Definability and decidability of binary predicates for time granularity

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Abstract

In this paper we study the definability and decidability of binary predicates for time granularity in monadic languages interpreted over finitely and infinitely layered structures. We focus our attention on the equi-level (resp. equi-column) predicate constraining two time points to belong to the same layer (resp. column) and on the horizontal (resp. vertical) successor predicate relating a time point to its successor within a given layer (resp. column). We give a number of positive and negative results by reduction to/from a wide spectrum of decidable/undecidable problems.
Key words time granularity, monadic theories, definability, decidability

1 Introduction

In this paper we systematically investigate the definability and decidability of several binary predicates in monadic languages interpreted over temporal structures for time granularity. The ability of providing and relating temporal representations at different 'grain levels' is widely recognized as an important research theme for temporal logic and a major requirement for many applications, including specification and verification of reactive system, knowledge representation and reasoning, temporal databases, and data mining [1, 4, 6, 13, 14]. We focus our attention on the area of formal specification and verification where the explicit representation of time granularity makes it possible to specify in a concise way reactive systems whose behaviour can be naturally modeled with respect to a possibly infinite set of differently-grained temporal domains/layers (we shall use the two terms interchangeably) [10, 16]. A logical specification framework incorporating a notion of *time granularity* has been systematically developed in [16] and later extended in [7]. It is based on a many-level view of temporal structures that replaces the flat temporal domain of standard linear and branching temporal logics by a temporal universe consisting of a possibly infinite set of differently-grained temporal layers. The monadic second-order (MSO for short) theory of the n-layered (there are exactly n layers) k-refinable (each time point can be refined into k time points of the immediately finer layer, if any) temporal structure for time granularity (n-LS), with matching decidability results, has been investigated in [19]. The MSO theory of the *k-refinable upward unbounded layered structure* (UULS), that is, the ω-layered structure consisting of a finest temporal layer together with an infinite number of coarser and coarser layers, and the MSO theory of the *k-refinable downward unbounded layered structure* (DULS), that is, the ω-layered structure consisting of a coarsest layer together with an infinite number of finer and finer layers, have been studied in [17]. The decidability of the theories of the UULS and the DULS has been proved by reducing their decision problems to the emptiness problem for systolic and Rabin tree automata, respectively. Such a connection between multi-level temporal logics and automata theory suggests a complementary point of view on time granularity: besides an important feature of a specification language, it can be viewed as a formal tool to investigate the definability of meaningful timing properties, such as periodicity, density, and exponential grow/decay, as well as the expressiveness and decidability of temporal logics [8, 9, 17, 18, 19]. In this respect, the number of layers (single vs. multiple, finite vs. infinite) of the underlying temporal structure, as well as the nature of their interconnections, play a major role: certain timing properties can be expressed using a single layer; others using a finite number of layers; others only exploiting an infinite number of layers.

In this paper, we deal with the definability and decidability of a set of binary predicates in monadic languages interpreted over the n-layered and ω-layered structures. We focus our attention on the equi-level (resp. equi-column) predicate constraining two time points to belong to the same layer (resp. column) and the horizontal (resp. vertical) successor predicate relating a time point to its successor within a given layer (resp. column), which allow one to express meaningful properties of time granularity [16]. As a matter of fact, definability and decidability problems for the equi-level and vertical successor predicates in monadic languages interpreted over the binary and k-ary trees have been already studied in the literature. More precisely, the decidability of the extension of the first-order theory of two successors, devoid of free set variables, over the infinite binary tree with the equi-level predicate was first proved by Elgot and Rabin in [3]. Thomas extended this result by showing that the monadic chain logic extended with the equi-level predicate over the infinite k-ary tree is decidable [23], while Läuchli and Savoiz proved the undecidability of the (weak) MSO theory of k successors over the infinite k-ary tree extended with either the equi-level or the vertical successor predicate [15]. Our paper generalizes these results in two directions: on the one hand, we interpret the monadic languages over more general structures; on the other hand, we take into consideration a larger set of predicates, including the equi-column and the horizontal successor predicates (these predicates are intimately related to the vertical successor and equi-level predicates, respectively, but in general they are not inter-definable).

The paper is organized as follows. In Section 2 we provide background knowledge about monadic theories of time granularity. Then, in Section 3 we introduce the relevant binary predicates for time granularity. In the subsequent sections, we study the definability and decidability of these predicates in monadic languages over the UULS, n-LS, and DULS. Conclusions provide an assessment of the work.

