RPO, Second-order Contexts, and λ-calculus*

Pietro Di Gianantonio   Furio Honsell   Marina Lenisa

Dipartimento di Matematica e Informatica, Università di Udine
via delle Scienze 206, 33100 Udine, Italy.
{digianantonio,honsell,lenisa}@dimi.uniud.it

Abstract. We apply Leifer-Milner *RPO approach* to the λ-calculus, endowed with *lazy* and *call by value reduction strategies*. We show that, contrary to process calculi, one can deal directly with the λ-calculus syntax and apply Leifer-Milner technique to a category of contexts, provided that we work in the framework of *weak bisimilarities*. However, even in the case of the transition system with minimal contexts, the resulting bisimilarity is *infinitely branching*, due to the fact that, in standard context categories, parametric rules such as β can be represented only by infinitely many ground rules. To overcome this problem, we introduce the general notion of *second-order context category*. We show that, by carrying out the RPO construction in this setting, the *lazy (call by value) observational equivalence* can be captured as a *weak bisimilarity equivalence* on a *finitely branching* transition system. This result is achieved by considering an encoding of λ-calculus in Combinatory Logic.

1 Introduction

Recently, much attention has been devoted to derive *labelled transition systems* and *bisimilarity congruences* from *reactive systems*, in the context of process languages and graph rewriting, [Sew02,LM00,SS03,GM05,BGK06,BKM06,EK06]. In the theory of process algebras, the operational semantics of CCS was originally given via a labelled transition system (lts), while more recent process calculi have been presented via reactive systems plus structural rules. Reactive systems naturally induce behavioral equivalences which are congruences w.r.t. contexts, while lts's naturally induce bisimilarity equivalences with coinductive characterizations. However, such equivalences are not congruences in general, or else it is an heavy, ad-hoc task to prove that they are congruences.

Generalizing [Sew02], Leifer and Milner [LM00] presented a general categorical method for deriving a transition system from a reactive system, in such a way that the induced bisimilarity is a congruence. The labels in Leifer-Milner's transition system are those contexts which are *minimal* for a given reaction to fire. Minimal contexts are identified via the categorical notion of *relative pushout (RPO)*. Leifer-Milner's central result guaranties that, under a suitable categorical condition, the induced bisimilarity is a *congruence* w.r.t. all contexts.

In the literature, some case studies have been carried out in the setting of process calculi, for testing the expressivity of Leifer-Milner's approach. Some difficulties have arisen in applying the approach directly to such languages, viewed

* Work supported by ART PRIN Project prot. 2005015824 (funded by MIUR).

as Lawvere theories, because of structural rules. Thus more complex categorical constructions have been introduced in [Lei01], and by Sassone and Sobocinski in [SS03,SS05]. Moreover, often intermediate encodings have been considered, in graph theory, for which the approach of "borrowed contexts" has been developed [EK06], and in Milner's bigraph theory [Mil07].

In this paper, we focus on the prototypical example of reactive system given by the λ-calculus, endowed with lazy and call by value (cbv) reduction strategies. We show that, in principle, contrary to process calculi, one could deal directly with the λ-calculus syntax and apply Leifer-Milner technique to the category of term contexts induced by the λ-terms, provided that we work in the setting of *weak bisimilarities*. However, even in the case of the transition system with minimal contexts, the lts and the induced bisimilarity turn out to be infinitely branching. This is mainly due to the fact that, in the category of contexts, the β-rule cannot be described parametrically, but it needs to be described extensionally using an infinite set of pairs of ground terms. In order to overcome this problem, we consider the combinatory logic and we introduce the general notion of *category of second-order term contexts*. Our main result amounts to the fact that, by carrying out Leifer-Milner's construction in this setting, the *lazy (cbv) contextual equivalence* can be captured as a *weak bisimilarity equivalence* on a (finitely branching) transition system. Technically, this result is achieved by considering an encoding of the lazy (cbv) λ-calculus in KS Combinatory Logic (CL), endowed with a lazy (cbv) reduction strategy, and by showing that the lazy (cbv) contextual equivalence on λ-calculus can be recovered as a lazy (cbv) equivalence on CL. It is necessary to consider such encoding, since the approach of second-order context categories proposed in this paper works for reaction rules which are "local", that is the reaction does not act on the whole term, but only locally. But the substitution operation on λ-calculus is not local.

Finally, the second-order approach carried out in this paper for the λ-calculus suggests a new general technique for dealing with any calculus with parametric rules, alternative to the one of luxes in [KSS05]. Moreover, the correspondence results obtained in this paper about the observational equivalences on λ-calculus and CL are interesting *per se* and, although natural and ultimately elementary, had not appeared previously in the literature.

Summary. In Section 2, we summarize the theory of reactive systems of [LM00]. In Section 3, we present the λ-calculus together with lazy and cbv reduction strategies and observational equivalences, and we discuss the RPO approach applied to the λ-calculus endowed with a structure of context category. In Section 4, we focus on Combinatory Logic (CL), we show how to recover on CL the lazy and cbv strategies and observational equivalences, and we discuss the RPO approach applied to CL, viewed as a context category. In Section 5, we introduce the notion of second-order context category, and we apply the RPO approach to CL viewed as a second-order rewriting system, thus obtaining characterizations of lazy and cbv observational equivalences as weak bisimilarities on finitely branching lts's. Final remarks and directions for future work appear in Section 6. For lack of space, proofs are omitted in this paper, however they are available in [DHL08].

2 The Theory of Reactive Systems

In this section, we summarize the theory of reactive systems proposed in [LM00] to derive lts's and bisimulation congruences from a given reduction semantics. Moreover, we discuss weak variants of Leifer-Milner's bisimilarity equivalence.

The theory of [LM00] is based on a categorical formulation of the notion of *reactive system*, whereby *contexts* are modeled as arrows of a category, *terms* are arrows having as domain 0 (a special object which denotes no holes), and reaction rules are pairs of terms.

Definition 1 (Reactive System). *A* reactive system $\mathbf{C}$ *consists of:*
– *a category* $\mathcal{C}$*;*
– *a distinguished object* $0 \in |\mathcal{C}|$*;*
– *a composition-reflecting subcategory* $\mathcal{D}$ *of* reactive contexts*;*
– *a set of pairs* $\mathbf{R} \subseteq \bigcup_{I \in |\mathcal{C}|} \mathcal{C}[0, I] \times \mathcal{C}[0, I]$ *of* reaction rules*.*

Reactive systems on term languages can be viewed as a special case of reactive systems in the sense of Leifer-Milner by instantiating $\mathcal{C}$ as a suitable category of term and contexts, also called the (free) Lawvere category, [LM00].

