Amending and enhancing electoral laws through mixed integer programming: the case of Italy

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Abstract: In this paper we discuss how a mathematical approach can be used to solve a serious drawback in the current Italian electoral system for the election of representatives at the Chamber of Deputies and suggest a methodology which is close to the one traditionally adopted in the Italian electoral history, but able to guarantee a more transparent, logical and fairer solution to the problem of transforming votes into seats.

1. The Italian electoral system and some of its drawbacks

The Italian electoral law, like many others (for example, Mexico, Germany, Switzerland), wishes to achieve a double proportionality: on the one hand, Parliament seats should be assigned to parties, within each regional constituency, proportionally to the votes cast for the individual parties in the constituency; on the other hand, the amount of national seats obtained by any given party should be distributed among the different constituencies proportionally to the votes obtained by the party in the single constituencies.

According to the Italian Constitution the size of the Chamber of Deputies is equal to 630 seats. The country is partitioned into 27 multi-member regional constituencies and the number of seats at stake in each regional constituency is proportional to the number of inhabitants, as provided by the latest population census. The only exception is the region of Valle dAosta which is a single-member district. Finally, 12 seats are assigned to a constituency of Italian citizens who are resident abroad.

The current Italian system was introduced in 2005 and it allocates seats proportionally to the votes obtained by each party (and coalition of parties) at the national level and within multi-member regional constituencies. A majority prize is meant to ensure that the party or coalition with the greatest number of total votes wins a substantial majority of seats in the Chamber of Deputies (i.e., at least 340 seats), no matter how many votes the other parties receive. There is a single ballot and candidates are elected on the basis of a blocked list. Moreover, a complex scheme of thresholds is adopted to select which parties and coalitions are eligible to compete in the seat allocation. Despite these special features (majority prize and thresholds) the system was advertised as a proportional one.

In fact, the principle of proportionality is adopted by many electoral systems as it is interpreted as a good approximation of the idea of “one-man-one-vote”: the percentage of votes that parties obtain in elections should be as close as possible to the percentage of seats they receive in the legislative assembly [4, 5]. Proportional representation is actually used by more nations than the plurality voting system. A general principle stated in many fundamental laws of nations (and in some cases in their Constitutions) is that each constituency's weight in the national assembly should reflect its population size.

However, achieving double proportionality is not so simple and in some countries the procedure used is flawed. This is the case of the procedure implemented in Italy, where for some voting outcomes the system may end up by awarding a party more (or less) seats within the regional constituencies than those the same party is entitled to at the national level; or by awarding a constituency more (or less) seats than those apportioned to it [9]. In the recent 2006 Italian elections, for example, the constituency of Molise ended up with 2 seats in the Chamber of Deputies instead of the 3 it is entitled to on the basis of the Constitution, while the Trentino Alto Adige constituency got 11 seats instead of 10. The Italian paradox is not an isolated case: a similar technical flaw was identified by Balinski and Ramirez in the 1996 Mexican double-proportional electoral law [6].

The defect lies in the procedure adopted in the Italian system, which does not acknowledge the complexity of the problem it is trying to tackle. The Italian electoral law first allocates seats to parties at the national level and then distributes these seats to the parties within each regional constituency, considered one at a time[1]. Both steps are carried out on a proportional basis, according to a well-known method called *Hare* or *Largest Remainders*. A fundamental property of the Largest Remainders method is that the number of seats is always equal to the exact share of seats a party should receive on a strictly proportional basis, rounded either downwards or upwards (see,e.g., [7]).

Let the party with the greatest number of votes be the "majority list" and the quotient between the total number votes and the number of seats at stake (617) be called the fractional national coefficient. This number rounded downwards is called the *national coefficient* and represents the cost of a seat in terms of votes in the national contest.

The computation of the number of seats allocated to each party (or coalition of parties) at the national level is carried out first, discarding parties which have obtained a share of votes smaller than the fixed threshold. Each eligible party is first assigned its exact share (or *exact quota*) of seats rounded downwards, i.e. dividing the number of votes the party has obtained by the national coefficient and rounding this number downwards. Then the number of remaining seats (which must be still awarded) is calculated and an additional seat is assigned to those parties which have the greatest fractional remainders (in fact this is a slight variant of the typical statement of the Largest Remainders method).

