# On Fraïssé's conjecture for linear orders of finite Hausdorff rank

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**Abstract**

We prove that the maximal order type of the wqo of linear orders of finite Hausdorff rank under embeddability is $\varphi_2(0)$, the first fixed point of the $\varepsilon$-function. We then show that Fraïssé's conjecture restricted to linear orders of finite Hausdorff rank is provable in $\mathsf{ACA}_0^+$ + "$\varphi_2(0)$ is well-ordered" and, over $\mathsf{RCA}_0$, implies $\mathsf{ACA}_0'$ + "$\varphi_2(0)$ is well-ordered".

## 1 Introduction

Let $\mathsf{LO}$ be the class of countable linear orders. If $L, L' \in \mathsf{LO}$ let $L \preccurlyeq L'$ mean that $L$ is *embeddable* into $L'$, i.e. there exists an order preserving injective map from $L$ to $L'$. $L \sim L'$ abbreviates $L \preccurlyeq L'$ and $L' \preccurlyeq L$. In this case we say that $L$ and $L'$ are *equimorphic*. It is immediate that $\preccurlyeq$ is a quasi-order (i.e. a reflexive and transitive binary relation) and that $\sim$ is an equivalence relation.

*Fraïssé's conjecture* ($\mathsf{FRA}$) is the statement that $\mathsf{LO}$ is well-quasi-ordered by $\preccurlyeq$, i.e. that there are neither infinite descending chains nor infinite antichains. Roland Fraïssé formulated this conjecture in 1948 ([Fra48]). Richard Laver ([Lav71]) established $\mathsf{FRA}$ in 1971 by proving a stronger statement using Nash-Williams' notion of better-quasi-order ([NW68]). *Laver's Theorem* states that

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$\mathsf{LO}$ is better-quasi-ordered by $\preccurlyeq$ (actually Laver proved even more, considering $\sigma$-scattered linear orders with labels from a better-quasi-order). All known proofs of Fraïssé's conjecture (see e.g. [Sim85a] for a descriptive set-theoretic one) actually establish Laver's theorem.

It is easy to state $\mathsf{FRA}$ in the language of second order arithmetic, and it is a longstanding open problem in reverse mathematics to establish its exact axiomatic strength. ([Sim99] is the main reference for the reverse mathematics program, and [Mar05] includes a survey on the reverse mathematics of problems related to well- and better-quasi-orders.) Laver's proof can be carried out within the strong system $\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$, and Richard Shore ([Sho93]) proved that $\mathsf{FRA}$ implies $\mathsf{ATR}_0$. Since $\mathsf{FRA}$ is a $\mathbf{\Pi}^1_2$ statement, standard model theoretic arguments (see e.g. [Mar96, Corollary 1.10]) yield that $\mathsf{FRA}$ does not imply $\mathbf{\Pi}^1_1$-$\mathsf{CA}_0$ (and, a fortiori, $\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$) over $\mathsf{ATR}_0$. More recently the second author ([Mon06]) showed that $\mathsf{FRA}$ is equivalent (over $\mathsf{RCA}_0$ or slightly stronger theories) to other statements about linear orders.

An easy observation is that to establish $\mathsf{FRA}$ it suffices to consider *scattered* linear orders, i.e. linear orders $L \in \mathsf{LO}$ such that $\mathbb{Q} \not\preccurlyeq L$ (all non-scattered countable linear orders are equimorphic to $\mathbb{Q}$). Scattered linear orders were first studied by Hausdorff a century ago ([Hau08]), and his results lead to the notion of Hausdorff rank.

**Definition 1.1** *For every ordinal $\alpha$, $\mathbb{Z}^\alpha$ is the linear order with domain*

$$\{\, f : \alpha \to \mathbb{Z} \mid \{\, \beta < \alpha \mid f(\beta) \neq 0 \,\} \text{ is finite} \,\}$$

*ordered by $f \sqsubset g$ iff $f(\beta) < g(\beta)$ for the largest $\beta < \alpha$ such that $f(\beta) \neq g(\beta)$.*

*We say that a linear order $L$ has* Hausdorff rank less than $\alpha$, *and write* $\mathrm{rk}_H(L) < \alpha$, *if $L$ embeds into a proper segment of $\mathbb{Z}^\alpha$, or equivalently if $\mathbf{1} + L + \mathbf{1} \preccurlyeq \mathbb{Z}^\alpha$. We say that $L$ has* finite Hausdorff rank *if is has Hausdorff rank less than $\omega$.*

(See [Ros82, Chapter 5] for equivalent definitions of $\mathbb{Z}^\alpha$ and $\mathrm{rk}_H$. There are slightly different definitions of Hausdorff rank in the literature, but they differ by at most one.)

Hausdorff proved that $\mathrm{rk}_H(L)$ exists (i.e. $\mathrm{rk}_H(L) < \alpha$ for some ordinal $\alpha$) if and only if $L$ is scattered. Moreover if $L$ is countable scattered then $\mathrm{rk}_H(L)$ is less than a countable ordinal. The reverse mathematics of the properties of Hausdorff rank was studied by Clote in [Clo89].

If $\alpha$ is a countable ordinal let $\mathsf{LO}_\alpha = \{\, L \in \mathsf{LO} \mid \mathrm{rk}_H(L) < \alpha \,\}$. We denote by $\mathsf{FRA}_\alpha$ the statement that $\mathsf{LO}_\alpha$ is well-quasi-ordered by $\preccurlyeq$. Hence, $\mathsf{FRA}_\alpha$ is the restriction of $\mathsf{FRA}$ to a subset of $\mathsf{LO}$. Our long term goal is to obtain information on the strength of $\mathsf{FRA}$ by looking at the strength of various $\mathsf{FRA}_\alpha$. In this paper we carry out the first step in this project by studying $\mathsf{FRA}_\omega$, i.e. Fraïssé's conjecture for linear orders of finite Hausdorff rank.

A key ingredient for all known proofs of $\mathsf{FRA}$ is the notion of indecomposable linear order.

**Definition 1.2** *A linear order $L$ is* indecomposable *if whenever $L = L_1 + L_2$ then either $L \preccurlyeq L_1$ or $L \preccurlyeq L_2$. Let $\mathsf{ILO}$ (resp. $\mathsf{ILO}_\alpha$) be the class of countable indecomposable linear orders (resp. indecomposable linear orders of Hausdorff rank less than $\alpha$).*

It follows from $\mathsf{FRA}$ that every countable scattered linear order is the finite sum of indecomposable linear orders. The second author showed in [Mon06] that the latter statement is indeed equivalent to $\mathsf{FRA}$ over $\mathsf{RCA}_0$.

We use signed trees ([Mon06,Mon07], see Section 2 below for the definition and the basic properties of these objects) to study indecomposable scattered linear orders. We denote by $\mathsf{ST}$ be the set of all signed trees. Using the appropriate notion of homomorphism between signed tree we define equimorphism between signed trees. The key fact about $\mathsf{ST}$ is that signed trees represent indecomposable countable linear orders in such a way that homomorphism of signed trees is equivalent to embeddability of the corresponding linear orders. Moreover for each indecomposable countable linear order $L$ there exists a signed tree representing a linear order equimorphic to $L$. (Indeed, in [Mon06] the second author proved that, over $\mathsf{RCA}_0$, $\mathsf{FRA}$ is equivalent to the statement that $\mathsf{ST}$ is well-quasi-ordered by homomorphism.) In this paper we exploit the fact that indecomposable linear orders of finite Hausdorff rank are represented by signed trees of finite height. Since each of these trees is equimorphic to a finite signed tree, to study $\mathsf{LO}_\omega$ we consider $\mathsf{ST}_\omega$, the set of finite signed trees.

It is well-known that a quasi-order $(Q, \leq)$ is a well-quasi-order (wqo for short) if and only if all its linear extensions are well-orders (see [CMS04] for an analysis of the logical strength of the equivalence between this and other characterizations of wqo). Here a linear extension of $(Q, \leq)$ is a quasi-order $\sqsubseteq$ which is linear (i.e. either $q \sqsubseteq r$ or $r \sqsubseteq q$ holds for every $q, r \in Q$), and satisfies $q \leq r \Rightarrow q \sqsubseteq r$ for all $q, r \in Q$. The linear extension is a well-order if it is well-founded and in this case its order type is the unique ordinal $\alpha$ isomorphic to it.

**Definition 1.3** *If $(Q, \leq)$ is a well-quasi-order the* maximal order type *of $(Q, \leq)$ is*

$$o(Q, \leq) = \sup\{\, \alpha \mid \alpha \text{ is the order type of a linear extension of } (Q, \leq)\,\}.$$

*Whenever the quasi-order $\leq$ is clear from the context we write $o(Q)$.*

De Jongh and Parikh ([JP77]) showed that the supremum in the definition of maximal order type is actually a maximum, i.e. every well-quasi-order $(Q, \leq)$ has a linear extension of order type $o(Q)$. Schmidt continued the study of maximal order types in her Habilitationsschrift ([Sch79]). There, she computed the maximal order type of the wqo investigated by Higman ([Hig52]), and gave upper bounds for the maximal order types of the wqo's investigated by Kruskal ([Kru60]) and Nash-Williams ([NW65]). Harvey Friedman (see [Sim85b]) used the maximal order type of the class of finite trees with embeddability preserving g.l.b.s to prove that Kruskal's theorem cannot be proved in $\mathsf{ATR}_0$. Further extensions of Friedman's method were then used to show that Robertson and Seymour's celebrated result about graph minors is not provable in $\mathbf{\Pi}^1_1\text{-}\mathsf{CA}_0$ ([FRS87]).

The starting point of our results is the computation of $o(\mathsf{LO}_\omega, \preccurlyeq)$. Recall that the Veblen functions give an ordinal notation system for ordinals below $\Gamma_0$. They are defined by letting $\varphi_0(\alpha) = \omega^\alpha$ and, for $\beta > 0$, $\varphi_\beta(\alpha) =$ the $\alpha$-th common fixed point of all $\varphi_\gamma$ with $\gamma < \beta$. In particular $\varphi_1$ enumerates the $\varepsilon$-numbers, i.e. the fixed points of the function $\beta \mapsto \omega^\beta$, and we will write $\varepsilon_\alpha$ in place of $\varphi_1(\alpha)$.

Using finite signed trees in Section 3 we prove:

**Theorem 1.4** $o(\mathsf{LO}_\omega, \preccurlyeq) = \varphi_2(0)$, *i.e. the least fixed point of the* $\varepsilon$ *function.*

Up to and including Section 3 the paper does not deal with subsystems of second order arithmetic and can be read also by those interested only in wqo theory.

In Section 4 we introduce the subsystems of second order arithmetic we will need and in particular $\mathsf{ACA}_0^+$ and $\mathsf{ACA}_0'$. $\mathsf{ACA}_0^+$ consists of $\mathsf{RCA}_0$ plus the statement "for every $X$, $X^{(\omega)}$ (the arithmetic jump of $X$) exists" (see [Sho06] for recent results about $\mathsf{ACA}_0^+$). $\mathsf{ACA}_0^+$ is strictly weaker than $\mathsf{ATR}_0$ and the ordinal $\varphi_2(0)$ mentioned in Theorem 1.4 is precisely the proof-theoretic ordinal of $\mathsf{ACA}_0^+$ ([Rat91]). This implies that $\mathsf{ACA}_0^+$ does not prove that $\varphi_2(0)$ is a well-order. In Section 4 we make the latter statement precise by introducing, in $\mathsf{RCA}_0$, an ordinal notation system for ordinals below $\varphi_2(0)$. $\mathsf{ACA}_0'$ consists of $\mathsf{RCA}_0$ plus the statement "for every $X$ and $k$, $X^{(k)}$ exists". $\mathsf{ACA}_0'$ is strictly weaker than $\mathsf{ACA}_0^+$ and strictly stronger than $\mathsf{ACA}_0$.

