What is decidable about Halpern and Shoham's modal logic of intervals?

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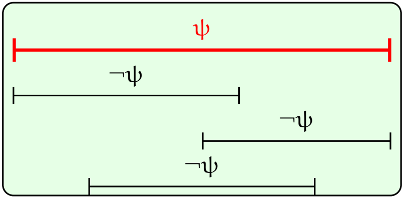
Outline

- Introduction
- A geometric account of interval temporal logics
- Decidability of $\mathsf{A}\mathsf{B}\overline{\mathsf{B}}$ over the class of all linear orders
- Generalization to $\mathsf{A}\mathsf{B}\overline{\mathsf{B}}\overline{\mathsf{L}}$

Interval temporal logics

Truth of formulae is defined over intervals (not points).



Interval temporal logics are very expressive (compared to point-based temporal logics).

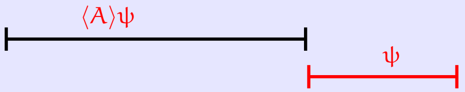
In particular, formulas of interval logics express properties of pairs of time points rather than of single time points, and are evaluated as sets of such pairs, i.e., as binary relations.

Thus, in general there is no reduction of the satisfiability/validity in interval logics to monadic second-order logic, and therefore Rabin's theorem is not applicable here.

An example: the future fragment of neighborhood logic

Formulas of the logic are recursively defined by the following grammar:

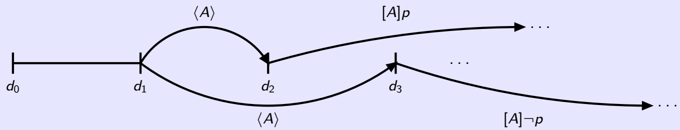
$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle A \rangle \varphi$$



We cannot abstract way from the left endpoint of intervals:

- contradictory formulas can hold over intervals with the same right endpoint and a different left endpoint.

$\langle A\rangle[A]p \wedge \langle A\rangle[A]\neg p$ is satisfiable ($[A] = \neg\langle A\rangle\neg$ as usual):



For any $d > d_3$, p holds over $[d_2, d]$ and $\neg p$ holds over $[d_3, d]$.

Binary Relations over intervals

The thirteen binary relations between two intervals on a linear ordering (those below and their inverses) form the set of *Allen's interval relations*:

current interval:

equals:

ends :

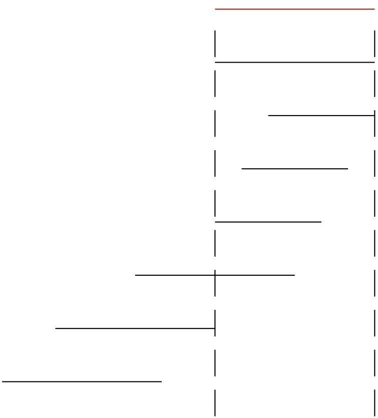
during:

begins:

overlaps:

meets:

before:



HS: the modal logic of Allen's interval relations

Allen's interval relations give rise to respective unary modal operators over frames where intervals are primitive entities, thus defining the multimodal logic HS introduced by Halpern and Shoham in 1991, interpreted over interval structures.

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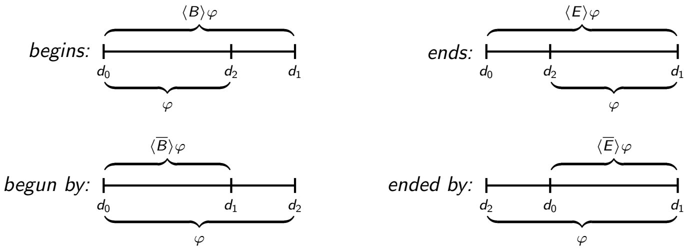
It suffices to choose as primitive the modalities $\langle B \rangle$, $\langle E \rangle$, $\langle \overline{B} \rangle$, $\langle \overline{E} \rangle$ corresponding to the relations begins, ends, and their inverses; the others are definable.