Once these steps have been carried out, if the majority list has not achieved at least 340 seats, it receives the majority prize, that is, a number of seats needed to reach a total of 340 in the Chamber. This means that the whole seat distribution must be re-calibrated, as only 277 are left in the chamber and they must be redistributed among the minority parties and coalitions. A majority *electoral coefficient* (i.e., the total majority list votes divided by 340 and rounded downwards) and a *minority electoral coefficient* (i.e., the sum of votes obtained by the other parties divided by 277 and rounded downwards) are the new references. The 277 minority seats are redistributed among the minority parties with the method of Largest Remainders described above, but using the minority electoral coefficient.

At this point the national seat allocation has been defined. Therefore, the allocation of seats to parties within the regional constituencies is bound to satisfy two types of sub-totals:

(a) the sum of the seats assigned to all parties within a given constituency must be equal to the number of

[1] The description given here is not complete as the Italian system also includes majority prizes and eligibility thresholds. For a more detailed analysis see [6].

seats actually at stake in the constituency;

(b) the sum of the seats awarded to a given party in all constituencies must be equal to the number of seats it was awarded on the basis of the national computation.

The procedure adopted to allocate seats to parties within the regional constituencies starts by computing the exact number of seats due to each party one constituency at a time (starting from the smallest one). This number is equal to the size of the constituency multiplied by the percentage of ballots the given party has obtained. This "exact quota of seats" is not necessarily an integer and usually carries a fractional part. Since a seat cannot be divided among different candidates, the law first assigns each party a number of seats equal to the exact quota rounded downwards. If more seats are at stake in the constituency, and until they are available, an additional seat is awarded to the party with the largest fractional remainder.

The problem with this procedure is it does not guarantee that, once all seats are assigned, the total amount awarded to each party is the same as the amount computed at the national level. By operating one constituency at a time, without worrying about the total amount of seats a party is entitled to at the national level, the logical constraint might not be satisfied. This is not a negligible defect and it has serious practical consequences: should such a paradoxical result occur, who will decide the final seat assignment? In Italy, the size of the Chamber of Deputies cannot be changed. Some parties will gain more seats with the regional allocation but others will with the national one. The failure of the Italian electoral law could trigger a serious controversy between political parties on whether the result of the national allocation should prevail on the results of the regional allocations. Claims of the different political groups would presumably vary according to which case is the most advantageous for them.

Acknowledging the possibility of such a paradoxical situation, the lawmakers introduced a correction mechanism which is executed whenever the sum of seats awarded to parties in the regional constituencies is not equal to the corresponding national seat allocation. It is applied starting from the party with the largest seat surplus, and following a decreasing order. Seats are transferred from the party with a surplus in those constituencies in which the party has obtained an additional seat thanks to its remainders, selecting the smallest remainders (the underlying idea is that seats are taken away from the party in those cases in which it was less entitled to them). The seats are transferred to one of the parties with a seat deficit in the same constituency, provided that such party has not already benefited from an additional seat on the basis of its own remainder. This transfer considers the party in the constituency with the largest unused remainder first, and then proceeds in decreasing order (the idea is to award the seat to the party which is next most entitled to it).

Although it is meant to correct the damage done, the mechanism does not always work because it operates only on seats rounded upwards, i.e., assigned to a given party because of its relatively "large" remainder. In other words the correction mechanism assumes that a paradox may occur, but only because of a party has benefited too much from its exact quotas being rounded upwards.

In fact, there are at least three types of paradoxes undermining the Italian electoral law and for which its correction mechanism is not sufficient to repair:

- *the surplus paradox for parties with exact regional quotas all rounded downwards*: when the sum of the seats assigned to a party (or coalition of parties) in the constituencies is greater than the number of seats it is entitled to at national level and all its regional seats are the result of exact quotas rounded downwards;
- *the deficit paradox for parties with exact regional quotas all rounded upwards*: when the sum of the seats assigned to a party (or coalition) in the constituencies is smaller than the number of seats it is entitled

to at the national level and it has already benefited from extra seats thanks to largest remainders in all constituencies where it has obtained votes.

