A Tour with Constructive Real Numbers *

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Abstract. The aim of this work is to characterize constructive real numbers through a minimal axiomatization. We introduce, discuss and justify 16 constructive axioms. Then we address their expressivity considering the alternative axiomatizations.

1 Overview of the work

This work tries to understand (again) constructive real numbers. Our main contribution is a new system of axioms, synthesized with the aim of being minimal, i.e. of assuming the least number of primitive notions and properties. Such a system is consistent with respect to reference models we have in mind — (equivalence classes of) Cauchy sequences [TvD88] and co-inductive streams of digits [CDG00] — and will be compared to other proposals of the literature [Bri99, GN01]. In particular we will prove that our axiomatization has a sufficient deductive power.

We have formalized and used our axioms inside the Logical Framework Coq [BB$^+$01]. However, the axioms can be stated and worked with in a general constructive logical setting, because we do not need all the richness of the Calculus of Constructions [CH88], the logic beneath Coq. In particular we do not require the use of dependent inductive types and universes. On the contrary, we should have available a logical system that accommodates second-order quantification (in order to axiomatize the existence of limit) and the Axiom of Choice (for defining the "reciprocal" function on reals different from zero).

We define constructive real numbers through sixteen axioms organized in four groups: arithmetic operations, ordering, Archimedes' postulate and completeness. Our axiomatization uses only three basic concepts: addition ($+$), multiplication ($\times$) and strict order ($<$).

In most of the constructive approaches to analysis [Bis67, Bee85, Wei00], real numbers are defined as a quotient of a set of representations (e.g. equivalence classes of Cauchy sequences, digit expansions, etc.). Hence, also in an axiomatic approach, it is necessary to see the reals as a set provided with an equivalence relation. In our proposal, this equivalence relation ($\sim$) is not a primitive notion, but it is derived together with its fundamental properties through the strict

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order relation. We define the equivalence by $(x \sim y) \;\triangleq\; \neg\,[(x < y) \vee (y < x)]$. Similarly, it is not necessary to assume as basic the apartness relation ($\#$) — which is a semi-decidable version of the inequality ($\neq$) — as it is definable in terms of the order relation.

The paper has the following structure. Section 2 introduces the constructive axioms, which are then explained and motivated. In Section 3 we start deducing some elementary consequences of the axioms. The following Section is devoted to a digression concerning possible models for the constructive real numbers. We conclude by articulating a detailed comparison between our axiomatization and other similar works in the literature.

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2 Constructive axioms

We introduce constructive real numbers as the mathematical entities satisfying four groups of axioms. The basic notions are the following:

- a representation set R, with two elements 0_R (zero) and 1_R (one);
- a binary relation $<$ (strict order) over R;
- two binary operations $+$ (addition) and $\times$ (multiplication) over R.

We do not assume the negation ($-$) and reciprocal ($^{-1}$) as primitive functions. The main reason for this choice is that the reciprocal function cannot be defined in Coq: in fact, in Coq, each function has to be totally defined; and, in a constructive setting, each function has to be continuous (w.r.t. the Euclidean topology). On the contrary, it is impossible to make continuous by extension the reciprocal function.

In order to state the axioms, it is convenient to define two relations and two functions:

- a binary relation $\sim$ (equivalence) over R tells that two names represent the same number. It expresses the redundancy of the representation;
- two recursively defined functions $inj, exp : \mathbb{N} \to R$ ($inj(n) = n$, $exp(n) = 2^n$) are used in the Archimedean and completeness axioms;
- a ternary relation $near \subseteq R \times R \times \mathbb{N}$ ($near(x, y, n) \Leftrightarrow |x - y| \leq 2^{-n}$) expresses the Euclidean metric.

Our axiomatization is parametric with respect to a set $\mathbb{N}$ of the natural numbers, that we suppose to be given. In our formalization in Coq, $\mathbb{N}$ is taken as the set of the inductive natural numbers. In a different context, $\mathbb{N}$ could be defined as a set satisfying the Peano's arithmetic axioms. Finally, we claim that constructive real numbers are captured by the following axioms.