Decidability of HS fragments: main parameters

More than four thousands fragments of HS can be identified by choosing suitable subsets of the set of basic modal operators.

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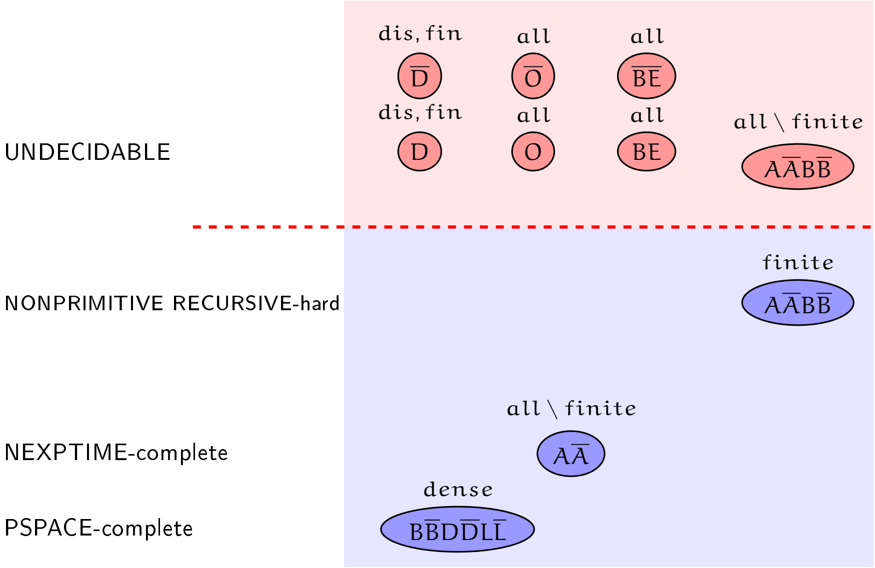
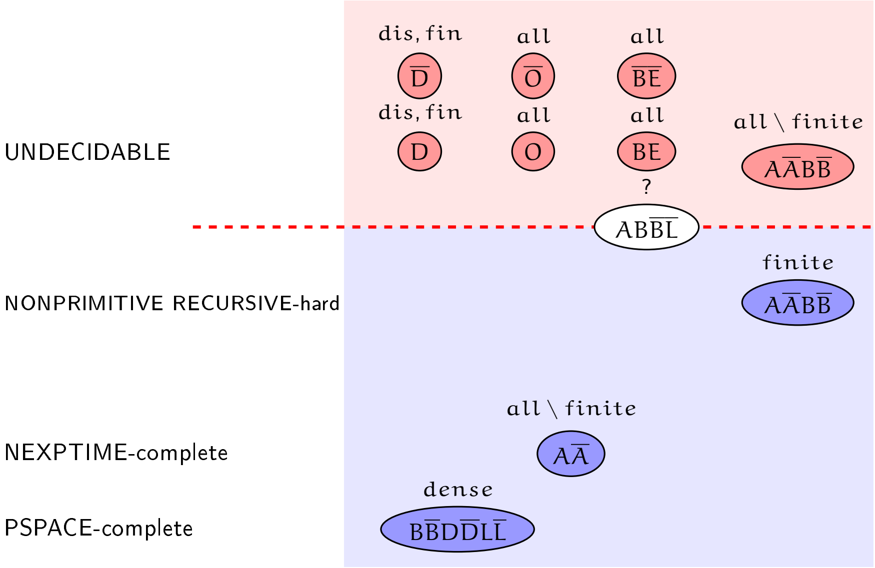
In principle, decidability of HS fragments depends on two factors:

- the set of interval modalities;
- the linear order over which the logic is interpreted.