Formalizing the proof of Theorem 1.4 in Section 5 we obtain:

**Theorem 1.5** $\mathsf{ACA}_0^+$ + *"$\varphi_2(0)$ is well-ordered" proves* $\mathsf{FRA}_\omega$.

In Section 6, building on ideas from [Sho93,Sho06], we obtain a lower bound for the complexity of $\mathsf{FRA}_\omega$.

**Theorem 1.6** $\mathsf{RCA}_0$ *proves that* $\mathsf{FRA}_\omega$ *implies* $\mathsf{ACA}_0'$ + "$\varphi_2(0)$ *is well-ordered*".

There is still a gap between the upper and lower bounds since $\mathsf{ACA}_0^+$ + "$\varphi_2(0)$ is well-ordered" is strictly stronger than $\mathsf{ACA}_0'$ + "$\varphi_2(0)$ is well-ordered", as the $\omega$-model consisting of all arithmetic sets is a model of the latter theory but not of the former.

## 2 Signed trees

Let us start by establishing our notation for finite sequences and trees.

**Definition 2.1** *We denote by* $X^{<\omega}$ *the set of* finite sequences *of elements of a set* $X$*. We use* $\emptyset$ *to denote the* empty sequence*. Let* $x \in X$ *and* $\sigma, \tau \in X^{<\omega}$*:* $\langle x \rangle$ *is the sequence of length* 1 *with sole element* $x$*;* $|\sigma|$ *is the* length *of* $\sigma$ *and for* $i < |\sigma|$*,* $\sigma(i)$ *is the* $i$*-th element of* $\sigma$*;* $\sigma \subseteq \tau$ *means that* $\sigma$ *is an* initial segment *of* $\tau$ *and* $\sigma \subset \tau$ *that* $\sigma$ *is a* proper initial segment *of* $\tau$*;* $\sigma^\frown\tau$ *is the sequence of length* $|\sigma| + |\tau|$ *obtained by concatenating* $\sigma$ *and* $\tau$*;* $\sigma * x$ *and* $x * \sigma$ *abbreviate* $\sigma^\frown\langle x \rangle$ *and* $\langle x \rangle^\frown\sigma$ *respectively.*

**Definition 2.2** *A nonempty set* $T \subseteq X^{<\omega}$ *is a* tree *if* $\sigma \in T$ *and* $\tau \subset \sigma$ *imply* $\tau \in T$*. We often refer to members of a tree as* nodes *and to* $\emptyset$ *(which belongs to every tree) as the* root*. A* subtree *of* $T$ *is a set of the form* $\{\, \tau \in X^{<\omega} \mid \sigma^\frown\tau \in T \,\}$ *for some* $\sigma \in T$*; when* $\sigma = \langle x \rangle$ *we have an* immediate subtree *of* $T$ *and we use* $T_x$ *to denote it.*

*A* child *of the node* $\sigma$ *of the tree* $T$ *is a node of* $T$ *of the form* $\sigma * x$*. A* leaf *of* $T$ *is a node of* $T$ *with no children. Let* $L(T) = \{\, \sigma \mid \sigma \text{ is a leaf of } T \,\}$*.*

*A tree* $T$ *is* well-founded *if there is no infinite sequence* $\sigma_0 \subset \sigma_1 \subset \dots$ *of elements of* $T$*. When* $T$ *is well-founded we can define* $\mathrm{ht}(\sigma, T) = \sup\{\, \mathrm{ht}(\sigma * n, T) + 1 \mid \sigma * n \in T \,\}$ *for every* $\sigma \in T$*. The* height *of* $T$ *is* $\mathrm{ht}(T) = \mathrm{ht}(\emptyset, T)$*.*

We can now introduce signed trees, which are our main tool in the study of indecomposable linear orders.

**Definition 2.3** *A* signed tree *is a pair* $(T, s_T)$ *where* $T \subseteq \omega^{<\omega}$ *is a well-founded tree and* $s_T : T \to \{+, -\}$*.*

*Let* $\mathsf{ST}$ *be the set of all signed trees and* $\mathsf{ST}_\omega = \{\, (T, s_T) \in \mathsf{ST} \mid T \text{ is finite} \,\}$*.*

*If* $(T, s_T) \in \mathsf{ST}$ *and* $\langle n \rangle \in T$ *let* $s_{T_n} : T_n \to \{+, -\}$ *be defined by* $s_{T_n}(\tau) = s_T(n * \tau)$*. Obviously* $(T_n, s_{T_n}) \in \mathsf{ST}$*.*

**Definition 2.4** *If* $(T, s_T), (T', s_{T'}) \in \mathsf{ST}$ *a map* $f : T \to T'$ *is a* homomorphism *if*

- $\sigma \subset \tau$ *implies* $f(\sigma) \subset f(\tau)$ *for every* $\sigma, \tau \in T$*;*
- $s_{T'}(f(\sigma)) = s_T(\sigma)$ *for every* $\sigma \in T$*.*

*Note that* $f$ *is not required to be one-to-one and that* $f(\sigma) \subset f(\tau)$ *might occur even when* $\sigma \not\subset \tau$*. If there exists a homomorphism from* $(T, s_T)$ *to* $(T', s_{T'})$ *we write* $(T, s_T) \preccurlyeq (T', s_{T'})$*.* $(T, s_T)$ *and* $(T', s_{T'})$ *are* equimorphic *(and we write* $(T, s_T) \sim (T', s_{T'})$*) if* $(T, s_T) \preccurlyeq (T', s_{T'})$ *and* $(T', s_{T'}) \preccurlyeq (T, s_T)$*.*

**Lemma 2.5** *Let* $(T, s_T), (T', s_{T'}) \in \mathsf{ST}$ *be such that* $s_T(\emptyset) = s_{T'}(\emptyset)$*. If for all* $n$ *with* $\langle n \rangle \in T$ *there exists* $m$ *such that* $\langle m \rangle \in T'$ *and* $(T_n, s_{T_n}) \preccurlyeq (T'_m, s_{T'_m})$ *then* $(T, s_T) \preccurlyeq (T', s_{T'})$*.*

**PROOF.** Immediate from the definition of homomorphism. □

**Lemma 2.6** *For every* $(T, s_T) \in \mathsf{ST}$ *with* $\mathrm{ht}(T) < \omega$ *there exists* $(T', s_{T'}) \in \mathsf{ST}_\omega$ *such that* $(T, s_T) \sim (T', s_{T'})$*.*

**PROOF.** Easy induction on $\mathrm{ht}(T)$, using Lemma 2.5. □

The following definition, which was introduced in [Mon06] and [Mon07], connects signed trees with indecomposable linear orders. Here and elsewhere $\omega^*$ denotes $\omega$ with the reverse order. Hence $\sum_{k \in \omega^*} L_k$ is $\cdots + L_2 + L_1 + L_0$.

**Definition 2.7** *To each* $(T, s_T) \in \mathsf{ST}$ *we associate a countable linear order* $\mathrm{lin}(T, s_T)$ *as follows (the definition is by recursion on the height of the tree):*

- *if* $T = \{\emptyset\}$ *and* $s_T(\emptyset) = +$ *let* $\mathrm{lin}(T, s_T) = \omega$*;*
- *if* $T = \{\emptyset\}$ *and* $s_T(\emptyset) = -$ *let* $\mathrm{lin}(T, s_T) = \omega^*$*;*
- *if* $T \neq \{\emptyset\}$ *and* $s_T(\emptyset) = +$ *let*

$$\mathrm{lin}(T, s_T) = \sum_{k \in \omega} \left( \sum_{n \in \{0, \dots, k\} : \langle n \rangle \in T} \mathrm{lin}(T_n, s_{T_n}) \right);$$

- *if* $T \neq \{\emptyset\}$ *and* $s_T(\emptyset) = -$ *let*

$$\mathrm{lin}(T, s_T) = \sum_{k \in \omega^*} \left( \sum_{n \in \{0, \dots, k\} : \langle n \rangle \in T} \mathrm{lin}(T_n, s_{T_n}) \right).$$

**Lemma 2.8** *([Mon06]) For each* $(T, s_T), (T', s_{T'}) \in \mathsf{ST}$ *we have*

*(a)* $\mathrm{lin}(T, s_T)$ *is a countable scattered indecomposable linear order;*

(b) $(T, s_T) \preccurlyeq (T', s_{T'})$ *if and only if* $\mathrm{lin}(T, s_T) \preccurlyeq \mathrm{lin}(T', s_{T'})$*, and in particular* $(T, s_T) \sim (T', s_{T'})$ *if and only if* $\mathrm{lin}(T, s_T) \sim \mathrm{lin}(T', s_{T'})$*;*

(c) $\mathrm{rk}_H(\mathrm{lin}(T, s_T))$ *and* $\mathrm{ht}(T)$ *differ by at most one.*

*Moreover each countable scattered indecomposable linear order different from* $\mathbf{1}$ *is equimorphic to* $\mathrm{lin}(T, s_T)$ *for some* $(T, s_T) \in \mathsf{ST}$*.*

**Corollary 2.9** $o(\mathsf{ILO}, \preccurlyeq) = o(\mathsf{ST}, \preccurlyeq)$ *and* $o(\mathsf{ILO}_\alpha, \preccurlyeq) = o(\mathsf{ST}_\alpha, \preccurlyeq)$ *whenever* $\alpha$ *is a limit ordinal. In particular* $o(\mathsf{ILO}_\omega, \preccurlyeq) = o(\mathsf{ST}_\omega, \preccurlyeq)$*.*

**PROOF.** Lemma 2.8 implies that lin is an isomorphism between $(\mathsf{ST}, \preccurlyeq)$ and $(\mathsf{ILO} \setminus \{\mathbf{1}\}, \preccurlyeq)$ viewed as partial orders (i.e. considered up to equimorphism). Thus $o(\mathsf{ILO}) = 1 + o(\mathsf{ST}) = o(\mathsf{ST})$ since $o(\mathsf{ST})$ is infinite. Since lin maps $\mathsf{ST}_\alpha$ onto $\mathsf{ILO}_\alpha \setminus \{\mathbf{1}\}$ for all limit ordinals $\alpha$, the same argument works for $o(\mathsf{ST}_\alpha)$. □

## 3 The ordinal of $(\mathsf{LO}_\omega, \preccurlyeq)$

In this section we prove Theorem 1.4, i.e. $o(\mathsf{LO}_\omega, \preccurlyeq) = \varphi_2(0)$. Our idea is to compute $o(\mathsf{ILO}_\omega)$ using Corollary 2.9: therefore we concentrate on $o(\mathsf{ST}_\omega)$. We will prove in Propositions 3.5 and 3.12 the two inequalities needed to establish $o(\mathsf{ST}_\omega) = \varphi_2(0)$. This yields $o(\mathsf{LO}_\omega) \geq o(\mathsf{ILO}_\omega) = \varphi_2(0)$. We will later use the fact that every countable linear order is the finite sum of indecomposable linear orders to show that $o(\mathsf{LO}_\omega) \leq \varphi_2(0)$.