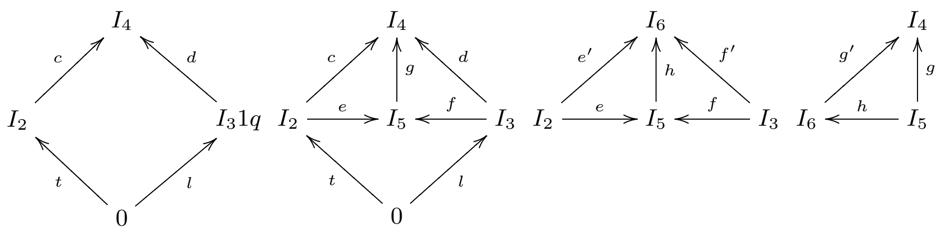
Given a reaction system with reactive contexts $\mathcal{D}$ and reaction rules $\mathbf{R}$, the *reaction relation* $\rightarrow$ is defined by: $t \rightarrow u$ iff $t = dl, \ u = dr$ for some $d \in \mathcal{D}$ and $\langle l, r \rangle \in \mathbf{R}$.

The behavior of a reactive system is expressed as an unlabelled transition system. On the other hand, many useful behavioral equivalences are only defined for lts's. The passage from reactive systems to lts's is obtained as follows.

Definition 2 (Context Label Transition System). *Given a reactive system* $\mathbf{C}$*, the associated context lts is defined as follows:*
– *states: arrows* $t : 0 \rightarrow I$ *in* $\mathcal{C}$*, for any* I*;*
– *transitions:* $t \stackrel{c}{\longrightarrow}_C u$ *iff* $c \in \mathcal{C}$ *and* $ct \rightarrow u$*.*

In the case of a reactive system defined on a category of contexts, a state is a term t, and an associated label is a context c such that ct reduces. In the following, we will consider also lts's obtained by reducing the set of transitions of the context lts. In the sequel, we will use the word lts to refer to any such lts obtained from a context lts. In the standard way, any lts induces a bisimilarity relation. In [LM00], the authors proposed a categorical criterion for identifying the "smallest context allowing a reaction".

Definition 3 (RPO/IPO).
i) Let $\mathcal{C}$ *be a category and let us consider the commutative diagram in Fig. 1(i). Any tuple* $\langle I_5, e, f, g \rangle$ *which makes diagram in Fig. 1(ii) commute is called a* candidate *for (i). A* relative pushout *is the smallest such candidate, i.e. it satisfies the universal property that given any other candidate* $\langle I_, e', f', g' \rangle$*, there exists a* unique *mediating morphism* $h : I_5 \rightarrow I_6$ *such that both diagrams in Fig. 1(iii) and Fig. 1(iv) commute.*
ii) A commuting square such as diagram in Fig 1(i) is an idem pushout *if* $\langle I_4, c, d, id_{I_4} \rangle$ *is its RPO.*



(i) (ii) (iii) (iv)

Fig. 1. Redex Square and Relative Pushout.

Definition 4 (IPO Transition System).
- *States: arrows* $t : 0 \rightarrow I$ *in* $\mathcal{C}$*, for any* I*;*
- *transitions:* $t \stackrel{c}{\longrightarrow}_I dr$ *iff* $d \in \mathcal{D}$*,* $\langle l, r\rangle \in \mathbf{R}$ *and the diagram in Fig. 1(i) is an IPO.*

That is, if inserting t into the context c matches dl, and c is the "smallest" such context (IPO condition), then t transforms to dr with label c, where r is the reduct of l. Let $\sim_I$ denote the bisimilarity induced by the IPO lts.

Definition 5 (Redex Square). *Let* $\mathbf{C}$ *be a reactive system and* $t : 0 \rightarrow I_2$ *an arrow in* $\mathcal{C}$*. A* redex square *(see Fig. 1(i)) consists of a left-hand side* $l : 0 \rightarrow I_3$ *of a reaction rule* $\langle l : 0 \rightarrow I_3, r : 0 \rightarrow I_3\rangle \in \mathbf{R}$*, a context* $c : I_2 \rightarrow I_4$ *and a reactive context* $d : I_3 \rightarrow I_4$ *such that* $ct = dl$*.*
A reactive system is said to have redex RPOs *if every redex square has an RPO.*

The following is Leifer-Milner's central result:

Theorem 1 ([LM00]). *Let* $\mathbf{C}$ *be a reactive system having redex RPOs. Then the IPO bisimilarity* $\sim_I$ *is a congruence w.r.t. all contexts, i.e. if* $a \sim_I b$ *then for all* c *of the appropriate type,* $ca \sim_I cb$*.*

2.1 Weak Bisimilarity

For dealing with the λ-calculus, it will be useful to consider the weak versions of the context and IPO lts's defined above, together with the corresponding notions of *weak bisimilarities.*

One can proceed in general, by defining a weak lts from a given lts:

Definition 6 (Weak lts and Bisimilarity). *Let* $\stackrel{\alpha}{\longrightarrow}$ *be a lts, and let* τ *be a label (identifying an unobservable action).*
i) We define the weak lts $\stackrel{\alpha}{\Longrightarrow}$ *by*

$$t \stackrel{\alpha}{\Longrightarrow} u \quad \textit{iff} \quad \begin{cases} t \stackrel{\tau}{\longrightarrow}^* u & \textit{if } \alpha = \tau \\ t \stackrel{\tau}{\longrightarrow}^* t' \stackrel{\alpha}{\longrightarrow} u' \stackrel{\tau}{\longrightarrow}^* u & \textit{otherwise .} \end{cases}$$

where $\stackrel{\tau}{\longrightarrow}^*$ *denotes the reflexive and transitive closure of* $\stackrel{\tau}{\longrightarrow}$*.*
ii) Let us call weak bisimilarity *the bisimilarity induced by the weak lts.*

The above definition differs from the one proposed in [LM00]. We cannot use that in [LM00], since it discriminates λ-terms which are equivalent in the usual semantics. The following easy lemma gives a useful characterization of the weak bisimilarity, whereby any $\stackrel{\alpha}{\longrightarrow}$-transition is mimicked by a $\stackrel{\alpha}{\Longrightarrow}$-transition:

Lemma 1. *Let $\stackrel{\alpha}{\longrightarrow}$ be a lts and let $\stackrel{\alpha}{\Longrightarrow}$ be the corresponding weak lts. The induced weak bisimilarity is the greatest symmetric relation $\mathcal{R}$ s.t.:*

$$\langle a, b\rangle \in \mathcal{R} \ \wedge \ a \stackrel{f}{\longrightarrow} a' \implies \exists b'.\ b \stackrel{f}{\Longrightarrow} b' \ \wedge \ \langle a', b'\rangle \in \mathcal{R}\ .$$

For dealing with the λ-calculus, we will consider a notion of *weak IPO bisimilarity*, where the identity context is unobservable. Such notions of weak IPO bisimilarities are not congruences w.r.t. all contexts, in general, however, as observed in [LM00] (end of Section 5), they are congruences at least w.r.t. reactive contexts:

Theorem 2. *Let* **C** *be a reactive system having redex RPOs. Then the weak IPO bisimilarity $\approx_I$, where the identity context is unobservable, is a congruence w.r.t. reactive contexts.*

3 The Lambda Calculus

First, we recall the λ-calculus syntax together with *lazy* and *cbv* reduction strategies and observational equivalences. Then, we show how to apply the RPO technique to λ-calculus, viewed as a context category, and we discuss some problematic issues.