Definition 1. ***(Axioms for constructive real numbers)***

$$Consts : R,\ \{0_R, 1_R\} \in R \quad < \subseteq R \times R \quad + : R \times R \to R \quad \times : R \times R \to R$$

$$\begin{array}{lll}
Defs: & \sim \subseteq R \times R & (x \sim y) \triangleq \neg((x < y) \vee (y < x)) \\
& inj : \mathbb{N} \to R & inj(0) \triangleq 0_R,\ inj(n+1) \triangleq inj(n) + 1_R \\
& exp : \mathbb{N} \to R & exp(0) \triangleq 1_R,\ exp(n+1) \triangleq exp(n) \times (1_R + 1_R) \\
& near \subseteq R \times R \times \mathbb{N} & near(x, y, n) \triangleq \forall \epsilon \in R.\ (1_R < \epsilon \times exp(n)) \to \\
& & \qquad (x < y + \epsilon) \wedge (y < x + \epsilon)
\end{array}$$

$$\begin{array}{lll}
Add: & +\text{-}associativity & \forall x, y, z \in R.\ (x + (y + z)) \sim ((x + y) + z) \\
& +\text{-}unit & \forall x \in R.\ (x + 0_R) \sim x \\
& negation & \forall x \in R.\ \exists y \in R.\ (x + y) \sim 0_R \\
& +\text{-}commutativity & \forall x, y \in R.\ (x + y) \sim (y + x)
\end{array}$$

$$\begin{array}{lll}
Mult: & \times\text{-}associativity & \forall x, y, z \in R.\ (x \times (y \times z)) \sim ((x \times y) \times z) \\
& \times\text{-}unit & \forall x \in R.\ (x \times 1_R) \sim x \\
& reciprocal & \forall x \in R.\ (0_R < x) \to \exists y \in R.\ (x \times y) \sim 1_R \\
& \times\text{-}commutativity & \forall x, y \in R.\ (x \times y) \sim (y \times x) \\
& distribuitivity & \forall x, y, z \in R.\ (x \times (y + z)) \sim (x \times y) + (x \times z)
\end{array}$$

$$\begin{array}{lll}
Order: & non\ triviality & 0_R < 1_R \\
& <\text{-}asymmetry & \forall x, y \in R.\ (x < y) \to \neg(y < x) \\
& <\text{-}co\text{-}transitivity & \forall x, y, z \in R.\ (x < y) \to (x < z) \vee (z < y) \\
& +\text{-}reflects\text{-}< & \forall x, y, z \in R.\ (x + z < y + z) \to (x < y) \\
& \times\text{-}reflects\text{-}< & \forall x, y \in R.\ (x \times z < y \times z) \to \\
& & \qquad (x < y) \vee ((y < x) \wedge (z < 0)) \\
& & \\
& Archimedean & \forall x \in R.\ \exists n \in \mathbb{N}.\ x < inj(n) \\
& & \\
& completeness & \forall f : \mathbb{N} \to R.\ \exists x \in R. \\
& & (\forall n \in \mathbb{N}.\ near(f(n), f(n+1), n+1)) \to \\
& & (\forall m \in \mathbb{N}.\ near(f(m), x, m))
\end{array}$$

Arithmetic operations. As the reader can see, the properties required for the arithmetic operations are just the same characterizing a classical abelian field; in [Bri99] this same set of properties is named "Heyting field". Note that it is sufficient to assume the existence of the reciprocal only for positive reals.

As we have already remarked, we do not assume the "negation" and the "reciprocal" functions: instead we assume the existence, for each real x, of its negation and, if $0 < x$, of its reciprocal elements. In this way we have to postulate the Axiom of Choice for extracting effectively from a number x its negation and its reciprocal.