We begin by proving or recalling some general facts about the function $o$. We start by making explicit a widely used technique (implicit in [JP77] and [Sch79]) for computing an upper bound for the maximal order type of a wqo.

**Definition 3.1** *Let* $(Q, \leq)$ *and* $(Q', \leq')$ *be quasi-orders. The function* $F : Q' \to Q$ embeds $(Q', \leq')$ *in* $(Q, \leq)$*, and we write* $F : (Q', \leq') \hookrightarrow (Q, \leq)$*, if*

$$\forall q, r \in Q' (F(q) \leq F(r) \Rightarrow q \leq' r).$$

*We write* $(Q', \leq') \hookrightarrow (Q, \leq)$ *(or* $Q' \hookrightarrow Q$ *when the binary relations are understood) if there exists a such function* $F$*.*

**Lemma 3.2** *Let* $(Q, \leq)$ *be a well-quasi-order and* $(Q', \leq')$ *be a quasi-order. If* $Q' \hookrightarrow Q$ *then* $(Q', \leq')$ *is a well-quasi-order and* $o(Q') \leq o(Q)$*.*

**PROOF.** Let $F$ witness $Q' \hookrightarrow Q$ and denote the range of $F$ by $\tilde{Q}$. Obviously $o(\tilde{Q}) \leq o(Q)$.

Let $\sqsubseteq'$ be any linear extension of $(Q', \leq')$. Define a binary relation $\sqsubseteq$ on $\tilde{Q}$ by setting, for $q, r \in Q'$,

$$F(q) \sqsubseteq F(r) \iff q \sqsubseteq' r.$$

This definition is well-posed because $F(q) = F(q')$ implies $q \sqsubseteq' q' \sqsubseteq' q$.

It is straightforward to check that $\sqsubseteq$ is a linear extension of $(\tilde{Q}, \leq)$ and hence (since $Q$ is a well-quasi-order) is a well-order. Hence $(Q', \sqsubseteq')$ is a well-order. This suffices to show that $(Q', \leq')$ is a well-quasi-order.

Moreover the order types of $(Q', \sqsubseteq')$ and $(\tilde{Q}, \sqsubseteq)$ are the same. Since the latter is not greater than $o(\tilde{Q})$ and $\sqsubseteq'$ is arbitrary we have $o(Q') \leq o(\tilde{Q}) \leq o(Q)$. □

**Definition 3.3** *If $(Q, \leq)$ is a quasi-order and $r \in Q$ let $Q_{\not\geq r} = \{\, q \in Q \mid q \not\geq r \,\}$.*

**Lemma 3.4** *Let $(Q, \leq)$ be a well-quasi-order and suppose that $\hat{Q} \subseteq Q$ is cofinal (i.e. $\forall q \in Q\, \exists r \in \hat{Q}\, q \leq r$). Then*

$$o(Q) = \sup\{\, o(Q_{\not\geq r}) + 1 \mid r \in \hat{Q} \,\}.$$

**PROOF.** This is essentially Lemma 2.6 of [JP77]. □

### *3.1 The lower bound*

**Proposition 3.5** $o(\mathsf{ST}_\omega) \geq \varphi_2(0)$.

**PROOF.** By Lemma 3.2 it suffices to define a function $F : \varphi_2(0) \to \mathsf{ST}_\omega$ witnessing $\varphi_2(0) \hookrightarrow \mathsf{ST}_\omega$.

We define $F(\alpha)$ by recursion on $\alpha$. For notational convenience let $\varepsilon_{-1} = \omega^{-1} = 0$ and let $F(-1)$ be the signed tree consisting only of the root labeled $+$. Now, for every $\alpha < \varphi_2(0)$ there exists a unique $\beta < \alpha$ such that $\varepsilon_\beta \leq \alpha < \varepsilon_{\beta+1}$. If $\alpha = \varepsilon_\beta + \delta$, we can write $\delta$ in Cantor normal form as $\omega^{\gamma_0} + \cdots + \omega^{\gamma_{k-1}}$ with the convention that if $\delta = 0$ we have $k = 1$ and $\gamma_0 = -1$, while if $\delta > 0$ then $\gamma_i \neq -1$ for every $i < k$. We thus have

$$\alpha = \varepsilon_\beta + \omega^{\gamma_0} + \cdots + \omega^{\gamma_{k-1}}$$

with $k > 0$, $\beta < \alpha$, and $\alpha > \gamma_0 \geq \cdots \geq \gamma_{k-1}$. We can assume that $F(\beta)$, $F(\gamma_0)$, ..., $F(\gamma_{k-1})$ are already defined.

Let $n = \mathrm{ht}(F(\beta))+3$, $m_0 = n+\mathrm{ht}(F(\gamma_0))+2$, and $m_i = m_{i-1}+\mathrm{ht}(F(\gamma_i))+1$ for $0 < i < k$. Then $F(\alpha)$ is the following signed tree. (A sample $F(\alpha)$ with $k = 2$

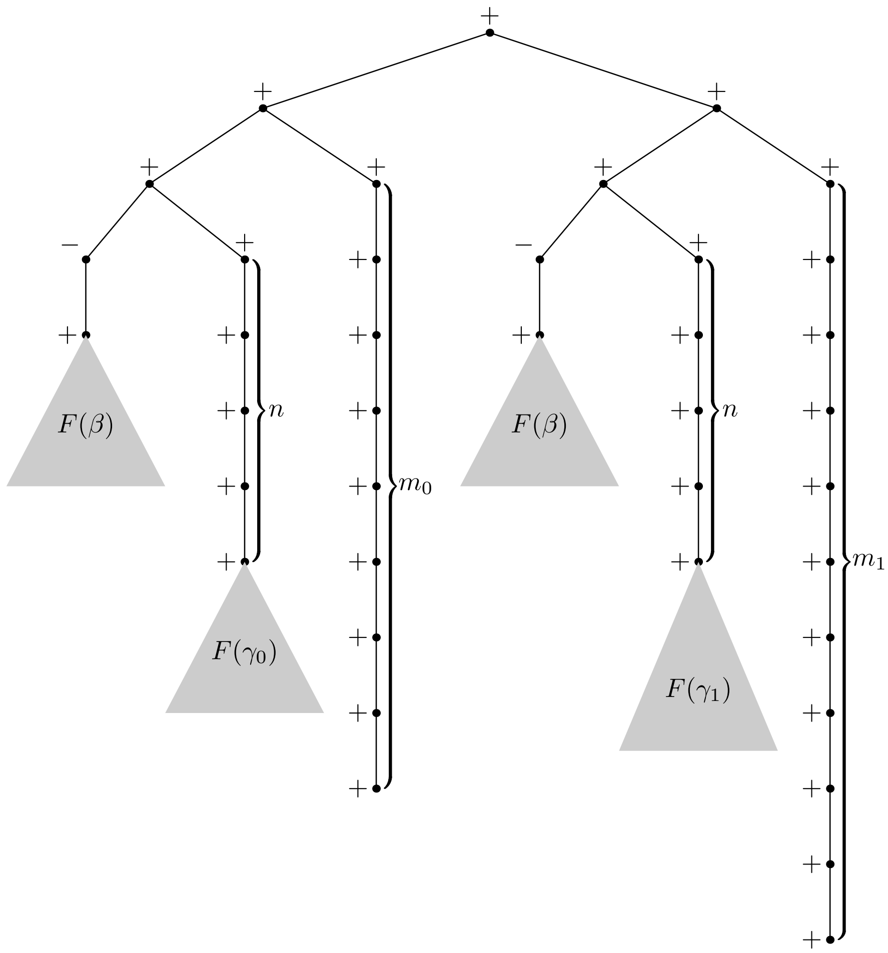


Fig. 1. A sample $F(\alpha)$ with $k = 2$.

is depicted in Figure 1.) The root of $F(\alpha)$ is labeled $+$, and has $k$ children, all labeled $+$. The $i$-th child of the root has two children, both labeled $+$: one of them is the starting point of a linear branch of $m_i$ nodes all labeled $+$ (i.e. it has a single child, which has a single child, etc. for $m_i$ times). The other child of the $i$-th child of the root has two children, one labeled $-$ and the other labeled $+$. The node labeled $-$ has a single child, which is the root of a copy of $F(\beta)$. The node labeled $+$ is the starting point of a linear branch of $n$ nodes all labeled $+$ and the last node of this branch is the root of a copy of $F(\gamma_i)$.

Notice that for every $\alpha$ the root of $F(\alpha)$ is labeled $+$. It is easy to show inductively that if a node labeled $-$ appears in $F(\alpha)$ then it has a single child which is the root of a copy of some $F(\xi)$ with $\varepsilon_\xi \leq \alpha$. Notice also that if $\alpha \neq -1$ then $F(\alpha)$ has at least one node labeled $-$.

Now we prove by induction on $\alpha$ that if $\alpha, \alpha' < \varphi_2(0)$ are such that $F(\alpha) \preccurlyeq F(\alpha')$ then $\alpha \leq \alpha'$. Fix $\alpha, \alpha'$ and a homomorphism $f$ witnessing $F(\alpha) \preccurlyeq F(\alpha')$.

We can write

$$\alpha = \varepsilon_\beta + \omega^{\gamma_0} + \cdots + \omega^{\gamma_{k-1}} \quad \text{and}$$
$$\alpha' = \varepsilon_{\beta'} + \omega^{\gamma'_0} + \cdots + \omega^{\gamma'_{k'-1}}$$

as above, with $\delta = \omega^{\gamma_0} + \cdots + \omega^{\gamma_{k-1}}$ and $\delta' = \omega^{\gamma'_0} + \cdots + \omega^{\gamma'_{k'-1}}$. Let also $n$, $m_0$, $\dots$, $m_{k-1}$, $n'$, $m'_0$, $\dots$, $m'_{k'-1}$ be defined as above for each of the two ordinals.

The nodes in $F(\alpha')$ labeled $-$ are the one immediately above the root of $F(\beta')$, and nodes in the interior of $F(\beta')$ or of some $F(\gamma'_i)$ with $i < k'$. Since the node in $F(\alpha)$ immediately above the root of one of the copies of $F(\beta)$ is mapped by $f$ to one of these nodes, the restriction of $f$ to any copy of $F(\beta)$ shows that either (1) $F(\beta) \preccurlyeq F(\beta')$, or (2) $F(\beta) \prec F(\beta')$, or (3) $F(\beta) \prec F(\gamma'_i)$. In case (1), by induction hypothesis, we have $\beta \leq \beta'$. Case (2) yields immediately $\beta < \beta'$. In case (3) we have $F(\beta) \leq F(\xi)$ for some $\xi$ with $\varepsilon_\xi \leq \gamma'_i$: since $\gamma'_i \leq \gamma'_0 < \alpha' < \varepsilon_{\beta'+1}$ we have $\xi \leq \beta'$. Moreover, by induction hypothesis, $\beta \leq \xi$ and hence $\varepsilon_\beta \leq \varepsilon_\xi \leq \varepsilon_{\beta'}$.