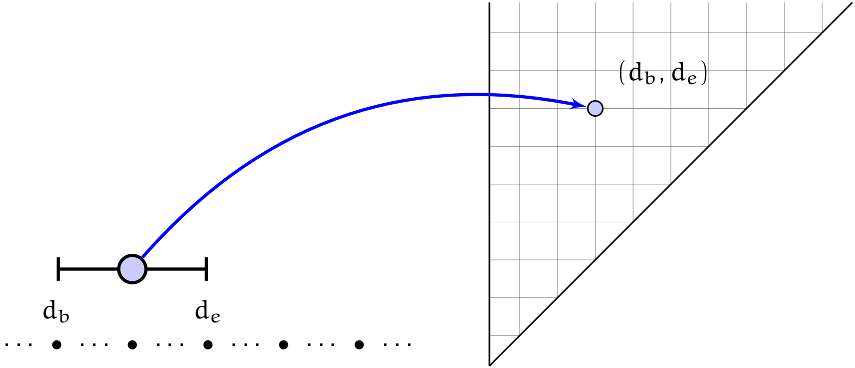
The existing landscape



The existing landscape

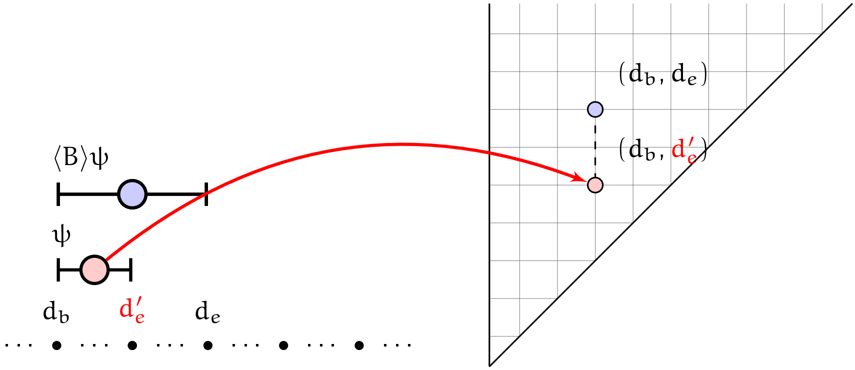


A geometrical account of interval logic: intervals



Every interval can be represented by a point in the second octant (in general, in the half plane $y \geqslant x$).

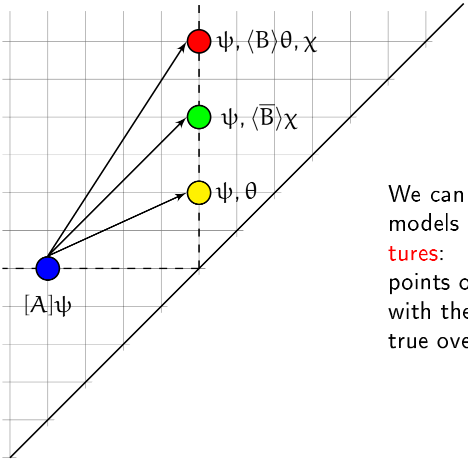
A geometrical account of interval logic: interval relations



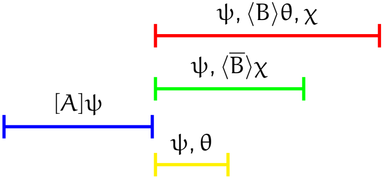
$$d_b < d'_e < d_e$$

Every interval relation has a spatial counterpart.

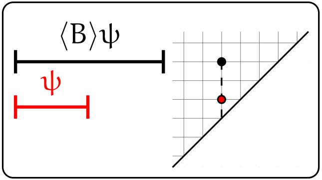
A geometrical account of interval logic: models

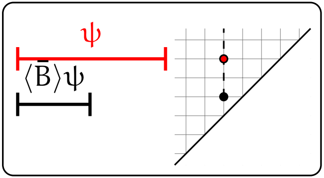


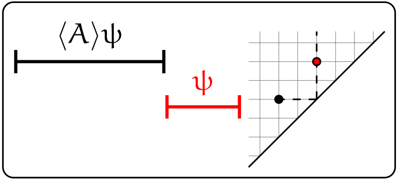
We can give a spatial interpretation to models of a formula φ as compass structures:
points of a compass structure are colored with the set of subformulas of φ that are true over the corresponding intervals