The necessity of the Axiom of Choice can be seen as a weakness of our axiomatization. However, there is no simple way to avoid it: in fact, without Choice, the reciprocal function cannot be defined inside Coq (whereas the negation function and the limit functional are definable).

An alternative axiomatization that does not require the Axiom of Choice could be obtained as follows. One postulates the existence of the negation and limit functions and, instead of a single inversion function, the existence of a series of approximations of the inversion function, $\texttt{inv} : (\mathbb{N} \times R) \to R$, satisfying the axiom:

$$\forall n \in \mathbb{N}.\ \forall x \in R.\ (1_R < x \times exp(n)) \to (x \times \texttt{inv}(n, x) \sim 1_R)$$

that is, the function $\lambda x.\ \texttt{inv}(n, x)$ behaves as the reciprocal function for all the real numbers bigger than 2^{-n}. Given a suitable representation for the reals, the function inv is Coq-definable, and allows the evaluation of the reciprocal of any real number x for which it is possible to find a natural number n such that $2^n < x$. We did not pursue this alternative axiomatization for simplicity reasons.

Order relation. Concerning the ordering, we make the following remarks.

First, the classical trichotomy of total order $(x < y) \vee (x = y) \vee (y < x)$ is not a constructive property: its substitute in the constructive setting is the property $(x < y) \to (x < z) \vee (z < y)$, which we name "co-transitivity".

Secondly, we have thought that it is cleaner to define only the relation of order: in fact, in constructive mathematics [TvD88, Bri99], the order is universally considered the most fundamental relation for the real numbers. The alternative would have been to start from the apartness relation — the constructive non equality — then to assume axioms for it, and further to introduce the order itself with its proper axioms. But this increases the length of the presentation of the constructive reals, and moreover introduces some redundancy, thus not permitting to carry out our declared purpose of being minimal.

The equivalence and the apartness relations are defined using the basic strict order, therefore their properties are derived from the axioms. We define the equivalence ($\sim$) and the apartness ($\#$) in the following way:

$$(x \sim y) \ \triangleq\ \neg\,[(x < y) \vee (y < x)] \qquad\qquad (x \# y) \ \triangleq\ (x < y) \vee (y < x)$$

There is still more, because we have been careful in the design of the relationship between the order and the operations. We are able to deduce all the basic properties relating the equivalence and the operations from the two reflection axioms:

$$(x + z < y + z) \to (x < y)$$
$$(x \times z < y \times z) \to (x < y) \vee [(y < x) \wedge (z < 0)]$$

The fact that the equivalence is preserved by the basic notions (addition, multiplication and order) is an immediate consequence of these two axioms and the $<$-*co-transitivity* one. Notice that, on the contrary, the preservation of the equivalence does not follow from the more usual preservation axioms [Bri99, GN01]:

$$(x < y) \to (x + z < y + z)$$
$$(0 < x) \wedge (0 < y) \to (0 < x \times y)$$

This particular phenomenon relies on the fact that the reflection of the order is more powerful than its preservation, as will be argued in Section 3.

Archimedean property. The Archimedean axiom links real numbers to natural numbers, stating that the reals are standard with respect to the naturals. This axiom does not exclude the existence of non-standard reals, but in this case also the naturals must be non-standard. That is, it is possible to conceive non-standard models for our axioms: these models would contain infinitary and infinitesimal real numbers as well as infinitary naturals.

Completeness. Finally, the completeness property for the field of the real numbers is postulated asking the existence of the limit of any Cauchy sequence $\langle s_n \rangle_{n \in \mathbb{N}}$ with an exponential convergency rate:

$$\forall n \in \mathbb{N}.\ |s_n - s_{n+1}| \leq 2^{-(n+1)}$$

Many others choices for capturing the completeness are possible, and our axiom could appear weak at a first glance. Anyway, in order to evaluate constructively the limit of a Cauchy sequence S, it is necessary to know its convergency rate: from this convergency rate it is possible to extract (constructively) a subsequence of S having an exponential convergency rate. It follows that starting from our axiom we are able to derive the completeness properties that are found in the literature [Bri99, GN01]. Our choice has been motivated by simplicity reasons.