Thus in every case we have $\beta \leq \beta'$: if $\beta < \beta'$ we are done because $\alpha < \varepsilon_{\beta+1} \leq \varepsilon_{\beta'} \leq \alpha'$. We now assume $\beta = \beta'$ and hence also $n = n'$. We need to show that $\delta \leq \delta'$. If $\delta = 0$ this is obvious. Hence we may assume $\gamma_0 \neq -1$.

Since the root of the copy of $F(\gamma_0)$ in $F(\alpha)$ has height $n + 2 = \mathrm{ht}(F(\beta)) + 5$, this node is not mapped by $f$ to a node in a copy of $F(\beta') = F(\beta)$ in $F(\alpha')$. Since $F(\gamma_0)$ has nodes labeled $-$, $f$ does not map $F(\gamma_0)$ into one of the linear branches of $F(\alpha')$ where all nodes are labeled $+$. Thus there exists $j_0 < k'$ such that the restriction of $f$ to $F(\gamma_0)$ witnesses $F(\gamma_0) \preccurlyeq F(\gamma'_{j_0})$. By induction hypothesis, $\gamma_0 \leq \gamma'_{j_0} \leq \gamma'_0$. If at least one of the inequalities is strict then $\gamma_0 < \gamma'_0$ and hence $\delta < \delta'$. Thus we assume $\gamma_0 = \gamma'_0$ and therefore $m_0 = m'_0$.

Repeating the same argument we obtain $j_1 < k'$ such that $F(\gamma_1) \preccurlyeq F(\gamma'_{j_1})$ and hence $\gamma_1 \leq \gamma'_{j_1}$. The linear branch with all nodes labeled $+$ which is in the same immediate subtree of $F(\alpha)$ as $F(\gamma_1)$ has a node of height $m_1 + 1 = m_0 + \mathrm{ht}(F(\gamma_0)) + 2$: this node cannot be mapped into the first immediate subtree of $F(\alpha')$, and this implies $j_1 > 0$. If either $\gamma_1 < \gamma'_{j_1}$ or $\gamma'_{j_1} > \gamma'_1$ then $\delta < \delta'$ follows immediately, and otherwise we iterate the argument. If we have to iterate the argument $k$ times, then $k \leq k'$ and $\gamma_i = \gamma'_i$ for every $i < k$, which implies $\delta \leq \delta'$. □

### *3.2 The upper bound*

There are various ways of proving upper bounds for maximal order type for some well-quasi-orders. We choose one that can be carried out in $\mathsf{RCA}_0$, but we will not worry about subsystems of second order arithmetic until the next section. The general format for this type of proof was used by Simpson in [Sim88] and by Rathjen and Weiermann in [RW93]. The main tool is the notion of reification of a quasi-order by an ordinal.

**Definition 3.6** *Let* $(Q, \leq)$ *be a quasi-order. We say that a finite sequence* $\langle q_0, \ldots, q_k \rangle \in Q^{<\omega}$ *is bad if* $\forall i < j \; q_i \nleq q_j$*. We define* $\mathrm{Bad}(Q) \subseteq Q^{<\omega}$ *to be the tree of finite bad sequences from* $Q$*.*

Notice that if $q * \sigma \in \mathrm{Bad}(Q)$ then $\sigma(i) \in Q_{\not\geq q}$ for all $i < |\sigma|$, so that $\sigma \in \mathrm{Bad}(Q_{\not\geq q})$.

Note that $Q$ is wqo if and only if $\mathrm{Bad}(Q)$ is a well-founded tree. The results of de Jongh and Parikh ([JP77]) imply that if $Q$ is a wqo, then $o(Q) = \mathrm{ht}(\mathrm{Bad}(Q))$.

**Definition 3.7** *Let* $(Q, \leq)$ *be a quasi-order. A* reification *of* $(Q, \leq)$ *by* $\alpha$ *is a map* $G : \mathrm{Bad}(Q) \to \alpha + 1$ *such that* $\sigma \subset \tau \Rightarrow G(\sigma) > G(\tau)$*.*

**Lemma 3.8** *If there exists a reification of* $(Q, \leq)$ *by* $\alpha$*, then* $(Q, \leq)$ *is a wqo and* $o(Q) \leq \alpha$*.*

We now define some operations on quasi-orders which preserve wqo. Some of these operations are well-known. See [JP77] and [Sch79] for more results and proofs.

**Disjoint union** If $(Q_1, \leq_1)$ and $(Q_2, \leq_2)$ are quasi-orders with $Q_1 \cap Q_2 = \emptyset$ we denote by $Q_1 \sqcup Q_2$ the quasi-order on $Q_1 \cup Q_2$ with no comparabilities between elements of $Q_1$ and $Q_2$.

**Finite parts** If $(Q, \leq)$ is a quasi-order, let $\mathcal{P}_f(Q)$ be the set of finite subsets of $Q$ with quasi-order defined by

$$X \leq' Y \iff \forall x \in X \, \exists y \in Y \, x \leq y.$$

**Finite sequences** Let $Q^{<\omega}$ be the set of finite sequences of elements of $Q$ with quasi-order $\leq'$ defined by

$$\langle x_0, \ldots, x_{n-1} \rangle \leq' \langle y_0, \ldots, y_{m-1} \rangle \iff \exists f : n \to m \text{ strictly increasing } \forall i < n \, x_i \leq y_{f(i)}.$$

**Finite trees with leaves labeled by** $Q$ Let $\mathcal{T}(Q)$ be the set of pairs $(T, l_T)$ where $T$ is a nonempty finite tree and $l_T : L(T) \to Q$ (recall that $L(T)$ is the set of the leaves of $T$).

The notion of homomorphism between elements of $\mathcal{T}(Q)$ is an adaption of the notion of homomorphism between signed trees (definition 2.4): if $(T, l_T), (T', l_{T'}) \in \mathcal{T}(Q)$ a map $f : T \to T'$ is a *homomorphism* if

- $\sigma \subset \tau$ implies $f(\sigma) \subset f(\tau)$ for every $\sigma, \tau \in T$;

- if $\sigma \in L(T)$ then $f(\sigma) \in L(T')$ and $l_T(\sigma) \leq l_{T'}(f(\sigma))$.

  If there exists a homomorphism from $(T, l_T)$ to $(T', l_{T'})$ we write $(T, l_T) \preccurlyeq (T', l_{T'})$.

The maximal order types of the wqo's obtained applying some of these operations has already been computed. We use Hessenberg's natural sum and exponential towers.

Recall that given ordinals written in Cantor normal form $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_{n-1}}$ and $\beta = \omega^{\beta_0} + \cdots + \omega^{\beta_{m-1}}$, Hessenberg's natural sum of ordinals is defined by

$$\alpha \oplus \beta = \omega^{\gamma_0} + \omega^{\gamma_1} + \cdots + \omega^{\gamma_{n+m-1}},$$

where $\gamma_0, \ldots, \gamma_{n+m-1}$ are such that $\gamma_0 \geq \gamma_1 \geq \cdots \geq \gamma_{n+m-1}$, and there exists a partition $\{\{a_0, \ldots, a_{n-1}\}, \{b_0, \ldots, b_{m-1}\}\}$ of $\{0, \ldots, n+m-1\}$ such that $\gamma_{a_i} = \alpha_i$ and $\gamma_{b_i} = \beta_i$. Hessenberg's natural sum is commutative and $\alpha + \beta \leq \alpha \oplus \beta$.

Let $\omega_i(\alpha)$ denote an exponential tower of $i$ $\omega$'s with $\alpha$ on top: $\omega_0(\alpha) = \alpha$ and $\omega_{i+1}(\alpha) = \omega^{\omega_i(\alpha)}$.

**Lemma 3.9** *([JP77]) Let $(Q, \leq)$, $(Q_1, \leq_1)$ and $(Q_2, \leq_2)$ be well-quasi-orders.*

*(1) If $Q_1 \cap Q_2 = \emptyset$ then $Q_1 \sqcup Q_2$ is a well-quasi-order and $o(Q_1 \sqcup Q_2) = o(Q_1) \oplus o(Q_2)$.*
*(2) $(Q^{<\omega}, \leq')$ is a well-quasi-order and $o(Q^{<\omega}) \leq \omega_2(o(Q)+1) = \omega^{\omega^{o(Q)+1}}$.*

It is not hard to see that $\mathcal{P}_f(Q) \hookrightarrow Q^{<\omega}$, and therefore $\mathcal{P}_f(Q)$ is a wqo. If $(Q, \leq)$ is a well-quasi-order it follows from well-known facts in wqo theory (e.g. Kruskal's Theorem) that $(\mathcal{T}(Q), \preccurlyeq)$ is also wqo.

To prove that $o(\mathsf{ST}_\omega) \leq \varphi_2(0)$, we simultaneously prove upper bounds for the maximal order type of various well-quasi-orders.

**Definition 3.10** *Let $\mathbb{W}$ be the smallest set of all quasi-orders such that:*

*(1) $\mathbf{1} = \{0\} \in \mathbb{W}$;*
*(2) $Q, P \in \mathbb{W} \Rightarrow P \sqcup Q \in \mathbb{W}$;*
*(3) $Q \in \mathbb{W} \Rightarrow \mathcal{P}_f(Q) \in \mathbb{W}$;*
*(4) $Q \in \mathbb{W} \Rightarrow \mathcal{T}(Q) \in \mathbb{W}$.*

*Let $F : \mathbb{W} \to \varphi_2(0)$ be defined by*

*(1) $F(\mathbf{1}) = 1$;*
*(2) $F(P \sqcup Q) = F(P) \oplus F(Q)$;*
*(3) $F(\mathcal{P}_f(Q)) = \omega^{F(Q)}$;*

*(4)* $F(\mathcal{T}(Q)) = \varepsilon_{F(Q)}$.

**Lemma 3.11** *For every* $Q \in \mathbb{W}$*,* $o(Q) \leq F(Q)$*.*

**PROOF.** By Lemma 3.8, for each $Q \in \mathbb{W}$, it suffices to define a reification $G_Q : \mathrm{Bad}(Q) \to F(Q)+1$ of $Q$. The idea used in the definition of $G_Q$ is the following. On the empty sequence, we define $G_Q(\emptyset) = F(Q)$. Now, if $\sigma \in \mathrm{Bad}(Q)$ is not empty, we have to define $G_Q(\sigma) < F(Q)$. Suppose that $\sigma = q * \tau$ where $q \in Q$ and $\tau \in \mathrm{Bad}(Q_{\not\geq q})$. The plan is to embed $Q_{\not\geq q}$ into some member of $\mathbb{W}$, and use recursion. So, to each $q \in Q \in \mathbb{W}$ we assign a partial order $H_{Q,q} \in \mathbb{W}$ and an embedding

$$h_{Q,q} : Q_{\not\geq q} \hookrightarrow H_{Q,q}.$$

The assignment of partial order has to satisfy $F(H_{Q,q}) < F(Q)$ so that, using recursion on $|\sigma|$, we can define

$$G_Q(q * \tau) = G_{H_{Q,q}}(h_{Q,q}(\tau)),$$

where $h_{Q,q}(\langle q_0, \ldots, q_{k-1}\rangle) = \langle h_{Q,q}(q_0), \ldots, h_{Q,q}(q_{k-1})\rangle \in \mathrm{Bad}(H_{Q,q})$. An easy induction on $|\sigma|$ shows that if $\gamma \subset \sigma$, then $G_Q(\gamma) > G_Q(\sigma)$.