The set of λ-*terms* Λ is defined by $(\Lambda \ni)\ M ::= \ x \mid MM \mid \lambda x.M$, where $x \in \mathit{Var}$ is an infinite set of variables. Let $FV(M)$ denote the set of free variables in M, and let us denote by Λ^0 the set of *closed* λ-terms.

As usual, λ-terms are taken up-to α-conversion, and application associates to the left. We consider the standard notions of β-rule $(\lambda x.M)N \to_\beta M[N/x]$ and β_V-rule $(\lambda x.M)N \to_{\beta_V} M[N/x]$. We denote by $=_\beta$ and $=_{\beta_V}$ the corresponding conversions.

A *reduction strategy* on the λ-calculus determines, for each term which is not a value, a suitable β-redex appearing in it to be contracted. The lazy and cbv reduction strategies are defined on closed λ-terms as follows:

Definition 7 (Reduction Strategies).
i) The lazy strategy $\to_l \subseteq \Lambda^0 \times \Lambda^0$ *reduces the leftmost β-redex, not appearing within a λ-abstraction. Formally, $\to_l$ is defined by the axiom:*

$$\frac{}{(\lambda x.M)N \to_l M[N/x]} \qquad\qquad \frac{N \to_l N'}{NP \to_l N'P}$$

ii) The call by value strategy $\to_v \subseteq \Lambda^0 \times \Lambda^0$ *reduces the leftmost β_V-redex, not appearing within a λ-abstraction. Formally, $\to_v$ is defined by the following rules:*

$$\frac{}{(\lambda x.M)V \to_v M[V/x]} \qquad \frac{N \to_v N'}{NP \to_v N'P} \qquad \frac{N \to_v N'}{(\lambda x.M)N \to_v (\lambda x.M)N'}$$

where V is a λ-abstraction.

We denote by $\to_\sigma^*$ the reflexive and transitive closure of a strategy $\to_\sigma$, for $\sigma \in \{l, v\}$, by Val_σ the set of values, i.e. the set of terms on which the reduction strategy halts (which coincides with the set of λ-abstractions in both cases), and by $M \Downarrow_\sigma$ the fact that there exists $V \in Val_\sigma$ such that $M \to_\sigma^* V$.

As we will see in Section 3.1 below, each strategy defines a reactive system on λ-terms in the sense of Definition 1. To this aim, it is useful to notice that the above reduction strategies can be alternatively determined by specifying suitable sets of *reactive contexts*, which are subsets of the following (closed) *unary contexts*, i.e. contexts with a single hole: $C[\,] ::= [\,] \mid PC[\,] \mid C[\,]P$.

Remark 1. i) The lazy strategy $\to_l$ is the closure of the β-rule under the reactive contexts, corresponding to the (closed) applicative contexts: $D[\,] ::= [\,] \mid D[\,]P$, ii) The cbv strategy $\to_v$ is the closure of the β_V-rule under the following (closed) reactive contexts: $D[\,] ::= [\,] \mid D[\,]P \mid (\lambda x.M)D[\,]$.

Each strategy induces an *observational (contextual) equivalence* à la Morris on closed terms, when we consider programs as *black boxes* and only observe their "halting properties".

Definition 8 (σ-observational Equivalence). *Let $\to_\sigma$ be a reduction strategy and let $M, N \in \Lambda^0$. The* observational equivalence *$\approx_\sigma$ is defined by*
$M \approx_\sigma N$ iff for any unary context $C[\,]$, $C[M] \Downarrow_\sigma \Leftrightarrow C[N] \Downarrow_\sigma$.

The definition of $\approx_\sigma$ can be extended to open terms by considering closing substitutions, i.e. for $M, N \in \Lambda$ s.t. $FV(M, N) \subseteq \{x_1, \ldots, x_n\}$, we define: $M \widehat{\approx}_\sigma N$ iff for all closing substitutions $\boldsymbol{P}$, $M[\boldsymbol{P}/\boldsymbol{x}] \approx_\sigma N[\boldsymbol{P}/\boldsymbol{x}]$.

Remark 2. Notice that the definition of unary contexts does not include λ-abstraction contexts, i.e. contexts where the hole appears under the scope of a λ-abstraction. Namely, such contexts are not relevant, since $\approx_\sigma$ is defined on closed terms. Moreover, often in the literature, the observational equivalence is defined by considering multi-holed contexts. However, it is easy to see that the two notions of observational equivalences, obtained by considering just unary or all multi-holed contexts, coincide.

The problem of reducing the set of contexts in which we need to check the behavior of two terms has been widely studied in the literature. In particular, for both strategies in Definition 7 above, a *Context Lemma* holds, which allows us to restrict ourselves to *applicative contexts* of the shape $[\,]\boldsymbol{P}$, where $\boldsymbol{P}$ denotes a list of terms. Let us denote by $\approx_\sigma^{app}$ the observational equivalence which checks the behavior of terms only in applicative contexts. This admits a coinductive characterization as follows:

Definition 9 (Applicative σ-bisimilarity).
i) A relation $\mathcal{R} \subseteq \Lambda^0 \times \Lambda^0$ is an applicative σ-bisimulation *if the following holds:*
$\langle M, N\rangle \in \mathcal{R} \implies (M \Downarrow_\sigma \Leftrightarrow N \Downarrow_\sigma) \wedge \forall P \in \Lambda^0.\ \langle MP, NP\rangle \in \mathcal{R}$.
ii) The applicative equivalence $\approx_\sigma^{app}$ is the largest applicative bisimulation.

The Context Lemma, a well-known resut [AO93,EHR92], states $\approx_\sigma = \approx_\sigma^{app}$. By the Context Lemma, the class of contexts in which we have to check the behavior of terms is smaller, however it is still infinite, thus the applicative bisimilarity is infinitely branching. In the following, we will study alternative coinductive characterizations of the observational equivalences, arising from the application of Leifer-Milner technique.

3.1 Lambda Calculus as a Reactive System

Both lazy and cbv λ-calculus can be endowed with a structure of reactive system in the sense of Definition 1, by considering a suitable variant of context category.

Definition 10 (Lazy, cbv λ-reactive Systems). $\mathbf{C}_\sigma^\lambda$, *for* $\sigma \in \{l, v\}$, *consists of*

- *the category whose objects are* $0, 1$, *where the morphisms from 0 to 1 are the closed terms (up-to α-equivalence), the morphisms from 1 to 1 are the unary contexts (up-to α-equivalence), and composition is context insertion;*
- *the subcategory of reactive contexts is determined by the reactive contexts for the lazy and cbv strategy, respectively, presented in Remark 1;*
- *the (infinitely many) reaction rules are* $(\lambda x.M)N \rightarrow_{\beta_\sigma} M[N/x]$, *for all* M, N.