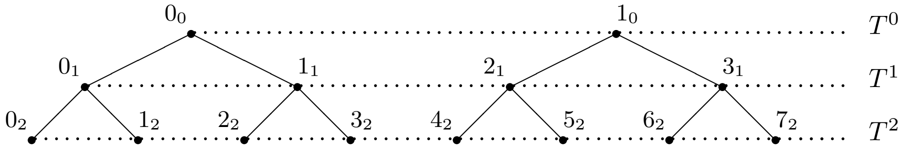

Figure 1: The 2-refinable 3-layered structure.

2 Monadic theories of layered structures

In this section we introduce the MSO theories of time granularity, and their first-order, path, and chain fragments (a more detailed presentation can be found in [5]). MSO theories are systems of MSO logic that allow quantification over arbitrary sets of elements. We show that some MSO theories of time granularity can be reduced to well-know classical theories, such as the MSO theory of one successor and the MSO theory of two successors, while other granularity theories are proper extensions of them.

Definition 2.1 (*The language of monadic second-order logic*)

Let $\tau = c_1, \ldots, c_r, u_1, \ldots, u_s, b_1, \ldots, b_t$ *be a finite alphabet of symbols, where* $c_1, \ldots, c_r$ *(resp.* $u_1, \ldots, u_s$*,* $b_1, \ldots, b_t$*) are constant symbols (resp. unary relational symbols, binary relational symbols), and let* $\mathcal{P}$ *be a finite set of uninterpreted unary relational symbols. The second-order language with equality* $\mathrm{MSO}[\tau \cup \mathcal{P}]$ *is built up as follows:*

1. atomic formulas *are of the forms* $x = y$*,* $x = c_i$*, with* $1 \leq i \leq r$*,* $u_i(x)$*, with* $1 \leq i \leq s$*,* $b_i(x,y)$*, with* $1 \leq i \leq t$*,* $x \in X$*,* $x \in P$*, where* x*,* y *are individual variables,* X *is a set variable, and* $P \in \mathcal{P}$*;*
2. formulas *are built up from atomic formulas by means of the Boolean connectives* $\neg$ *and* $\wedge$*, and the quantifier* $\exists$ *ranging over both individual and set variables.*

In the following, we shall write $\mathrm{MSO}_{\mathcal{P}}[\tau]$ for $\mathrm{MSO}[\tau \cup \mathcal{P}]$; in particular, we shall write $\mathrm{MSO}[\tau]$ when $\mathcal{P}$ is meant to be the empty set. The first-order fragment of $\mathrm{MSO}_{\mathcal{P}}[\tau]$ will be denoted by $\mathrm{MFO}_{\mathcal{P}}[\tau]$, while its path (resp. chain) fragment, which is obtained by interpreting second-order variables over paths (resp. chains), will be denoted by $\mathrm{MPL}_{\mathcal{P}}[\tau]$ (resp. $\mathrm{MCL}_{\mathcal{P}}[\tau]$).

Relational structures for time granularity consists of a (possibly infinite) number of distinct layers. We focus our attention on the n-layered structure, which include a fixed finite number n of layers, and ω-layered structures, which feature an infinite number of layers.

The n-layered structure. Let $n \geq 1$ and $k \geq 2$. For every $0 \leq i < n$, let $T^i = \{j_i \mid j \geq 0\}$. The *$n$-layered temporal universe* is the set $\mathcal{U}_n = \bigcup_{0 \leq i < n} T^i$. The ($k$-refinable) *$n$-layered structure* ($n$-LS for short) is the relational structure $\langle \mathcal{U}_n, (\downarrow_j)_{j=0}^{k-1}, < \rangle$. Such a structure can be viewed as an infinite sequence of complete (k-ary) trees of height $n-1$, each one rooted at a point of the coarsest layer T^0 (see Figure 1). The sets T^i, with $0 \leq i < n$, are the layers of the trees. For every $0 \leq j \leq k-1$, $\downarrow_j$ is the j-th *successor*