All is left is to define $H_{Q,q}$ and $h_{Q,q}$ for each $q \in Q \in \mathbb{W}$. The definition is by recursion on how many operations are needed to build $Q \in \mathbb{W}$. At the same time we will prove by induction that $F(H_{Q,q}) < F(Q)$.

(1) Case $Q = \mathbf{1} = \{0\}$: In this case we skip the definition of $H_{Q,q}$ and $h_{Q,q}$ and directly define $G_Q$. $G_{\mathbf{1}}(\emptyset) = 1$, $G_{\mathbf{1}}(\langle 0\rangle) = 0$.

(2) Case $Q = P_0 \sqcup P_1$: First consider $q \in P_0$. Let

$$H_{Q,q} = H_{P_0,q} \sqcup P_1$$

and $h_{Q,q}(r) = h_{P_0,q}(r) \in H_{P_0,q}$ if $r \in {P_0}_{\not\geq q}$ and $h_{Q,q}(r) = r$ if $r \in P_1$. Note that

$$F(H_{Q,q}) = F(H_{P_0,q}) \oplus F(P_1) < F(P_0) \oplus F(P_1) = F(Q).$$

If $q \in P_1$, we define $H_{Q,q}$ and $h_{Q,q}$ analogously.

(3) Case $Q = \mathcal{P}_f(P)$: Consider $q = \{q_0, \ldots, q_{k-1}\} \subseteq Q$. Let

$$H_{Q,q} = \bigsqcup_{i<k} \mathcal{P}_f(H_{P,q_i}).$$

Given $r = \{r_0, \ldots, r_{l-1}\} \in Q_{\not\geq q}$, there exists $i < k$ such that $\forall j < l\, q_i \not\leq r_j$. Consider the least such $i$. Then let $h_{Q,q}(r) = \{h_{P,q_i}(r_0), \ldots, h_{P,q_i}(r_{l-1})\} \in$

$\mathcal{P}_f(H_{P,q_i})$. Note that

$$F(H_{Q,q}) = \bigoplus_{i<k} \omega^{F(H_{P,q_i})} < \omega^{F(P)} = F(Q).$$

(4) Case $Q = \mathcal{T}(P)$: We need the following auxiliary operation. Let $(J, \leq)$ be a quasi-order. For every $k \in \omega$ let $\mathcal{T}_k(J) = \{ (T, l_T) \in \mathcal{T}(J) \mid \mathrm{ht}(T) \leq k \}$.

Let $\{ q_n \mid n \in \omega \} \subseteq P$ be such that $\forall q \in P \, \exists n \, q \leq q_n$. Let $(T_n, l_n) \in \mathcal{T}_n(P)$ consist of a single branch of $n$ nodes with the last node splitting into $n+1$ leaves labeled $q_0, \ldots, q_n$. Let

$$\mathcal{S}^n = \{ (T, l_T) \in \mathcal{T}(P) \mid (T_n, l_n) \not\preccurlyeq (T, l_T) \}.$$

For each $n$ we define $H_n \in \mathbb{W}$ with $F(H_n) < F(Q)$ and $h_n : \mathcal{S}^n \hookrightarrow H_n$.

Before defining $H_n$ and $h_n$ let us show how to use them to complete the proof. Notice that the set $\{ (T_n, l_n) \mid n \in \omega \}$ is cofinal in $\mathcal{T}(P)$. In fact given $(T, l_T) \in \mathcal{T}(P)$ pick $n$ so large that $\mathrm{ht}(T) \leq n$ and the finite range of $l_T$ is a subset of $\bigcup_{i \leq n} \{ q \in P \mid q \leq q_i \}$: then $(T, l_T) \preccurlyeq (T_n, l_n)$. Therefore, for every $q \in Q$, there exists a least $n$ such that $Q_{\not\geq q} \subseteq \mathcal{S}^n$, so it suffices to let $H_{Q,q} = H_n$ and $h_{Q,q} = h_n \restriction Q_{\not\geq q}$, the restriction of $h_n$ to $Q_{\not\geq q}$.

To construct $H_n$ let us start from

$$\hat{H}_n = \mathcal{T}_n(P \sqcup \bigsqcup_{i \leq n} \mathcal{T}(P_{\not\geq q_i})).$$

Consider $(T, l_T) \in \mathcal{S}^n$. Notice that if $\sigma \in T$ is such that $|\sigma| = n$ and $\sigma \notin L(T)$ then there exists $i \leq n$ such that $\forall \tau \in L(T)\ (\tau \supset \sigma \Rightarrow q_i \not\leq l_T(\tau))$, i.e. $l_T(\tau) \in P_{\not\geq q_i}$. Now let $\hat{T} = \{ \tau \in T \mid |\tau| \leq n \}$ so that $\mathrm{ht}(\hat{T}) \leq n$. Define also $l_{\hat{T}} : L(\hat{T}) \to P \sqcup \bigsqcup_{i<n} \mathcal{T}(P_{\not\geq q_i})$ by

$$l_{\hat{T}}(\sigma) = \begin{cases} l_T(\sigma), & \text{if } |\sigma| < n; \\ \text{the subtree of } (T, l_T) \text{ rooted at } \sigma, & \text{if } |\sigma| = n. \end{cases}$$

In the first case $\sigma \in L(T)$ and $l_{\hat{T}}(\sigma) \in P$, while in the second case $l_{\hat{T}}(\sigma) \in \mathcal{T}(P_{\not\geq q_i})$ for some $i \leq n$ by the observation above.

Set $\hat{h}_n(T, l_T) = (\hat{T}, l_{\hat{T}})$. It is not hard to check that $\hat{h}_n \colon \mathcal{S}^n \hookrightarrow \hat{H}_n$.

Since $\hat{H}_n \notin \mathbb{W}$, we have to modify our construction a bit. Let $J = P \sqcup \bigsqcup_{i \leq n} \mathcal{T}(H_{P,q_i}) \in \mathbb{W}$ and $j : P \sqcup \bigsqcup_{i \leq n} \mathcal{T}(P_{\not\geq q_i}) \hookrightarrow J$ be the map defined in the obvious way using the $h_{P,q_i}$'s. We can then extend $j$ to $j : \hat{H}_n \hookrightarrow \mathcal{T}_n(J)$.

Define $f : \mathcal{T}_n(J) \to J \cup \mathcal{P}_f(\mathcal{T}_{n-1}(J))$ as follows:

$$f(T, l_T) = \begin{cases} l_T(\emptyset) & \text{if } T = \{\emptyset\}; \\ \{ (T_n, l_{T_n}) \mid \langle n \rangle \in T \} & \text{otherwise.} \end{cases}$$

Checking that $f$ witnesses $\mathcal{T}_n(J) \hookrightarrow J \sqcup \mathcal{P}_f(\mathcal{T}_{n-1}(J))$ is straightforward. Iterating $n$ times $f$ we obtain

$$f^n : \mathcal{T}_n(J) \hookrightarrow J \sqcup \mathcal{P}_f(J \sqcup \mathcal{P}_f(\dots(J \sqcup \mathcal{P}_f(J))\dots)$$

Let

$$H_n = J \sqcup \mathcal{P}_f(J \sqcup \mathcal{P}_f(\dots(J \sqcup \mathcal{P}_f(J))\dots)$$

and $h_n = f^n \circ j \circ \hat{h}_n : \mathcal{S}^n \hookrightarrow H_n$.

Since $F(Q) = \varepsilon_{F(P)}$ is closed under $\oplus$ and $F(H_{P,q_i}) < F(P)$, we have $F(J) = F(P) \oplus \bigoplus_{i \leq n} \varepsilon_{F(H_{P,q_i})} < \varepsilon_{F(P)}$. Since $F(Q)$ is also closed under exponentiation with base $\omega$, we obtain $F(H_{Q,q}) < F(Q)$. $\square$

**Proposition 3.12** *$o(\mathsf{ST}_\omega) \leq \varphi_2(0)$.*

**PROOF.** We employ the same technique of the proof of Lemma 3.11: we keep the same notation and define, for $q \in Q = \mathsf{ST}_\omega$, $H_{Q,q} \in \mathbb{W}$ with $F(H_{Q,q}) < \varphi_2(0)$ and $h_{Q,q} : Q_{\not\geq q} \hookrightarrow H_{Q,q}$. As in the preceding proof this yields a reification of $\mathsf{ST}_\omega$ by $\varphi_2(0)$, and by Lemma 3.8 the proof is complete.

For every $n$ let $(T_n, s_n), (T_n, \hat{s}_n) \in \mathsf{ST}_\omega$ be defined as follows: $T_n$ consists of a single branch of $n+1$ nodes; $s_n$ labels the nodes of $T_n$ with even length $+$, and the nodes with odd length $-$; $\hat{s}_n$ acts dually, labeling nodes of $T_n$ with even length $-$, and nodes with odd length $+$. Let $\mathsf{ST}^n_\omega = \{\, (T, s_T) \in \mathsf{ST}_\omega \mid (T_n, s_n) \not\preccurlyeq (T, s_T) \,\}$ and similarly $\widehat{\mathsf{ST}}^n_\omega = \{\, (T, s_T) \in \mathsf{ST}_\omega \mid (T_n, \hat{s}_n) \not\preccurlyeq (T, s_T) \,\}$. It is obvious that $o(\mathsf{ST}^n_\omega) = o(\widehat{\mathsf{ST}}^n_\omega)$.

The set $\{\, (T_n, s_n) \mid n \in \omega \,\}$ is cofinal in $\mathsf{ST}_\omega$: if $\mathrm{ht}(T) = k$ then $(T, s_T) \preccurlyeq (T_{2k+1}, s_{2k+1})$. Therefore, for every $q = (T, s_T) \in \mathsf{ST}_\omega$ there exists $n$ such that

$$Q_{\not\geq q} \subseteq \mathsf{ST}^n_\omega.$$

For every $n$ we define $H_n \in \mathbb{W}$ with $F(H_n) < \varphi_2(0)$ and $h_n : \mathcal{S}^n \hookrightarrow H_n$. Once this is done the proof is completed exactly as in Case (4) of the previous proof.

When $n = 0$ notice that an element of $\mathsf{ST}^0_\omega$ is a tree with all nodes labeled $-$. Therefore $\mathsf{ST}^0_\omega$ is order isomorphic to $\mathcal{T}(\mathbf{1}) \in \mathbb{W}$. Thus we can set $H_0 = \mathcal{T}(\mathbf{1})$ (so that $F(H_0) = \varepsilon_1 < \varphi_2(0)$) and let $h_0$ be the order isomorphism mentioned above.

Fix $(T, s_T) \in \mathsf{ST}^{n+1}_\omega$. Since $(T_n, s_n) \not\preccurlyeq (T, s_T)$, $T$ does not contain a sequence $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_n$ with $s_T(\sigma_i) = +$ for even $i$, and $s_T(\sigma_i) = -$ for odd $i$. Therefore if $\sigma_0$ is such that $s_T(\sigma_0) = +$ the subtree of $(T, s_T)$ rooted at $\sigma_0$ belongs to $\widehat{\mathsf{ST}}^n_\omega$ (because it contains no sequence $\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_n$ with $s_T(\sigma_i) = +$ for even $i$, and $s_T(\sigma_i) = -$ for odd $i$). Let $U_T = \{\, \sigma \in T \mid \forall \tau \subset \sigma \; s_T(\tau) = - \,\}$. $U_T$ is a tree and if $\sigma \in L(U_T)$ then $\sigma \in L(T)$ or $s_T(\sigma) = +$.