The minimality of our axiomatization could be useful both for theoretical reasons — the mathematical curiosity about an essential characterization of the constructive reals is addressed — and practical ones — a simple test for possible models is provided. However, rather than pursuing a minimal set of axioms at all costs, we have chosen to axiomatize the different notions (order, addition, multiplication, etc.) separately, for the sake of the clarity of the axiomatization.

3 Axioms at work

In this Section we point out the elementary mathematical theory arising from the axioms. Such results can be used for deducing more complex properties of the constructive real numbers.

We begin with some consequences concerning the order; successively we will address the addition, the multiplication and other defined notions. Notice that all the logic steps used in the following proofs are constructive. These proofs have been also carried out formally in the proof assistant Coq: this and further work is documented in [CDG01].

Proposition 1. ***(Order)***
The following properties of the order and derived notions follow from the axioms:

1. *irreflexivity:* $\forall x.\ \neg(x < x)$
2. *transitivity:* $\forall x, y, z.\ (x < y) \wedge (y < z) \rightarrow (x < z)$
3. *preservation by equivalence:* $\forall x, y, z, w.\ (x < y) \wedge (x \sim z) \wedge (y \sim w) \rightarrow (z < w)$
4. *co-transitivity of apartness:* $\forall x, y, z.\ (x \# y) \rightarrow (x \# z) \vee (z \# y)$
5. *apartness preservation:* $\forall x, y, z, w.\ (x \# y) \wedge (x \sim z) \wedge (y \sim w) \rightarrow (z \# w)$

Proof. The proofs are quite easy, so we just sketch some of them. Point (2) is proved applying the $<$-*co-transitivity* to $(x < y)$; point (3) using twice the $<$-*co-transitivity*; point (5) by a double application of (4). $\square$

We state now the main fact concerning the addition, i.e. it preserves the equivalence. Moreover we deduce also the preservation of order, which is assumed as an axiom in the alternative axiomatizations [Bri99, GN01].

Proposition 2. ***(Addition)***
The following properties of the addition follow from the axioms:

1. *strong extensionality* : $\forall x, y, z, w.\ (x + y) \# (z + w) \to (x \# z) \vee (y \# w)$
2. *equivalence preservation* : $\forall x, y, z, w.\ (x \sim y) \wedge (z \sim w) \to (x + z) \sim (y + w)$
3. *order preservation* : $\forall x, y, z.\ (x < y) \to (x + z < y + z)$
4. *equivalence reflection* : $\forall x, y, z.\ (x + z) \sim (y + z) \to (x \sim y)$

Proof. (1) The goal follows immediately from the two judgments $(x+y) \# (z+y) \to (x \# z)$ (left extensionality) and $(x + y) \# (x + z) \to (y \# z)$ (right extensionality). Left extensionality can be written $(x+y < z+y) \vee (z+y < x+y) \to (x < z) \vee (z < x)$ and proved by cases using the $+$-*reflects*-$<$ axiom. Right extensionality can be reduced to left extensionality by *co-transitivity of apartness*, *apartness preservation* and the $+$-*commutativity* axiom.

(3) Using *negation*, point (2), $+$-*unit* and $+$*associativity* we can derive $x \sim ((x+z)+(-z))$. By Proposition 1.3 it is then possible to deduce $(x+z)+(-z) < (y + z) + (-z)$, and from this the thesis via $+$-*reflects*-$<$.

Point (2) follows from (1); point (4) from (3). $\square$

We consider similar results for the multiplication: we prove it preserves the equivalence and deduce additional consequences.