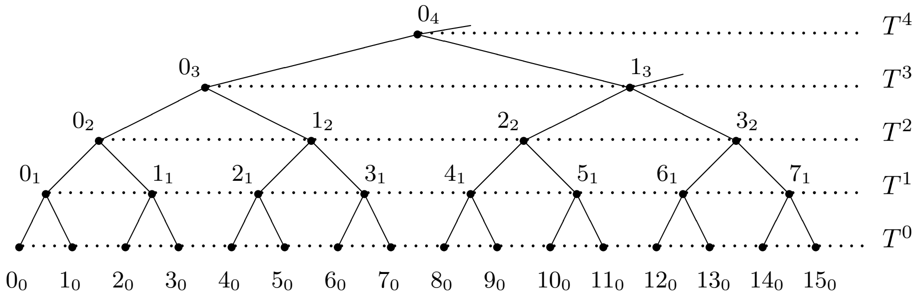

Figure 2: The 2-refinable upward unbounded layered structure.

relation over $\mathcal{U}_n$ such that $\downarrow_j (x, y)$ (also denoted by $\downarrow_j (x) = y$) if y is the j-th son of x. Note that for all x belonging to the finest layer T^{n-1} there exist no $0 \leq j \leq k-1$ and $y \in \mathcal{U}_n$ such that $\downarrow_j (x) = y$. Finally, $<$ is a total ordering over $\mathcal{U}_n$ given by the *pre-order* (root-left-right in the binary trees) visit of the nodes (for elements belonging to the same tree) and by the total linear ordering of trees (for elements belonging to different trees). Formally, for any pair $a_b, c_d \in \mathcal{U}_n$, we have that $\downarrow_j (a_b) = c_d$ if $b < n-1$, $d = b+1$, and $c = a \cdot k + j$. The total ordering $<$ is defined as follows:

1. if $x = a_0$, $y = b_0$, and $a < b$ over $\mathbb{N}$, then $x < y$;
2. for all $x \in \mathcal{U}_n \setminus T^{n-1}$, $x <\downarrow_0 (x)$, and $\downarrow_j (x) <\downarrow_{j+1} (x)$, for all $0 \leq j < k-1$;
3. if $x \in \mathcal{U}_n \setminus T^{n-1}$, $x < y$, and not $\mathtt{ancestor}(x, y)$, then $\downarrow_{k-1} (x) < y$;
4. if $x < z$ and $z < y$, then $x < y$,

where $\mathtt{ancestor}(x, y)$ if there exists $0 \leq j \leq k-1$ such that $\downarrow_j (x) = y$ or there exist $0 \leq j \leq k-1$ and z such that $\downarrow_j (z) = y$ and $\mathtt{ancestor}(x, z)$. A *path* over the n-LS is a subset of the domain whose elements can be written as a sequence $x_0, x_1, \ldots x_m$, with $m \leq n-1$, in such a way that, for every $i = 1, \ldots m$, there exists $0 \leq j < k$ for which $x_i =\downarrow_j (x_{i-1})$. A *full path* is a maximal path with respect to set inclusion. A *chain* is any subset of a path. A $\mathcal{P}$-labeled n-LS is a relational structure $\langle \mathcal{U}_n, (\downarrow_i)_{i=0}^{k-1}, <, (P)_{P \in \mathcal{P}} \rangle$, where the tuple $(\mathcal{U}_n, (\downarrow_i)_{i=0}^{k-1}, <)$ is the n-LS and, for every $P \in \mathcal{P}$, $P \subseteq \mathcal{U}_n$ is the set of points labeled with P. The decidability of $\mathrm{MSO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ over the n-LS has been proved in [19] by reducing it to the decidability of the MSO theory of one successor, which is known to be (non-elementarily) decidable [22].

Theorem 2.2 $\mathrm{MSO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ *over the* n-LS *is (non-elementarily) decidable.*

The upward unbounded layered structure. The UULS is a relational structure $\langle \mathcal{U}, \langle \downarrow_i \rangle_{i=0}^{k-1}, < \rangle$. It can be viewed as a complete (k-ary) infinite tree generated from the leaves (Figure 2). The sets T^i, with $i \geq 0$, are the layers of the tree. For every $0 \leq j \leq k-1$, $\downarrow_j$ is the j-th *successor relation* over $\mathcal{U}$ such that $\downarrow_j (x, y)$ (also denoted by $\downarrow_j (x) = y$) if y is the j-th son of x. The total ordering $<$ over $\mathcal{U}$ is induced by the *in-order* (left-root-right in the binary tree) visit of the treelike structure. Formally, for