Let $l_T(\sigma)$ be the subtrees of $(T, s_T)$ rooted at $\sigma$. By the observation above $l_T(\sigma) \in \widehat{\mathsf{ST}^n_\omega}$ when $s_T(\sigma) = +$. When $s_T(\sigma) = -$ then $\sigma \in L(T)$ and $l_T(\sigma)$ is the tree consisting of the root labeled $-$: if $n > 0$ we have $l_T(\sigma) \in \widehat{\mathsf{ST}^n_\omega}$ also in this case, while when $n = 0$ we identify this tree with the lone element of $\mathbf{1}$. Therefore $l_T : L(U_T) \to \widehat{\mathsf{ST}^n_\omega}$ when $n > 0$ and $l_T : L(U_T) \to \mathbf{1} \sqcup \widehat{\mathsf{ST}^n_\omega}$ when $n = 0$.

If we set $f_n(T, s_T) = (U_T, l_T)$ we have a function $f_n : \mathsf{ST}^{n+1}_\omega \to \mathcal{T}(\widehat{\mathsf{ST}^n_\omega})$ or $f_n : \mathsf{ST}^{n+1}_\omega \to \mathcal{T}(\mathbf{1} \sqcup \widehat{\mathsf{ST}^n_\omega})$, depending on whether $n > 0$ or not. It is easy to check that $f_n$ witnesses $\mathsf{ST}^{n+1}_\omega \hookrightarrow \mathcal{T}(\widehat{\mathsf{ST}^n_\omega})$ (or $\mathsf{ST}^{n+1}_\omega \hookrightarrow \mathcal{T}(\mathbf{1} \sqcup \widehat{\mathsf{ST}^n_\omega})$). Then

$$f_0 \circ \cdots \circ f_n : \mathsf{ST}^{n+1}_\omega \hookrightarrow Q_n.$$

where $Q_n = \mathcal{T}(\mathcal{T}(\mathcal{T}(\ldots(\mathbf{1} \sqcup \mathcal{T}(\mathbf{1}))\ldots)))$ with $n+1$ occurrences of $\mathcal{T}$. Since $Q_n \in \mathbb{W}$ and $F(Q_n) = \varepsilon_{\varepsilon_{\ldots\varepsilon_1+1}} < \varphi_2(0)$ the proof is complete. $\square$

### *3.3 Maximal order type of* $\mathsf{LO}_\omega$

We are now ready to prove Theorem 1.4 that says that $o(\mathsf{LO}_\omega, \preccurlyeq) = \varphi_2(0)$. Since $o(\mathsf{ST}_\omega) = o(\mathsf{ILO}_\omega) \leq o(\mathsf{LO}_\omega)$, by Proposition 3.5 we have that $o(\mathsf{LO}_\omega, \preccurlyeq) \geq \varphi_2(0)$. The other inequality requires the following observation.

Let $L_n = \mathbb{Z}^n \cdot \omega$. Note that for every $L \in \mathsf{LO}_\omega$, there exists $n$ such that $L \preccurlyeq L_n$. Therefore $o(\mathsf{LO}_\omega) = \sup_n o((\mathsf{LO}_\omega)_{\not\succcurlyeq L_n})$. Every $L \in (\mathsf{LO}_\omega)_{\not\succcurlyeq L_n}$ can be decomposed as a finite sum of indecomposable linear orders, say $J_0 + \cdots + J_k$. Since $L_n$ is indecomposable, we have that for every $i \leq k$, $J_i \in (\mathsf{ILO}_\omega)_{\not\succcurlyeq L_n}$. This gives us an embedding $(\mathsf{LO}_\omega)_{\not\succcurlyeq L_n} \hookrightarrow ((\mathsf{ILO}_\omega)_{\not\succcurlyeq L_n})^{<\omega}$. Since $o((\mathsf{ILO}_\omega)_{\not\succcurlyeq L_n}) < \varphi_2(0)$, we have that $o(((\mathsf{ILO}_\omega)_{\not\succcurlyeq L_n})^{<\omega}) < \varphi_2(0)$. It follows that $o((\mathsf{LO}_\omega)_{\not\succcurlyeq L_n}) < \varphi_2(0)$ and hence $o(\mathsf{LO}_\omega, \preccurlyeq) \leq \varphi_2(0)$.

## 4 Subsystems of second order arithmetic

We refer the reader to [Sim99] for background information on subsystems of second order arithmetic. From now on, when we write the name of a subsystems of second order arithmetic in parenthesis at the beginning of a definition or of a theorem, it means that the definition or the theorem is being carried out in that system.

Recall that $\mathsf{RCA}_0$ is the basis system: it consists of axioms stating that the natural numbers form an ordered semi-ring, plus the axiom schemes of $\mathbf{\Sigma}^0_1$-induction and $\mathbf{\Delta}^0_1$-comprehension.

**Definition 4.1** *($\mathsf{RCA}_0$) We say that $X = Y'$ if $\forall e(e \in X \iff \{e\}^Y(e)\downarrow)$, where $\{e\}^Y$ is the e-th Turing functional with oracle $Y$.*

*We say that $X = Y^{(k)}$ if $X = \langle X_0, \ldots, X_k \rangle$, $X_0 = Y$ and for every $n < k$, $X_{n+1} = (X_n)'$. We say that $X = Y^{(\omega)}$ if $X = \langle X_0, \ldots \rangle$, $X_0 = Y$ and for every $n$, $X_{n+1} = (X_n)'$.*

Recall that $\mathsf{ACA}_0$ (the system obtained by adding arithmetic comprehension to $\mathsf{RCA}_0$) is equivalent to $\mathsf{RCA}_0$ plus the statement $\forall Y\, \exists X\, X = Y'$.

**Definition 4.2** *$\mathsf{ACA}_0'$ is $\mathsf{RCA}_0$ plus the statement $\forall Y\, \forall k\, \exists X\, X = Y^{(k)}$. $\mathsf{ACA}_0^+$ is $\mathsf{RCA}_0$ plus the statement $\forall Y\, \exists X\, X = Y^{(\omega)}$.*

As in the proof of [Mon06, Lemma 3.4] we define Hausdorff rank using powers of $\mathbb{Z}$. Since we are interested only in finite Hausdorff rank we do not need any transfinite recursion.

**Remark 4.3** *($\mathsf{RCA}_0$) A linear order $L$ has finite Hausdorff rank if $L \preccurlyeq \mathbb{Z}^n$ for some $n$.*

The proof-theoretic ordinals of $\mathsf{ACA}_0$, $\mathsf{ACA}_0'$ and $\mathsf{ACA}_0^+$ are $\varepsilon_0$, $\varepsilon_\omega$ and $\varphi_2(0)$ respectively. Since $\mathsf{ACA}_0$ is conservative over Peano Arithmetic the first result follows from Gentzen's analysis of that theory. The result for $\mathsf{ACA}_0'$ is due originally to Jäger (unpublished notes) and a proof appears in [McA85]; recently Michael Rathjen gave a new proof sketch: details will appear in the forthcoming Ph.d. thesis of Bahareh Afshari at the University of Leeds. The computation of the proof-theoretic ordinal of $\mathsf{ACA}_0^+$ is due to Rathjen ([Rat91]).

In particular, the consistency of $\mathsf{ACA}_0^+$ is equivalent to the statement saying that $\varphi_2(0)$ is well-ordered, and hence this statement cannot be proved in $\mathsf{ACA}_0^+$. To make this statement more precise, we now explain how "$\varphi_2(0)$ is well-ordered" is expressed within $\mathsf{RCA}_0$.

Let us introduce a system of notations for the ordinals below $\varphi_2(0)$. The idea is to formalize the way of writing an ordinal $\alpha < \varphi_2(0)$ used in the proof of Proposition 3.5: with the convention $\varepsilon_{-1} = \omega^{-1} = 0$ we have

$$\alpha = \varepsilon_\beta + \omega^{\gamma_0} + \cdots + \omega^{\gamma_{k-1}}$$

with $k > 0$, $\beta < \alpha < \varepsilon_{\beta+1}$, and $\alpha > \gamma_0 \geq \cdots \geq \gamma_{k-1}$. We can then write $\beta$, and $\gamma_0, \ldots, \gamma_{k-1}$ the same way, and this process will eventually stop when $\alpha$ is written in *normal form*.

The formal definition is the following.

**Definition 4.4** *($\mathsf{RCA}_0$) Let $\mathcal{L}$ be the language consisting of the constant $-1$, the unary operations $\omega^{(\cdot)}$ and $\varepsilon_{(\cdot)}$, and the binary operation $(\cdot) + (\cdot)$.*

*We simultaneously define a set of terms $\Phi$ of $\mathcal{L}$ and an order on $\Phi$ as follows:*

- $-1 \in \Phi$ *and* $-1 \leq t$ *for every* $t \in \Phi$;
- $\varepsilon_t + \omega^{t_0} + \cdots + \omega^{t_{k-1}} \in \Phi$ *whenever* $k > 0$, $t, t_0, \ldots, t_{k-1} \in \Phi$, *and* $\varepsilon_t + \omega^{-1} \geq t_0 \geq \cdots \geq t_{k-1}$. *If* $t_i = -1$ *then we require* $k = 1$ *and* $i = 0$;
- $$\varepsilon_{t'} + \omega^{t'_0} + \cdots + \omega^{t'_{k'-1}} \leq \varepsilon_t + \omega^{t_0} + \cdots + \omega^{t_{k-1}}$$

  *if and only if either* $t' < t$ *or* $t' = t$ *and (either there exists* $i$ *such that* $t'_i \neq t_i$, *and for the first such* $i$, $t'_i < t_i$, *or* $t'_i = t_i$ *for every* $i < k'$ *and* $k' \leq k$*).*

Notice that the order $(\Phi, \leq)$ is primitive recursive. When we write a term in $\Phi$, we are thinking of an ordinal below $\varphi_2(0)$. Conversely, as in Proposition 3.5 we can prove informally that for every ordinal below $\varphi_2(0)$ there is a term representing it. It is straightforward to show informally that the order relation between elements of $\Phi$ is isomorphic to $\varphi_2(0)$.

The relationships between $\Phi$ and $\varphi_2(0)$ mentioned above cannot even be stated in $\mathsf{RCA}_0$, because $\varphi_2(0)$ does not exist in the language of second order arithmetic. When we refer to the statement "$\varphi_2(0)$ is well-ordered" in the language of second order arithmetic, we are actually saying "$(\Phi, \leq)$ is well-ordered". In terms of reverse mathematics, $\mathsf{RCA}_0$ suffices to show that $(\Phi, \leq)$ is a linear order. It is much harder to prove that is a well-order (as noticed above, $\mathsf{ACA}_0^+$ does not suffice).

Within $\mathsf{RCA}_0$ it is straightforward to define the set $\mathsf{ST}_\omega$ and the notion of homomorphism between finite signed trees. We can also define $\mathsf{ST}$, $\mathsf{LO}_\omega$ and $\mathsf{ILO}_\omega$, but notice that these are classes and not sets. Within $\mathsf{RCA}_0$ we also define the function lin and prove the following Lemma ([Mon06, Proposition 2.13]), showing that Lemma 2.8.b is provable in $\mathsf{ACA}_0$.