Proposition 3. ***(Multiplication)***
The following properties of the multiplication follow from the axioms:

1. *strong extensionality* : $\forall x, y, z, w.\ (x \times y) \# (z \times w) \to (x \# z) \vee (y \# w)$
2. *equivalence preservation* : $\forall x, y, z, w.\ (x \sim y) \wedge (z \sim w) \to (x \times z) \sim (y \times w)$
3. *positivity reflection* : $\forall x, y, z.\ ((x \times z) < (y \times z)) \wedge (0 < z) \to (x < y)$
4. *zero annuls multiplication:* $\forall x.\ (x \times 0 \sim 0)$
5. *reciprocal preserves positivity:* $\forall x.\ (0 < x) \to (0 < x^{-1})$
6. *positivity preservation* : $\forall x, y.\ (0 < x) \wedge (0 < y) \to (0 < x \times y)$

Proof. The proofs of points (1) and (2) use the same arguments of the corresponding proofs for the addition. Point (3) follows through axiom $\times$-*reflects*-$<$.

(4) From $(0 + 0) \sim 0$ we derive $((x \times 0) + (x \times 0)) \sim (0 + (x \times 0))$, and then the thesis through the Proposition 2.4.

(5) From the hypothesis $(0 < x)$ and axiom $(0 < 1)$ we obtain $0 < (x^{-1} \times x)$; then we have $(0 \times x) < (x^{-1} \times x)$, from which we conclude by point (3).

(6) Using the hypotheses we derive $x \sim ((x \times y) \times y^{-1})$, from which we deduce $(0 \times y^{-1}) < ((x \times y) \times y^{-1})$; the thesis follows by points (5) and (3). $\square$

It is easy to prove that the $+$-*reflects*-$<$ axiom is equivalent to the *equivalence preservation* plus *order preservation* of Proposition 2. In [Bri99] and [GN01] these two properties are taken as axioms: we prefer our choice for minimality reasons. A similar consideration applies to multiplication.

We also remark that two possible candidates for the $+$-*reflects*-$<$ axiom, namely $(x\times z < y\times z)\wedge(0 < z) \rightarrow (x < y)$ and $(x\times z < y\times z) \rightarrow (x < y)\vee(z < 0)$, are too weak.

We list now other typical and useful properties of the constructive real numbers. They involve also the auxiliary notion of non-strict order ($\leq$), which is formalized as follows:

$$(x \leq y) \;\triangleq\; \neg\,(y < x)$$

Proposition 4. ***(Other properties)***
The following judgments can be derived from the axioms and their corollaries:

$$\begin{array}{ll}
\mathit{Order}: & (x \leq y) \wedge (y \leq x) \rightarrow (x \sim y) \\
 & (x \leq y) \wedge (y \leq z) \rightarrow (x \leq z) \\
 & ((z < x) \rightarrow (z < y)) \rightarrow (x \leq y) \\
 & \\
\mathit{Addition}: & (0 < x) \rightarrow (-x < 0) \\
 & (0 < x + y) \rightarrow ((0 < x) \vee (0 < y)) \\
 & (x \leq y) \leftrightarrow (x + z \leq y + z) \\
 & \\
\mathit{Multiplication}: & (x \times (-y)) \sim -(x \times y) \\
 & (x < y) \wedge (0 < z) \rightarrow (x \times z < y \times z) \\
 & (x < y) \wedge (z < 0) \rightarrow (y \times z < x \times z) \\
 & (0 < x \times y) \rightarrow (x \# 0) \wedge (y \# 0)
\end{array}$$

4 Consistency and completeness

The usual way for defining real numbers is the use of (equivalence classes of) Cauchy sequences of rational numbers [Bee85, BB85, TvD88], but there exist other constructions that can be easily proved equivalent to this one. An example is the approach that introduces the reals as infinite sequences of digits [PEE97, Wei00, CDG00]: in this case the equivalence follows from the fact that it is possible to transform effectively a representation of a number through a Cauchy sequence in a representation of the same number through an infinite stream of digits, and vice-versa.

In the distillation of our set of axioms we have used as reference the definition of the real numbers via Cauchy sequences of rationals. This construction has been used as a model for testing the consistency of the axioms: in order to accept a judgment as an axiom we have first informally verified that it is satisfied by the Cauchy model. We are now developing a formal proof in Coq [CDG01] that our axioms are satisfied by a construction of the real numbers through “streams of digits”.

