Modal Deduction in Second-Order Logic and Set Theory - II*

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Abstract

In this paper, we generalize the set-theoretic translation method for polymodal logic introduced in [11] to *extended modal logics*. Instead of devising an ad-hoc translation for each logic, we develop a general framework within which a number of extended modal logics can be dealt with. We first extend the basic set-theoretic translation method to *weak monadic second-order logic* through a suitable change in the underlying set theory that connects up in interesting ways with constructibility; then, we show how to tailor such a translation to work with specific cases of extended modal logics.

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1 Introduction

The investigations presented in this paper are related with the set-theoretic translation method introduced in [11], the $\Box$-as-Pow translation from now on. One of the basic ideas behind the $\Box$-as-Pow translation was to formalize the notion of validity in Kripke frames by a set-theoretic formula. Such a formula is then set-theoretically provable if and only if the original formula is modally derivable. A peculiarity of the technique was the weakness of the underlying set theory Ω: four axioms describing most rudimentary and basic operators of naive set theory. As a matter of fact, the possibility to use a "toy" (and hence computationally usable) set theory, such as Ω, was one of the original motivations. On the other hand, the weakness of Ω is reflected by the fact that the modal notion that the $\Box$-as-Pow translation is able to capture is very weak: modal derivability for Kripke-complete extensions of K.

In [7], a stronger link between modal logic and axiomatic set theory was established, by showing how to modify the translation in order to capture the semantics of general frames—and hence modal derivability in its full strength. However, the general trend in the field is now toward extended modal logics and, in this

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respect, there are at least two directions that one can explore. One is to leave the standard modal language (or, equivalently, its standard translation in monadic second-order logic) unchanged, and to strengthen the deductive power of the logic over its intended frames; the other is to enhance the expressive power of the logic by extending the basic language (for instance, by adding new propositional operators). In the limit, the first direction leads us to standard monadic second-order theories of frames, of which a prime example is the logic L_2 [5, 6]. In particular, such a logic suffices for carrying out the usual Sahlqvist-style substitution arguments that are so characteristic for much modern work in modal logic (e.g., [16]). The semantics of L_2 is obtained by considering general frames in which the existence of all valuations defined by means of suitable first-order formulas is required. For the case of "pure" general-frame semantics considered in [7]–that is, for the case in which no condition is imposed on the set of possible valuations–the underlying set theory is still Ω. This cannot be the case with L_2, whose expressive power forces a vigorous departure from Ω. In this paper, we establish a deeper link between second-order and set-theoretic derivability to capture modal logical consequence in general frames closed under first-order definitions. Our first main result is the proof that modal derivability in L_2 can be captured via (a suitable adaptation of) the $\Box$-as-Pow translation. The underlying set theory is the extension of Ω obtained by adding closure axioms for the so-called Gödel constructible operations [2, 9, 14]. An interesting by-product of this result is that the analysis presented in [7, 11] becomes a special case of a general correspondence between first-order definable substitutions and Gödel-style constructible operations on sets. Also the second direction, according to which we extend the modal logic language, requires the introduction of an appropriate deductive apparatus. However, in this case, the emphasis is not on making such an apparatus as strong as possible, and even conservative extensions over the original language are quite significant. Our second main result is that we can extend the $\Box$-as-Pow translation to deal with this class of extended modal logics as well. As paradigmatic cases, we considered the modal logic of inequality and the minimal tense logic.

The paper is organized as follows. In Section 2, we briefly review the notation and present some background results on modal logic and translation methods. In particular, we report the basic results relative to the $\Box$-as-Pow translation, the logical consequence in general frames closed under first-order definitions, and the notion of L_2-derivability. In Section 3, we begin presenting the theory Ω_c obtained from Ω adding one axiom for each of the operations introduced by Gödel to describe the constructible universe, and then we rewrite the $\Box$-as-Pow translation within the language of Ω_c. Next, we prove the completeness of the translation with respect to L_2-derivability, showing that whenever there exists a model of Ω_c containing a set-theoretic frame satisfying a formula, then a true frame satisfying the same formula can be built. Subsequently, we prove the correctness of the translation showing that a constructible universe (a model of Ω_c) can always be generated starting from a general frame closed for L_0-definitions. We conclude the section showing the adequacy of Ω_c to capture the full monadic second-order theory of L_2. In Section 4, we discuss the possibility of adapting the proposed technique to deal with extensions of modal logic and some sample cases are considered.

2 Background results

In this section we provide the reader with some background results on modal logic and translation methods. We first recall some notation and results about modal logic (cf. [6] for details); then, we describe the standard translation method for modal logic and report the basic results about the weak second-order logic L_2 [5, 6]; finally, we present the main features of the set-theoretic translation for polymodal logic [11], and compare it with the standard one [7].

2.1 Modal Logic

Propositional modal formulae are constructed using propositional letters $P_1, P_2, \ldots$, boolean operators $\vee, \neg$ and the $\Box$ operator. The minimal modal logic K_s consists of a set of propositional axioms complete for classical logic, the modal axiom $\Box(P \to Q) \to (\Box P \to \Box Q)$, and the rules of substitution, modus ponens and necessitation (infer $\Box\phi$ from ϕ). Deducibility of ψ from ϕ and the axioms, using the rules, is denoted by $\phi \vdash_{K_s} \psi$. A *frame* F is a pair (W, R), where W is a set (of *worlds*) and R is a binary relation on W (the *accessibility relation*). A *valuation* $\Vdash$ in the frame (W, R) is a function from $\{P_1, P_2, \ldots\}$ into $Pow(W)$. For all $w \in W$, $w \Vdash \phi$ is defined by induction on the structural complexity of the modal formula ϕ, as usual. A formula ϕ is *valid* in a frame (W, R) ($(W, R) \models \phi$) if and only if, for all $w \in W$ and for all valuations $\Vdash$, $w \Vdash \phi$ holds. A formula ψ is a *frame logical consequence* of a formula ϕ ($\phi \models_f \psi$) if and only if, for all frames F, if ϕ is valid in F, then ψ is valid in F. A formula ϕ is said to be *complete* if and only if, for all ψ, $\phi \vdash_{K_s} \psi \Leftrightarrow \phi \models_f \psi$. Examples of incomplete formulae can be found in the literature (for instance, in [6]). A semantic characterization of K_s is obtained via general frames. A *general frame* is a pair $(F, \mathcal{W})$, where F is a frame and $\mathcal{W}$ is a set of subsets of W, which is closed under the boolean operations of union, complementation w.r.t. W, and $\Box(X) = \{w \in W : \forall v\ (wRv \to v \in X)\}$. Validity in general frames is defined as for frames except for the fact that we only consider valuations taking values on $\mathcal{W}$. Logical consequence in general frames ($\phi \models_{gf} \psi$) is defined accordingly. For all modal formulae ϕ, ψ we have $\phi \vdash_{K_s} \psi \Leftrightarrow \phi \models_{gf} \psi$.

2.2 The standard translation method

Let (the language of) L_2 be a second-order language containing a binary predicate R and equality. The *standard translation* ST of a modal formula $\theta(P_1, \ldots, P_n)$ is the L_2-formula $ST(\theta)(x, \mathtt{P_1}, \ldots, \mathtt{P_n})$, where $\mathtt{P_1}, \ldots, \mathtt{P_n}$ are set variables corresponding to propositional letters $P_1, \ldots, P_n$, defined as follows:

- $ST(P_i) = \mathtt{P_i}(x_0)$;
- $ST(\phi \vee \psi) = ST(\phi) \vee ST(\phi)$;
- $ST(\neg\phi) = \neg ST(\phi)$;
- $ST(\Box\phi) = \forall y\ (x_0 R y \to ST(\phi)(y|x_0))$,

where y does not occur in $ST(\phi)$ and $y|x_0$ denotes uniform substitution of y for x_0.

The *closed standard translation* $\overline{ST}(\phi)$ of a modal formula ϕ is defined as the L_2-sentence $\forall \mathtt{P_1} \ldots \forall \mathtt{P_n} \forall x\ ST(\phi)$, and it is easy to see that

$$\phi \models_f \psi \text{ if and only if } \overline{ST}(\phi) \models \overline{ST}(\psi),$$

where the $\models$ on the right-hand side denotes second-order logical consequence.

In [4], axiomatic theories for deduction in the L_2-language are introduced. In particular, a system of weak second-order logic is defined by means of a suitable form of substitution. Let an L_0-formula be an L_2-formula without occurrences of second-order quantifiers. Given an L_2-formula α and an L_0-formula $\gamma(x)$, denote by $\alpha(\gamma|\mathtt{P})$ the L_2-formula obtained from α by replacing the occurrences of $\mathtt{P}(u)$ in which the $\mathtt{P}$ is free in α by $\gamma(u|x)$ (after changing bound variables as necessary [4]). L_0-substitution is expressed by the following schema, where α is an L_2-formula and γ is an L_0-formula:

$$\forall \mathtt{P} \alpha \to \alpha(\gamma|\mathtt{P}).$$

Weak second-order logic contains a set of axioms which is complete for first-order predicate logic plus L_0-substitution. It is possible to show that weak second-order logic is a non-conservative extension of K_s, namely, $\phi \vdash_{K_s} \psi \Rightarrow \overline{ST}(\phi) \vdash_{L_2} \overline{ST}(\psi)$, but there exist ϕ and ψ such that $\overline{ST}(\phi) \vdash_{L_2} \overline{ST}(\psi)$ and $\phi \nvdash_{K_s} \psi$ [4].

A semantic counterpart of deducibility in weak second-order logic can be obtained by means of closure under L_0-definitions in general frames. A general frame $(F, \mathcal{W})$ is *closed under L_0-definitions* if, for all L_0-formulae γ, with free world-variables $x, x_1, \ldots, x_n$ and free set-variables $X_1, \ldots, X_m$, and for all $w_1, \ldots, w_n \in W$ and $A_1, \ldots, A_m \in \mathcal{W}$, the set $\{w \in W : F \models \gamma(w, w_1, \ldots, w_n, A_1, \ldots, A_m)\}$ belongs to $\mathcal{W}$. Given two L_0-sentences α and β, it holds that $\alpha \vdash_{L_2} \beta$ if and only if for all general frames $(F, \mathcal{W})$ closed under L_0-definitions, if $(F, \mathcal{W}) \models \alpha$, then $(F, \mathcal{W}) \models \beta$.

2.3 The set-theoretic translation method

The set-theoretic translation is based on the idea of representing any Kripke frame as a set, with the accessibility relation modeled using the membership relation $\in$[1]. The theory driving the translation is a very weak, (first-order) finitely axiomatizable set theory called Ω [11, 12]. The axioms, in the language with relational symbols $\in$ and $\subseteq$, and functional symbols $\cup, \setminus$, and Pow, are the following:

$$\begin{aligned} x \in y \cup z &\leftrightarrow x \in y \vee x \in z; \\ x \in y \setminus z &\leftrightarrow x \in y \wedge x \notin z; \\ x \subseteq y &\leftrightarrow \forall z(z \in x \rightarrow z \in y); \\ x \in Pow(y) &\leftrightarrow x \subseteq y. \end{aligned}$$

Notice that neither the extensionality axiom nor the axiom of foundation are in Ω. Given a modal formula $\phi(P_1, ..., P_n)$, we define its translation as the *set-theoretic term* $\phi^*(x, x_1, ..., x_n)$, with variables $x, x_1, ..., x_n$, built using $\cup, \setminus$, and Pow. Intuitively, the term $\phi^*(x, x_1, ..., x_n)$ represents the set of those worlds (in the frame x) in which the formula ϕ holds. The inductive definition of $\phi^*(x, x_1, ..., x_n)$ is the following:

- $P_i^* = x_i$;
- $(\phi \vee \psi)^* = \phi^* \cup \psi^*$;
- $(\neg\phi)^* = x \setminus \phi^*$;
- $(\Box\phi)^* = Pow(\phi^*)$.

For all modal formulae ϕ, ψ, the following results hold, showing the completeness and the soundness of the translation, respectively [11]:

$$\phi \vdash_{K_s} \psi \;\Rightarrow\; \Omega \vdash \forall x(Trans(x) \wedge \forall \vec{z}(x \subseteq \phi^*(x, \vec{z})) \rightarrow \forall \vec{z}(x \subseteq \psi^*(x, \vec{z}))),$$

$$\Omega \vdash \forall x(Trans(x) \wedge \forall \vec{z}(x \subseteq \phi^*(x, \vec{z})) \rightarrow \forall \vec{z}(x \subseteq \psi^*(x, \vec{z}))) \;\Rightarrow\; \phi \models_f \psi,$$

where $Trans(x)$ stands for $\forall y\ (y \in x \rightarrow y \subseteq x)$ and $\vec{z}$ stands for $z_1 \ldots z_n$. It is worth noting that, for frame-complete theories, the above translation captures exactly the notion of K_s-derivability.

In [11], we also showed how to generalize the set-theoretic translation for modal logic to polymodal logic, using the membership relation $\in$ to mimic the work of some finite collection of accessibility relations.