**Lemma 4.5** *($\mathsf{ACA}_0$) Let $T, T' \in \mathsf{ST}$. Then*

$$T \preccurlyeq T' \iff \mathrm{lin}(T) \preccurlyeq \mathrm{lin}(T').$$

When we say within $\mathsf{RCA}_0$ that a quasi-order $(Q, \leq)$ is wqo we mean that there exists no infinite bad sequences (i.e. for every $f : \omega \to Q$ there exist $i < j$ with $f(i) \leq f(j)$): in [CMS04] it is proved that in $\mathsf{RCA}_0$ this implies, but is not equivalent to, the statement that $(Q, \leq)$ has no infinite descending chains and antichains. When we deal with a quasi-order which is a class some simple coding formalizes the same definition within $\mathsf{RCA}_0$, yielding statements of $\mathsf{FRA}$ and $\mathsf{FRA}_\omega$ in $\mathsf{RCA}_0$.

## 5 Upper bound for $\mathsf{FRA}_\omega$

In this section we prove Theorem 1.5, i.e. that $\mathsf{ACA}_0^+$ together with the statement "$\varphi_2(0)$ is well-ordered" suffices to prove $\mathsf{FRA}_\omega$.

**Proposition 5.1** *($\mathsf{RCA}_0$) The following are equivalent:*

*(i)* $\mathsf{ST}_\omega$ *is well-quasi-ordered by* $\preccurlyeq$*;*
*(ii)* $\varphi_2(0)$ *is well-ordered.*

**PROOF.** The proof consists of formalizing in $\mathsf{RCA}_0$ the proofs of Section 3.

For the implication (i)⇒(ii), we need to look at the proof of Proposition 3.5. There we defined a function $F : \varphi_2(0) \to \mathsf{ST}_\omega$ witnessing $\varphi_2(0) \hookrightarrow \mathsf{ST}_\omega$. The definition of $F$ and the proof that it satisfies the required properties are by $\mathbf{\Delta}^0_0$-recursion and $\mathbf{\Delta}^0_0$-induction on the length of the normal form of the ordinal notations below $\varphi_2(0)$ and hence they can be easily be carried out in $\mathsf{RCA}_0$. Then, if we had an infinite descending sequence in $\varphi_2(0)$, we could map it through $F$ and get an infinite bad sequence in $\mathsf{ST}_\omega$.

For the implication (ii)⇒(i), we first need to look at the proof of Lemma 3.11. We can easily name all the partial orders that are in $\mathbb{W}$. So, in $\mathsf{RCA}_0$ we can work with $\mathbb{W}$ as a set, by using a set of names for its elements. Then, we can think of $F : \mathbb{W} \to \varphi_2(0)$ as a second order object which is defined by recursion. The definition of $H_{Q,q}$ and $h_{Q,q}$ is computably uniform in $q \in Q \in \mathbb{W}$ and is done by recursion on the number of operations needed to define $Q \in \mathbb{W}$ (i.e. on the complexity of the name for $Q$). $G_Q(\sigma)$ is also computably uniform in $Q \in \mathbb{W}$ and is defined by recursion on $|\sigma|$. By induction on number of operations needed to define $Q \in \mathbb{W}$ we can prove that for every $q \in Q \in \mathbb{W}$ we have $F(H_{Q,q}) < F(Q)$. Then, in $\mathsf{RCA}_0$, we can prove that $\sigma \subset \tau \Rightarrow G_Q(\sigma) > G_Q(\tau)$. Therefore, for all $Q \in \mathbb{W}$, if $F(Q)$ is well-ordered $\mathrm{Bad}(Q)$ is well-founded and hence $Q$ is well-quasi-ordered.

The argument applies also to $\mathsf{ST}_\omega$ (proof of Proposition 3.12) and shows that if $\varphi_2(0)$ is well-ordered then $\mathsf{ST}_\omega$ is well-quasi-ordered. □

**Lemma 5.2** *($\mathsf{ACA}_0^+$) Given a sequence $\{ L_i \mid i \in \omega \}$ of linear orders of finite rank, there is a sequence $\{ \langle T_{i,1}, \dots, T_{i,k_i} \rangle \mid i \in \omega \}$ of finite sequences of finite signed trees such that for every $i$*

$$L_i \sim \mathrm{lin}(T_{i,1}) + \cdots + \mathrm{lin}(T_{i,k_i}).$$

**PROOF.** In [Mon06, Lemma 3.4] it is proved that if $\mathsf{ST}_\alpha$ is well-quasi-ordered, then every computable linear order $L$ of rank $\alpha$ is equimorphic to a finite sum of linear orders of the form $\mathrm{lin}(T)$ and of rank $\leq \alpha$ via an equimorphism computable in $0^{(2(\alpha+1)^2)}$.

When $\alpha < \omega$, $\mathsf{RCA}_0$ proves that $\mathsf{ST}_\alpha$ is well-quasi-ordered just because it is finite. Noting that the proof of Lemma 3.4 in [Mon06] is uniform, we obtain that given a computable sequence $\{ L_i \mid i \in \omega \}$ of linear orders of finite rank there is a sequence $\{ \langle T_{i,1}, \dots, T_{i,k_i} \rangle \mid i \in \omega \}$ computable in $0^{(\omega)}$ which satisfies the statement of the Lemma.

If the sequence we are given is not computable, it suffices to relativize the proof, and $\mathsf{ACA}_0^+$ suffices. $\square$

**Proposition 5.3** *($\mathsf{ACA}_0^+$) If $\mathsf{ST}_\omega$ is well-quasi-ordered by $\preccurlyeq$ then $\mathsf{FRA}_\omega$ holds.*

**PROOF.** Consider an infinite sequence $\{ L_i \mid i \in \omega \}$ of linear orders of finite rank; we want to show it is not a bad sequence. By Lemma 5.2 there exists a sequence $\{ \langle T_{i,1}, \dots, T_{i,k_i} \rangle \mid i \in \omega \}$ of members of ${\mathsf{ST}_\omega}^{<\omega}$ such that for every $i$

$$L_i \sim \mathrm{lin}(T_{i,1}) + \cdots + \mathrm{lin}(T_{i,k_i}).$$

By Higman's theorem, which is provable in $\mathsf{ACA}_0$ ([Sim88,Clo90,Mar05]), we have that ${\mathsf{ST}_\omega}^{<\omega}$ is well-quasi-ordered by $\preccurlyeq$. Therefore, there exists $i < j$ such that $\langle T_{i,1}, \dots, T_{i,k_i} \rangle \preccurlyeq \langle T_{j,1}, \dots, T_{j,k_j} \rangle$. By Lemma 4.5 we have

$$L_i \sim \mathrm{lin}(T_{i,1}) + \cdots + \mathrm{lin}(T_{i,k_i}) \preccurlyeq \mathrm{lin}(T_{j,1}) + \cdots + \mathrm{lin}(T_{j,k_j}) \sim L_j.$$

Thus $\{ L_i \mid i \in \omega \}$ is not a bad sequence. $\square$

Propositions 5.1 and 5.3 obviously imply Theorem 1.5.

## 6 Lower bound for $\mathsf{FRA}_\omega$

In this section we prove Theorem 1.6, i.e. that $\mathsf{RCA}_0$+$\mathsf{FRA}$ implies $\mathsf{ACA}_0'$ plus the statement "$\varphi_2(0)$ is well-ordered".

We start by noticing that some of the intermediate steps in Shore's proof ([Sho93]) that $\mathsf{FRA}$ implies $\mathsf{ATR}_0$ show that $\mathsf{RCA}_0 \vdash \mathsf{FRA}_\omega \Rightarrow \mathsf{ACA}_0$. Indeed the proofs of [Sho93, Theorem 3.1] (establishing that $\mathsf{RCA}_0 \vdash \mathsf{FRA} \Rightarrow \mathbf{\Sigma}^0_2$-induction) and of [Sho93, Theorem 1.1] (showing that $\mathsf{RCA}_0 + \mathbf{\Sigma}^0_2$-induction $\vdash \mathsf{FRA} \Rightarrow \mathsf{ACA}_0$) make use only of linear orders (actually, well-orders) of finite Hausdorff rank. We thus have:

**Proposition 6.1** *($\mathsf{RCA}_0$) $\mathsf{FRA}_\omega$ implies $\mathsf{ACA}_0$.*

We will now build on Shore's ideas to prove the following Proposition.

**Proposition 6.2** *($\mathsf{ACA}_0$) $\mathsf{FRA}_\omega$ implies $\mathsf{ACA}_0'$.*

**PROOF.** Fix $k$. We want to show that $0^{(k)}$ exists. By relativizing the proof as usual, we will get that for every $Y$, $Y^{(k)}$ exists.

First, for each $n \leq k$, we construct a sequence of linear orders $\{ A_{j,n} \mid j \in \omega \}$, such that, if $0^{(n)}$ exists and $h : A_{j,n} \to A_{l,n}$ is an embedding for some $j < l$, then $h$ computes $0^{(n+1)}$. Unfortunately, this induction step is not enough to prove $\mathsf{ACA}_0'$ as $\mathbf{\Sigma}^1_1$-induction would be required. So we will do the following. For each $j$ we define

$$A_j = \sum_{n \leq k} (\omega^* + \omega + A_{j,n}).$$

Then, we will show that if we have an embedding $A_j \preccurlyeq A_l$ for some $j < l$, we can recover embeddings $h_n : A_{j,n} \to A_{l,n}$ for each $n \leq k$. We use these embeddings together to define a sequence of sets $X_0, X_1, \ldots, X_k$. Then, using arithmetic induction on $n \leq k$, we show that for every $n$, $X_{n+1} \geq_T (X_n)'$, and thus $\langle X_0, \ldots, X_k \rangle$ witnesses the existence of $0^{(k)}$.