The above definition is well-posed, in particular the subcategory of reactive contexts is composition-reflecting.

One can easily check that the system $\mathbf{C}_\sigma^\lambda$ admits RPOs; since there are no abstraction contexts, this fact can be proved by repeating the corresponding proof for the category of term contexts, [Sew02].

Lemma 2. *The reactive system* $\mathbf{C}_\sigma^\lambda$, *for* $\sigma \in \{l, v\}$, *has redex RPOs.*

The IPO contexts of a closed term for the lazy and cbv reactive systems are summarized in the first two tables of Fig. 2. The applicative IPO contexts appear in the third table. This class of contexts is interesting, since it is sufficient for determining the observational equivalence (see Theorem 3 below).

Lazy lts

term	IPO contexts
$\lambda x.M$	$[\,]P$, $PC[\,]$
$(\lambda x.M)N\boldsymbol{P}$	$[\,]$, $PC[\,]$

Cbv lts

term	IPO contexts
$\lambda x.M$	$[\,]P$, $RC[\,]$, $(\lambda x.Q)[\,]$
$(\lambda x.M)N\boldsymbol{P}$	$[\,]$, $RC[\,]$

where R is not a cbv value.

Lazy/cbv appl. lts's

term	IPO cont.
$\lambda x.M$	$[\,]P$
$(\lambda x.M)N\boldsymbol{P}$	$[\,]$

Fig. 2. IPO contexts for the lazy/cbv lts's and for their applicative restrictions.

The strong versions of context and IPO bisimilarities are too fine, since they take into account reaction steps, and tell apart β-convertible terms. Trivially, I and II, where $I = \lambda x.x$, are equivalent neither in the context bisimilarity nor in the IPO bisimilarity, since $I \stackrel{[\,]}{\not\to}$, while $II \stackrel{[\,]}{\to}$ (both in the lazy and cbv case). On the other hand, one can easily check that the weak context bisimilarity, where the identity context [ ] is unobservable, equates all closed terms.

The main result of this section, whose proof can be found in [DHL08], is the following:

Theorem 3. *Both for lazy and cbv strategies, the observational equivalence, the weak IPO bisimilarity (where the identity context is unobservable), and the applicative weak IPO bisimilarity (where only applicative contexts are considered) coincide.*

Theorem 3 above gives us interesting characterizations of lazy and cbv observational equivalences, in terms of lts's where the labels are significantly reduced. However, such lts's (and bisimilarities) are still infinitely branching, e.g. $\lambda x.M \stackrel{P}{\to}_I$, for all $P \in \Lambda^0$. This is due to the fact that the context categories underlying the reactive systems $\mathbf{C}_l^\lambda$ and $\mathbf{C}_v^\lambda$ allow only for a ground representation of the β-rule through infinitely many ground rules. In order to overcome this problem, one should look for alternative categories which allow for a parametric representation of the β-rule as $(\lambda x.X)Y \to X[Y/x]$, where X, Y are parameters. To this aim, we introduce the category of *second-order term contexts* (see Section 5 below). However, as we will see, this approach works only if the reaction rules are "local", that is they do not act on the whole term, but only locally. In particular, the operation of substitution on the λ-calculus is not local and thus it is not describable by a finite set of reaction rules. To avoid this problem, in the following section we consider encodings of the λ-calculus into Combinatory Logic (CL) endowed with suitable strategies and equivalences, which turn out to correspond to lazy and cbv equivalences.

4 Combinatory Logic

In this section, we focus on *Combinatory Logic* [HS86] with Curry's combinators $\mathbf{K}, \mathbf{S}$, and we study its relationships with the λ-calculus endowed with lazy and cbv reduction strategies. An interesting result that we prove is that we can define suitable reduction strategies on CL-terms, inducing observational equivalences which correspond to lazy and cbv equivalences on λ-calculus. As a consequence, we can safely shift our attention from the reactive system of λ-calculus to the simpler reactive system of CL. In this section, we apply Leifer-Milner construction to CL viewed as a (standard) context category, and we study weak versions of context and IPO bisimilarities. Our main result is that we can recover lazy and cbv observational equivalences as weak IPO equivalences on CL*, a variant of standard CL. Here the approach is first-order, thus the IPO equivalences are still infinitely branching. However, the results in this section are both interesting in themselves, and useful for our subsequent investigation of Section 5, where CL is viewed as a second-order rewriting system, and characterizations of the observational equivalences as finitely branching IPO bisimilarities are given.

Definition 11 (Combinatory Terms). *The set of combinatory terms is defined by:* $(CL \ni)\ M\ ::=\ x \mid \mathbf{K} \mid \mathbf{S} \mid MM$ *, where* $\mathbf{K}$*,* $\mathbf{S}$ *are combinators.*
The set of combinatory terms is endowed with the following reaction rules:

$$\mathbf{K}MN \to M \qquad\qquad \mathbf{S}MNP \to (MP)(NP)$$

Let CL^0 *denote the set of* closed *CL-terms.*

4.1 Correspondence with the λ-calculus

Let $\Lambda(\mathbf{K}, \mathbf{S})$ denote the set of λ-terms built over constants $\mathbf{K}, \mathbf{S}$. The following is a well-known encoding, [HS86]:

Definition 12 (λ-encoding). *Let* $\mathcal{T} : \Lambda(\mathbf{K}, \mathbf{S}) \to CL$ *be the transformation defined as follows:*

$$\begin{array}{ll} \mathcal{T}(x) = x & \mathcal{T}(C) = C \ \ \textit{if } C \in \{\mathbf{K}, \mathbf{S}\} \\ \mathcal{T}(MN) = \mathcal{T}(M)\mathcal{T}(N) & \mathcal{T}(\lambda x.MN) = \mathbf{S}\mathcal{T}(\lambda x.M)\mathcal{T}(\lambda x.N) \\ \mathcal{T}(\lambda x.x) = \mathbf{SKK} & \mathcal{T}(\lambda x.\lambda y.M) = \mathcal{T}(\lambda x.\mathcal{T}(\lambda y.M)) \\ \mathcal{T}(\lambda x.y) = \mathbf{K}y & \mathcal{T}(\lambda x.C) = \mathbf{K}\mathcal{T}(C) \ \ \textit{if } C \in \{\mathbf{K}, \mathbf{S}\} \end{array}$$

In particular, if we restrict the domain of $\mathcal{T}$ *to* Λ*, we get an encoding of the* λ*-calculus into CL.*
Vice versa, there is a natural embedding of CL into the λ*-calculus* $\mathcal{E} : CL \to \Lambda$*:*
$\mathcal{E}(\mathbf{K}) = \lambda xy.x \quad \mathcal{E}(\mathbf{S}) = \lambda xyz.(xz)(yz) \quad \mathcal{E}(x) = x \quad \mathcal{E}(MN) = \mathcal{E}(M)\mathcal{E}(N)$