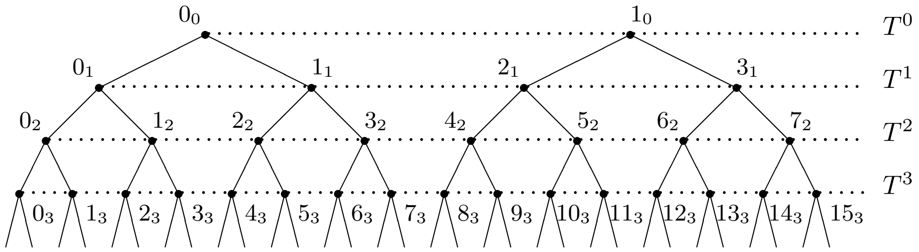

Figure 3: The 2-refinable downward unbounded layered structure.

every $a_b, c_d \in \mathcal{U}$, $\downarrow_j (a_b) = c_d$ if $b > 0$, $d = b - 1$, and $c = a \cdot k + j$. The total ordering $<$ is defined as follows:

1. for all $x \in \mathcal{U} \setminus T^0$, $\downarrow_0 (x) < x$, $x <\downarrow_1 (x)$, and $\downarrow_j (x) <\downarrow_{j+1} (x)$, for every $0 < j < k-1$;
2. if $x < y$ and not $\texttt{ancestor}(x, y)$, then $\downarrow_{k-1} (x) < y$;
3. if $x < y$ and not $\texttt{ancestor}(y, x)$, then $x <\downarrow_0 (y)$;
4. if $x < z$ and $z < y$, then $x < y$.

A *path* over the UULS is a subset of the domain whose elements can be written as an infinite sequence $x_0, x_1, \ldots$ such that, for every $i \geq 1$, there exists $0 \leq j < k$ such that $x_{i-1} =\downarrow_j (x_i)$. A *full path* is a maximal (infinite) path with respect to set inclusion. A *chain* is any subset of a path. It is worth noting that every pair of paths over the UULS may differ on a finite prefix only. A $\mathcal{P}$-labeled UULS is obtained by expanding the UULS with a set $P \subseteq \mathcal{U}$, for any $P \in \mathcal{P}$. The decidability of $\mathrm{MSO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ over the UULS has been proved in [17] by reducing it to the decidability of a proper extension of the MSO theory of one successor [20].

Theorem 2.3 $\mathrm{MSO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ *over the UULS is (non-elementarily) decidable.*

The downward unbounded layered structure. Let $\mathcal{U} = \bigcup_{i \geq 0} T^i$ be the *ω-layered temporal universe*. The DULS is a relational structure $\langle \mathcal{U}, (\downarrow_i)_{i=0}^{k-1}, < \rangle$. It can be viewed as an infinite sequence of complete (k-ary) infinite trees, each one rooted at a point of the coarsest domain T^0 (see Figure 3). The sets T^i, with $i \geq 0$, are the layers of the trees. The successor relations $\downarrow_j$, with $0 \leq j \leq k-1$, and the total ordering $<$ over $\mathcal{U}$ are defined as for the n-LS. Formally, for any pair $a_b, c_d \in \mathcal{U}$, we have that $\downarrow_j (a_b) = c_d$ if and only if $d = b + 1$ and $c = a \cdot k + j$, while the total ordering $<$ is defined as follows:

1. if $x = a_0$, $y = b_0$, and $a < b$ over $\mathbb{N}$, then $x < y$;
2. for all $x \in \mathcal{U}$, $x <\downarrow_0 (x)$, and $\downarrow_j (x) <\downarrow_{j+1} (x)$, for all $0 \leq j < k-1$;
3. if $x < y$ and not $\texttt{ancestor}(x, y)$, then $\downarrow_{k-1} (x) < y$;

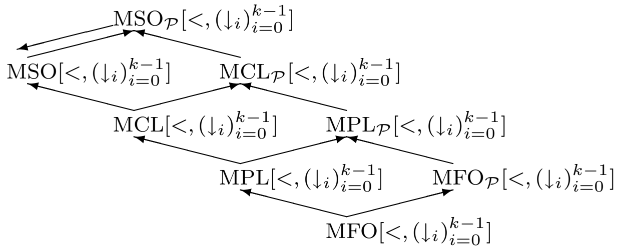

Figure 4: A hierarchy of monadic formalisms over layered structures.