What was already known: $A B \overline{B}$

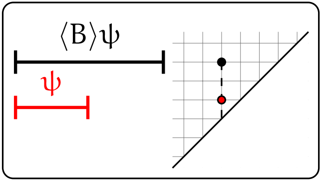


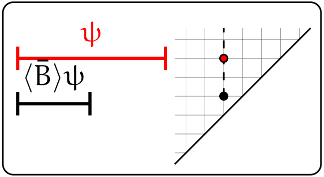


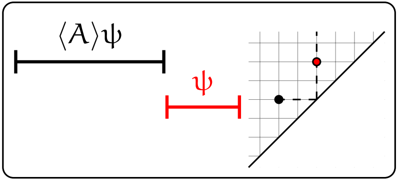


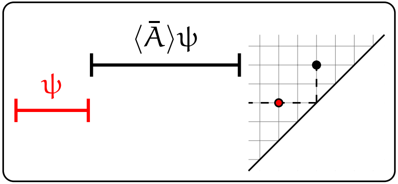
$A B \overline{B}$ is EXPSPACE-complete over the natural numbers.

What was already known: $A\overline{A}B\overline{B}$



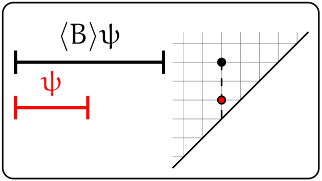


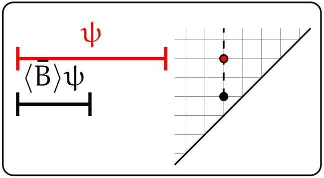


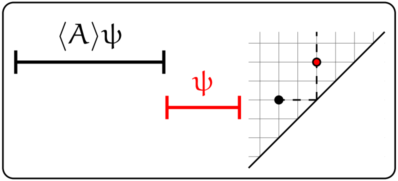


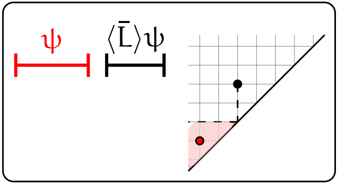
$A\overline{A}B\overline{B}$ is NONPRIMITIVE RECURSIVE-hard over finite linear orders; undecidable elsewhere.

What is this paper about: $A B \overline{B} \overline{L}$

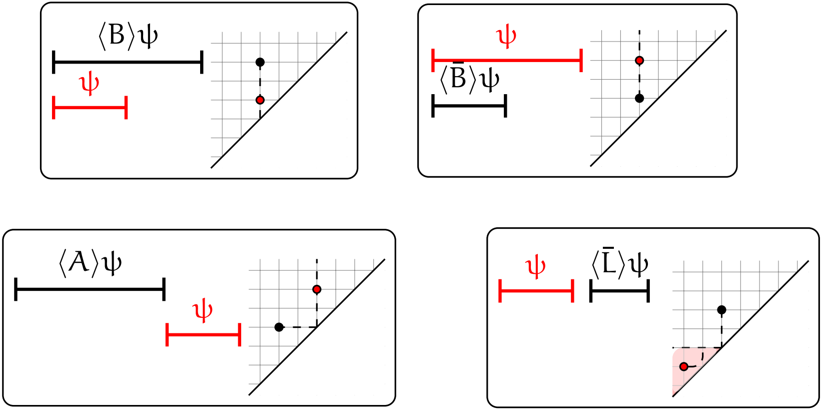






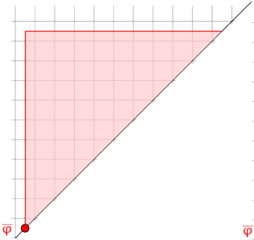


What is this paper about: $AB\overline{BL}$

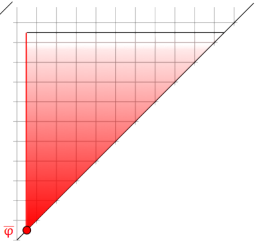


$\overline{L}$ is easily definable in terms of $\overline{A}$ $(\langle\overline{L}\rangle = \langle\overline{A}\rangle\langle\overline{A}\rangle)$