[1] It is worth mentioning that a similar idea has been recently exploited by Barwise and Moss to "model" non-standard set theories (e.g., non-well-founded set theory) as infinitary modal logics [3].

2.4 A Comparison of the two methods

The set-theoretic translation method and the standard one are compared in [7], where we completely cleared up the relationship between the two methods, and we managed to extend the adequacy result to all modal logics (frame-complete or not) under a slightly modified translation, using a standard model-theoretic proof not involving any non-well-founded set theory. More precisely, in [11], using a set-theoretic translation method, we proved that derivability in the minimal modal logic K_s corresponds precisely to derivability in Ω. In [7], this approach was shown equivalent to working with standard first-order translations of modal formulae in a theory of general frames.

Standard modal deduction via the standard translation is supported by a simple two-sorted first-order theory μ of general frames. Consider a two-sorted first-order language, with *worlds* (denoted by small letters) and *sets* (denoted by capital letters), with binary predicates R (on worlds) and $\in$ (between worlds and sets), as well as operations $-, \cup$ and $\Box$ on the set sort. This language is adequate for the description of the general frame semantics. The minimal logic describing it is obtained by considering the following theory μ:

- First-Order Principles for both sorts;
- $\forall \mathtt{P} \forall w (w \in -\mathtt{P} \leftrightarrow \neg w \in \mathtt{P})$;
- $\forall \mathtt{PQ} \forall w (w \in \mathtt{P} \cup \mathtt{Q} \leftrightarrow w \in \mathtt{P} \vee w \in \mathtt{Q})$;
- $\forall \mathtt{P} \forall w (w \in \Box \mathtt{P} \leftrightarrow \forall v (wRv \rightarrow v \in \mathtt{P}))$
- $\forall \mathtt{PQ} (\forall w (w \in \mathtt{P} \leftrightarrow w \in \mathtt{Q}) \rightarrow \mathtt{P} = \mathtt{Q})$ (extensionality).

Comparing μ with Ω, we established an effective equivalence between the latter and a suitable extension of μ, that we call μ^+. Such an extension is obtained adding to μ an axiom reflecting the "uniformity" of our set-theoretic approach which uses just $\in$ for both set-membership and the accessibility relation. Formally, μ^+ is the theory obtained by adding to μ the axiom

- $\forall w \exists \mathtt{P} \forall v \ (v \in \mathtt{P} \leftrightarrow wRv)$

that links R and $\in$, guaranteeing that, for each world w, there exists the set of its R-successors.

In [7], we showed that the standard translation method with respect to μ^+ and the set-theoretic one with respect to Ω are equivalent. More precisely, we proved that, for all modal formulae ϕ, ψ,

$$\mu^+ \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi) \Longleftrightarrow \Omega \vdash \forall x \ (Trans(x) \wedge \forall \vec{z}(x \subseteq \phi^*(x, \vec{z})) \rightarrow \forall \vec{z}(x \subseteq \psi^*(x, \vec{z}))).$$

We then proved that μ^+ is non-conservative over μ. On the ground of the equivalence between Ω and μ^+, this allows us to conclude that the $\Box$-as-Pow translation method given in [11] is not adequate for logics which are not frame complete. By varying our translation, however, we managed to obtain a full equivalence between general modal deduction in K_s and Ω-deduction [7]. This is the desired general result, which is based on standard set-theoretic methods, including a suitable adaptation of Fraenkel-Mostowski permutations of the universe [15].

3 Capturing L_2-derivability in Constructible Universes

In this section, we present the main result of this paper, namely, the generalization of the set-theoretic translation to weak monadic second-order logic.

The section is divided in three parts: we first introduce the axioms of the underlying theory we will consider, as well as the adaptation of the basic translation method to the present context; then, we prove the soundness and completeness of the method introducing the notions of *generated frame* and *generated model*; finally, we show how the revised translation allows us to deal with the full monadic L_2-language.

3.1 The axiomatic theory and the translation

We will work with a set theory whose axioms are closely related to the so-called Gödel operations for defining the constructible universe. Among all possible approaches, we choose a weaker version of the one proposed by Barwise in [2] with the aim of stating our results for a theory as weak as possible. We introduce functions defining the singleton operator, suitable cartesian products ($\times, \times_=, \times_\in$), together with the associated projections (Dom, Rng), plus some operations allowing us to manipulate argument positions in ordered sequences (C_1, C_2).

Let us call Ω_c the resulting theory. Its language consists of $=, \in$, and $\subseteq$ as predicate symbols, $\{\}$, Dom, and Rng as unary functional symbols, and $\cup, \setminus, \times, \times_\in, \times_=, C_1$, and C_2 as binary functional symbols. The axioms for Ω_c are the identity axioms and the axioms, already in Ω, describing $\subseteq, \cup$, and $\setminus$ in terms of $\in$ (cf. Section 2.3), plus the axioms defining $\{\}, \times, \times_\in, \times_=, Dom, Rng, C_1$, and C_2, listed below:

$$
\begin{aligned}
t \in \{x\} &\leftrightarrow t = x;\\
t \in x \times y &\leftrightarrow \exists a \in x \exists b \in y\ (t = \langle a, b\rangle);\\
t \in x \times_\in y &\leftrightarrow \exists a \in x \exists b \in y\ (t = \langle a, b\rangle \wedge a \in b);\\
t \in x \times_= y &\leftrightarrow \exists a \in x \exists b \in y\ (t = \langle a, b\rangle \wedge a = b));\\
t \in Dom(x) &\leftrightarrow \exists s\ (\langle t, s\rangle \in x);\\
t \in Rng(x) &\leftrightarrow \exists s\ (\langle s, t\rangle \in x);\\
t \in C_1(x,y) &\leftrightarrow \exists a \exists b \exists c\ (\langle a, b\rangle \in x \wedge c \in y \wedge t = \langle a, \langle b, c\rangle\rangle);\\
t \in C_2(x,y) &\leftrightarrow \exists a \exists b \exists c\ (\langle a, c\rangle \in x \wedge b \in y \wedge t = \langle a, \langle b, c\rangle\rangle),
\end{aligned}
$$

where $\langle x, y\rangle$ is the usual a shorthand for $\{\{x\}\} \cup \{\{x\} \cup \{y\}\}$. Inductively, we denote by $\langle x_1, \dots, x_n\rangle$ the pair $\langle x_1, \langle x_2, \dots, x_n\rangle\rangle$.

The theory Ω_c differs from the classical approach to constructible sets (cf. [9]) mainly in the following points:

1. we do not introduce classes;
2. we do not introduce the axioms of choice, infinity, replacement, and powerset, since they are not needed;
3. we do not introduce any form of extensionality, since our aim is to use the membership relation to mimic the accessibility relation in frames;
4. the existence of singletons replaces the existence of pairs, and binary union replaces (the stronger) unary union.

The translation function $(\cdot)^*$ is the same as that given in Section 2.3, except for the $\Box$ operator. This is obviously the case, because we do not have Pow among the symbols in the language of Ω_c. Moreover, $Trans(x)$ disappears from the antecedent of the translated sentence.

To understand these changes, it can be easily checked that, in the case of Ω, it would have made no difference to work with $(\Box\phi)^*$ defined either as $Pow(\phi^*)$ or as $Pow(\phi^*) \cap x$ (cf. Section 2.3). Actually, we chose the first alternative only to maintain the translated terms simpler. It is also easy to see that $Pow(\phi^*) \cap x = \{y \in x : y \cap x \subseteq \phi^*\}$, whenever x is transitive; as a matter of fact, we will see that the set $\{y \in x : y \cap x \subseteq \phi^*\}$ can always be used to translate $\Box\phi$ (even in the case in which x is not transitive). Finally, since the following holds:

$$\{y \in x : y \cap x \subseteq \phi^*\} = x \setminus Rng((x \setminus \phi^*) \times_\in x),$$

we put:

$$(\Box\phi)^* = x \setminus Rng((x \setminus \phi^*) \times_\in x).$$

In the next two sections, we will prove the soundness and completeness of the translation by showing that for any pair of modal formulae ϕ, ψ,

$$L_2 \vdash \overline{ST}(\phi) \to \overline{ST}(\psi) \Leftrightarrow \Omega_c \vdash \forall x(\forall \vec{z}\,(x \subseteq \phi^*(x, \vec{z})) \to \forall \vec{z}\,(x \subseteq \psi^*(x, \vec{z}))).$$

3.2 Completeness of the translation

We start proving the completeness of the translation, that is,

$$L_2 \vdash \overline{ST}(\phi) \to \overline{ST}(\psi) \Rightarrow \Omega_c \vdash \forall x(\forall \vec{z}\,(x \subseteq \phi^*(x, \vec{z})) \to \forall \vec{z}\,(x \subseteq \psi^*(x, \vec{z}))).$$

Given a model of Ω_c and an element in the model, we preliminary construct a general frame, and show that it is closed under L_0-definitions. To this end, it is useful to introduce the notion of *generated general frame*.

DEFINITION 3.1 *Let $(U, \in^U, \ldots)$ be a model for Ω_c, and let x_0 be an element of U. The* general frame generated *by x_0 in U is the triplet $(W, R, \mathcal{W})$ where:*

- $W = \{w \in U : w \in^U x_0\}$,
- *for all $w, w' \in W$, $wRw' \Leftrightarrow w' \in^U w$,*
- $\mathcal{W} = \{\{w \in U : w \in^U y\} : y \in U \wedge y \subseteq^U x_0\}$.

In order to show that a generated general frame is a general frame closed under L_0-definitions, we first prove that for any L_0-formula $\alpha(x_1, \ldots, x_n)$, the set of n-tuples of worlds defined by it in $(W, R, \mathcal{W})$ can be obtained in U by means of the set-theoretic operations of Ω_c.

In the following, given $y \in U$ such that $y \subseteq^U x_0$, we will identify it with the element $\{w \in U : w \in^U y\}$ of $\mathcal{W}$ in the generated general frame; in particular, we will identify x_0 with W. Hence, it makes sense to evaluate set-terms on elements of $\mathcal{W}$, as shown by the following lemma.

LEMMA 3.2 *For every L_0-formula $\gamma(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$, whose free variables are among $x_1, \ldots, x_n$ (world variables) and $Y_1, \ldots, Y_k$ (set variables), there exists a term $F_\gamma(X_1, \ldots, X_n, Y_1, \ldots, Y_k)$ in the language of Ω_c such that, for all $V_1, \ldots, V_n, W_1, \ldots, W_k$ in $\mathcal{W}$, and $u \in U$,*

$$U \models u \in F_\gamma(V_1, \ldots, V_n, W_1, \ldots, W_k)$$

if and only if
there exist $w_1 \in V_1, \ldots, w_n \in V_n$ such that

a) $U \models u = \langle w_n, w_{n-1} \ldots, w_1\rangle$,
b) $(W, R) \models \gamma(w_1, \ldots, w_n, W_1, \ldots, W_k)$.

The proof of Lemma 3.2 is given in the Appendix.

On the ground of Lemma 3.2, we can now prove that any generated general frame is indeed a general frame closed under L_0-definitions.

LEMMA 3.3 *Let $(U, \in^U, \ldots)$ be a model for Ω_c, and let x_0 be an element of U. The general frame $(W, R, \mathcal{W})$ generated by x_0 in U is a general frame closed under L_0-definitions.*

Proof. Let $\gamma(x_1,\ldots,x_n,Y_1,\ldots,Y_k)$ be an L_0-formula, $a_2,\ldots,a_n \in W$, and $W_1,\ldots,W_k \in \mathcal{W}$. We have to show that the set $\{a_1 \in W : (W,R) \models \gamma(a_1,a_2\ldots,a_n,W_1,\ldots,W_k)\}$ belongs to $\mathcal{W}$. Since the sets $\{a_2\},\ldots,\{a_n\}$ belong to $\mathcal{W}$, by applying Lemma 3.2 with $V_1 = W, V_2 = \{a_2\},\ldots,V_n = \{a_n\}$, we have that, for every $u \in U$,

$$U \models u \in F_\gamma(W,\{a_2\},\ldots,\{a_n\},W_1,\ldots,W_k)$$

if and only if

there exists $a_1 \in W$ such that

a) $U \models u = \langle a_n, a_{n-1}\ldots,a_1\rangle$,
b) $(W,R) \models \gamma(a_1,\ldots,a_n,W_1,\ldots,W_k)$.