4. if $x < z$ and $z < y$, then $x < y$.

A *path* over the DULS is a subset of the domain whose elements can be written as an infinite sequence $x_0, x_1, \ldots$ such that, for every $i \geq 1$, there exists $0 \leq j < k$ for which $x_i = \downarrow_j (x_{i-1})$. A *full path* is a maximal (infinite) path with respect to set inclusion. A *chain* is any subset of a path. A $\mathcal{P}$-labeled DULS is a relational structure $\langle \mathcal{U}, (\downarrow_i)_{i=0}^{k-1}, <, (P)_{P\in\mathcal{P}} \rangle$, where the tuple $(\mathcal{U}, (\downarrow_i)_{i=0}^{k-1}, <)$ is the DULS and, for every $P \in \mathcal{P}$, $P \subseteq \mathcal{U}$ is the set of points labeled with P. The decidability of $\mathrm{MSO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ over the DULS has been proved in [17] by reducing it to the decidability of the MSO theory of k successors, which is known to be (non-elementarily) decidable [22].

Theorem 2.4 $\mathrm{MSO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ *over the DULS is (non-elementarily) decidable.*

We conclude the section by introducing some notations and basic properties that will help us in comparing expressive power and logical properties of the various formal systems. Definitions and results are given for full second-order languages with uninterpreted unary relational symbols, but they immediately transfer to their fragments, possibly devoid of uninterpreted unary relational symbols. Let $\mathcal{M}(\varphi)$ be the set of models of the formula φ. We say that $\mathrm{MSO}_{\mathcal{P}}[\tau_1]$ can be *embedded* into $\mathrm{MSO}_{\mathcal{P}}[\tau_2]$, denoted $\mathrm{MSO}_{\mathcal{P}}[\tau_1] \rightarrow \mathrm{MSO}_{\mathcal{P}}[\tau_2]$, if there is an *effective* translation tr of $\mathrm{MSO}_{\mathcal{P}}[\tau_1]$-formulas into $\mathrm{MSO}_{\mathcal{P}}[\tau_2]$-formulas such that, for every formula $\varphi \in \mathrm{MSO}_{\mathcal{P}}[\tau_1]$, $\mathcal{M}(\varphi) = \mathcal{M}(tr(\varphi))$. Moreover, we say that $\mathrm{MSO}_{\mathcal{P}}[\tau_1]$ is *as expressive as* $\mathrm{MSO}_{\mathcal{P}}[\tau_2]$, written $\mathrm{MSO}_{\mathcal{P}}[\tau_1] \rightleftarrows \mathrm{MSO}_{\mathcal{P}}[\tau_2]$, if both $\mathrm{MSO}_{\mathcal{P}}[\tau_1] \rightarrow \mathrm{MSO}_{\mathcal{P}}[\tau_2]$ and $\mathrm{MSO}_{\mathcal{P}}[\tau_2] \rightarrow \mathrm{MSO}_{\mathcal{P}}[\tau_1]$. It is immediate to see that if $\mathrm{MSO}_{\mathcal{P}}[\tau_1] \rightarrow \mathrm{MSO}_{\mathcal{P}}[\tau_2]$ and $\mathrm{MSO}_{\mathcal{P}}[\tau_2]$ is decidable (resp. $\mathrm{MSO}_{\mathcal{P}}[\tau_1]$ is undecidable), then $\mathrm{MSO}_{\mathcal{P}}[\tau_1]$ is decidable (resp. $\mathrm{MSO}_{\mathcal{P}}[\tau_2]$ is undecidable) as well. Besides decidability issues, we are interested in definability ones. Let β be a relational symbol. We say that β is *definable* in $\mathrm{MSO}_{\mathcal{P}}[\tau]$ if $\mathrm{MSO}_{\mathcal{P}}[\tau \cup \{\beta\}] \rightarrow \mathrm{MSO}_{\mathcal{P}}[\tau]$. If the addition of β to a decidable formalism $\mathrm{MSO}_{\mathcal{P}}[\tau]$ makes the resulting formalism $\mathrm{MSO}_{\mathcal{P}}[\tau \cup \{\beta\}]$ undecidable, we can conclude that β is not definable in $\mathrm{MSO}_{\mathcal{P}}[\tau]$. The opposite does not hold in general: the predicate β may not be definable in $\mathrm{MSO}_{\mathcal{P}}[\tau]$, but the extension of $\mathrm{MSO}_{\mathcal{P}}[\tau]$ with β may preserve decidability. In such a case, we obviously cannot reduce the decidability of $\mathrm{MSO}_{\mathcal{P}}[\tau \cup \{\beta\}]$ to that of $\mathrm{MSO}_{\mathcal{P}}[\tau]$.