Definition 13 (Lazy/cbv Reduction Strategy on CL).
i) The lazy reduction strategy $\to_l \subseteq CL^0 \times CL^0$ *reduces the leftmost outermost CL-redex. Formally:*

$$\frac{}{\mathbf{S}M_1M_2M_3 \to_l (M_1M_3)(M_2M_3)} \qquad \frac{}{\mathbf{K}M_1M_2 \to_l M_1} \qquad \frac{M \to_l M'}{MP \to_l M'P}$$

ii) The cbv strategy $\to_v \subseteq CL^0 \times CL^0$ *is defined by*

$$\frac{}{\mathbf{S}V_1V_2V_3 \to_v (V_1V_3)(V_2V_3)} \qquad \frac{}{\mathbf{K}V_1V_2 \to_v V_1} \qquad \frac{M_1 \to_v M_1'}{\mathbf{K}M_1 \to_v \mathbf{K}M_1'}$$
$$\frac{M_2 \to_v M_2'}{\mathbf{K}V_1M_2 \to_v \mathbf{K}V_1M_2'} \qquad \frac{M_1 \to_v M_1'}{\mathbf{S}M_1 \to_v \mathbf{S}M_1'} \qquad \frac{M_2 \to_v M_2'}{\mathbf{S}V_1M_2 \to_v \mathbf{S}V_1M_2'}$$
$$\frac{M_3 \to_v M_3'}{\mathbf{S}V_1V_2M_3 \to_v \mathbf{S}V_1V_2M_3'} \qquad \frac{M \to_v M'}{MP \to_v M'P}$$

where V_1, V_2, V_3 *are values, i.e. non* $\to_v$*-reducible CL-terms:*

$$V\ ::=\ \mathbf{K} \mid \mathbf{S} \mid \mathbf{K}V \mid \mathbf{S}V \mid \mathbf{S}VV\ .$$

Alternatively we could define the lazy strategy $\to_l$ as the closure of CL-reaction rules under the following reactive contexts (which coincide with the applicative ones): $D[\,]\ ::=\ [\,] \mid D[\,]P$.
Similarly, by considering the restriction to values of the reaction rules of Definition 11, we could define the cbv strategy $\to_v$ as the closure of CL-reaction rules under the following reactive contexts:

$$D[\,]\ ::=\ [\,] \mid D[\,]P \mid \mathbf{K}D[\,] \mid \mathbf{K}VD[\,] \mid \mathbf{S}D[\,] \mid \mathbf{S}VD[\,] \mid \mathbf{S}V_1V_2D[\,].$$

Let $\downarrow_\sigma$ denote the convergence relation on CL, for $\sigma \in \{l, v\}$.

Definition 14 (Lazy/cbv Equivalence on CL).
i) A relation $\mathcal{R} \subseteq CL^0 \times CL^0$ *is a* CL lazy/cbv bisimulation *if:*
$\langle M, N \rangle \in \mathcal{R} \implies (M \downarrow_\sigma \Leftrightarrow N \downarrow_\sigma) \wedge \forall P \in \Lambda^0. \langle MP, NP \rangle \in \mathcal{R}$.
ii) Let $\simeq_\sigma \subseteq CL^0 \times CL^0$ *be the largest CL lazy/cbv bisimulation.*
iii) Let $\widehat{\simeq}_\sigma \subseteq CL \times CL$ *denote the extension of* $\simeq_\sigma$ *to open terms defined by: for* $M, N \in CL$ *s.t.* $FV(M, N) \subseteq \{x_1, \ldots, x_n\}$, $M \widehat{\simeq}_\sigma N$ *iff for all closing substitutions* $\boldsymbol{P}$, $M[\boldsymbol{P}/\boldsymbol{x}] \simeq_\sigma N[\boldsymbol{P}/\boldsymbol{x}]$.

The following theorem, whose proof is in [DHL08], is interesting *per se*:

Theorem 4. *For all* $M, N \in \Lambda$, $M \widehat{\approx}_\sigma N \iff \mathcal{T}(M) \widehat{\simeq}_\sigma \mathcal{T}(N)$.

4.2 The First-order Approach: CL as a Context Category

In the lazy case, where the reactive contexts coincide with the applicative ones, we can endow CL with a structure of reactive system in the sense of [LM00], by considering the smaller context category consisting of just applicative contexts. This allows us to obtain directly an lts with only applicative labels. In the cbv case, where the set of reactive contexts is larger, one can reduce the labels of the IPO bisimilarity only a posteriori, see [DHL08] for more details. For lack of space and in order to focus on other important aspects, we work out in detail only the lazy case.

Definition 15 (Lazy CL Reactive System). *The* lazy CL reactive system $\mathbf{C}_l^1$ *consists of:*

- *the context category whose objects are* $0, 1$, *where the morphisms from 0 to 1 are the closed terms, the morphisms from 1 to 1 are the closed applicative contexts, and composition is context substitution;*
- *the reactive contexts are all the closed applicative contexts;*
- *the reaction rules are* $\mathbf{K}M_1M_2 \to M_1$ *and* $\mathbf{S}M_1M_2M_3 \to (M_1M_2)(M_1M_3)$, *for all* M_1, M_2, M_3.

It is easy to prove that the reactive system $\mathbf{C}_l^1$ has redex RPOs. One can easily check that the minimal contexts are of the shape $[\;]\boldsymbol{P}$, where $\boldsymbol{P}$ has the minimal length for the top-level reaction to fire.

The strong versions of context and IPO bisimilarities are too fine, since, as in the λ-calculus case, they take into account reduction steps, and tell apart β-convertible terms. Thus we consider weak variants of such equivalences, where the identity context $[\;]$ is unobservable. Weak context bisimilarity is too coarse, since it equates all terms. However, we will prove that the weak IPO bisimilarity "almost" coincides with the lazy equivalence. Moreover, we will show how to recover the exact correspondence by considering a suitable variant of CL.

First of all, let $\simeq_{lI}$ denote the *lazy weak IPO bisimilarity* obtained by considering the identity context as unobservable. By Theorem 2, $\simeq_{lI}$ is a congruence w.r.t. reactive contexts, i.e.:

Proposition 1. *For all* $M, N, P \in CL^0$, $M \simeq_{lI} N \implies MP \simeq_{lI} NP$.

The rest of this section is devoted to compare the lazy weak IPO bisimilarity $\simeq_{lI}$ with the lazy equivalence on CL $\simeq_l$ defined in Definition 14.

Using coinduction and Proposition 1 one can easily prove that $\simeq_{lI}\subseteq\simeq_l$.