It follows that, for every $a_1 \in U$,

$$U \models a_1 \in Rng^{n-1}(F_\gamma(W,\{a_2\},\ldots,\{a_n\},W_1,\ldots,W_k))$$

if and only if

$$a_1 \in W \text{ and } (W,R) \models \gamma(a_1,a_2\ldots,a_n,W_1,\ldots,W_k).$$

Hence, $Rng^{n-1}(F_\gamma(W,\{a_2\},\ldots,\{a_n\},W_1,\ldots,W_k)) \subseteq^U W$, and the set

$$\{a_1 \in W : (W,R) \models \gamma(a_1,a_2\ldots,a_n,W_1,\ldots,W_k)\}$$

is equal to the set $\{a_1 \in U : a_1 \in^U Rng^{n-1}(F_\gamma(W,\{a_2\},\ldots,\{a_n\},W_1,\ldots,W_k))\}$. By definition of generated general frame, it follows that the set

$$\{a_1 \in W : (W,R) \models \gamma(a_1,a_2\ldots,a_n,W_1,\ldots,W_k)\}$$

belongs to $\mathcal{W}$. Hence $(W,R,\mathcal{W})$ is closed under L_0-definability (we do not need to verify the closure under union, relative complementation, and $\Box$ because, if $W_1,W_2 \in \mathcal{W}$, then $W_1 \cup W_2$, $W \setminus W_1$, and $\Box(W_1)$ are characterized by L_0-definitions with parameters in $\{W_1,W_2\}$).

■

On the basis of the previous results, we can now prove the completeness of the translation.

THEOREM 3.4 (Completeness of the translation method) *For any pair of formulae ϕ,ψ,*

$$L_2 \vdash \overline{ST}(\phi) \to \overline{ST}(\psi) \Rightarrow \Omega_c \vdash \forall x(\forall \vec{z}\ (x \subseteq \phi^*(x,\vec{z})) \to \forall \vec{z}\ (x \subseteq \psi^*(x,\vec{z}))).$$

Proof. Let $(U,\in^U,\ldots)$ be a model for Ω_c, and let x_0 be an element of U. We already proved that the general frame $(W,R,\mathcal{W})$ generated by x_0 is closed under L_0-definitions. To prove the thesis, it suffices to show that a modal formula θ is valid in $(W,R,\mathcal{W})$ (equivalently, $(W,R,\mathcal{W}) \models \overline{ST}(\theta)$) if and only if $U \models \forall \vec{z}\ (x_0 \subseteq \theta^*(x_0,\vec{z}))$ (equivalently, $U \models \forall \vec{z}\ (W \subseteq \theta^*(W,\vec{z}))$, by the identification of x_0 with W).

We can basically proceed as in [7]. It is not difficult to prove, indeed, that for all $w \in W$ and $z_1,\ldots,z_n \in U$,

$$U \models w \in \theta^*(W,z_1,\ldots,z_n) \Leftrightarrow (W,R,\mathcal{W}) \models ST(\theta)(w,z_1\cap W,\ldots,z_n \cap W),$$

where, for $i = 1,\ldots,n$, $z_i \cap W$ stands for the element $\{v \in W : v \in^U z_i\}$ of $\mathcal{W}$. From this, the thesis easily follows. We just check the property for the $\Box$ operator, leaving the remaining cases to the reader:

$$U \models w \in (\Box\theta)^*(W,z_1,\ldots,z_n) \Leftrightarrow U \models w \in W \setminus Rng((W\setminus\theta^*)\times_\in W) \Leftrightarrow$$

$$U \models w \notin Rng((W\setminus\theta^*)\times_\in W) \Leftrightarrow U \models \forall s\ (\langle s,w\rangle \notin (W\setminus\theta^*)\times_\in W) \Leftrightarrow$$

$$U \models \forall s\ (s \in W \wedge s \in w \to s \in \theta^*) \Leftrightarrow \forall s \in W\ (wRs \to (W,R,\mathcal{W}) \models ST(\theta)(s,z_1\cap W,\ldots,z_n\cap W)) \Leftrightarrow$$

$$(W,R,\mathcal{W}) \models ST(\Box\theta)(w,z_1\cap W,\ldots,z_n\cap W).$$

■

3.3 Soundness of the translation

Let us prove now the soundness of the translation. We will first define a suitable notion of (Ω_c-)model *generated* by a general frame $(W, R, \mathcal{W})$ closed under L_0-definability; then, we will prove the following two claims:

- the generated (Ω_c-)model is actually a model of Ω_c, which contains W as an element;
- a modal formula θ is valid in the general frame $(W, R, \mathcal{W})$ if and only if $\forall\vec{z}(W \subseteq \theta^*(W, \vec{z}))$ holds in the generated Ω_c-model.

In the following, let $(W, R, \mathcal{W})$ denote a general frame closed under L_0-definitions. By definition, W is a set of the standard universe. Without loss of generality, we suppose that

1) all the elements of W are sets of same rank k,

2) there exists a set X_0 of rank k which is not in W,

3) no element in $W \cup \{X_0\}$ is the empty set, a singleton, a pair, or contains such elements.

The above three assumptions allow us to build the domain of our model in such a way that all its elements have rank greater than or equal to k. More precisely, at rank k we will only have the elements of W and the set X_0 playing the role of the empty-set. The remaining elements of the model will be obtained closing the set $(\mathcal{W} \setminus \{\emptyset\}) \cup \{X_0\}$ under the singleton and the following operations:

$\cup'$ and $\setminus'$, defined as follows:

$z \cup' r$	$r = \emptyset \vee r = X_0$	$r \neq \emptyset \wedge r \neq X_0$
$z = \emptyset \vee z = X_0$	X_0	r
$z \neq \emptyset \wedge z \neq X_0$	z	$z \cup r$

$z \setminus' r$	$r = \emptyset \vee r = X_0$	$r \neq \emptyset \wedge r \neq X_0$
$z = \emptyset \vee z = X_0$	X_0	X_0
$z \neq \emptyset \wedge z \neq X_0$	z	u

where $u = \begin{cases} z \setminus r & \text{if } z \setminus r \neq \emptyset \\ X_0 & \text{otherwise} \end{cases}$;

for $\circ \in \{\times, \times_=, C_1, C_2\}$, $z \circ' r = \begin{cases} z \circ r & \text{if } z \circ r \neq \emptyset \text{ and } z \neq X_0, r \neq X_0 \\ X_0 & \text{otherwise} \end{cases}$;

$$z \times'_{\in} r = \begin{cases} z \times_{\in,R} r & \text{if } z \times_{\in,R} r \neq \emptyset \text{ and } z \neq X_0, r \neq X_0 \\ X_0 & \text{otherwise} \end{cases},$$

where $z \times_{\in,R} r = \{\langle a, b\rangle : a \in z, b \in r, a \in b \vee bRa\}$;

$$Dom'(z) = \begin{cases} Dom(z) & \text{if } Dom(z) \neq \emptyset \text{ and } z \neq X_0 \\ X_0 & \text{otherwise} \end{cases};$$

$$Rng'(z) = \begin{cases} Rng(z) & \text{if } Rng(z) \neq \emptyset \text{ and } z \neq X_0 \\ X_0 & \text{otherwise} \end{cases}.$$

The model that we are defining can be seen as an Herbrand model built starting from the elements of the general frame $(W, R, \mathcal{W})$. Its domain U_c is defined as follows:

$\mathcal{W}_0 = (\mathcal{W} \setminus \{\emptyset\}) \bigcup \{X_0\};$

$\mathcal{W}_{n+1} = \{\{t_1\}, Dom'(t_1), Rng'(t_1), t_1 \circ' t_2 : \circ \in \{\cup, \setminus, \times, \times_{\in}, \times_{=}, C_1, C_2\}, t_1, t_2 \in \bigcup_{i=0}^{n} \mathcal{W}_i\};$

$\overline{\mathcal{W}} = \bigcup_{i=0}^{\omega} \mathcal{W}_i;$

$U_c = W \cup \overline{\mathcal{W}}.$

In the following (cf. Corollary 3.10), we will show that the sets W and $\overline{\mathcal{W}}$ are disjoint.

DEFINITION 3.5 *The* model generated *by $(W, R, \mathcal{W})$ is defined as follows.*

The domain of interpretation is U_c.

The interpretation $\in^{U_c}$ of the membership relation is:

$$x \in^{U_c} y \text{ iff } x \in F(y),$$

where

$$F(y) = \begin{cases} y & \text{if } y \in \overline{\mathcal{W}}, \\ \{w \in W : yRw\} = R(y) & \text{if } y \notin \overline{\mathcal{W}}, y \in W, R(y) \neq \emptyset, \\ X_0 & \text{if } y \notin \overline{\mathcal{W}}, y \in W, R(y) = \emptyset. \end{cases}$$

The interpretation of function symbols is:

$$\begin{aligned} \{z\}^{U_c} &= \{z\}, \\ z \circ^{U_c} r &= F(z) \circ' F(r), \text{ for } \circ \in \{\cup, \setminus, \times, \times_{=}, \times_{\in}, C_1, C_2\}, \\ Dom^{U_c}(z) &= Dom'(F(z)), \\ Rng^{U_c}(z) &= Rng'(F(z)). \end{aligned}$$

OBSERVATION 3.6 *Since $F(u) \in \overline{\mathcal{W}}$ for all $u \in U_c$, it can be easily seen that the range of the defined interpretation of the function symbols is contained in $\overline{\mathcal{W}}$. Moreover, $\emptyset \notin U_c$, and, from the inductive definition of $\overline{\mathcal{W}}$, it follows that $u \in \overline{\mathcal{W}}$ if and only if there exist a term $\tau(X_1, \ldots, X_n)$ and $A_1, \ldots, A_n \in \mathcal{W}_0$, such that $U_c \models u = \tau(A_1, \ldots, A_n)$. In the following, we denote by $\vec{A}$ the sequence $A_1, \ldots, A_n$, and $\vec{A} \in \mathcal{W}_0$ stands for $A_1, \ldots, A_n \in \mathcal{W}_0$.*

In the next definition, we will introduce a notion of *height* of a term that will play an important role in the rest of the proof.

DEFINITION 3.7 *The* height *h of a term is defined as follows:*

$h(x) = 1$ *if x is a variable,*

$h(t \cup s) = h(t \setminus s) = max(h(t), h(s)) + 1,$

$h(\{t\}) = h(Dom(t)) = h(Rng(t)) = h(t) + 1,$

$h(t \times s) = h(t \times_{\in} s) = h(t \times_{=} s) = max(h(t), h(s)) + 5,$

$h(C_1(t, s)) = h(C_2(t, s)) = max(h(t), h(s)) + 9.$

The height *of an element* $u \in \overline{\mathcal{W}}$ *is defined as follows:*

$$h(u) = min\{h(\tau) : \exists \vec{A} \in \mathcal{W}_0, U_c \models u = \tau(\vec{A})\}.$$

The following lemma shows that U_c is transitive outside $W \cup \{X_0\}$ and that the height decreases according to the standard membership relation.

LEMMA 3.8 *For all* $u \in \overline{\mathcal{W}} \setminus \{X_0\}$, *if* $a \in u$ *then* $a \in U_c$, *and either* $a \in W$ *or* $a \in \overline{\mathcal{W}}$ *and* $h(a) < h(u)$.

Proof. The proof is by induction on $h(u)$, starting from $h(u) = 1$, which implies $u \in \mathcal{W}_0 = (\mathcal{W} \setminus \{\emptyset\}) \cup \{X_0\}$. Since $u \neq X_0$, we have that $u \in \mathcal{W} \setminus \{\emptyset\}$ and $a \in u$ implies $a \in W$.

Suppose that the thesis is true for all $u' \in \overline{\mathcal{W}} \setminus \{X_0\}$ with $h(u') < h(u)$, and let $\tau(\vec{X})$ be a term such that $\exists \vec{A} \in \mathcal{W}_0$ with $U_c \models u = \tau(\vec{A})$ and $h(u) = h(\tau)$.