In the following, we shall explore the definability and decidability of relevant binary predicates for time granularity with respect to $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}]$ and its first-order, chain, and path fragments $\mathrm{MFO}[<, (\downarrow_i)_{i=0}^{k-1}]$, $\mathrm{MPL}[<, (\downarrow_i)_{i=0}^{k-1}]$, and $\mathrm{MCL}[\tau]$ of $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}]$ (as well as their $\mathcal{P}$-variants $\mathrm{MFO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$, $\mathrm{MPL}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$, and $\mathrm{MCL}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$). Figure 4 summarizes the relationships between the expressive powers

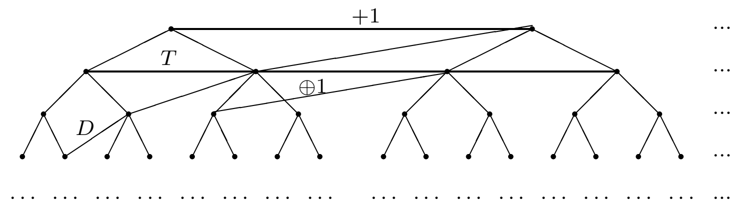

Figure 5: The global predicates for time granularity.

of such formal systems (an arrow from $\mathcal{T}$ to $\mathcal{T}'$ stands for $\mathcal{T} \rightarrow \mathcal{T}'$). From Theorems 2.2, 2.3, and 2.4, it immediately follows that all the formalisms in Figure 4, when interpreted over the n-LS, the UULS, and the DULS, are decidable.

3 Binary predicates for time granularity

In this section, we introduce the binary predicates for time granularity we are interested in; in the subsequent sections, we shall investigate definability and decidability issues about them. More precisely, we shall investigate the possibility of defining such predicates within the given systems for time granularity, and, whenever this is not possible, the possibility of adding them preserving decidability.

Let $\langle \mathcal{U}, (\downarrow_i)_{i=0}^{k-1}, < \rangle$ be a layered structure and let n_r and m_s be two elements of $\mathcal{U}$. We focus our attention on the following set of predicates:

1. *equi-level predicate* T, such that $T(n_r, m_s)$ iff $r = s$;
2. *i-th equi-level predicate* T^i, such that $T^i(n_r, m_s)$ iff $r = s = i$;
3. *equi-column predicate* D, such that $D(n_r, m_s)$ iff $n = m$;
4. *i-th equi-column predicate* D^i, such that $D^i(n_r, m_s)$ iff $n = m = i$;
5. *horizontal successor* $+1$, such that $+1(n_r, m_s)$ iff $r = s$ and $m = n + 1$;
6. *i-th horizontal successor*$+_i1$, such that $+_i1(n_r, m_s)$ iff $r = s = i$ and $m = n + 1$;
7. *vertical successor* $\oplus 1$, such that $\oplus 1(n_r, m_s)$ iff $n = m$ and $s = r + 1$;
8. *i-th vertical successor* $\oplus_i 1$, such that $\oplus_i 1(n_r, m_s)$ iff $n = m = i$ and $s = r + 1$.

T^i, D^i, $+_i1$, and $\oplus_i 1$, and T, D, $+1$, and $\oplus 1$ are respectively called *local* and *global* predicates. Global predicates are depicted in Figure 5. In [16], among the many possible relations between time points belonging to the temporal universe, the above local predicates are identified as the *primitives* for time granularity, that is, relations that any specification language for time granularity should be able to express. Global predicates are a natural generalization of local ones. In particular, the *equi-level* predicate allows one to check whether or not two elements belong to the same layer, while the *equi-column* predicate allows one to verify whether two elements are at the same distance from the origin of the layer they belong to.