However, the converse inclusion, i.e. $\simeq_l\subseteq\simeq_{lI}$, does not hold, since e.g. $\mathbf{K} \simeq_l \mathbf{S}(\mathbf{KK})(\mathbf{SKK})$, while $\mathbf{K} \not\simeq_{lI} \mathbf{S}(\mathbf{KK})(\mathbf{SKK})$. Namely $\mathbf{S}(\mathbf{KK})(\mathbf{SKK})\xrightarrow{[\,]P}_I$, while $\mathbf{K} \not\xrightarrow{[\,]P}_I$. The problem arises since the equivalence $\simeq_{lI}$ tells apart terms whose top-level combinators expect a different number of arguments to reduce. In order to overcome this problem, we consider an extended calculus, CL*, where the combinators **K** and **S** become unary, at the price of adding new intermediate combinators and intermediate reductions (the reactive contexts are the ones in Definition 15).

Definition 16. *Let CL^* be the combinatory calculus defined by*

- *Terms:* $M \ ::= \ x \mid \mathbf{K} \mid \mathbf{S} \mid \mathbf{K}'M \mid \mathbf{S}'M \mid \mathbf{S}''MN \mid MN$
  where $\mathbf{K}$, $\mathbf{K}'$, $\mathbf{S}$, $\mathbf{S}'$, $\mathbf{S}''$ *are combinators.*
- *Rules:* $\mathbf{K}M \to \mathbf{K}'M \qquad \mathbf{K}'MN \to M$
  $\mathbf{S}M \to \mathbf{S}'M \qquad \mathbf{S}'MN \to \mathbf{S}''MN \qquad \mathbf{S}''MNP \to (MP)(NP)$

Notice that the calculus in the above definition is well-defined, since the set of terms is closed under the reaction rules.

Now let use denote by $\simeq^*_{lI}$ the weak IPO bisimilarity obtained by considering the lazy reactive system over CL*. Then, we have $\mathbf{K} \simeq^*_{lI} \mathbf{S}(\mathbf{KK})(\mathbf{SKK})$. More in general, by considering the reactive system over CL*, the induced weak IPO bisimilarity $\simeq^*_{lI}$ coincides with the lazy equivalence on CL:

Theorem 5. *For all $M, N \in CL^0$, $\quad \simeq^*_{lI}=\simeq_l$.*

As a consequence of Theorem 4 and Theorem 5 above, we can recover the lazy observational equivalence on λ-terms as weak IPO bisimilarity on CL*.

Proposition 2. *For all $M, N \in \Lambda^0$, $M \approx_l N \iff \mathcal{T}(M) \simeq^*_{lI} \mathcal{T}(N)$.*

However, such notion of weak IPO bisimilarity still suffers of the problem of being infinitely branching, since IPO contexts are either $[\,]$ or $[\,]P$, for all $P \in (CL^*)^0$. This problem will be solved in Section 5.1, where CL* is endowed with a structure of second-order context category.

5 Second-order Term Contexts

The definition of term context category [LM00] can be generalized to a definition of second-order term context. The generalization is obtained by extending the term syntax with function (second-order) variables, that is variables not standing for terms but instead for functions on terms. The formal definition is the following

Definition 17 (Category of Second-order Term Contexts). *Let Σ be a signature for a term language. The category of* second-order term contexts *over Σ is defined by: objects are finite lists of naturals $\langle n_1, \ldots, n_k\rangle$, an arrow*

$\langle m_1, \ldots, m_h \rangle \to \langle n_1, \ldots, n_k \rangle$ *is a k-tuple* $\langle t_1, \ldots, t_k \rangle$*, where the term* t_i *is defined over the signature* $\Sigma \cup \{F_1^{m_1}, \ldots, F_h^{m_h}\} \cup \{X_{i,1} \ldots, X_{i,n_i}\}$*, where* $F_i^{m_i}$ *is a function variable of arity* m_i*,* $X_{i,j}$ *is a ground variable. The category of* second-order linear term context*,* $T_2^*(\Sigma)$*, is the subcategory whose arrows are n-tuples of terms, satisfying the condition that the n-tuples have to contain exactly one use of each function variable* $F_i^{m_i}$*.*

One can check that the standard category of term contexts over Σ coincides with the subcategory whose objects are the lists containing only copies of the natural number 0; in fact this subcategory uses function variables with no arguments and the ground variables do not appear.

Since the simplest way to define composition in category $T_2^*(\Sigma)$, and more generally in the category of second-order term context, is in terms of β-reduction, it is useful to represent morphisms, i.e. terms on the signature $\Sigma \cup \{F_1^{m_1}, \ldots, F_h^{m_h}\} \cup \{X_{i,1}, \ldots, X_{i,n_i}\}$, using a λ-notation for binding variables, that is, instead of writing a term with free variables, we write its lambda closure. To avoid ambiguities we use a different symbol $\boldsymbol{\lambda}$ for this form of lambda abstraction. With this notation a term t on the signature $\Sigma \cup \{F_1^{m_1}, \ldots, F_h^{m_h}\} \cup \{X_1, \ldots, X_n\}$ is written as: $\boldsymbol{\lambda} F_1^{m_1} \ldots F_h^{m_h}. \boldsymbol{\lambda} X_1 \ldots X_n.t$, or as $\boldsymbol{\lambda} \boldsymbol{F}.\boldsymbol{\lambda} \boldsymbol{X}.t$ for brevity. In general a second-order context $\langle t_1, \ldots, t_k \rangle : \langle m_1, \ldots, m_h \rangle \to \langle n_1, \ldots, n_k \rangle$ is written as $\boldsymbol{\lambda} \boldsymbol{F}.\langle \boldsymbol{\lambda} \boldsymbol{X}_1.t_1, \ldots, \boldsymbol{\lambda} \boldsymbol{X}_k.t_k \rangle$.

(i) The identity on $\langle n_1, \ldots, n_k \rangle$ is: $\boldsymbol{\lambda} \boldsymbol{F}.\langle \boldsymbol{\lambda} \boldsymbol{X}_1.F_1^{n_1}(\boldsymbol{X}_1), \ldots, \boldsymbol{\lambda} \boldsymbol{X}_k.F_k^{n_k}(\boldsymbol{X}_k) \rangle$.