- If $\tau = \mu \cup \nu$, then $u = u_1 \cup^{U_c} u_2$, with $U_c \models u_1 = \mu(\vec{A})$, $U_c \models u_2 = \nu(\vec{A})$, and $u_1, u_2 \in \overline{\mathcal{W}}$. Moreover, both u_1, u_2 are different from X_0: if u_1 were equal to X_0, then $u = X_0 \cup^{U_c} u_2 = F(X_0) \cup' F(u_2) = X_0 \cup' u_2 = u_2$, and $h(u) = h(u_2) \leq h(\nu) < h(\tau))$ (the same argument shows that $u_2 \neq X_0$). Since $\emptyset \notin U_c$, u_1, u_2 are also different from $\emptyset$, and $u = F(u_1) \cup' F(u_2) = u_1 \cup' u_2 = u_1 \cup u_2$. Moreover, $h(\mu), h(\nu) < h(\tau) = h(u)$ and since $h(u_1) \leq h(\mu)$ and $h(u_2) \leq h(\nu)$, it follows that $h(u_1), h(u_2) < h(u)$. If $a \in u$, then $a \in u_1$ or $a \in u_2$ and the thesis follows by the inductive hypothesis.
- The case $\tau = \mu \setminus \nu$ is similar to the previous case, and it is left to the reader.
- If $\tau = \{\mu\}$, then $u = \{u'\}^{U_c} = \{u'\}$, with $U_c \models u' = \mu(\vec{A})$, $u' \in \overline{\mathcal{W}}$, and $h(u') < h(u)$. Then, $a \in u$ implies $a = u'$, and the thesis follows.
- If $\tau = Dom(\mu)$, then $u = Dom^{U_c}(u')$, with $U_c \models u' = \mu(\vec{A})$, $u' \in \overline{\mathcal{W}}$, and $h(u') < h(u)$. Since $u \neq X_0$ and $u = Dom^{U_c}(u') = Dom'(F(u'))$, it follows that $u' \neq X_0$ and $u = Dom(u')$. If $a \in u$, then there exists b such that $\langle a, b\rangle \in u'$. By induction, $\langle a, b\rangle \in U_c$ and since no element of $W \cup \{X_0\}$ is a pair, it follows that $\langle a, b\rangle \in \overline{\mathcal{W}} \setminus \{X_0\}$, and $h(\langle a, b\rangle) < h(u') < h(u)$. Then, we can apply the inductive hypothesis to $\langle a, b\rangle$. Since $\{a\} \in \langle a, b\rangle$, we have that $\{a\} \in U_c$, and since no element of $W \cup \{X_0\}$ is a singleton, we obtain $\{a\} \in \overline{\mathcal{W}} \setminus \{X_0\}$, and $h(\{a\}) < h(\langle a, b\rangle) < h(u)$. Another application of the inductive hypothesis to $\{a\}$ guarantees that $a \in U_c$ and either $a \in W$ or $h(a) < h(\{a\}) < h(u)$.
- The case $\tau = Rng(\mu)$ is similar to the previous case, and it is left to the reader.
- If $\tau = \mu \times \nu$, then $u = u_1 \times^{U_c} u_2$, with $u_1, u_2 \in \overline{\mathcal{W}}$, and $h(u_1), h(u_2) < h(u)$. Since $u \neq X_0$, we have that $u_1, u_2 \neq X_0$, and $u = u_1 \times^{U_c} u_2 = u_1 \times u_2$. If $a \in u$, there exist $b \in u_1$ and $c \in u_2$ such that $a = \langle b, c\rangle$. Moreover, $a = \langle b, c\rangle = \langle b, c\rangle^{U_c}$, because no element of $W \cup \{X_0\}$ is a singleton or a pair. Hence, it follows that $a \in \overline{\mathcal{W}}$, because, by induction, we have $b \in U_c$ and $c \in U_c$.

  To see that $h(a) < h(u)$, suppose that neither b nor c are in W. Then, $h(b) < h(u_1)$, $h(c) < h(u_2)$, and $h(a) \leq max(h(b), h(c)) + 4 < max(h(u_1), h(u_2)) + 4 < h(u)$. Suppose that b is in W, but c is not. Hence $\{b\}$ is in $\mathcal{W}_0$ and $h(\{b\}) = 1$. Therefore, $h(a) \leq h(c) + 4 < max(h(u_1), h(u_2)) + 4 < h(u)$. The remaining cases are similar.
- If $\tau = \mu \times_= \nu$ or $\tau = \mu \times_\in \nu$, we can proceed as in the previous case.
- If $\tau = C_1(\mu, \nu)$, then $u = C_1^{U_c}(u_1, u_2)$, with $u_1, u_2 \in \overline{\mathcal{W}}$, and $h(u_1), h(u_2) < h(u)$. Since $u \neq X_0$, we have that $u_1, u_2 \neq X_0$ and $u = C_1^{U_c}(u_1, u_2) = C_1(u_1, u_2)$. If $a \in u$, then there are b, c, and d such that $a = \langle b, \langle c, d\rangle\rangle$ and $\langle b, c\rangle \in u_1$, $d \in u_2$. From the inductive hypothesis and the fact that no element of $W \cup \{X_0\}$ is a pair, it follows that $\langle b, c\rangle \in \overline{\mathcal{W}} \setminus \{X_0\}$, $h(\langle b, c\rangle) < h(u_1)$, $d \in U_c$, and either $d \in W$

or $d \in \overline{\mathcal{W}}$ and $h(d) < h(u_2)$. Using the inductive hypothesis twice on $b \in \{b\} \in \langle b, c\rangle$ as in the case of Dom, we can prove that either b is in W or $b \in \overline{\mathcal{W}}$ and $h(b) < h(u_1)$, and the same is true for c. Suppose that b, c, and d are not in W. Thus, $h(b), h(c) < h(u_1)$ and $h(a) \leq max(h(b), h(c), h(d)) + 8 < max(h(u_1), h(u_2)) + 8 < h(u)$. The case in which one of b, c, and d is in W is proved as in the preceding case.

- The case of $\tau = C_2(\mu, \nu)$ is similar to the previous case, and thus it is left to the reader.

■

From the above lemma, using the hypothesis on the rank of the elements in W, it follows that any element in $\overline{\mathcal{W}} \setminus \{X_0\}$ has rank strictly greater than k:

LEMMA 3.9 *For all u in $\overline{\mathcal{W}} \setminus \{X_0\}$, $rk(u) > k$.*

Proof. By induction on the height of u. First of all we recall that U_c does not contain the empty set.

If $h(u) = 1$, then $u \in \mathcal{W}$ and $a \in u$ implies $a \in W$. The thesis follows from the fact that all elements of W have rank equal to k.

If $h(u) > 1$ and $a \in u$, then from Lemma 3.8 either $a \in W$ or $a \in \overline{\mathcal{W}}$ and $h(a) < h(u)$. If $a \in W$ or $a = X_0$, then $rk(a) = k$. Otherwise, $a \in \overline{\mathcal{W}} \setminus \{X_0\}$, and the thesis follows from the inductive hypothesis.

■

COROLLARY 3.10 *$U_c \models \forall x(x \notin X_0)$, and $W \cap \overline{\mathcal{W}} = \emptyset$.*

Proof. The proof is by contradiction. Since $F(X_0) = X_0$ and $rk(X_0) = k$, it follows that if $u \in U_c$ and $u \in^{U_c} X_0$, then $rk(u) < k$. Moreover, since any element of W has rank equal to k, u does not belong to W. Therefore, $u \in \overline{\mathcal{W}} \setminus \{X_0\}$, which, together with $rk(u) < k$, contradicts Lemma 3.9.

As for the second part of the corollary, since any element of W is different from X_0 and has rank equal to k, we have that $W \cap \overline{\mathcal{W}} = \emptyset$.

■

On the ground of the above results, we can prove the first of our initial claims.

LEMMA 3.11 *The model generated by $(W, R, \mathcal{W})$ is a model of Ω_c. Moreover, this model is extensional on elements of $\overline{\mathcal{W}}$, i.e. if $u, u' \in \overline{\mathcal{W}}$ are such that $a \in^{U_c} u \leftrightarrow a \in^{U_c} u'$ for all $a \in U_c$, then $u = u'$.*

Proof. In the following equivalences, we will exploit the fact that the range of an interpreted function is a subset of $\overline{\mathcal{W}}$ (Observation 3.6), and hence the function F is the identity on this range. Let a, z, and r be elements of U_c.

- $a \in^{U_c} \{z\}^{U_c}$ iff $a \in F(\{z\}^{U_c})$ iff $a \in \{z\}^{U_c}$ iff $a \in \{z\}$ iff $a = z$.
- $a \in^{U_c} z \cup^{U_c} r$ iff $a \in F(z \cup^{U_c} r)$ iff $a \in z \cup^{U_c} r$ iff $a \in F(z) \cup' F(r)$.

  If $F(z) \cup F(r) = \emptyset$, then $F(z) \cup' F(r) = X_0$; since no element of U_c belongs to X_0 (cf. Corollary 3.10), we have $a \in^{U_c} z \cup^{U_c} r$ iff $a \in^{U_c} z$ or $a \in^{U_c} r$.

  If $F(z) \cup F(r) \neq \emptyset$, and $F(z), F(r) \neq X_0$, then $a \in^{U_c} z \cup^{U_c} r$ iff $a \in F(z) \cup F(r)$ iff $a \in F(z)$ or $a \in F(r)$ iff $a \in^{U_c} z$ or $a \in^{U_c} r$.

  If $F(z) = X_0$ and $F(r) \neq \emptyset$, or vice versa, then $z = X_0$ and $F(z) \cup' F(r) = F(r)$. Hence $a \in^{U_c} z \cup^{U_c} r$ iff $a \in F(r)$ iff $a \in^{U_c} r$. On the other hand, Corollary 3.10 says that $a \notin^{U_c} z$, and the thesis follows.

  The remaining cases are left to the reader.

- The case of $z \setminus^{U_c} r$ is similar to the above one.
- $a \in^{U_c} z \times_{\in}^{U_c} r$ iff $a \in F(z \times_{\in}^{U_c} r)$ iff $a \in z \times_{\in}^{U_c} r$ iff $a \in F(z) \times'_{\in} F(r)$.

  If $F(z) \times_{\in,R} F(r) = \emptyset$ or $F(z) = X_0$ or $F(r) = X_0$, then there is no pair $\langle b, c\rangle$ of elements of U_c such that $b \in^{U_c} z, c \in^{U_c} r$ and $b \in^{U_c} c$. On the other hand, no element of U_c belongs to $X_0 = z \times_{\in}^{U_c} r$, and the axiom is verified.

  If $F(z) \times_{\in,R} F(r) \neq \emptyset$, $F(z) \neq X_0$, and $F(r) \neq X_0$, then $a \in^{U_c} z \times_{\in}^{U_c} r$ iff $a \in F(z) \times_{\in,R} F(r)$ iff $\exists b, c\ (a = \langle b, c\rangle \wedge b \in F(z) \wedge c \in F(r) \wedge (b \in c \vee cRb))$ iff (use Lemma 3.8) $\exists b, c \in U_c\ (a = \langle b, c\rangle^{U_c} \wedge b \in^{U_c} z \wedge c \in^{U_c} r \wedge b \in^{U_c} c)$.
- The cases of the axioms for $\times, \times_{=}, C_1, C_2$ are similar.
- $a \in^{U_c} Dom^{U_c}(z)$ iff $a \in F(Dom^{U_c}(z))$ iff $a \in Dom^{U_c}(z)$ iff $a \in Dom'(F(z))$.

  If $Dom(F(z)) = \emptyset$ or $F(z) = X_0$, then there is no pair $\langle a, b\rangle$ in $F(z)$ as well as $a \notin X_0 = Dom'(F(z))$, and the axiom is verified.

  If $Dom(F(z)) \neq \emptyset$ and $F(z) \neq X_0$, we have $a \in^{U_c} Dom^{U_c}(z)$ iff $a \in Dom(F(z))$ iff $\exists b\ (\langle a, b\rangle \in F(z))$ iff (use Lemma 3.8) $\exists b \in U_c\ (\langle a, b\rangle^{U_c} \in F(z))$ iff $\exists b \in U_c\ (\langle a, b\rangle^{U_c} \in^{U_c} z)$.
- In a similar way it is possible to verify the axiom for Rng.

Finally, to see that extensionality holds, suppose we have $u, u' \in \overline{\mathcal{W}}$ such that for all $a \in U_c$ it holds $a \in^{U_c} u$ iff $a \in^{U_c} u'$. Then, either $u = u' = X_0$, or, for example, $u \neq X_0$, u is not empty, and, by Lemma 3.8, if $d \in u$, then $d \in U_c$. Hence $u \neq X_0$ implies $u' \neq X_0$ and $u = u'$.

■

Let us now extend the definition of height to the elements of W by putting $h(w) = 0$ for all $w \in W$. In the following lemma, we show that we can depict the elements of any element $\tau(\vec{A})$ of $\overline{\mathcal{W}}$ by using smaller terms and Δ_0-formulae. Using the closure of $\overline{\mathcal{W}}$ under L_0-definitions, we will then prove that with elements in U_c we do not get more subsets of W that the ones already in $\overline{\mathcal{W}}$ (cf. Lemma 3.13).

LEMMA 3.12 *For any set term $\tau(\vec{y})$ there are:*

1. *terms $\tau_1(\vec{x}, \vec{y}), \ldots, \tau_{k_\tau}(\vec{x}, \vec{y},)$, such that $h(\tau_i) \leq h(\tau)$ and if $h(\tau) > 1$, then $h(\tau_i) < h(\tau)$,*
2. *Δ_0-formulae in the language $\{=, \in\}$, $\alpha_1^\tau(\vec{x}, \vec{y}), \ldots, \alpha_{k_\tau}^\tau(\vec{x}, \vec{y})$, with possibly W as parameter,*

such that, for all $\vec{A} \in \mathcal{W}_0 \cup W$, and for all $u \in U_c$,

$$U_c \models u \in \tau(\vec{A}) \leftrightarrow \bigvee_{i=1}^{k_\tau} \exists \vec{w} \in W\ (u = \tau_i(\vec{w}, \vec{A}) \wedge \alpha_i^\tau(\vec{w}, \vec{A})).$$

The proof of Lemma 3.12 is given in the Appendix.