Some of the above predicates are functional in nature, and we shall sometimes use a functional notation for them. For instance, we shall write $+1(x) = y$ for $+1(x, y)$ (the same for $\oplus 1$). Moreover, we shall write $T^i(x)$ as a shorthand for $T^i(x, x)$, which states that x belongs to the i-th layer (the same for $D^i(x, x)$). It is worth pointing out that the predicates $+_i 1$ and T^i are inter-definable in MFO$[<, (\downarrow_i)_{i=0}^{k-1}]$ over layered structures as follows:

$$\begin{array}{lcl} +_i 1(x, y) & = & x < y \wedge T^i(x, y) \wedge \forall z((T^i(x, z) \wedge x < z) \rightarrow y \leq z); \\ T^i(x, y) & = & \exists w(+_i 1(x, w)) \wedge \exists w(+_i 1(y, w)). \end{array}$$

Similarly, $+1$ and T are inter-definable in MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ (as a matter of fact, $+1$ is first-order definable in terms of T, while T is second-order definable in terms of $+1$) as follows:

$$\begin{array}{lcl} +1(x, y) & = & x < y \wedge T(x, y) \wedge \forall z((T(x, z) \wedge x < z) \rightarrow y \leq z); \\ T(x, y) & = & \forall X(x \in X \wedge \forall z(z \in X \rightarrow \exists w(+1(z, w) \wedge w \in X)) \rightarrow y \in X) \vee \\ & & \forall X(y \in X \wedge \forall z(z \in X \rightarrow \exists w(+1(z, w) \wedge w \in X)) \rightarrow x \in X). \end{array}$$

It follows that $+_i 1$ (resp. $+1$) is definable in MFO$[<, (\downarrow_i)_{i=0}^{k-1}]$ (resp. MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$) if and only if T^i (resp. T) is definable in MFO$[<, (\downarrow_i)_{i=0}^{k-1}]$ (resp. MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$). In a similar way, it can be shown the inter-definability of the pairs of predicates $(\oplus_i 1, D^i)$ (resp. $(\oplus 1, D)$) in MFO$[<, (\downarrow_i)_{i=0}^{k-1}]$ (resp. MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$).

In the following, we first focus on the upward unbounded layered structure, then we move to the n-layered structure, and finally we consider the downward unbounded one. We study the upward unbounded layered structure before than the n-layered one because some results for the latter can be directly obtained from those for the former.

4 Definability and decidability over the UULS

In this section, we investigate the definability and decidability of the given binary predicates in monadic languages interpreted over the UULS. The i-th equi-level T^i can be defined as follows:

$$\begin{array}{lcl} T^0(x, y) & = & \neg \exists z_1(\downarrow_0 (x) = z_1) \wedge \neg \exists z_2(\downarrow_0 (y) = z_2); \\ T^{i+1}(x, y) & = & \exists z_1 \exists z_2(T^i(z_1, z_2) \wedge \downarrow_0 (x) = z_1 \wedge \downarrow_0 (y) = z_2). \end{array}$$

As we have already shown, the horizontal successor $+_i$ can be defined in terms of T^i. As for D^i, the predicate D^0 can be expressed as follows:

$$\begin{array}{lcl} \mathtt{D}^0(x, y) & = & \exists X(x \in X \wedge y \in X \wedge 0_0 \in X \wedge \forall z((T^0(z) \wedge z \neq 0_0) \rightarrow \\ & & z \notin X) \wedge \forall z(z \in X \rightarrow \exists w(\downarrow_0 (w) = z \wedge w \in X))) \wedge \\ & & \forall z((z \in X \wedge z \neq 0_0) \rightarrow \exists w(\downarrow_0 (z) = w \wedge w \in X)))). \end{array}$$

where 0_0 is the first-order definable origin of layer T^0.
Let $a_n k^n + \ldots a_0 k^0$ be the k-ary representation of i, for any $i > 0$. D^i can be defined as follows:

$$D^i(x, y) \quad = \quad \exists z(D^0(z) \wedge \downarrow_{a_0, \ldots, a_n} (z) = x) \wedge \exists z(D^0(z) \wedge \downarrow_{a_0, \ldots, a_n} (z) = y).$$