A similar result is presented in [GN01]: the constructive reals are built as Cauchy sequences of rational numbers, and a different axiomatization is introduced. That work gives also a proof that the axiomatization is categorical — i.e. any two models are isomorphic. Since in the next Section we will show that our axiomatization is equivalent to that one [GN01], we can deduce that our axiomatization is complete; and we can claim that it is also categorical.

5 Comparison to the related literature

In this Section we compare our axioms to the other approaches of the literature [TvD88, Bri99, GPWZ00, GN01]. In particular, we will prove that our axiomatization has the same deductive power of the alternative ones.

The work of Troelstra and van Dalen [TvD88] is a contribution which gives a constructive treatment of the theory of the real numbers in the context of a constructive approach to mathematics. Although the authors do not address the quest for an axiomatization of the constructive reals, their approach focuses on aspects strictly related to the present work. They build the constructive reals as equivalence classes of fundamental (Cauchy) sequences of rationals; then they introduce the primitive strict order relation ($<$) and the arithmetic functions ($+, \cdot$). The basic properties of these notions are the same we have proved in Section 3, or follow simply from those results.

We will mainly refer our work to other two contributions.

5.1 FTA

The FTA approach [GPWZ00] is similar to ours in regard to the tool used for the formalization, i.e. the proof assistant Coq. There, constructive real numbers are just seen as a parameter contextually to a more extensive project concerning the mechanical certification of a theorem of constructive mathematics. In fact, the main aim is just to dispose of a collection of properties for describing the reals which is sufficiently powerful for proving the Fundamental Theorem of Algebra. The axiomatization is separately presented and focused in [GN01]: this work first introduces the algebraic structure of constructive setoids via an apartness relation; successively, step by step, it gains the notion of constructive real number. We are proving the equivalence between our axiomatization and this one.

The FTA approach uses 28 axioms. The essential differences with respect to ours are the introduction of the apartness relation as primitive and the assumption of strongly extensionality for the arithmetic functions of addition and multiplication.

The structure. Constructive reals are introduced by the tuple:

$$\langle \mathbb{R}, 0, 1, +, *, -, ^{-1}, =, <, \# \rangle$$

Field. The following axioms are assumed in order to have the structure of constructive setoids:

$$
\begin{aligned}
&\texttt{ap\_irr} : \forall x.\ \neg(x \# x)\\
&\texttt{ap\_sym} : \forall x, y.\ (x \# y) \to (y \# x)\\
&\texttt{ap\_cot} : \forall x, y.\ (x \# y) \to \forall z.(x \# z) \vee (z \# y)\\
&\texttt{ap\_tight} : \forall x, y.\ \neg(x \# y) \leftrightarrow (x = y)
\end{aligned}
$$

The above properties can easily be derived from our axioms: ap_irr (irreflexivity) follows from the irreflexivity of the order $\neg(x < x)$; ap_cot (co-transitivity) is a consequence of the $<$-*co-transitivity* axiom. Instead ap_sym (symmetry) and ap_tight (tightness) are deduced just from the definitions. Some of the properties which lead to a constructive field coincide with ours:

$$
\begin{aligned}
&\texttt{add\_assoc} : \forall x, y, z.\ (x + (y + z)) = ((x + y) + z)\\
&\texttt{add\_unit} : \forall x.\ (x + 0) = x\\
&\texttt{add\_commut} : \forall x, y.\ (x + y) = (y + x)\\
&\texttt{minus\_proof} : \forall x.\ x + (-x) = 0\\
&\texttt{mult\_assoc} : \forall x, y, z.\ (x * (y * z)) = ((x * y) * z)\\
&\texttt{mult\_unit} : \forall x.\ (x * 1) = x\\
&\texttt{mult\_commut} : \forall x, y.\ (x * y) = (y * x)\\
&\texttt{dist} : \forall x, y, z.\ (x * (y + z)) = ((x * y) + (x * z))\\
&\texttt{rcpcl\_proof} : \forall x.\ (x \# 0) \to (x * (x^{-1})) = 1
\end{aligned}
$$