LEMMA 3.13 *For all $u \in U_c$, $F(u) \cap W \in \mathcal{W}$.*

Proof. Consider first the case in which $u \in W \cup \{X_0\}$; in this case $F(u) \cap W = R(u)$ or $F(u) \cap W = X_0 \cap W = \emptyset$. In both cases, $F(u) \cap W \in \mathcal{W}$, since $\mathcal{W}$ is closed under L_0-definitions. If $u \in \overline{\mathcal{W}} \setminus X_0$, then the element $u' = u \setminus' (u \setminus' W)$ is in $\overline{\mathcal{W}}$. Moreover, for all $a \in U_c$, it holds that $a \in^{U_c} u'$ if and only if $a \in F(u) \cap W$, and we can consider u' instead of $F(u) \cap W$ (recall that $x \cap y = x \setminus (x \setminus y)$). Since $u' \in \overline{\mathcal{W}}$, there exists $\vec{A}$ in $\mathcal{W}_0$ such that $U_c \models u' = \tau(\vec{A})$, where τ is a term in the language of Ω_c. From Lemma 3.12, it follows that there exist

- terms $\tau_1(\vec{x},\vec{Y}),\ldots,\tau_k(\vec{x},\vec{Y})$,
- Δ_0-formulae $\alpha_1^\tau(\vec{x},\vec{Y}),\ldots,\alpha_k^\tau(\vec{x},\vec{Y})$

such that for all $a \in U_c$,

$$U_c \models a \in \tau(\vec{A}) \leftrightarrow \bigvee_{i=1}^{k_\tau} \exists \vec{w} \in W \left(a = \tau_i(\vec{w},\vec{A}) \wedge \alpha_i^\tau(\vec{w},\vec{A})\right).$$

Notice that if $h(\tau_i) > 1$, then, for any $\vec{w}$ in W, to build the element $\tau_i(\vec{w},\vec{A})$ we apply at least once a function of the model, and $\tau_i(\vec{w},\vec{A}) \notin W$. Since all the elements of $\tau(\vec{A})$ are in W and $\overline{W} \cap W = \emptyset$, by Corollary 3.10, we must have $h(\tau_i) = 1$ or $U_c \models \neg\alpha_i^\tau(\vec{w},\vec{A})$, for all $\vec{w}$. Moreover, if $\tau_i = y_j$, then $\tau_i(A_1,\ldots A_n) = A_j \notin W$. Thus, if the variables $\vec{x}$ used in the terms $\tau_1,\ldots,\tau_k$ are $x_1,\ldots,x_m$, then

$$U_c \models a \in \tau(\vec{A}) \leftrightarrow \bigvee_{\substack{1 \le i \le k_\tau \\ \tau_i \in \{x_1,\ldots,x_m\}}} \exists \vec{w} \in W \left(a = \tau_i(\vec{w},\vec{A}) \wedge \alpha_i^\tau(\vec{w},\vec{A})\right).$$

Notice now that the validity of the right-hand side formula above is equivalent to the validity in (W,R) of a corresponding L_0-formula (just read the atomic subformulae of type $x_i \in x_j$ in α_i^τ as x_jRx_i, and the parameter X_0 as the empty set). Since $\mathcal{W}$ is closed under L_0-definitions, this implies that the set $\{a \in U_c : a \in^{U_c} u'\} = F(u) \cap W$ belongs to $\mathcal{W}$.

■

On the basis of the previous results we can now prove the soundness of the translation.

THEOREM 3.14

$$\Omega_c \vdash \forall x(\forall \vec{z}\ (x \subseteq \phi^*(x,\vec{z})) \rightarrow \forall \vec{z}\ (x \subseteq \psi^*(x,\vec{z}))) \Rightarrow L_2 \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi).$$

Proof. Consider a general frame $(W,R,\mathcal{W})$ closed under L_0-definitions, and the generated general model of Definition 3.5. We only need to show that a modal formula θ is valid in $(W,R,\mathcal{W})$ if and only if $U_c \models \forall \vec{z}\ (W \subseteq \theta^*(W,\vec{z}))$. This follows from Lemma 3.13. Indeed, for all $w \in W$, $u_1,\ldots,u_n \in U_c$, we can prove by induction on θ that

$$(\star) \qquad U_c \models w \in \theta^*(W,u_1,\ldots,u_n) \Leftrightarrow (W,R,\mathcal{W}) \models ST(\theta)(w,F(u_1)\cap W,\ldots,F(u_n)\cap W).$$

We prove $(\star)$ only for the cases $\theta = P_i$ and $\theta = \Box\psi$, leaving the remaining cases to the reader.

- If $\theta = P_i$ then $U_c \models w \in \theta^*(W,u_1,\ldots,u_n)$ iff $U_c \models w \in u_i$ iff $w \in^{U_c} u_i$ iff $w \in F(u_i)$ iff $w \in F(u_i) \cap W$ iff $(W,R,\mathcal{W}) \models \mathtt{P_i(x)}(w,F(u_i)\cap W)$ iff $(W,R,\mathcal{W}) \models ST(P_i)(w,F(u_i)\cap W)$ iff $(W,R,\mathcal{W}) \models ST(\theta)(w,F(u_i)\cap W)$.
- If $\theta = \Box\psi$, then $U_c \models w \in \theta^*(W,u_1,\ldots,u_n)$ iff $U_c \models w \in W \setminus Rng((W\setminus\psi^*)\times_\in W)$ iff $U_c \models \forall u \langle u,w\rangle \notin (W\setminus\psi^*)\times_\in W$ iff $U_c \models \forall u(u \in W \wedge u \in w \rightarrow u \in \psi^*)$ iff $(W,R,\mathcal{W}) \models \forall u(wRu \rightarrow ST(\psi)(u,F(u_1)\cap W,\ldots,F(u_n)\cap W)$ iff $(W,R,\mathcal{W}) \models ST(\Box\psi)(u,F(u_1)\cap W,\ldots,F(u_n)\cap W)$ iff $(W,R,\mathcal{W}) \models ST(\theta)(u,F(u_1)\cap W,\ldots,F(u_n)\cap W)$.

Since by Lemma 3.13 we have $F(u)\cap W \in \mathcal{W}$, for all $u \in U_c$, it easily follows that

$$U_c \models \forall \vec{z}\ (W \subseteq \theta^*(W,\vec{z})) \Leftrightarrow (W,R,\mathcal{W}) \models \overline{ST}(\theta).$$

■

3.4 Full monadic L_2

Let us consider now the case of the full monadic L_2-language. The translation proposed in Section 3.1 encoded L_2-derivability of formulae of the form $\overline{ST}(\phi) \to \overline{ST}(\psi)$ into derivability in Ω_c. It is not difficult to see, however, that the proofs of Theorems 3.4 and 3.14 have a wider scope and allow us to deal with the full monadic second-order language of L_2.

Let $\phi(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ be a generic L_2-formula. Consider the translation τ, introducing in some case a new free variable x, which is inductively defined as follows:

- $\tau(x_i R x_j) = x_j \in x_i$;
- $\tau(x_i = x_j) = \ x_i = x_j$;
- $\tau(x_i \in Y_j) = x_i \in y_j$;
- $\tau(\alpha \vee \beta) = \tau(\alpha) \vee \tau(\beta)$;
- $\tau(\neg\alpha) = \neg\tau(\alpha)$;
- $\tau(\forall x_i \alpha) = \forall x_i\ (x_i \in x \to \tau(\alpha))$;
- $\tau(\forall Y_i \alpha) = \forall y_i \tau(\alpha)$.

It is possible to prove the following theorem:

THEOREM 3.15 *For every L_2-sentence ϕ, the following holds:*

$$L_2 \vdash \phi \Leftrightarrow \Omega_c \vdash \forall x\ \tau(\phi)(x).$$

The completeness and soundness of the translation can be proved as in Sections 3.2 and 3.3, respectively. Consider, for example, the case of completeness: given an Ω_c-model U and an element x_0 of U, we first build the generated general frame of Definition 3.1; then, we prove that, for each L_2-formula $\phi(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$, $w_1, \ldots, w_n \in W$, and $y_1, \ldots y_k \in U$,

$$U \models \tau(\phi)(W, w_1, \ldots, w_n, y_1, \ldots, y_k) \ \text{ iff } \ (W, R, \mathcal{W}) \models \phi(w_1, \ldots, w_n, y_1 \cap W, \ldots, y_k \cap W).$$

The same kind of correspondence holds for the proof of soundness. Moreover, it is possible to show that τ is a *generalization* of the translation given in Section 3.1: if ψ is the formula $\overline{ST}(\phi)$, where $\phi(P_1, \ldots, P_n)$ is a modal formula, then it holds that: $\Omega_c \vdash \forall x\ (\tau(\phi)(x) \leftrightarrow \forall \vec{z}\ (x \subseteq \phi^*(x, \vec{z}))$.

4 A set-theoretic translation for extended modal logics

In this section we will consider an important family of contexts to which one can extend the field of applicability of the set-theoretic translation method, namely, modal logics extended with new modal operators. We will focus on the cases of the modal logic of inequality [17], the so-called irreflexivity rule [13], and minimal tense logic [10]. However, many other cases can be treated analogously and we will conclude the section with some comments on a general theory relative to such a technique.

In [8], we have shown that there exists a straightforward solution to the problem of generalizing the set-theoretic translation to any (reasonable) extension of modal logic: for each new operator of the extended modal logic under consideration, add a new set-theoretic term, and extend Ω with an axiom governing its behavior. In the following, we will present an alternative, more systematic approach to the problem of translating extended modal logics into (suitable) set theories, based on the results proved in the previous section.

4.1 Modal logic of inequality

Consider the difference operator D, whose Kripke-frame semantics can be expressed as follows:

$$w \Vdash D\varphi \quad \Leftrightarrow \quad \exists w'(w' \neq w \ \wedge \ w' \Vdash \varphi).$$

The introduction of the D operator allows one to develop a modal logic of inequality and has been extensively studied. Our reference for technical aspects relative to the modal logic of D is [18], where the interested reader can find more details and proofs of the modal results mentioned in this section.

An axiomatization of the basic modal logic of inequality DL is given by the axioms and inference rules of propositional logic plus the axioms:

A_1 $\bar{D}(p \rightarrow q) \rightarrow (\bar{D}p \rightarrow \bar{D}q)$ (normality),

A_2 $p \rightarrow \bar{D}Dp$ (symmetry),

A_3 $DDp \rightarrow (p \vee Dp)$ (pseudotransitivity),

and the rules of inference:

R_1 $\vdash \varphi \ \Rightarrow \ \vdash \bar{D}\varphi$,

R_2 $\vdash (p \wedge \bar{D}\neg p) \rightarrow \phi$ and p is not in $\varphi \ \Rightarrow \ \vdash \varphi$,

where $\bar{D}$ is the dual of D, that is, $\bar{D} = \neg D \neg$.

Let $\mathcal{L}(D)$ be the language containing D as the only modal operator and let $\varphi \in \mathcal{L}(D)$. The logic DL is complete with respect to $\mathcal{L}(D)$ as well as to all its extensions (even though this is not the case if R_2 is eliminated). Also the logic $DL_m = K + DL + (p \wedge \bar{D}p \rightarrow \Box p)$ in the language $\mathcal{L}(\Box, D)$ is complete and will be considered as basic hereafter.

Our aim is to introduce a term (using Gödel constructible functions) which will allow us to extend the set-theoretic translation to the D operator. The term corresponding to D must guarantee the following fact:

$$(D\varphi)^{*,D} \quad = \quad \begin{cases} \varphi^{*,D} & \text{if } \varphi^{*,D} = \emptyset, \\ x \setminus \varphi^{*,D} & \text{if } x \cap \varphi^{*,D} = \{w\}, \\ x & \text{otherwise.} \end{cases}$$

The following definition is immediately seen to satisfy our requirements:

$$(D\varphi)^{*,D} \quad = \quad Dom(x \times_{\neq} \varphi^{*,D}),$$

where $A \times_{\neq} B = \{\langle a, b\rangle \mid a \in A \wedge b \in B \wedge a \neq b\}$ can be defined in terms of our original operators as $A \times B \setminus A \times_{=} B$.

The following theorem shows that the translation $(.)^{*,D}$ is sound and complete with respect to Ω_c.

THEOREM 4.1 *Let φ and ψ be formulae in the language $\mathcal{L}(\Box, D)$,*

(completeness) $\varphi \vdash_{DL_m} \psi \ \Rightarrow \ \Omega_c \vdash \forall x(\forall \vec{y}(x \subseteq \varphi^{*,D}(x, \vec{y})) \rightarrow \forall \vec{z}(x \subseteq \psi^{*,D}(x, \vec{z}))),$

(soundness) $\Omega_c \vdash \forall x(\forall \vec{y}(x \subseteq \varphi^{*,D}(x, \vec{y})) \rightarrow \forall \vec{z}(x \subseteq \psi^{*,D}(x, \vec{z}))) \ \Rightarrow \ \varphi \models_f \psi.$

Proof. The completeness of the translation can be proved by induction on the length of derivations in DL_m. For each axiom, we need to show that Ω_c proves the corresponding translated sentence. For example, for A_2 we need to show that:

$$\Omega_c \quad \vdash \quad \forall x, x_1 \left(x \subseteq (x \setminus x_1) \cup x \setminus Dom\left(x \times_{\neq} (x \setminus Dom(x \times_{\neq} x_1))\right)\right).$$

This fact essentially follows from the correctness of axiom A_2 with respect to the semantics of D and from the fact that $(.)^{*,D}$ correctly models such a semantics. More specifically:

$$x \subseteq (x \setminus x_1) \cup x \setminus Dom\left(x \times_{\neq} (x \setminus Dom(x \times_{\neq} x_1))\right) \quad \Leftrightarrow \quad (x \cap x_1) \subseteq x \setminus Dom\left(x \times_{\neq} (x \setminus Dom(x \times_{\neq} x_1))\right).$$

If $x_1 = \emptyset$, the result is obvious. If $x_1 = \{w\}$, then $x \setminus Dom\left(x \times_{\neq} (x \setminus Dom(x \times_{\neq} x_1))\right) = x_1$ and hence, again, the inclusion holds. In all the other cases, $Dom(x \times_{\neq} x_1) = x$ and $(x \cap x_1) \subseteq x \setminus Dom(x \times_{\neq} \emptyset) = x$.