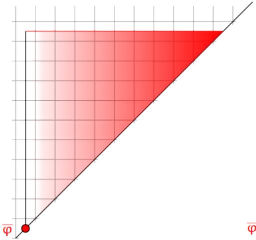
From Compass Structures to Bounded Compass Structures



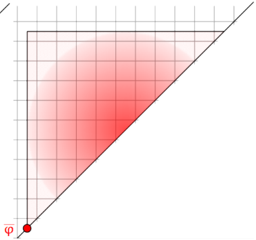
bounded



unbounded in the future



unbounded in the past



unbounded

Given an input $\mathsf{A}\mathsf{B}\bar{\mathsf{B}}\bar{\mathsf{L}}$-formula φ, we translate it into a formula $\overline{\varphi}$ such that φ is satisfiable if and only if $\overline{\varphi}$ is satisfiable at the initial point of a (possibly infinite but) bounded compass structure.

Decidability of $A B \overline{B} \overline{L}$ over all linear orders

We prove our decidability result in two steps:

- first, we prove that the satisfiability problem for the simpler fragment $A B \overline{B}$ over all linear orders is decidable
  - we first define a suitable notion of pseudo-model for a satisfiable formula of $A B \overline{B}$
  - then, we prove that the problem of establishing whether or not such a pseudo-model exists is decidable
- then, we show how to generalize the proof to $A B \overline{B} \overline{L}$

Basic ingredients

Let $\mathcal{G}$ be a bounded compass structure for a formula φ

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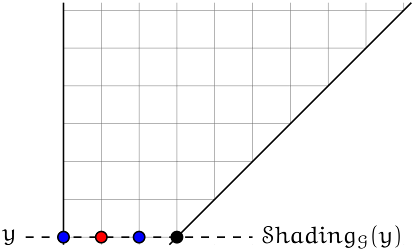
Shading

The shading of a row y of $\mathcal{G}$ $\mathcal{S}hading_{\mathcal{G}}(y)$ is the set of all and only the atoms associated with points in y



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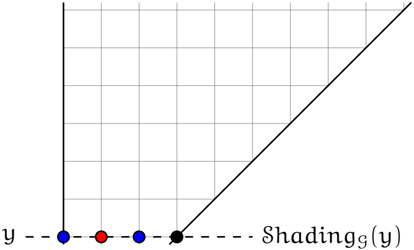

$\mathcal{S}hading_{\mathcal{G}}(y) = \{ \bullet, \bullet, \bullet \}$

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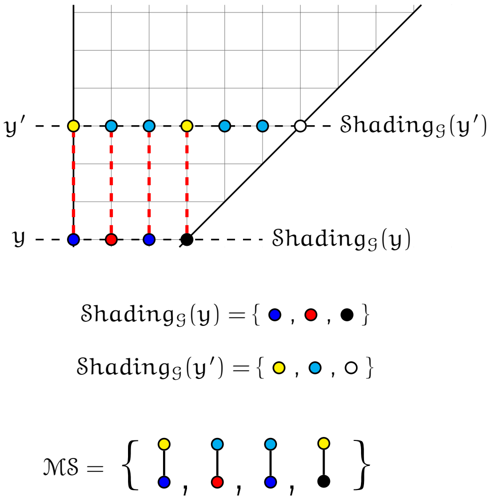
Matching set

Given two shadings S_1 and S_2 of $\mathcal{G}$, a matching set is a finite of set of pairs of corresponding atoms, respectively belonging to S_1 and S_2, that satisfy suitable matching properties