Moreover, FTA uses the extra axioms:

$$
\begin{aligned}
&\texttt{add\_strext} : \forall x, y, z, u.\ (x + y) \# (z + u) \to (x \# z) \vee (y \# u)\\
&\texttt{minus\_strext} : \forall x, y.\ ((-x) \# (-y)) \to (x \# y)\\
&\texttt{mult\_strext} : \forall x, y, z, u.\ (x * y) \# (z * u) \to (x \# z) \vee (y \# u)\\
&\texttt{non\_triv} : 1 \# 0\\
&\texttt{rcpcl\_ap\_zero} : \forall x.\ (x \# 0) \to (x^{-1} \# 0)\\
&\texttt{rcpcl\_strext} : \forall x, y.\ (x^{-1}) \# (y^{-1}) \to (x \# y)
\end{aligned}
$$

We have proved add_strext and mult_strext (strong extensionality of addition and multiplication) in Propositions 2.1 and 3.1. We establish here rcpcl_strext (strong extensionality of the "reciprocal" function); simpler arguments suffice for proving the property minus_strext.
First we show rcpcl_ap_zero, i.e. the reciprocal respects the apartness with respect to zero. Starting from $x \# 0$ and the non triviality $1 \# 0$ — which follows from our axiom $0 < 1$ — we have both $(x * (x^{-1})) = 1$ by the $\times$-*unit* axiom and $(x * 0) = 0$ by Proposition 3.4. Next we have $(x * x^{-1}) \# (x * 0)$ by the Proposition 1.5, and then we conclude through the strong extensionality of the multiplication.
The main proof of rcpcl_strext works as follows. Since $x^{-1} = x^{-1} * (y * y^{-1})$ and $y^{-1} = y^{-1} * (x * x^{-1})$, by Proposition 1.5 and $\times$-*associativity* we deduce $(x^{-1} * y) * y^{-1} \# (x^{-1} * x) * y^{-1}$. Using the strong extensionality of the multiplication, we have first $(x^{-1} * y) \# (x^{-1} * x)$ and further $y \# x$, thus concluding via ap_sym.

Ordered field. The only axiom shared with ours is the asymmetry:

$$\texttt{less\_asym} : \forall x, y.\ (x < y) \to \neg(y < x)$$

The extra axioms are the following:

$$\begin{array}{l}
\texttt{less_strext} : \forall x, y, z, u.\ (x < y) \to (z < u) \vee (x \# z) \vee (y \# u) \\
\texttt{less_trans} : \forall x, y, z.\ (x < y) \wedge (y < z) \to (x < z) \\
\texttt{less_irr} : \forall x.\ \neg(x < x) \\
\texttt{add_resp_less} : \forall x, y, z.\ (x < y) \to (x + z < y + z) \\
\texttt{times_resp_pos} : \forall x, y.\ (0 < x) \wedge (0 < y) \to (0 < x * y) \\
\texttt{less_conf_ap} : \forall x, y, z.\ (x \# y) \leftrightarrow (x < y) \vee (y < x).
\end{array}$$

The properties `less_irr` (irreflexivity) and `less_trans` (transitivity) have been derived in Proposition 1.2; `less_strext` (strong extensionality of order) follows from $<$-*co-transitivity*; `less_conf_ap` just coincides with our definition of apartness. Finally the `add_resp_less` and `times_resp_pos` axioms have been derived in the Propositions 2.3 and 3.6.