The verification of all the remaining axioms is left to the reader.

For R_1, assuming that $\Omega_c \vdash \forall x, \vec{y}(x \subseteq \varphi^{*,D}(x, \vec{y}))$, we have that $\Omega_c \vdash \forall x, \vec{y}(x \setminus \varphi^{*,D}(x, \vec{y}) = \emptyset)$, and hence

$$\Omega_c \quad \vdash \quad \forall x, \vec{y}\left(x \subseteq x \setminus Dom\left(x \times_{\neq} \left(x \setminus \varphi^{*,D}(x, \vec{y})\right)\right)\right).$$

For R_2 assume $\Omega_c \vdash \forall x, \vec{y}, z(x \subseteq (x \setminus z) \cup Dom(x \times_{\neq} z) \cup \varphi^{*,D}(x, \vec{y}))$. It is easy to see that $(x \setminus z) \cup Dom(x \times_{\neq} z)$ is equal to $x \setminus z$ when $x \cup z$ is a singleton, while it is equal to x otherwise. Ω_c proves the above sentence for all possible z, and z does not occur in $\varphi^{*,D}(x, \vec{y})$. Moreover, for any element $w \in x$, we can consider the case in which $z = \{w\}$, where we obtain that $x \subseteq (x \setminus \{w\}) \cup \varphi^{*,D}(x, \vec{y})$, namely, $w \in \varphi^{*,D}(x, \vec{y})$. Hence

$$\Omega_c \quad \vdash \quad \forall x, \vec{y}\left(x \subseteq \varphi^{*,D}(x, \vec{y})\right).$$

In order to show the soundness of the translation, let $\mathcal{F} = \langle W, R\rangle$ be a frame such that $\mathcal{F} \models \varphi$ and $\mathcal{F} \not\models \psi$. In the model of Ω_c generated by the general frame $\langle W, R, Pow(W)\rangle$ there exists an element that plays the role of $\mathcal{F}$ and hence

$$\Omega_c \quad \not\vdash \quad \forall x(\forall \vec{y}(x \subseteq \varphi^{*,D}(x, \vec{y})) \rightarrow \forall \vec{z}(x \subseteq \psi^{*,D}(x, \vec{z}))).$$

■

The reader can easily check that a weaker set theory for which Theorem 4.1 can be reformulated is[2]:

$$\Omega^D \quad = \quad \Omega + \{\} + \times_{\neq} + Dom.$$

Being able to treat DL_m has many interesting by-products coming from the fact that important notions, which cannot be expressed in $\mathcal{L}(\Box)$, become expressible in $\mathcal{L}(\Box, D)$. An example is that of irreflexivity, that corresponds to $\bar{D}p \rightarrow \Box p$.

4.2 The irreflexivity rule

An alternative approach to the treatment of irreflexivity in frames consists in adding the following *irreflexivity rule* (IRR)

$$\vdash (\neg p \wedge \Box p \rightarrow \varphi) \text{ with } p \notin \varphi \quad \Rightarrow \quad \vdash \varphi,$$

[2]It is worth noting that, since $\times_{\neq}$ is perfectly symmetric, Dom can possibly be replaced by Rng.

that allows us to capture logical consequence in irreflexive frames via modal deduction (cf. [13]).

The $\Box$-as-Pow translation characterizes exactly logical consequence in irreflexive frames as long as the underlying set theory is well-founded. As a matter of facts a very weak form of foundation is necessary, namely the requirement that no set belongs to itself. In the following theorem let $(.)^*$ denote the $\Box$-as-Pow translation relative to the language $\mathcal{L} = \{\cup, \setminus, Pow\}$ (cf. [11]).

THEOREM 4.2 *Let φ and ψ be formulae in the language $\mathcal{L}(\Box)$ and let $\models_f^{irr}$ denote logical consequence in irreflexive frames.*

(completeness) $\varphi \vdash_{K+IRR} \psi \;\Rightarrow\; \Omega + \forall y(y \notin y) \vdash \forall x(Trans(x) \wedge \forall \vec{y}(x \subseteq \varphi^*(x, \vec{y})) \rightarrow \forall \vec{z}(x \subseteq \psi^*(x, \vec{z}))),$

(correctness) $\Omega + \forall y(y \notin y) \vdash \forall x(Trans(x) \wedge \forall \vec{y}(x \subseteq \varphi^*(x, \vec{y})) \rightarrow \forall \vec{z}(x \subseteq \psi^*(x, \vec{z}))) \;\Rightarrow \varphi \models_f^{irr} \psi.$

Notice that the theory $\Omega + \forall y(y \notin y)$ used in the above theorem does not contain any more comprehension than Ω (as a matter of fact, moving the axiom $\forall y(y \notin y)$ to the antecedent of the translation as $\forall y \in x(y \notin y)$, one could work with Ω). On the other hand, we have seen that the set theory in which one can carry out the translation for the minimal modal logic of inequality is $\Omega^D = \Omega + \{\} + \times_{\neq} + Dom$. This is not surprising, as we know that irreflexivity can be expressed using the D operator, and it also tells us that the comprehension we need to capture the notion of irreflexivity is as much as we need for the basic machinery of the translation and strictly less of what we need for D.

4.3 Minimal tense logic

As another example, we consider the case of minimal tense logic TL [10]. TL extends propositional logic with two one-place operators: F (future-tense) and P (past-tense). A sound and complete axiomatic system for TL consists of the usual axioms and rules of propositional logic plus the axioms:

$$G(p \rightarrow q) \rightarrow (Gp \rightarrow Gq) \qquad H(p \rightarrow q) \rightarrow (Hp \rightarrow Hq)$$
$$p \rightarrow GPp \qquad p \rightarrow HFp$$

where G and H stand for $\neg F \neg$ and $\neg P \neg$, respectively, and the rules of (temporal) necessitation: from ϕ to infer $G\phi$ and $H\phi$.

As shown in [19], it is possible to reduce frame validity in tense logic to that in modal logic. Even though somewhat complicated, the composition of such a reduction and our translation method for modal logic would allow us to encode minimal tense logic into Ω as it stands.

In [7], we devised a specific translation method for TL that slightly extends Ω, but keeps the translation simpler. In view of the one-way character of $\in$, the main issue to consider is the treatment of the *backward looking* operator H. To this end, we extended the language of Ω with a binary operator $Pow^{-y}(x)$, and added the following axiom (defining it) to Ω:

$$x \in Pow^{-y}(z) \quad \text{if and only if} \quad x \in y \wedge]x, y[\, \subseteq z$$

where $]x, y[\, \subseteq z$ is a shorthand for the formula $\forall t\ (x \in t \wedge t \in y \rightarrow t \in z)$. Let $\overline{\Omega}$ be the resulting theory.

The translation of propositional letters and boolean connectives is the usual one. The translation of G is the same as the translation of $\Box$ in basic modal logic. The translation of H can be defined in terms of the $Pow^{-y}(x)$ operator as follows:

$$(H\phi)^{*,TL} \;\;=\;\; Pow^{-W}(\phi^{*,TL}).$$

In [7], we proved that for any TL-formula $\phi(P_1, \ldots, P_n)$, it holds that:

$$\vdash_{TL} \phi \;\Leftrightarrow\; \overline{\Omega} \vdash \forall x(Trans(x) \rightarrow \forall x_1 \ldots x_n(x \subseteq \phi^{*,TL}(x, x_1, \ldots, x_n))).$$

As in the case of the logic of inequality, we show that it is possible to find two terms, built using Gödel constructible functions, which allow us to translate the modal operators G and H.

The following definitions are immediately seen to satisfy our requirements:

$$\begin{array}{rcl}(G\varphi)^{*,TL} & = & x \setminus Rng((x \setminus \varphi^{*,TL}) \times_{\in} x) \\ (H\varphi)^{*,TL} & = & x \setminus Dom(x \times_{\in} (x \setminus \varphi^{*,TL})).\end{array}$$

It is worth noting that the constructible terms translating G and H, even though not extremely compact, result entirely symmetric, thereby reflecting the symmetry of G and H

The following theorem shows that the translation $(.)^{*,TL}$ is sound and complete with respect to Ω_c.

THEOREM 4.3 *Let φ be a TL-formula,*

(completeness) $\vdash_{TL} \varphi \;\Rightarrow\; \Omega_c \vdash \forall x \forall \vec{y}(x \subseteq \varphi^{*,TL}(x, \vec{y})$,

(soundness) $\Omega_c \vdash \forall x \forall \vec{y}(x \subseteq \varphi^{*,TL}(x, \vec{y}) \;\Rightarrow\; \models_f \varphi$.

The proof is quite similar to the one for the logic of inequality, and thus omitted.

The reader can easily check that a weaker set theory for which Theorem 4.3 can be reformulated is:

$$\Omega^{TL} \quad = \quad \Omega + \{\} + \times_{\in} + Dom + Rng.$$

Concluding remark. The fact that L_2-derivability can be translated into derivability from a set theory like Ω_c hints at the possibility of systematically mapping (reasonable) modal operators into set-theoretic terms. The cases studied in this section give interesting and rather natural examples of how such a translation goes, and also point out the possibility of tailoring the background set theory precisely — starting from Ω_c. A strong argument in favour of the existence of a systematic mapping technique, is the fact that the constructible operators certainly offer a great — and 'tunable' — variety of possibilities for searching the set-theoretic counterparts of a given modal operator together with its semantics. The literature on extensions of modal logics offers a wealth of cases to study: counting modalities, terminological logics, metric temporal logics, etc..

Conclusions and further directions

In this paper, we addressed the problem of generalizing the set-theoretic translation method developed in [7, 11] to extended modal logics. The link we established between the fragment of second-order logic which corresponds to L_2 and the theory of constructible sets is the main outcome of the paper. In the last part of the paper, we showed how this result can be easily specialized to work with various extended modal logics inside the full L_2 language, such as the modal logic of inequality and the minimal tense logic. The achieved results can be generalized in two directions. On the modal side, one can try to explicitly characterize the class of extended modal logics which are embeddable in L_2. On the set-theoretic side, one can try to establish whether the set theories obtained by tailoring Ω_c, e.g. Ω^D for the modal logic of inequality and Ω^{TL} for the minimal tense logic, are somehow *minimal*. An interesting consequence of such a minimality result would be the ability of comparing the relative expressive strength of (translatable) extended modal logics. Finally, we are considering possible uses of the proposed translation as a bridge between the meta-theory of second-order logic and set theory.

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Appendix

LEMMA 3.2 *For every L_0-formula $\alpha(x_1,\dots,x_n,Y_1,\dots,Y_k)$, whose free variables are among $x_1,\dots,x_n$ (world variables) and $Y_1,\dots,Y_k$ (set variables), there exists a term $F_\alpha(X_1,\dots,X_n,Y_1,\dots,Y_k)$ in the language of Ω_c such that, for all $V_1,\dots,V_n,W_1,\dots,W_k$ in $\mathcal{W}$, and $u\in U$,*

$$U\models u\in F_\alpha(V_1,\dots,V_n,W_1,\dots,W_k)$$

if and only if
there exist $w_1\in V_1,\dots,w_n\in V_n$ such that

a) $U\models u=\langle w_n,w_{n-1}\dots,w_1\rangle$,
b) $(W,R)\models\alpha(w_1,\dots,w_n,W_1,\dots,W_k)$.

Proof. The proof is by induction on the structural complexity of α, and it is very similar to the proof of Lemma 6.1, pg. 64, in [2].

By possibly renaming bound variables, we can assume that the formula α verifies the following property: if $\forall x_i$ or $\exists x_i$ occur in α, then i is the largest index among the indices of the free variables in the scope of the quantifier. Moreover, notice that a formula with free variables among $x_1,\dots,x_i,Y_1,\dots,Y_j$ can be considered either as a formula $\alpha(x_1,\dots,x_i,Y_1,\dots,Y_j)$ or as a formula $\alpha(x_1,\dots,x_n,Y_1,\dots,Y_k)$, for any $n\geq i$, $k\geq j$. Statement (1) below shows that we can consider every possible denotation for the same formula without loosing the property expressed by the Lemma.