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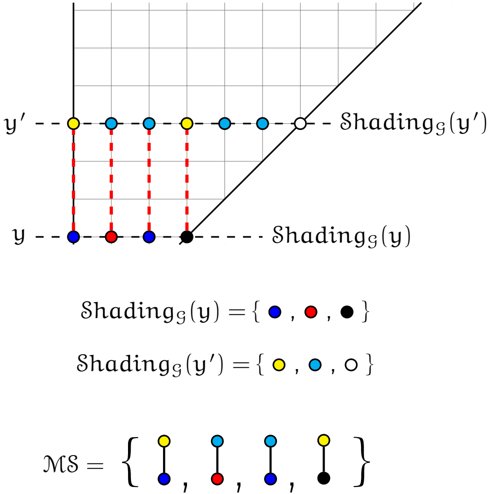
Matching graph

A matching graph is the composition of a sequence of matching sets



The notion of decomposition tree

Matching set and matching graph allow us to define the key notion of decomposition tree

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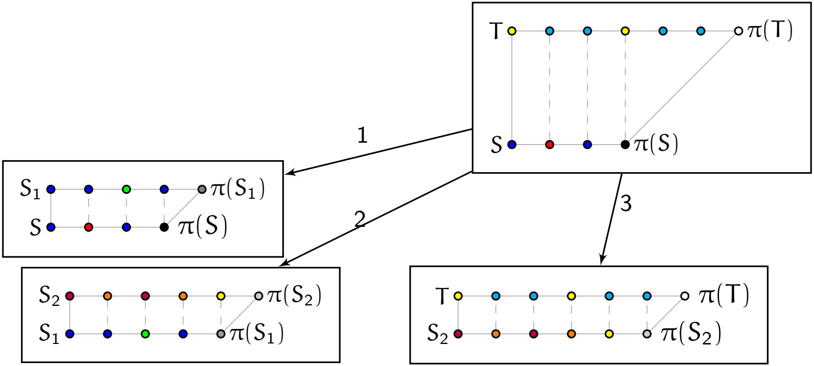

A decomposition tree for a formula φ can be viewed as the unfolding of a finite graph, which provides a finite representation of a (possibly infinite) bounded compass structure

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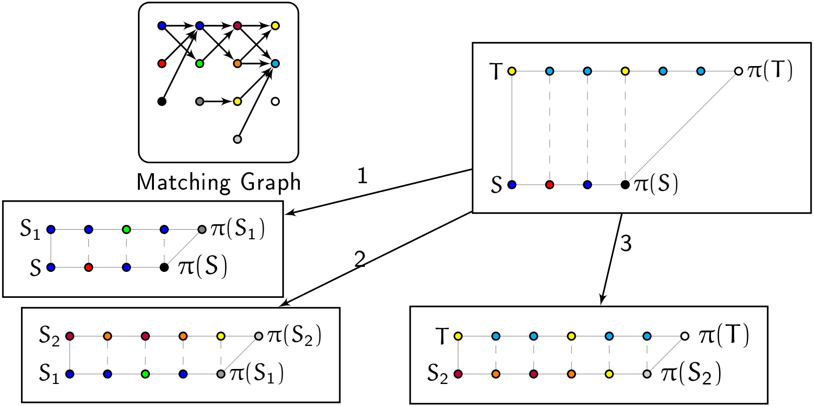


A decomposition tree for a formula φ can be viewed as the unfolding of a finite graph, which provides a finite representation of a (possibly infinite) bounded compass structure

Decidability of $A B \overline{B}$

Completeness

Let φ be an $A B \overline{B}$ -formula and $\mathcal{G} = \langle \mathbb{P}_{\mathbb{O}}, \mathcal{L} \rangle$ be a bounded compass structure for φ. Then, there exists a decomposition tree $T_\varphi = \langle \mathcal{T}, \nu \rangle$ for φ with rank $\leqslant 4 \cdot |\varphi| \cdot 2^{18|\varphi|+2} + 2^{9|\varphi|+1} + 1$.