Archimedes. This axiom coincide with ours:

$$\texttt{arch\_proof} : \forall x.\ \exists n \in \mathbb{N}.\ (x < (\texttt{nreal}\ \ n))$$

Limit. The axiom of completeness asks that every Cauchy sequence has a limit:

$$\begin{array}{l}
\texttt{lim_proof} : \forall s : \mathbb{N} \to \mathbb{R},\ \text{cauchy}.\ \forall \epsilon > 0.\ \exists n \in \mathbb{N}. \\
\qquad\qquad (\forall m \in \mathbb{N}.\ (n \leq m) \to |\,(s\ m) - \texttt{lim}\ s\,| < \epsilon)
\end{array}$$

It is not difficult to prove that this axiom is at least as strong as the one we have used in our axiomatization. We can deduce the FTA axiom from ours as follows: in order to calculate the limit of a generic Cauchy sequence S we need to be able to extract a subsequence of S converging with an exponential convergency rate. This subsequence can be obtained easily once we know the convergency rate of S, which in turn can be extracted, using the Axiom of Choice, from the proposition stating that S satisfies the Cauchy condition.

Theorem 1. ***(Equivalence between axiomatizations)***
We conclude that our axiomatization and the FTA one are equivalent.

5.2 Bridges

Bridges [Bri99] uses the framework of Bishop's constructive mathematics for presenting a constructive axiomatization of the real line. His main motivation coincides partially with the ours: the curiosity about the properties that suffice to characterize the real numbers and to develop the real analysis. The way chosen is to capture the idea that a real could be approximated by arbitrarily close rational numbers.

The constructive axiomatization given by Bridges collects 20 axioms. The axioms are quite similar to ours apart the completeness one: we discuss briefly the main differences.

The equivalence relation $=$ is defined by $(x = y) \triangleq (x \geq y) \wedge (y \geq x)$, where $(x \geq y) \triangleq (\forall z.\ (y > z) \rightarrow (x > z))$. By Propositions 4.4 and 4.5, we have that $\forall z.\ (y > z) \rightarrow (x > z)$ if and only if $\neg(x < y)$. The same could be proved using the axioms of Bridges. And so, Bridges' approach is equivalent to ours up to the use of the more involved definition of the equivalence relation "$=$".

The axioms concerning the field structure coincide with ours. Bridges assumes as additional implicit axioms the "extensionality" of relations and operations, i.e. they preserve the equivalence. As already remarked, in our approach these are just derived properties.

As far as the axioms for the order are concerned, the main difference is that Bridges requires that the operations "preserve" the order, whereas we require that the operations "reflect" the order.

The completeness axiom chosen by Bridges is quite different from ours. In [Bri99] the completeness of the real line is postulated through a "least-upper-bound principle", which requires that every "strongly bounded" set of reals has a least upper bound; from such a principle, it is then derived that every Cauchy sequence has limit. In [BR99] and [GPWZ00] it is proved that the existence of the l.u.b. of "strongly bounded" sets can be deduced from the existence of the limit for Cauchy sequences. Hence Bridges' approach to completeness is equivalent to ours. We have preferred to state the completeness in terms of Cauchy sequences, because it is simpler.

6 Conclusion

In this work we have focused on the only two existing axiomatizations of the constructive real numbers and we have proposed a third one.

All the three axiomatizations have the same deductive power. As far as we know, Bridges [Bri99] and the FTA group [GPWZ00] have proposed their axioms independently; we too have stated our axioms without being aware of the work by Bridges.

We claim that our axiomatization has the advantage of being simpler and of using a minimal set of notions. In particular, we give a more direct treatment of the equivalence "$\sim$" and the non-strict order "$\leq$" relations than Bridges. Also our completeness axiom is simpler than the corresponding one in any of the other proposals, and in general the whole axiomatization is more compact.

A possible direction for future work is to consider an axiomatization which does not require the Axiom of Choice. In this perspective, it would be interesting to consider also an axiomatization of the constructive reals obtained by Dedekind cuts: in fact, Cauchy sequences and Dedekind cuts are equivalent constructions for the reals only if the Axiom of Choice is available [TvD88]. Results in this sense would help to characterize the fundamental differences between the two constructions.

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