(1) For any m,r,n,k, if $\beta(x_1,\dots,x_m,Y_1,\dots,Y_r)=\alpha(x_1,\dots,x_n,Y_1,\dots,Y_k)$ and the thesis is true for $\alpha(x_1,\dots,x_n,Y_1,\dots,Y_k)$, then it is true for $\beta(x_1,\dots,x_m,Y_1,\dots,Y_r)$.

Statement (1) follows from (1a)–(1c) below.

(1a) For any $r\geq 1$, if $\beta(x_1,\dots,x_n,Y_1,\dots,Y_k)=\alpha(x_1,\dots,x_n,Y_1,\dots,Y_r)$ and the thesis is true for the formula $\alpha(x_1,\dots,x_n,Y_1,\dots,Y_r)$, then it is true for $\beta(x_1,\dots,x_n,Y_1,\dots,Y_k)$, with

$$F_\beta(X_1,\dots,X_n,Y_1,\dots,Y_k)=F_\alpha(X_1,\dots,X_n,Y_1,\dots,Y_r)$$

(the verification is left to the reader).

(1b) If $\beta(x_1,\dots,x_n,x_{n+1},Y_1,\dots,Y_k)=\alpha(x_1,\dots,x_n,Y_1,\dots,Y_k)$ and the thesis is true for the formula $\alpha(x_1,\dots,x_n,Y_1,\dots,Y_k)$, then it is true for $\beta(x_1,\dots,x_n,x_{n+1},Y_1,\dots,Y_k)$, with

$$F_\beta(X_1,\dots,X_{n+1},Y_1,\dots,Y_k)=X_{n+1}\times F_\alpha(X_1,\dots,X_n,Y_1,\dots,Y_k).$$

Indeed, for all $V_1,\dots,V_{n+1},W_1,\dots,W_k$ in $\mathcal{W}$, and $u\in U$,

$$U\models u\in F_\beta(V_1,\dots,V_{n+1},W_1,\dots,W_k)$$

if and only if

$$U\models u\in V_{n+1}\times F_\alpha(V_1,\dots,V_n,W_1,\dots,W_k)$$

if and only if

$$U\models\exists w_{n+1}\exists b\ (w_{n+1}\in V_{n+1}\wedge b\in F_\alpha(V_1,\dots,V_n,W_1,\dots,W_k)\wedge u=\langle w_{n+1},b\rangle)$$

if and only if
there exist $w_1\in V_1,\dots,w_{n+1}\in V_{n+1}$ such that

a) $U \models u = \langle w_{n+1}, w_n, \ldots, w_1 \rangle$,
b) $(W, R) \models \beta(w_1, \ldots, w_{n+1}, W_1, \ldots, W_k)$.

(1c) If $\beta(x_1, \ldots, x_{n-1}, Y_1, \ldots, Y_k) = \alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ and the thesis is true for the right-hand side formula $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$, then it is true for $\beta(x_1, \ldots, x_{n-1}, Y_1, \ldots, Y_k)$, with

$$F_\beta(X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_k) = Rng(F_\alpha(X_1, \ldots, X_{n-1}, W, Y_1, \ldots, Y_k))$$

(we leave the easy verification to the reader).

Two further preliminary steps are needed before we can start the inductive proof. In each step, we just determine the term F_β, leaving the verification of the thesis to the reader. The notation $\alpha(\ldots, x_i|x_j, \ldots, \ldots, \ldots)$ is used to denote uniform substitution of x_i for x_j in the formula α.

(2) If $\beta(x_1, \ldots, x_{n+1}, Y_1, \ldots, Y_k) = \alpha(x_1, \ldots, x_{n+1}|x_n, Y_1, \ldots, Y_k)$ and the thesis is true for α, then it is true for β. If $n = 1$, it is enough to take $F_\beta(X_1, X_2, Y_1, \ldots, Y_k) = F_\alpha(X_2, Y_1, \ldots, Y_k) \times X_1$. If $n > 1$, let $F_\beta(X_1, \ldots, X_{n+1}, Y_1, \ldots, Y_k) = C_2(F_\alpha(X_1, \ldots, X_{n-1}, X_{n+1}, Y_1, \ldots, Y_k), X_n)$.

(3) If $\beta(x_1, \ldots, x_n, Y_1, \ldots, Y_k) = \alpha(x_{n-1}|x_1, x_n|x_2, Y_1, \ldots, Y_k)$, with $n \geq 2$, and the thesis is true for α, then it is true for β, with $F_\beta(X_1, \ldots, X_n, Y_1, \ldots, Y_k) = C_1(F_\alpha(X_{n-1}, X_n, Y_1, \ldots, Y_k), X_{n-2} \times \cdots \times X_1)$.

Let us prove now the thesis by induction on the structural complexity of $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$. We start with atomic formulae.

(4a) If $\alpha(x_1, x_2)$ is the formula $x_1 = x_2$, then let $F_\alpha = X_2 \times_= X_1$. If we need to consider $x_1 = x_2$ as a formula in the variables $x_1, \ldots, x_n, Y_1, \ldots, Y_k$, then the thesis follows from (1).

(4b) If $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ is the formula $x_{n-1} = x_n$, then the thesis follows from (4a) and (3).

(4c) If $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ is the formula $x_m = x_r$, with $m \leq r \leq n$, there are two possibilities: if $m = r$, then $F_\alpha = X_n \times \ldots \times X_1$; if $m < r$, then the thesis follows from (4a) and (2) by induction on r.

(5a) If $\alpha(x_1, x_2)$ is the formula $x_1 R x_2$, then $F_\alpha = X_2 \times_\in X_1$. If we need to consider $x_1 R x_2$ as a formula in the variables $x_1, \ldots, x_n, Y_1, \ldots, Y_k$, then the thesis follows from (1).

(5b) If $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ is the formula $x_{n-1} R x_n$, the thesis follows from (5a) and (3).

((5c) If $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ is the formula $x_i R x_j$, consider the formulae

$$\begin{aligned}
\gamma_1(x_1, \ldots, x_{n+2}, Y_1, \ldots, Y_k) &= x_i = x_{n+1}; \\
\gamma_2(x_1, \ldots, x_{n+2}, Y_1, \ldots, Y_k) &= x_j = x_{n+2}; \\
\gamma_3(x_1, \ldots, x_{n+2}, Y_1, \ldots, Y_k) &= x_{n+1} R x_{n+2}.
\end{aligned}$$

From (4c) and (5b), it follows that the thesis is true for γ_i, $i = 1, 2, 3$. Then, it is easy to verify that the thesis is true for α, with F_α equal to $Rng(Rng(F_{\gamma_1}(X_1, \ldots, X_n, X_i, X_j, Y_1, \ldots, Y_k) \cap F_{\gamma_2}(X_1, \ldots, X_n, X_i, X_j, Y_1, \ldots, Y_k) \cap F_{\gamma_3}(X_1, \ldots, X_n, X_i, X_j, Y_1, \ldots, Y_k)))$.

6) If $\alpha(x_1, Y_1)$ is the formula $x_1 \in Y_1$, then $F_\alpha(X_1, Y_1) = X_1 \cap Y_1$. It can be easily generalized to the case of $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k) = x_i \in Y_j$ as in (5c).

In the inductive step, the case of boolean combinations of formulae is dealt with $\cup$ and $\setminus$ (whit respect to $X_n \times \ldots \times X_1$, where n is the number of world variables displayed in the formula), and left to the reader.

Consider the case in which $\beta(x_1, \ldots, x_n, Y_1, \ldots, Y_k) = \exists x_{n+1}\alpha(x_1, \ldots, x_{n+1}, Y_1, \ldots, Y_k)$. Let $F_\beta(X_1, \ldots, X_n, Y_1, \ldots, Y_k) = Rng(F_\alpha(X_1, \ldots, X_n, W, Y_1, \ldots, Y_k))$. Then, for all $V_1, \ldots, V_n, W_1, \ldots, W_k$ in $\mathcal{W}$, and $u \in U$,

$$U \models u \in F_\beta(V_1, \ldots, V_n, W_1, \ldots, W_k)$$

if and only if

there exist $w_1 \in V_1, \ldots, w_n \in V_n, s \in W$ such that

a) $U \models u = \langle w_n, w_{n-1}, \ldots, w_1\rangle$,
b) $(W, R) \models \alpha(w_1, \ldots, w_n, s, W_1, \ldots, W_k)$

if and only if

there exist $w_1 \in V_1, \ldots, w_n \in V_n$ such that

a) $U \models u = \langle w_n, w_{n-1}, \ldots, w_1\rangle$,
b) $(W, R) \models \exists x_{n+1}\alpha(w_1, \ldots, w_n, x_{n+1}, W_1, \ldots, W_k)$.

■

LEMMA 3.12 *For any set term $\tau(\vec{y})$ there are:*

1. *terms $\tau_1(\vec{x}, \vec{y}), \ldots, \tau_{k_\tau}(\vec{x}, \vec{y},)$, such that $h(\tau_i) \leq h(\tau)$ and if $h(\tau) > 1$, then $h(\tau_i) < h(\tau)$,*
2. *Δ_0-formulae in the language $\{=, \in\}$, $\alpha_1^\tau(\vec{x}, \vec{y}), \ldots, \alpha_{k_\tau}^\tau(\vec{x}, \vec{y})$, with possibly W as parameter,*

such that, for all $\vec{A} \in \mathcal{W}_0 \cup W$, and for all $u \in U_c$,

$$U_c \models u \in \tau(\vec{A}) \leftrightarrow \bigvee_{i=1}^{k_\tau} \exists \vec{w} \in W \ (u = \tau_i(\vec{w}, \vec{A}) \wedge \alpha_i^\tau(\vec{w}, \vec{A})).$$

Proof. The proof is by induction on the height of the term τ.

If $h(\tau) = 1$ and $\tau = y$, then put $k_\tau = 1$, $\tau_1 = x$, and $\alpha_1^\tau = x \in y$.

Suppose $h(\tau) > 1$, and let μ, ν be terms such that $h(\mu), h(\nu) < h(\tau)$. By induction, the lemma holds for μ, ν; let $\mu_1, \ldots, \mu_{k_\mu}$ and $\nu_1, \ldots, \nu_{k_\nu}$ be the terms, described in (1), for μ and ν, respectively.

We begin by showing the following four facts:

(a) there exists a Δ_0-formula $\alpha_=^{\mu,\nu}$ such that for all $\vec{A} \in \mathcal{W}_0 \cup W$,

$$U_c \models \Big(\mu(\vec{A}) = \nu(\vec{A})\Big) \leftrightarrow \alpha_=^{\mu,\nu}(\vec{A});$$

(b) there exists a Δ_0-formula $\alpha_\in^{\mu,\nu}$ such that for all $\vec{A} \in \mathcal{W}_0 \cup W$,

$$U_c \models \Big(\mu(\vec{A}) \in \nu(\vec{A})\Big) \leftrightarrow \alpha_\in^{\mu,\nu}(\vec{A});$$

(c) there exist Δ_0-formulae $\beta_{1,1}^\mu(\vec{x}, \vec{x}', \vec{y}), \beta_{1,2}^\mu(\vec{x}, \vec{x}', \vec{y}) \ldots, \beta_{k_\mu, k_\mu}^\mu(\vec{x}, \vec{x}', \vec{y},)$, such that for all $\vec{A} \in \mathcal{W}_0 \cup W$ and for all $a, b \in U_c$,

$$U_c \models \{a, b\} = \mu(\vec{A}) \leftrightarrow \bigvee_{i,j=1}^{k_\mu} \exists \vec{w}', \vec{w}'' \in W \ (a = \mu_i(\vec{w}', \vec{A}) \wedge b = \mu_j(\vec{w}'', \vec{A}) \wedge \beta_{i,j}^\mu(\vec{w}', \vec{w}'', \vec{A}));$$

(d) there exist Δ_0-formulae $\sigma_1^\mu(\vec{x},\vec{y}),\ldots,\sigma_{k_\mu}^\mu(\vec{x},\vec{y})$ such that for all $\vec{A}\in\mathcal{W}_0\cup W$ and $a\in U_c$,

$$U_c \models \{a\} = \mu(\vec{A}) \leftrightarrow \bigvee_{i=1}^{k_\mu} \exists\vec{w}\in W\ (a=\mu_i(\vec{w},\vec{A})\wedge\sigma_i^\mu(\vec{w},\vec{A})).$$

Each of the above facts is proved by induction using the main inductive hypothesis.

(a) if $h(\mu)=h(\nu)=1$, the formula $\alpha_=^{\mu,\nu}$ is $\mu=\nu$.