- *the constituency paradox*: when the sum of seats assigned to the parties within a certain constituency does not match the total number of seats apportioned, according to the Italian Constitution, to that constituency.

These paradoxes are not connected to the application of the majority prize, although when the majority and minority coefficients tend to be very different and different from the national vote/seat ratio, they may be more likely to occur.

2. A formal description of the Italian electoral bug

From a mathematical point of view, the electoral procedure adopted in Italy and in other countries wishing to achieve double proportionality is meant to solve the following problem: find a matrix of non-negative integers (the seats), whose sums of rows (the constituencies) and sums of columns (the political parties) are fixed and whose entries are "proportional" to a given matrix (the matrix of votes). This is the well-known *biproportional discrete allocation problem* which is in itself of great interest and has many applications, not only in the electoral field (see for example [1]).

We use the following notation:

M = a set of regional *constituencies*;

N = a set of political *parties*;

m = number of constituencies;

n = number of parties;

v_{ij} = number of votes obtained by party j in constituency i,;

$Z = \{(i,j) : v_{ij} = 0, i \in M; j \in N\}$;

V = total number of votes;

S = total number of seats;

r_i = number of seats at stake in constituency i;

s_j = total number of seats awarded to party j, at the national level;

We assume that

$$\sum_{i \in M} r_i = \sum_{j \in N} s_j = S \tag{1}$$

Then v_{iN} and v_{Mj} are the sum of the votes cast in constituency i (across all parties) and the sum of the votes cast for party j (across all constituencies), respectively:

$$v_{iN} = \sum_{j \in N} v_{ij}$$

$$v_{Mj} = \sum_{i \in M} v_{ij}$$

$$v_{MN} = \sum_{j \in N} \sum_{i \in M} v_{ij} = V$$

The *biproportional allocation problem* in integers [2,3,4,5,10] is to find a matrix of seats $\mathbf{x} = [x_{ij}]$ for each constituency $i \in M$ and each party $j \in N$ such that the following constraints hold:

$$x_{iN} = r_i \quad \text{for every constituency } i \tag{2}$$

$$x_{Mj} = s_j \quad \text{for every party } j \tag{3}$$

$$x_{ij} = 0 \quad (i,j) \in Z \tag{4}$$

$$x_{ij} \geq 0 \quad \text{and integer } \forall (i,j) \tag{5}$$

Finally, we would like x_{ij} to be "as proportional as possible" tp v_{ij} for all $i \in M$ and $j \in N$.

Let $q_{ij} = v_{ij}\, r_i / v_{iN}$ be the *exact share* of seats for party j in constituency i. Now $q_{MN} = S$. Perfect proportionality is achieved by letting $x_{ij} = q_{ij}$ for all i, j. If there are no further constraints, this is the obvious solution to the problem, but x_{ij} must be integer and $x_{Mj} = s_j$ must hold as well.

The idea underlying the Italian method is to consider the exact quotas each party is entitled to in the regional constituency and to round these numbers upwards or downwards (in the case the majority prize is assigned to some party; these quotas are not the exact ones but a modified version based on the majority or minority seats). The Italian electoral law adopts the method of Largest Remainders both at the national and regional level; therefore all resulting seat allocations comply with a property called *quota satisfaction*, according to which

$$\lfloor v_{ij}\, r_i / v_{iN} \rfloor \leq x_{ij} \leq \lceil v_{ij}\, r_i / v_{iN} \rceil \tag{6}$$

must hold for every party j and constituency i (at the regional level) but also:

$$\lfloor v_{Mj}\, S/V \rfloor \leq x_{ij} \leq \lceil v_{Mj}\, S/V \rceil \tag{7}$$

must hold for every party j at the national level.