Soundness

Let φ be an $A B \overline{B}$ -formula and $T_\varphi = \langle \mathcal{T}, \nu \rangle$ be a decomposition tree for φ. Then, there exists a bounded compass structure $\mathcal{G} = \langle \mathbb{P}_{\mathbb{O}}, \mathcal{L} \rangle$ for φ.

Theorem

Let φ an $A B \overline{B}$ -formula. Then, φ is satisfiable in the class of all linear orders if and only if there exists a decomposition tree $T_\varphi = \langle \mathcal{T}, \nu \rangle$ for φ with rank $\leqslant 4 \cdot |\varphi| \cdot 2^{18|\varphi|+2} + 2^{9|\varphi|+1} + 1$.

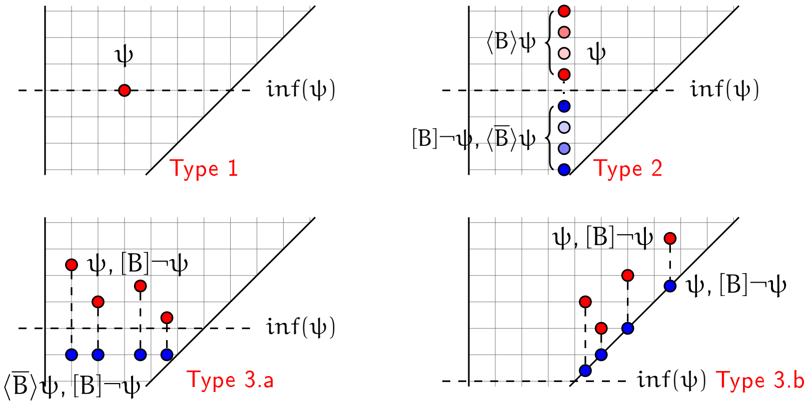
The addition of $\overline{\mathrm{L}}$: complications

To deal with $A\mathrm{B}\overline{\mathrm{BL}}$, the notion of decomposition tree must be suitably generalized.

The addition of $\overline{L}$: complications

To deal with $AB\overline{BL}$, the notion of decomposition tree must be suitably generalized.

Given a bounded compass structure $\mathcal{G}$ and a formula $\langle \overline{L} \rangle \psi$ that occurs in $\mathcal{G}$, we must distinguish among the following three cases for ψ:



The main result: decidability of $\mathsf{AB\overline{B}\overline{L}}$

Theorem

Let φ be an $\mathsf{AB\overline{B}\overline{L}}$-formula. Then, φ is satisfiable in the class of all linear orders if and only if there exists an *extended decomposition tree* T_φ for φ with rank $m \leqslant 4 \cdot |\varphi| \cdot 2^{18|\varphi|+2} + 2^{9|\varphi|+1} + |\varphi| + 1$.

We reduced the problem of establishing whether an *(extended) decomposition tree* for φ exists to the nonemptiness problem for a suitable regular tree language $\mathcal{T}_\varphi$.

Nonemptiness for $\mathcal{T}_\varphi$ can be checked using exponential-space in the size of the formula.

By previous results for $\mathsf{AB\overline{B}}$, we know that the satisfiability problem for $\mathsf{AB\overline{B}\overline{L}}$ is EXPSPACE-hard, and thus EXPSPACE-completeness immediately follows.