(ii) The composition between the morphisms $\boldsymbol{\lambda} \boldsymbol{F}.\langle \boldsymbol{\lambda} \boldsymbol{X}_1.s_1, \ldots, \boldsymbol{\lambda} \boldsymbol{X}_k.s_k \rangle : \langle l_1, \ldots, l_h \rangle \to \langle m_1, \ldots, m_k \rangle$ and $\boldsymbol{\lambda} \boldsymbol{G}.\langle \boldsymbol{\lambda} \boldsymbol{Y}_1.t_1, \ldots, \boldsymbol{\lambda} \boldsymbol{Y}_j.t_j \rangle : \langle m_1, \ldots, m_k \rangle \to \langle n_1, \ldots, n_j \rangle$ is the β-normal form of the expression
$\boldsymbol{\lambda} \boldsymbol{F}.(\boldsymbol{\lambda} \boldsymbol{G}.\langle \boldsymbol{\lambda} \boldsymbol{Y}_1.t_1, \ldots, \boldsymbol{\lambda} \boldsymbol{Y}_j.t_j \rangle)\, (\boldsymbol{\lambda} \boldsymbol{X}_1.s_1, \ldots, \boldsymbol{\lambda} \boldsymbol{X}_k.s_k) : \langle l_1, \ldots, l_h \rangle \to \langle n_1, \ldots, n_j \rangle$.
Informally, the composition is given by a j-tuple of expressions t_i in which every function variable G_l is substituted by the corresponding expression s_l, with the ground variables of s_l substituted by the corresponding parameters of G_l in t_i. For example, considering the signature for natural numbers $\{0, S, +\}$, the composition between $\boldsymbol{\lambda} F. \boldsymbol{\lambda} X_1.F(X_1, 0) : \langle 2 \rangle \to \langle 1 \rangle$ and $\boldsymbol{\lambda} G. \boldsymbol{\lambda} Y_1 Y_2.(G(S(Y_1)) + Y_2) : \langle 1 \rangle \to \langle 2 \rangle$ is the second order context: $\boldsymbol{\lambda} F. \boldsymbol{\lambda} Y_1 Y_2.F(S(Y_1), 0) + Y_2 : \langle 2 \rangle \to \langle 2 \rangle$.

Note that the identity morphism is defined as a λ-term implementing the identity function, while composition on morphisms is defined by the function composition in the λ-setting. Given this correspondence, it is easy to prove that the categorical properties for the identity hold, while the associativity of composition essentially follows from the unicity of the normal form.

The main general result on second-order term contexts is the following:

Proposition 3. *For any signature* Σ*, the category of second-order linear term contexts over* Σ *admits RPOs.*

5.1 CL as Second-order Rewriting System

In this section, we consider the second-order context category for the combinatory calculus CL* and we show that the weak IPO bisimilarity thus obtained coincides with the observational equivalence on λ-calculus. Interestingly, the second-order open bisimilarity gives a uniform characterization also on open terms. For simplicity, we work out in detail only the lazy case. However, the main result holds also in the cbv case.

Note that the terms of CL are defined by signature $\Sigma_{CL} = \{K, S, \mathtt{app}\}$, where $\mathtt{app}$ is the binary operation of application that is usually omitted. So the term $\mathbf{SKK}$ actually stands for $\mathtt{app}(\mathtt{app}(\mathbf{S}, \mathbf{K}), \mathbf{K})$.

Definition 18 (Lazy CL* Second-order Reactive System). *The* lazy second-order Reactive system $\mathbf{C}_l^{2*}$ *consists of:*

- *the category whose objects are the lists with at most one element, and whose arrows $\epsilon \to \langle n \rangle$ are the terms of CL^* with, at most, n (first order) variables, and the whose $\langle m \rangle \to \langle n \rangle$ are the second-order applicative contexts of the shape $F(M_1, \ldots, M_m)N_1 \ldots N_k$, with, at most, n (first order) variables;*
- *the reactive contexts are all the second-order applicative contexts;*
- *the reaction rules are:*
  $\mathbf{K}X_1 \to \mathbf{K}'X_1 \qquad \mathbf{K}'X_1X_2 \to X_1$
  $\mathbf{S}X_1 \to \mathbf{S}'X_1 \qquad \mathbf{S}'X_1X_2 \to \mathbf{S}''X_1X_2 \qquad \mathbf{S}''X_1X_2X_3 \to (X_1X_2)(X_1X_3).$

To maintain the notation for contexts used in Sections 3, 4, in the sequel a second-order applicative context $F(M_1, \ldots, M_m)N_1 \ldots N_k : \langle m \rangle \to \langle n \rangle$ will be more conveniently written as $[\ ]_\theta N_1 \ldots N_k$, where θ is a substitution s.t. $\theta(X_i) = M_i$ for all $i = 1, \ldots, m$, moreover we write $M \stackrel{[\ ]_\theta \boldsymbol{P}}{\to} M'$ iff $(M\theta)\boldsymbol{P} \to M'$. Given Proposition 3, and the underlined RPOs construction, we have:

Corollary 1. *The reactive system $\mathbf{C}_l^{2*}$ has redex RPOs.*

Using Lemma 1, one can check that the lazy weak IPO bisimilarity for $\mathbf{C}_l^{2*}$, obtained by considering the identity context as unobservable, can be characterized as follows:

Lemma 3 (Second-order Lazy Weak IPO bisimilarity).
i) A symmetric relation $\mathcal{R}$ on terms of CL^* is a second-order lazy weak IPO bisimulation *if the following holds*
$\langle M, N \rangle \in \mathcal{R}$ and $M \stackrel{[\ ]_\theta \boldsymbol{P}}{\longrightarrow}_I M'$ then there exists N' such that $N \stackrel{[\ ]_\theta \boldsymbol{P}}{\Longrightarrow}_I N'$ and $\langle M', N' \rangle \in \mathcal{R}$.
ii) The second-order lazy weak IPO bisimilarity $\simeq_{lI}^{2*}$ *is the greatest second-order lazy weak IPO bisimulation.*

Notice that, for a given term, there could be infinitely many second-order IPO reductions, where the redex is entirely contained in the context. E.g. for the term XM_1, $XM_1 \stackrel{[\ ]_{\{\mathbf{K}Y/X\}}}{\longrightarrow} \mathbf{K}'YM_1$ and $XM_1 \stackrel{[\ ]_{\{\mathbf{K}Y_1Y_2/X\}}}{\longrightarrow} \mathbf{K}'Y_1Y_2M_1$ are both IPO reductions of this shape. However, there are only finitely many IPO contexts s.t. the redex is not entirely contained in the context. One can show that such IPO reductions are sufficient for determining the weak IPO bisimilarity, thus getting a finitely branching characterization.

Example: Let $M = XM_1$. The "relevant" IPO reductions of M (i.e. those which are not entirely contained in the context) are the following:

$XM_1 \overset{[\,]_{\{\mathbf{K}/X\}}}{\longrightarrow} \mathbf{K}'M_1$; $XM_1 \overset{[\,]_{\{\mathbf{K}'Y/X\}}}{\longrightarrow} Y$; $XM_1 \overset{[\,]_{\{\mathbf{K}'/X\}}Y}{\longrightarrow} M_1$; $XM_1 \overset{[\,]_{\{\mathbf{S}/X\}}}{\longrightarrow} \mathbf{S}'M_1$; $XM_1 \overset{[\,]_{\{\mathbf{S}'Y/X\}}}{\longrightarrow} \mathbf{S}''YM_1$; $XM_1 \overset{[\,]_{\{\mathbf{S}'/X\}}Y}{\longrightarrow} \mathbf{S}''M_1Y$; $XM_1 \overset{[\,]_{\{\mathbf{S}''YZ/X\}}}{\longrightarrow} (YZ)$ (YM_1); $XM_1 \overset{[\,]_{\{\mathbf{S}''Y/X\}}Z}{\longrightarrow} (YM_1)(YZ)$; $XM_1 \overset{[\,]_{\{\mathbf{S}''/X\}}YZ}{\longrightarrow} (M_1Y)(M_1Z)$.