It is fairly easy to build realistic examples for which, however the rounding is carried out, it is impossible to satisfy both row- (constituency) and column- (party) sum constraints (2) and (3).

The *surplus paradox* certainly occurs if there is a party j such that:

$$\sum_{i \in M} \lfloor v_{ij}\, r_i / v_{iN} \rfloor > \lceil v_{Mj}\, S/V \rceil \tag{8}$$

The *deficit paradox* certainly occurs if there is a party j such that:

$$\sum_{i \in M} \lceil v_{ij}\, r_i / v_{iN} \rceil > \lfloor v_{Mj}\, S/V \rfloor \tag{9}$$

In the second case the correction mechanism will get stuck because the law never considers the possibility that a lack of seats can occur although a party's exact quota of seats has already been rounded upwards in all constituencies (and therefore the party is never eligible to receive additional seats).

Finally, the *constituency paradox* occurs if $x_{iN} \neq r_i$ for some i.

Although the problem the Italian electoral law attempts to solve is not an easy one, a "sound" solution to the biproportional discrete allocation problem always exists, as proved by Balinski and Demange [2, 3], who also provide an algorithm to solve the problem, which resembles the out-of-kilter one for minimum cost network flows. Thus, appropriate and correct procedures exist and they have actually been adopted in practice as, for example, in the Zurich Canton electoral law [10]. The simplest such procedure [4], [8], [11] can be viewed as a discrete version of the well-known RAS algorithm [1]: starting from the matrix of votes cast, a double proportional integer matrix of seats can be obtained by alternating row and column scalings, followed at each iteration by rounding the resulting entries according to a given divisor threshold.

In Italy, a simple variant of the above procedure could be implemented, relying, rather than on a divisor method, on the Largest Remainders rule, traditionally employed for the Chamber elections.

In this paper we propose an alternative optimization approach to solve the biproportional discrete allocation problem. Such approach could be helpful in practice to find a solution closer to the Italian electoral tradition, where the principle underlying the idea of proportionality is to minimize a given measure of deviation between the resulting seat allocation and the perfectly proportional one[2]. For example, a possible measure that could be taken into account is the largest absolute difference (maximum absolute error) between the number of seats assigned to each party j within each constituency i and the exact fractional share q_{ij}. Even if the absolute error is a nonlinear function, the problem can be easily formulated as a mixed integer linear program (MILP), where the integer variables x_{ij} represent the number of seats to assign to each party j in each constituency i.

Then the MILP formulation is (MILP1):

$$
\begin{aligned}
\tau^* \; = \; & \min \tau \\
\text{s.t.} \quad & q_{ij} - \tau \leq x_{ij} \leq q_{ij} + \tau && \forall\,(i,j) && (10)\\
& \sum_{j=1}^{s} x_{ij} = r_i && \forall\, i \in M && (11)\\
& \sum_{i=1}^{r} x_{ij} = s_j && \forall\, j \in N && (12)\\
& x_{ij} = 0 && (i,j) \in Z && (13)\\
& x_{ij} \geq 0 \text{ and integer} && \forall\,(i,j) && (14)
\end{aligned}
$$

In MILP1 the maximum absolute error is treated as a variable and there are bound constraints (10) over the errors; assignment constraints (11) and (12), relative to the row- and column- sums, respectively; zero-vote zero-seat constraints (13), whereby a party should receive no seats in those constituencies where it gets no votes; integrality and non negativity constraints (14). The difficulty is, of course, the integral nature of the variables. However, in the electoral case this problem usually has a small size and can be easily solved in few seconds by a personal computer using standard mathematical programming software[14]. In any case, special purpose network algorithms can also be designed in order to solve the problem more efficiently [12]. The key observation is that, for any fixed value of τ, the constraints (10) can always be strengthened to

$$\lceil q_{ij} - \tau \rceil^+ \leq x_{ij} \leq \lfloor q_{ij} + \tau \rfloor, \qquad \forall (i,j) \tag{15}$$

where $z^+ = \max\{0, z\}$.