Dense and discrete linear orders

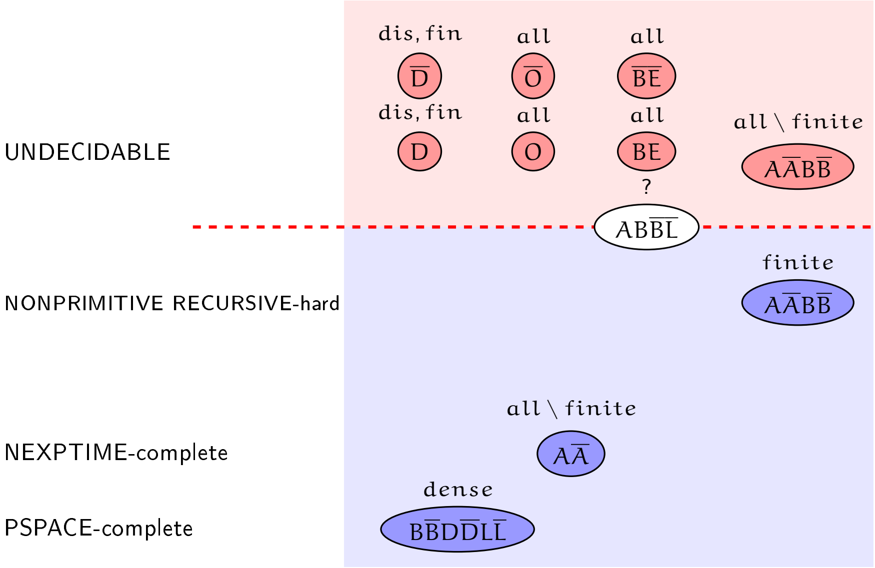
The decidability of $\mathrm{AB\overline{BL}}$ over the class of dense linear orders immediately follows, as density can be defined in $\mathrm{AB\overline{BL}}$ by a constant formula:

an $\mathrm{AB\overline{BL}}$ -formula φ is satisfiable over the class of dense linear orders if and only if the (constant) formula $\varphi \wedge [G](\neg\pi \rightarrow \langle B \rangle \neg\pi)$ is satisfiable over the class of all linear orders

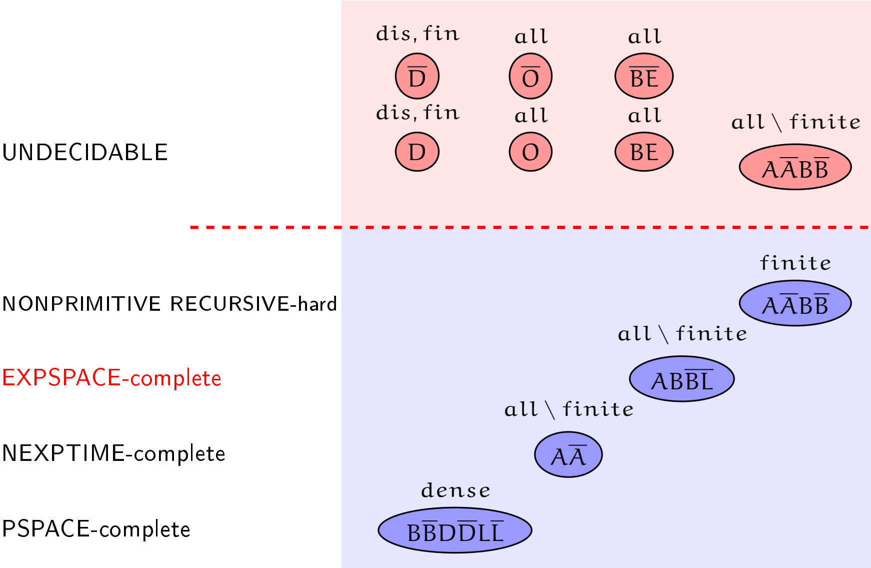
A similar argument cannot be applied to (weakly) discrete linear orders. However, it is possible to tailor the decidability proof for the class of all linear orders to them

Thus, we can conclude that the $\mathrm{A\bar{A}B\bar{L}}$ is a maximal fragment of HS with respect to the decidability over the class of all linear orders, dense orders, and discrete orders.

The updated landscape: the maximal fragment $AB\overline{BL}$



The updated landscape: the maximal fragment $AB\overline{B}\overline{L}$



Thank You