If $h(\mu)=1$ and $h(\nu)>1$ (or vice versa), then μ is a variable, say y_i. For all $\vec{A}\in\mathcal{W}_0\cup W$, $\mu(\vec{A})=A_i$, $\nu(\vec{A})\notin W$ (cf. Corollary 3.10). Since U_c is extensional on $\overline{\mathcal{W}}$ we obtain:

$$U_c\models A_i=\nu(\vec{A})\leftrightarrow A_i\notin W\wedge\forall t\ (t\in A_i\leftrightarrow t\in\nu(\vec{A})).$$

Since $1<h(\nu)<h(\tau)$, the main inductive hypothesis holds for ν, with terms ν_j such that $h(\nu_j)<h(\nu)$, and

$$U_c\models t\in\nu(\vec{A})\leftrightarrow\bigvee_{j=1}^{k_\nu}\exists\vec{w}\in W\left(t=\nu_j(\vec{w},\vec{A})\wedge\alpha_j^\nu(\vec{w},\vec{A})\right).$$

Thus $\forall t\ (t\in A_i\leftrightarrow t\in\nu(\vec{A}))$ is equivalent in U_c to

$$\forall t\in A_i\left(\bigvee_{j=1}^{k_\nu}\exists\vec{w}\in W\left(t=\nu_j(\vec{w},\overline{A})\wedge\alpha_j^\nu(\vec{w},\vec{A})\right)\right)\wedge\bigwedge_{j=1}^{k_\nu}\forall\vec{w}\in W\left(\alpha_j^\nu(\vec{w},\vec{A})\to\nu_j(\vec{w},\vec{A})\in A_i\right).$$

Notice now that if $h(\nu_j)>1$, then the element $\nu_j(\vec{w},\vec{A})$ does not belong to A_i in the model U_c (since it does not belong to W), and thus $\alpha_j^\nu(\vec{w},\vec{A})$ is false in U_c. This implies that $A_i=\nu(\vec{A})$ is equivalent in U_c to

$$A_i\notin W\wedge\forall t\in A_i\left(\bigvee_{j=1}^{k_\nu}\exists\vec{w}\in W\left(t=\nu_j(\vec{w},\vec{A})\wedge\alpha_j^\nu(\vec{w},\vec{A})\right)\right)\wedge$$
$$\wedge\bigwedge_{\substack{1\le j\le k_\nu\\ h(\nu_j)=1}}\forall\vec{w}\in W\left(\alpha_j^\nu(\vec{w},\vec{A})\to\nu_j(\vec{w},\vec{A})\in A_i\right),$$

which is a Δ_0-formula except for the subformula $t=\nu_j(\vec{w},\vec{A})$. Since $h(\nu_j)<h(\nu)$, we can proceed inductively on $t=\nu_j(\vec{w},\vec{A})$ to obtain the required Δ_0-formula.

If $h(\mu),h(\nu)>1$, then for all $\vec{A}\in\mathcal{W}_0\cup W$, $\mu(\vec{A}),\nu(\vec{A})\notin W$ and by Lemma 3.11 (extensionality),

$$U_c\models\left(\mu(\vec{A})=\nu(\vec{A})\right)\leftrightarrow\forall t\ (t\in\mu(\vec{A})\to t\in\nu(\vec{A}))\ \wedge\ \forall t\ (t\in\nu(\vec{A})\to t\in\mu(\vec{A})).$$

By the inductive hypothesis, the lemma holds for μ,ν with terms μ_i,ν_j such that $h(\mu_i)<h(\mu)$ and $h(\nu_i)<h(\nu)$. Thus, the first conjunct on the right-hand side of the above formula is equivalent to

$$\bigwedge_{i=1}^{k_\mu}\forall\vec{w}\in W\left[\alpha_i^\mu(\vec{w},\vec{A})\to\bigvee_{j=1}^{k_\nu}\exists\vec{w}'\in W\ (\mu_i(\vec{w},\vec{A})=\nu_j(\vec{w}',\vec{A})\wedge\alpha_j^\nu(\vec{w}',\vec{A}))\right].$$

Consider the subformula $\mu_i(\vec{w},\vec{A})=\nu_j(\vec{w}',\vec{A})$. Since $h(\mu_i)<h(\mu)$ and $h(\nu_j)<h(\nu)$, we can proceed inductively to obtain the required Δ_0-formula. The same argument applies to the second conjunct.

(b) It follows directly from (a) and the inductive hypothesis of the lemma.

(c) If $h(\mu) = 1$, then $\mu = y_i$ and we can take $\beta_{1,1}(x, x', y)$ equal to the formula

$$y \notin W \wedge x \in y \wedge x' \in y \wedge \forall v \in W \ (v \in y \rightarrow v = x \vee v = x').$$

If $h(\mu) > 1$, then $\mu(\vec{A}) \notin W$ and, by Lemma 3.11, we have

$$U_c \models \{a, b\} = \mu(\vec{A}) \leftrightarrow a \in \mu(\vec{A}) \wedge b \in \mu(\vec{A}) \wedge \forall t \ (t \in \mu(\vec{A}) \rightarrow t = a \vee t = b).$$

Since $h(\mu) < h(\tau)$, by the main inductive hypothesis this is easily seen to be equivalent (in U_c) to the formula

$$\bigvee_{i,j=1}^{k_\mu} \exists \vec{w}'\vec{w}'' \in W \left[a = \mu_i(\vec{w}', \vec{A}) \wedge b = \mu_j(\vec{w}'', \vec{A}) \wedge \alpha_i^\mu(\vec{w}', \vec{A}) \wedge \alpha_j^\mu(\vec{w}'', \vec{A}) \wedge \right.$$

$$\left. \wedge \bigwedge_{l=1}^{k_\mu} \forall \vec{s} \in W \left(\alpha_l^\mu(\vec{s}, \vec{A}) \rightarrow \mu_l(\vec{s}, \vec{A}) = \mu_i(\vec{w}', \vec{A}) \vee \mu_l(\vec{s}, \vec{A}) = \mu_j(\vec{w}'', \vec{A}) \right) \right].$$

Since $h(\mu_i) < h(\mu) < h(\tau)$ for any $i \in \{1, \ldots, k_\mu\}$, from (a) it follows that the equalities between the terms above are equivalent to Δ_0-formulae.

(d) It easily follows from (c).

To complete the proof of the lemma, we consider the various cases corresponding to the structure of the term τ. To simplify the notation, we sometime avoid to explicitly indicate the elements $\vec{A}$. We suppose that the Lemma holds for terms μ, ν with $h(\mu), h(\nu) < h(\tau)$, and we denote by μ_i, ν_j, $\alpha_i^\mu, \alpha_j^\nu$ the corresponding terms and Δ_0-formulae.

- The case $\tau = \mu \cup \nu$ directly follows from the inductive hypothesis.
- Let $\tau = \mu \setminus \nu$. In U_c we have $a \in \tau(\vec{A}) \leftrightarrow a \in \mu(\vec{A}) \wedge a \notin \nu(\vec{A})$, which is easily seen to be equivalent to

$$\bigvee_{i=1}^{k_\mu} \exists \vec{w} \in W \left[a = \mu_i(\vec{w}) \wedge \alpha_i^\mu(\vec{w}) \wedge \bigwedge_{j=1}^{k_\nu} \forall \vec{w}' \in W \left(\alpha_j^\nu(\vec{w}') \rightarrow \mu_i(\vec{w}) \neq \nu_j(\vec{w}') \right) \right].$$

  The last formula can be reduced to a Δ_0-formula applying (a) to each subformula $\mu_i(\vec{w}) \neq \nu_j(\vec{w}')$. Notice that $h(\mu_i) \leq h(\mu) < h(\tau)$, $h(\nu_j) \leq h(\nu) < h(\tau)$.

- Let $\tau = \{\mu\}$. In this case, $U_c \models a \in \tau(\vec{A}) \leftrightarrow a = \mu(\vec{A})$. Since $h(\mu) < h(\tau)$, it is enough to take $\tau_1(\vec{x}, \vec{y}) = \mu(\vec{y})$ and $y_1 = y_1$ as Δ_0-formula.
- Let $\tau = \mu \times \nu$. By the inductive hypothesis, $a \in \tau(\vec{A})$ is equivalent in U_c to

$$\bigvee_{i=1}^{k_\mu} \bigvee_{j=1}^{k_\nu} \exists \vec{w}\vec{w}' \in W \ \left[a = \langle \mu_i(\vec{w}), \nu_j(\vec{w}') \rangle \wedge \alpha_i^\mu(\vec{w}) \wedge \alpha_j^\nu(\vec{w}') \right].$$

  Since $h(\langle \mu_i, \nu_j \rangle) < h(\tau)$, the thesis follows.

- The case of $\tau = \mu \times_= \nu$ is proved as the case of $\tau = \mu \times \nu$ using (a).
- The case $\tau = \mu \times_\in \nu$ is proved as the case of $\tau = \mu \times \nu$ using (b).
- Let $\tau = Dom(\mu)$. Suppose that $h(\mu) = 1$. In this case, $\mu = y_i$ and $U_c \models \forall x(x \notin Dom(A_i))$, because no element of an element of $\mathcal{W}_0 \cup W$ is a pair. It is then enough to take $\tau_1 = x$, and $x \neq x$ as α_1^τ.

  Consider now the case in which $h(\mu) > 1$. In U_c, $a \in \tau(\vec{A})$ is equivalent to

$$\exists b \ \left(\langle a, b\rangle = \{\{a\}, \{a, b\}\} \in \mu(\vec{A})\right),$$

  which, in turn, is equivalent to

$$\exists b, c, d \ \left(c = \{a\} \wedge d = \{a, b\} \wedge \{c, d\} \in \mu(\vec{A})\right).$$

  By the inductive hypothesis and (c), this can be rewritten as

$$\exists b, c, d \left[c = \{a\} \wedge d = \{a, b\} \wedge \right.$$

$$\left. \wedge \bigvee_{i=1}^{k_\mu} \exists \vec{w} \in W \left(\alpha_i^\mu(\vec{w}) \wedge \bigvee_{l,m=1}^{k_{\mu_i}} \exists \vec{w}', \vec{w}'' \in W \left(c = \mu_{i,l}(\vec{w}', \vec{w}) \wedge d = \mu_{i,m}(\vec{w}'', \vec{w}) \wedge \beta_{l,m}^{\mu_i}(\vec{w}', \vec{w}'', \vec{w}) \right) \right) \right],$$

  in which c and d can be eliminated obtaining

$$\exists b \bigvee_{i=1}^{k_\mu} \bigvee_{l,m=1}^{k_{\mu_i}} \exists \vec{w}, \vec{w}', \vec{w}'' \in W \left[\mu_{i,l}(\vec{w}', \vec{w}) = \{a\} \wedge \mu_{i,m}(\vec{w}'', \vec{w}) = \{a, b\} \wedge \alpha_i^\mu(\vec{w}) \wedge \beta_{l,m}^{\mu_i}(\vec{w}', \vec{w}'', \vec{w}) \right].$$

  Using (c) again, we can eliminate b, obtaining the following formula:

$$\bigvee_{i=1}^{k_\mu} \bigvee_{l,m=1}^{k_{\mu_i}} \bigvee_{r,s=1}^{k_{\mu_{i,m}}} \exists \vec{w}, \vec{w}', \vec{w}'', \vec{v}', \vec{v}'' \in W \left[a = \mu_{i,m,r}(\vec{v}', \vec{w}'', \vec{w}) \wedge \beta_{r,s}^{\mu_{i,m}}(\vec{v}', \vec{v}'', \vec{w}'', \vec{w}) \wedge \right.$$

$$\left. \wedge \mu_{i,l}(\vec{w}', \vec{w}) = \{\mu_{i,m,r}(\vec{v}', \vec{w}'', \vec{w})\} \wedge \alpha_i^\mu(\vec{w}) \wedge \beta_{l,m}^{\mu_i}(\vec{w}', \vec{w}'', \vec{w}) \right].$$

  Since $h(\mu) > 1$, we have that $h(\mu_i) < h(\mu)$, for all $i \in \{1, \ldots, k_\mu\}$. It follows that $h(\mu_{i,l}), h(\{\mu_{i,m,r}\}) < h(\tau)$, and a further application of (a) to the subformula $\mu_{i,l}(\vec{w}', \vec{w}) = \{\mu_{i,m,r}(\vec{v}', \vec{w}'', \vec{w})\}$ gives the desired result.
- The case of $\tau = Rng(\mu)$ is proved as the case of Dom.
- Let $\tau = C_1(\mu, \nu)$. In U_c, we have

$$a \in \tau(\vec{A}) \leftrightarrow \exists b, c, d \ (a = \langle b, \langle c, d\rangle\rangle \wedge \langle b, c\rangle \in \mu(\vec{A}) \wedge d \in \nu(\vec{A})).$$

  Let $\mu' = Dom(\mu)$ and $\mu'' = Rng(\mu)$. Notice that $\langle b, c\rangle \in \mu(\vec{A})$ implies $b \in \mu'(\vec{A})$ and $c \in \mu''(\vec{A})$. Hence the above formula is equivalent to

$$\exists b, c, d \ (a = \langle b, \langle c, d\rangle\rangle \wedge b \in \mu'(\vec{A}) \wedge c \in \mu''(\vec{A}) \wedge \langle b, c\rangle \in \mu(\vec{A}) \wedge d \in \nu(\vec{A})).$$

  Since $h(\mu'), h(\mu'') < h(\tau)$, we can conclude as in the case of $\times$.
- The case of $\tau = C_2(\mu, \nu)$ is proved as the case of C_1.

■