The integrality theorem of network flows ensures that for any fixed t, if a feasible solution to the system (11), (12), (13) and (15) exists at all, then an integral solution also exists and it can be found by a maximum flow algorithm.

A matrix $\mathbf{x} = [x_{ij}]$ satisfying constraints (11)–(14) will be called an *apportionment*.

Although the apportionment obtained from the optimal solution of the above MILP1 – or from the alternating scaling procedure based on Largest Remainders – might not satisfy the Balinski-Demange axioms for proportionality (whose fulfillment calls for the use of divisor methods), it might be a reasonable choice if one wishes to adhere as much as possible to the traditional Largest Remainders logic. Nevertheless, some practical problems remain, for example the possible non-uniqueness of the optimal solution and the need of an easy to check "certificate of optimality" of the solution. It is important to underline that, in addition to

[2] Many measures can be adopted, each one representing a different idea of proportionality and each one usually minimized by a different class of algorithms (for a more detailed discussion see [6]).

the seat distribution, the above MILP1 is able to provide also the maximum deviation between the number of seats assigned to each party within each constituency and their exact share. This is in itself a certificate of validity of the solution that, along with constraints (11), (12), and (13), can be easily checked by everybody without solving any MILP. A stronger certificate can be also provided by showing that no better solution can exist, again without solving any MILP. This certificate is based on the following relationship valid for any subsets $\bar{I}$ of constituencies and $\bar{J}$ of parties:

$$\sum_{i\in\bar{I}} r_i = \sum_{i\in\bar{I}}(\sum_{j\in\bar{J}} x_{ij} + \sum_{j\notin\bar{J}} x_{ij}), \qquad \sum_{j\in\bar{J}} s_j = \sum_{j\in\bar{J}}(\sum_{i\in\bar{I}} x_{ij} + \sum_{i\notin\bar{I}} x_{ij})$$

so

$$\alpha := \sum_{i\in\bar{I}} r_i - \sum_{j\in\bar{J}} s_j = \sum_{i\in\bar{I}}\sum_{j\notin\bar{J}} x_{ij} - \sum_{i\notin\bar{I}}\sum_{j\in\bar{J}} x_{ij} =: \beta$$

This identity may be worded as: for every subset of constituencies and for every subset of parties, the difference α between the total number of seats in the constituencies and the total number of seats for the parties is equal to the difference β between the total seats assigned in the constituencies to the other parties and the total seats assigned to the parties in the other constituencies.

The number of seats assigned to constituency-party pairs is bounded above by a number γ, depending on the maximum error, so that $\beta \leq \gamma$. Hence if $\alpha > \gamma$ we have an infeasibility proof since $\alpha = \beta$. The subsets $\bar{I}$ and $\bar{J}$ for which $\alpha > \gamma$ are a byproduct of MILP1. Essentially, the certificate is a consequence of the well-known max flow–min cut theorem [14].

As an alternative, one may wish to minimize the maximum relative error rather than the maximum absolute one. The minimum value of the maximum relative error is the optimum σ^* of the following optimization problem:

$$\begin{aligned} \sigma^* &= \min \max_{ij} \sigma_{ij} \\ \text{s.t.} & \quad (11)\text{-}(14) \\ & \quad \left| \frac{x_{ij} - q_{ij}}{q_{ij}} \right| \leq \sigma_{ij} \qquad \forall\, (i,j) \notin Z \end{aligned} \tag{16}$$

The above nonlinear integer optimization problem can be re-written as a MILP (MILP2):

$$\begin{aligned} \sigma^* &= \min \sigma \\ \text{s.t.} & \quad (11)\text{-}(14) \\ & \quad 0 \leq \sigma_{ij} \leq \sigma \qquad \forall (i,j) \\ & \quad (1-\sigma_{ij})\, q_{ij} \leq x_{ij} \leq (1+\sigma_{ij})\, q_{ij} \qquad \forall\, (i,j) \end{aligned} \tag{17}$$

A slightly different MILP is obtained when one wants to minimize the *average* relative error. In this case the objective function is simply the sum of all σ_{ij}'s, and neither the auxiliary variable σ nor the inequalities $\sigma_{ij} \leq \sigma$ in constraints (17) are needed. We denote such program by MILP3.