The idea of the construction is as follows. We want to define $\{ A_{j,n} \mid j \in \omega \}$ such that any embedding $h_n : A_{j,n} \to A_{l,n}$ for $j < l$ produces a function that dominates the set of true stages for the enumeration of $0^{(n+1)}$ from $0^{(n)}$. (Recall that $t$ is a true stage for $f : \omega \to \omega$ if $\forall s > t \; f(s) > f(t)$. As noticed in [Sho93], $\mathsf{RCA}_0$ suffices to prove that for any one-to-one $f$ there exists infinitely many true stages, while $\mathsf{RCA}_0 + \mathbf{\Sigma}^0_2$-induction proves that for any one-to-one $f$ and any $i$ there exist the $i$-th true stage for $f$.) Let $t_i$ be the $i$-th true stage for the enumeration of $0^{(n+1)}$ from $0^{(n)}$. Given $j$, we will let $A_{n,j,t_{j+i}}$ be isomorphic to $\omega^{n+1} + 1$ and, for $s$ not of the form $t_{j+i}$, $A_{j,n,s}$ be isomorphic to $\omega^n \cdot p + 1$ for some $p \in \omega$. Then, we define

$$A_{j,n} = \sum_s A_{j,n,s}.$$

Working in $\mathsf{ACA}_0$, we have to be careful in defining the linear orders $A_{j,n,s}$. We will use sequences of linear orders $\{ L^\Sigma_{n,e} \mid n \in \omega \}$ and $\{ L^\Pi_{n,e} \mid n \in \omega \}$ (the main idea for this construction is taken from a similar construction in [Sho06, Section 5]): the reader should keep in mind that we would like to have, for every $n, e \in \omega$,

$$L^\Sigma_{n,e} \cong \begin{cases} \omega^{n+1} & \text{if } e \in 0^{(n+1)} \\ \omega^n & \text{if } e \notin 0^{(n+1)} \end{cases} \tag{6.1}$$

and

$$L^{\Pi}_{n,e} \cong \begin{cases} \omega^n \cdot p \text{ for some } p \in \omega, p > 0 & \text{if } e \in 0^{(n+1)} \\ \omega^{n+1} & \text{if } e \notin 0^{(n+1)}. \end{cases} \tag{6.2}$$

For each $n$ denote by $\Omega^{n+1}$ the natural presentation of the linear order of order type $\omega^{n+1}$ consisting of the ordered $n$-tuples of natural numbers ordered antilexicographically. $L^{\Sigma}_{n,e}$ and $L^{\Pi}_{n,e}$ will be subsets of $\Omega^{n+1}$ under this order. Since $\mathsf{ACA}_0$ proves that $\Omega^{n+1}$ is a well-order for each $n$, we have that $L^{\Sigma}_{n,e}$ and $L^{\Pi}_{n,e}$ are well-orders.

We define $L^{\Sigma}_{n,e}$ and $L^{\Pi}_{n,e}$ by recursion on $n$. Recall the existence of a computable function $f$ such that for all $n \in \omega$

(a) for all $e \in \omega$, $e \in 0^{(n+1)}$ if and only if there exists $x$ such that $f(n,e,x) \notin 0^{(n)}$;
(b) for all $e, x, y \in \omega$, if $x < y$ and $f(n,e,x) \notin 0^{(n)}$ then $f(n,e,y) \notin 0^{(n)}$.

The function $f$ can be defined in $\mathsf{RCA}_0$, and for each $n$, if $0^{(n)}$ exists then $\mathsf{RCA}_0$ proves (a) and (b) above (here, since $0^{(n+1)}$ might not exist, $e \in 0^{(n+1)}$ is to be understood as an abbreviation for $\{e\}^{0^{(n)}}(e)\downarrow$).

Let $L^{\Sigma}_{0,e}$ (resp. $L^{\Pi}_{0,e}$) be the set $\{0\} \cup \{\, s \mid \{e\}_s(e)\downarrow \}$ (resp. $\{0\} \cup \{\, s \mid \{e\}_s(e)\uparrow \}$), which is indeed a subset of $\Omega^1$. If we have already defined $L^{\Sigma}_{n,e}$ and $L^{\Pi}_{n,e}$ for every $e$, let

$$\begin{aligned} L^{\Sigma}_{n+1,e} &= \{\, (\vec{y}, x) \in \Omega^{n+2} \mid \vec{y} \in L^{\Pi}_{n,f(n,e,x)} \} \quad \text{and} \\ L^{\Pi}_{n+1,e} &= \{\, (\vec{y}, x) \in \Omega^{n+2} \mid \vec{y} \in L^{\Sigma}_{n,f(n,e,x)} \}, \end{aligned}$$

so that the order types of $L^{\Sigma}_{n+1,e}$ and $L^{\Pi}_{n+1,e}$ are respectively

$$\sum_{x \in \omega} L^{\Pi}_{n,f(n,e,x)} \quad \text{and} \quad \sum_{x \in \omega} L^{\Sigma}_{n,f(n,e,x)}.$$

**Claim 6.2.1** *For every $n, e \in \omega$, $\mathrm{rk}_H(L^{\Sigma}_{n,e}) \le n+1$ and $\mathrm{rk}_H(L^{\Pi}_{n,e}) \le n+1$.*

**PROOF.** This is obvious, since $L^{\Sigma}_{n,e} \preccurlyeq \Omega^{n+1} \preccurlyeq \mathbb{Z}^{n+1}$ and the same for $L^{\Pi}_{n,e}$. $\square$

**Claim 6.2.2** *Let $m \in \omega$ be such that $0^{(m)}$ exists. Then for every $n \le m$ and $e \in \omega$ (6.1) and (6.2) hold.*

**PROOF.** Fix $m$ and assume $0^{(m)}$ exists. We use induction on $n \le m$ to prove an apparently stronger statement: for every $e$, $L^{\Sigma}_{n,e}$ (resp. $L^{\Pi}_{n,e}$) satisfies

(6.1) (resp. (6.2)) with an isomorphism computable in $0^{(n)}$. This statement is arithmetical, so we have enough induction in $\mathsf{ACA}_0$ to carry out the proof.

The base and the induction steps are both straightforward using that $f$ satisfies (a) and (b). $\square$

When we do not know that $0^{(n)}$ exists, we know very little about the order type of $L^{\Sigma}_{n,e}$ and $L^{\Pi}_{n,e}$. However, as noticed above, we do know they are well-ordered.

Now we are ready to formally define sequentially the orders $A_{j,n,s}$, $A_{j,n}$ and $A_j$. Notice that "$s$ is a true stage for $f$ and there are at least $j-1$ true stages for $f$ smaller than $s$" is a $\mathbf{\Pi}^0_1$ formula. Hence there exists a computable function $g$ such that for each $n, s, j$, $g(n,j,s) \notin 0^{(n+1)}$ if and only if $s$ is the $(j+i)$-th true stage for the enumeration of $0^{(n+1)}$ from $0^{(n)}$, for some $i$. Define $A_{j,n,s} = L^{\Pi}_{n,g(n,j,s)} + 1$. We may assume that the domains of the $A_{j,n,s}$'s are pairwise disjoint and denote by $m_{j,n,s}$ the first element of $A_{j,n,s}$.

As announced let $A_{j,n} = \sum_s A_{j,n,s}$ and $A_j = \sum_{n \leq k}(\omega^* + \omega + A_{j,n})$. By Claim 6.2.1 we have $\mathrm{rk}_H(A_{j,n}) \leq n+2$ and hence $\mathrm{rk}_H(A_j) \leq k+3$ for every $j$. Let $\pi_{j,n} : A_{j,n} \to \omega$ be the function which assigns to every $z \in A_{j,n}$ the unique $s$ such that $z \in A_{j,n,s}$.

By $\mathsf{FRA}_\omega$ there exists $j < l$ and an embedding $h : A_j \to A_l$. For $n \leq k$ let $h_n$ be the restriction of $h$ to $A_{j,n}$

**Claim 6.2.3** *For each $n \leq k$, $h_n$ maps $A_{j,n}$ into $A_{l,n}$.*

**PROOF.** For every $j$ we have $\omega^* \cdot (k+1) \preccurlyeq A_j$, while the fact that each $L^{\Pi}_{n,e}$ is well-ordered implies that the $A_{j,n}$'s are well-ordered and hence $\omega^* \cdot (k+2) \not\preccurlyeq A_j$. In other words, both $A_j$ and $A_l$ contain exactly $k+1$ copies of $\omega^*$.

Thus, for each $n \leq k$, an initial segment of the copy of $\omega^*$ in $A_j$ immediately preceding $\omega + A_{j,n}$ is mapped by $h$ coinitially to the copy of $\omega^*$ in $A_l$ which precedes immediately $\omega + A_{l,n}$. Therefore the copy of $\omega$ immediately preceding $A_{j,n}$ is mapped to $\omega^* + \omega + A_{l,n}$, and this implies that $h_n$ maps $A_{j,n}$ into $A_{l,n}$. $\square$

**Claim 6.2.4** *Suppose $0^{(n)}$ exists. If $j < l$ and $h_n : A_{j,n} \to A_{l,n}$ is an embedding, then $0^{(n+1)} \leq_T h_n \oplus 0^{(n)}$.*

**PROOF.** Recursively in $h_n$ we define $k : \omega \to \omega$ by setting $k(0) = t_j$ and $k(x+1) = \pi_{l,n}(h_n(m_{j,n,k(x)}))$.

We prove by induction that $k(x) \geq t_{j+x}$ for every $x \in \omega$. The case $x = 0$ is trivial. Assuming $k(x) \geq t_{j+x}$ notice that $\omega^{n+1} \cdot (x+1)$ embeds into the initial segment of $A_{j,n}$ bounded by $m_{j,n,k(x)}$, and hence also into the initial segment of $A_{l,n}$ bounded by $h_n(m_{j,n,k(x)})$. This implies that $t_{l+x} \leq \pi_{l,n}(h_n(m_{j,n,k(x)})) = k(x+1)$. Since $j < l$ we have $t_{j+x+1} \leq t_{l+x}$ and the induction step is complete.

Hence $x \in 0^{(n+1)}$ if and only if it appears among the first $k(x)$ elements of the enumeration of $0^{(n+1)}$ from $0^{(n)}$. Thus $0^{(n+1)} \leq_T h_n \oplus 0^{(n)}$. □

Let $X_0 = \emptyset$, and, for $n < k$, $X_{n+1} = h_n \oplus X_n$. Using the previous claim, we can prove by arithmetic induction on $n$, that for every $n \leq k$ there exist an index $e_n$ such that $\{e_n\}^{X_n} = 0^{(n)}$. Therefore $\langle \{e_0\}^{X_0}, \dots, \{e_k\}^{X_k} \rangle$ witnesses the existence of $0^{(k)}$. □

**Remark 6.3** *Proposition 6.2 cannot be strengthened by replacing* $\mathsf{ACA}_0'$ *with* $\mathsf{ACA}_0^+$*. In fact* $\mathsf{FRA}_\omega$ *holds in the* $\omega$*-model consisting of all arithmetic sets, where* $\mathsf{ACA}_0^+$ *fails. The reason is that if we are given a sequence* $\{L_n : n \in \omega\}$ *of arithmetic linear orders of finite Hausdorff rank, then the exists* $i < j$ *such that* $L_i \preccurlyeq L_j$*. At first, we do not know that this embedding is in the model of arithmetic sets. But it follows from the proofs of [Mon06, Lemmas 3.4 and 2.5] that, if there is an embedding between two linear orders of finite Hausdorff rank, then the embedding is arithmetic in the linear orders and hence belongs to the model.*

**Proposition 6.4** *(*$\mathsf{ACA}_0$*)* $\mathsf{FRA}_\omega$ *implies that* $\mathsf{ST}_\omega$ *is well-quasi-ordered by* $\preccurlyeq$*.*

**PROOF.** Consider an infinite sequence $\{ (T_i, s_{T_i}) \mid i \in \omega \}$ of elements of $\mathsf{ST}_\omega$; we want to show that it is not a bad sequence. The sequence $\{ \mathrm{lin}(T_i, s_{T_i}) \mid i \in \omega \}$ is a sequence in $\mathsf{LO}_\omega$. By $\mathsf{FRA}_\omega$, there exists $i < j$ such that $\mathrm{lin}(T_i, s_{T_i}) \preccurlyeq \mathrm{lin}(T_j, s_{T_j})$. By Lemma 4.5, we have $(T_i, s_{T_i}) \preccurlyeq (T_j, s_{T_j})$. □

Propositions 6.1, 6.2, 6.4, and 5.1 imply Theorem 1.6.

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