb{Q}$ appearing in $F(M)$. Therefore g maps $L^{(\ell+1)}$ into $r_{2\ell} \cdot \mathbb{Q} + M^{(\ell+1)} + r_{2\ell+2} \cdot \mathbb{Q}$. Since the colors used by $L^{(\ell+1)}$ have second component greater than $2\ell + 2$, g cannot map any element of $L^{(\ell+1)}$ into either $r_{2\ell} \cdot \mathbb{Q}$ or $r_{2\ell+2} \cdot \mathbb{Q}$.

Therefore (the restriction of) g witnesses $L^{(\ell+1)} \preceq_{\mathsf{R}} M^{(\ell+1)}$, and hence we have also $L \preceq_{\mathsf{R}} M$. $\square$

Proof of Theorem 5. We have

$$\begin{aligned} \preceq_{\mathsf{R}} \leq_B (\preceq_{\mathsf{R}})^{\mathrm{cf}} & \quad \text{by Remark 2} \\ \leq_B (\preceq_{\mathsf{R}}^*)^{\mathrm{cf}} & \quad \text{by Lemma 9 and Remark 2} \\ \leq_B \preceq_{\mathsf{R}} & \quad \text{by Lemma 8.} \quad \square \end{aligned}$$

4. A class of non-Borel P's such that $P <_B P^{\mathrm{cf}}$

The cofinal sequence (P_ξ) of section 2 is built using the operation $P \mapsto P^{\mathrm{cf}}$ at successor steps, allowing to obtain Borel preorders of increasing complexity. Here we build a $\leq_B$-downward closed class of non-Borel preorders P's such that $P <_B P^{\mathrm{cf}}$.

With a straightforward modification of a construction for equivalence relations due to John Clemens ([Cle01, §3.3]) we define two analytic preorders P_S and P'_S. These preorders have the property that $P \leq_B P_S$ and $P \leq_B P'_S$ if and only if P is a Borel preorder.

We further show the following: suppose P is an analytic non-Borel preorder such that there exist equivalence relations E, F on standard Borel spaces with $P \leq_B (E \times P_S) \oplus (F \times P'_S)$; then we have $P <_B P^{\mathrm{cf}}$.

Let $B \subseteq \omega^\omega$ be the set of codes for Borel preorders on ω^ω. B is $\mathbf{\Pi}^1_1$ by [Kec95, Theorem 35.5] and the fact that the set of codes for reflexive and transitive Borel relations is $\mathbf{\Pi}^1_1$. Fixing a coanalytic rank on B, for each $\alpha \in \omega_1$ let $B_\alpha \subseteq B$ be the Borel set of elements of rank less than α. Let $S \in \mathbf{\Sigma}^1_1((\omega^\omega)^3)$, $S' \in \mathbf{\Pi}^1_1((\omega^\omega)^3)$ be such that, for $z \in B$, (z, y_1, y_2) is in S if and only if it is in S' if and only if y_1 is related to y_2 in the preorder coded by z. Define on $(\omega^\omega)^2$ the analytic non-Borel preorders P_S, P'_S by:

$$\begin{aligned} (z_1, y_1) \, P_S \, (z_2, y_2) & \iff z_1 = z_2 \wedge (z_1 \notin B \vee (z_1, y_1, y_2) \in S), \\ (z_1, y_1) \, P'_S \, (z_2, y_2) & \iff (z_1 \notin B \wedge z_2 \notin B) \vee \\ & \qquad \vee (z_1 = z_2 \wedge (z_1, y_1, y_2) \in S). \end{aligned}$$

Notice that when $z_1, z_2 \in B$ each of $(z_1, y_1) \, P_S \, (z_2, y_2)$ and $(z_1, y_1) \, P'_S \, (z_2, y_2)$ is equivalent to

$$z_1 = z_2 \wedge (z_1, y_1, y_2) \in S',$$

which is coanalytic. Therefore for each α the restrictions of P_S and P'_S to $B_\alpha \times \omega^\omega$ are coanalytic and thus Borel.

Proposition 10. *For any preorder P on a standard Borel space, P is Borel if and only if $P \leq_B P_S$ and $P \leq_B P'_S$.*

Proof. The proof is a straightforward adaptation of the argument given by Clemens for equivalence relations.

For the forward implication we can assume that P is defined on ω^ω; let z be one of its codes. Then $x \mapsto (z, x)$ witnesses both $P \leq_B P_S$ and $P \leq_B P'_S$.

Conversely, if $P \leq_B P_S$ all P-equivalence classes are Borel, because this is the case with P_S. If $P \leq_B P'_S$ let f witness this. Let $A = f^{-1}(B \times \omega^\omega)$. The complement of A is either empty or is a single equivalence class of P; hence A is Borel. Consequently, $f(A)$ is analytic and its projection onto the first coordinate must be included in some B_α. As noticed above, the restriction of P'_S to the Borel set $B_\alpha \times \omega^\omega$ is Borel. So the restriction of P to A is Borel reducible to a Borel relation, and thus is itself Borel. It follows that P too is Borel. □

Theorem 11. *Let P be a non-Borel preorder and E, F be arbitrary equivalence relations on standard Borel spaces. Then $P^{\mathrm{cf}} \not\leq_B (E \times P_S) \oplus (F \times P'_S)$.*

Proof. Notice that P^{cf} is directed. This implies that the image of any reduction of P^{cf} to $(E \times P_S) \oplus (F \times P'_S)$ is included in some $[e]_E \times (\omega^\omega)^2$ or in some $[f]_F \times (\omega^\omega)^2$, and therefore we have a reduction of P^{cf} to either P_S or P'_S.

First assume f is a Borel reduction of P^{cf} to P'_S. Let $\pi : (\omega^\omega)^2 \to \omega^\omega$ be the projection on the first coordinate. Using again the fact that P^{cf} is directed we have either that $\pi f(\vec{x}) \notin B$ for all $\vec{x}$ or that there exists $z \in B$ such that $\pi f(\vec{x}) = z$ for all $\vec{x}$. In the first case $\vec{x}\, P^{\mathrm{cf}}\, \vec{y}$ for all $\vec{x}, \vec{y}$. In the second case P^{cf} Borel reduces to the Borel preorder coded by z. Since P, and thus P^{cf}, is not Borel, in either case we reach a contradiction.

A similar argument shows that $P^{\mathrm{cf}} \not\leq_B P_S$ and completes the proof. □

Corollary 12. *If P is a non-Borel preorder such that $P \leq_B (E \times P_S) \oplus (F \times P'_S)$ for some equivalence relations E, F on standard Borel spaces, then $P <_B P^{\mathrm{cf}}$. In particular $P_S <_B {P_S}^{\mathrm{cf}}$ and $P'_S <_B {P'_S}^{\mathrm{cf}}$.*

Proof. Immediate by Remark 2 and Theorem 11. □

Corollary 13. *Let E and F be arbitrary equivalence relations on standard Borel spaces. Then $\preceq_{\mathsf{R}} \not\leq_B (E \times P_S) \oplus (F \times P'_S)$.*

Proof. By Theorem 5 and Corollary 12. □

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