3. A multistage approach

We propose a hierarchical two-stage optimization procedure in which, in the first stage, MILP1 is solved in order to find sharp bounds on the maximum error (and also on the individual variables x_{ij}); in the second stage, another MILP, namely MILP2 or MILP3 with the additional constraints

$$q_{ij} - \tau^* \leq x_{ij} \leq q_{ij} + \tau^*, \qquad \forall (i,j) \tag{18}$$

is formulated in order to minimize the maximum or the average relative error, respectively, taking into account the already computed bounds on the variables.

The second-stage optimal solution (whose components are double-starred here) still yields an optimal apportionment $\mathbf{x}^{**}$ which is still optimal for the first-stage; moreover, it provides new and sharper bounds over the variables

$$q_{ij} - \min\left\{\tau^*, q_{ij}\,\sigma_{ij}^{**}\right\} \leq x_{ij} \leq q_{ij} + \min\left\{\tau^*, q_{ij}\,\sigma_{ij}^{**}\right\}, \qquad \forall (i,j) \tag{19}$$

and finally, it minimizes the maximum or the average relative error. A third stage criterion that may be invoked is "least deviation from law": one looks, subject to the above bounds and assignment constraints, for an apportionment $\mathbf{x} = [x_{ij}]$ that differs from the (incorrect) institutional seat allocation $\mathbf{y} = [y_{ij}]$ in as few components as possible. Here too one can obtain a MILP formulation. Let $\mathbf{u}$ be the $r \times s$ matrix with components

$$u_{ij} = \begin{cases} 0 & \text{if } x_{ij} = y_{ij} \\ 1 & \text{otherwise} \end{cases}$$

Then the MILP formulation (MILP4) is:

$$\begin{aligned} \min\; & \sum_{i=1}^{r}\sum_{j=1}^{s} u_{ij} \\ \text{s.t.}\; & (11)\text{-}(14) \\ & y_{ij} - M\; u_{ij} \leq x_{ij} \leq y_{ij} + M\; u_{ij} \qquad && \forall\,(i,j) \\ & u_{ij} \in \{0,1\} && \forall\,(i,j) \end{aligned}$$

where M is a large positive constant – actually, it is enough to choose M as the total number of seats S in the Parliament.

Finally, in order to achieve the uniqueness of the optimal solution, additional steps can be implemented so as to select exactly one solution for a given input vote distribution. This can be done in different ways, according to different criteria. For example, in the set of all optimal solutions (with the same maximum absolute error and the same average relative error) one can always choose the lexicographic best solution according to a suitable total order of the variables x_{ij}, e.g., the order of nonincreasing votes v_{ij}, or nonincreasing exact shares q_{ij}.

An alternative lexicographic optimization strategy is the so called *Unordered Lexico Optimization*. An Unordered Lexico minimum is defined in the following way: given a vector $a \in R^n$ let $\theta(a) \in R^n$ be the vector obtained from a by permuting its entries so that the entries in $\theta(a)$ are arranged in non-increasing order (in case of equal entries break the tie in any fixed way). Then given two vectors $a, b \in R^n$ we say that a is *Unordered Lexico better* than b if $\theta(a)$ is lexicographically smaller than $\theta(b)$, i.e. there exists an integer k such that $\theta_i(a) = \theta_i(b)$ for $i < k$ and $\theta_k(a) < \theta_k(b)$. A vector $a \in A \subset R^n$ is an Unordered Lexico Optimum in A if there is no $b \in A$ which is Unordered Lexico better than a.

In our case we want to find Unordered Lexico Optima for the vectors $|q_{ij} - x_{ij}|$ with respect to all feasible x in MILP1. This can be easily done by "freezing", after the above described first stage, the (usually unique) x_{ij} with largest deviation from the corresponding exact share q_{ij}, then minimizing again the largest deviation over the remaining x_{ij}'s, and iterating. Appropriate tie-breaking rules can be defined, although they are unnecessary in most cases.

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