In general, the relevant IPO contexts are summarized in the following table:

term	IPO contexts
X	$[\,]_{\{\overline{\mathbf{C}}Y/X\}}$, $[\,]_{\{\overline{\mathbf{C}}/X\}}Y$
$XN\boldsymbol{P}$	$[\,]_{\{\overline{\mathbf{C}}/X\}}$
$\mathbf{C}$	$[\,]_{\emptyset}X$
$\mathbf{C}N\boldsymbol{P}$	$[\,]_{\emptyset}$

In order to prove that the IPO contexts in the table are sufficient for determining the weak IPO bisimilarity, one can show (by "coinduction up-to") that the bisimilarity obtained by considering only such contexts is a congruence w.r.t. substitutions.

$\mathbf{C} \in \{\mathbf{K}, \mathbf{S}, \mathbf{K}'M, \mathbf{S}'M, \mathbf{S}''MN \mid M, N \in CL\}$
$\overline{\mathbf{C}} \in \{\mathbf{K}, \mathbf{S}, \mathbf{K}'Z, \mathbf{S}'Z, \mathbf{S}''Z_1Z_2 \mid Z, Z_1, Z_2$ fresh variables$\}$.

More in general, by Theorem 2, $\simeq_{lI}^{2*}$ is a congruence w.r.t. reactive contexts:

Proposition 4. *For all terms of CL^* M, N, for all substitutions θ, for all CL^*-terms $P_1, \ldots, P_n$,* $M \simeq_{lI}^{2*} N \implies (M\theta)\boldsymbol{P} \simeq_{lI}^{2*} (N\theta)\boldsymbol{P}$.

The following is the main result of this section:

Proposition 5. *For all $M, N \in \Lambda$, $M \widehat{\approx}_l N \iff \mathcal{T}(M) \simeq_{lI}^{2*} \mathcal{T}(N)$.*

The proof of the above proposition, which appears in [DHL08], proceeds by showing that $\simeq_{lI}^{2*}$ coincides with the natural extension of $\simeq_{lI}^{*}$ to open terms.

Notice that Proposition 5 gives a uniform finitely branching characterization also on open terms.

6 Final Remarks and Directions for Future Work

– There are several other attempts to deal with parametric rules in the literature. In particular, in [KSS05], the authors introduce the notion of *luxes* to generalize the RPO approach to cases where the rewriting rules are given by pairs of arrows having a domain different from 0. When instantiated to the category of contexts, the luxes approach allows to express rewriting rules not formed by pairs of ground terms but, instead formed by pairs of contexts (open terms), and so allowing parametricity. Compared to our approach, based on the notion of second-order context, the approach of luxes is more abstract and can be applied to a wider range of cases (categories). However, if we compare the two approaches in the particular case of context categories, we find that luxes approach has a more restricted way to instantiate a given parametric rule. This restriction results in a not completely satisfactory treatment of the λ-calculus. It remains the open question whether substituting the notion of second-order contexts with a more abstract one, that should look like a general second-order Lawvere theory. In this way, it should be possible to recover the extra generality of luxes.
– A possible alternative approach for dealing with the λ-calculus in Leifer-Milner's RPO setting, it that of using suitable encodings in the (bi)graph framework [Mil07]. However, we feel that our term solution based on second-order

context categories and CL is simpler and more direct. Alternatively, in place of CL, one could also consider a λ-calculus with explicit substitutions, in order to obtain a convenient encoding of the β-rule, allowing for a representation as a second-order reactive system. This is an experiment to be done. Here we have choosen CL, since it is simpler; moreover, the correspondence between the standard λ-calculus and the one with explicit substitutions deserves further study.
– We have considered lazy and cbv strategies, however also other strategies, e.g. *head* and *normalizing* could be dealt with, possibly at the price of some complications due to the fact that such strategies are usually defined on open terms. It would be also interesting to explore non-deterministic strategies on λ-calculus.

References

AO93. S. Abramsky, L. Ong. Full Abstraction in the Lazy Lambda Calculus, *Information and Computation*, **105**(2), 159–267, 1993.

BKM06. F. Bonchi, B.Konig, U. Montanari. Saturated Semantics for Reactive Systems, *LICS06*, 2006.

BGK06. F. Bonchi, F. Gadducci, B. Konig. Process Bisimulation via a Graphical Encoding, *ICGT'06*, LNCS **4178**, 168–183, 2006.

DHL08. P. Di Gianantonio, F. Honsell, M. Lenisa. RPO, Second-order Contexts, and λ-calculus, Extended version, 2008, available at: `www.dimi.uniud.it/pietro/papers/socl.pdf`.

EHR92. L. Egidi, F. Honsell, S. Ronchi Della Rocca. Operational, Denotational and Logical Descriptions: a Case Study, *Fundamenta Inf.*, **16**(2), 149–169, 1992.

EK06. H. Ehrig, B. König. Deriving bisimulation congruences in the DPO approach to graph rewriting with borrowed contexts, *Mathematical Structure in Computer Science*, **16**(6), 1133–1163, 2006.

GM05. F. Gadducci, U. Montanari. Observing Reductions in Nominal Calculi via a Graphical Encoding of Processes, LNCS **3838**, 106–126, 2005.

HS86. R. Hindley, J. Seldin. Introduction to combinators and l-calculus, Cambridge University Press, 1986.

KSS05. B. Klin, V. Sassone, P. Sobocinski. Labels from reductions: Towards a general theory, *CALCO'05*, LNCS **3629**, 30–50, 2005.

Lei01. J. Leifer. Operational congruences for reactive systems, PhD thesis, University of Cambridge Computer Laboratory, 2001.

LM00. J. Leifer, R. Milner. Deriving bisimulation congruences for reactive systems, *Proc. CONCUR'00*, LNCS **1877**, 243–258, 2000.

Mil07. R. Milner. Local bigraphs and confluence: two conjectures, *Proc. Express'06*, ENTCS **175**, 65–73, 2007.

SS03. V. Sassone, P. Sobocinski. Deriving bisimulation congruences: 2-categories vs precategories, *FoSSaCS 2003*, LNCS **2620**, 409–424, 2003.

SS05. V. Sassone, P. Sobocinski. Reactive systems over cospans, *LICS'05*, IEEE, 311–320, 2005.

Sew02. P. Sewell. From rewrite rules to bisimulation congruences, *Theoretical Computer Science*, **274**(1-2), 183–230, 2002.

Sob04. P. Sobocinski. Deriving process congruences from reduction rules, *PhD thesis*, University of Aarhus, 2004.