where, for any $w \in \{0, \ldots, k-1\}^*$, $\downarrow_w (x)$ is inductively defined as follows: if $w = \epsilon$, then $\downarrow_w (x) = x$, otherwise, if $w = av$, with $a \in \{0, \ldots, k-1\}$ and $v \in \{0, \ldots, k-1\}^*$, then $\downarrow_w (x) = \downarrow_a (\downarrow_v (x))$. The vertical successor $\oplus_i 1$ can be defined in terms of D^i. Notice that second-order quantification comes into play in the definition of D^0 only; furthermore, the semantics of D^0 does not change if we interpret the second-order variable X as a path. Hence, the i-th equi-column D^i and the i-th vertical successor $\oplus_i 1$ can be encoded in MPL$[<, (\downarrow_i)_{i=0}^{k-1}]$, while the i-th equi-level T^i and the i-th horizontal successor $+_i 1$ can be encoded in MFO$[<, (\downarrow_i)_{i=0}^{k-1}]$.

Consider now the global predicates. We start by showing that the addition of the vertical predicates $\oplus 1$ or D to MFO$_\mathcal{P}[<, (\downarrow_i)_{i=0}^{k-1}]$ makes it undecidable. The proof reduces a suitable undecidable version of the tiling problem to the satisfiability problem for MFO$_\mathcal{P}[<, (\downarrow_i)_{i=0}^{k-1}, \oplus 1]$-formulas.

Theorem 4.1 *Both* MFO$_\mathcal{P}[<, (\downarrow_i)_{i=0}^{k-1}, D]$ *and* MFO$_\mathcal{P}[<, (\downarrow_i)_{i=0}^{k-1}, \oplus 1]$ *over the k-ary UULS are undecidable.*

Proof. We prove the theorem for the binary UULS; the generalization to the k-refinable UULS is straightforward. We show that MFO$_\mathcal{P}[<, \downarrow_0, \downarrow_1, \oplus 1]$ over the binary UULS is undecidable by embedding the *octant tiling problem* into it [12]. Since MFO$_\mathcal{P}[<, \downarrow_0, \downarrow_1, \oplus 1] \to$ MFO$_\mathcal{P}[<, \downarrow_0, \downarrow_1, D]$, it follows that MFO$_\mathcal{P}[<, \downarrow_0, \downarrow_1, D]$ is undecidable as well. The octant tiling problem is the problem of establishing whether, given a finite set of tile types $\mathcal{T}$, $\mathcal{T}$ can tile $\mathcal{O} = \bigcup_{i \geq 0}\{(i, j) \mid 0 \leq j \leq i\}$. For every tile type $t \in \mathcal{T}$, let $\texttt{right}(t)$, $\texttt{left}(t)$, $\texttt{up}(t)$, and $\texttt{down}(t)$ be the colors of the corresponding sides of t. The octant tiling problem consists in finding a function $f : \mathcal{O} \to \mathcal{T}$ such that $\texttt{right}(f(n, m)) = \texttt{left}(f(n + 1, m))$ and, whenever $m < n$, $\texttt{up}(f(n, m)) = \texttt{down}(f(n, m + 1))$. We reduce the octant tiling problem to the satisfiability problem for MFO$_\mathcal{P}[<, \downarrow_0, \downarrow_1, \oplus 1]$ over the binary UULS. Let $\mathcal{T} = \{T_1, \dots, T_k\}$ be the set of tile types. We construct a monadic chain formula $\varphi_\mathcal{T}$ such that $\mathcal{T}$ tiles $\mathcal{O}$ if and only if $\varphi_\mathcal{T}$ is satisfiable over the binary UULS.

The first step is forcing the octant grid over the binary UULS $\langle \mathcal{U}, \downarrow_0, \downarrow_1, < \rangle$ (cf. Figure 6). The octant grid domain is the set $\mathcal{G} = \bigcup_{i \geq 0}\{(2^{(i-j)} - 1)_j \mid 0 \leq j \leq i\} \subset \mathcal{U}$. We have that $x \in \mathcal{G}$ if and only if x is reachable through a rightmost path rooted at some point in $\{0_i \mid i \geq 0\}$. The horizontal grid successor $\mathtt{s_0}$ is such that, for every $n_r \in \mathcal{G}$, $\mathtt{s_0}(n_r) = n_{r+1}$, while the vertical grid successor $\mathtt{s_1}$ is such that, for every $n_r \in \mathcal{G}$, with $r > 0$, $\mathtt{s_1}(n_r) = (2n + 1)_{r-1}$. For every $n_r \in \mathcal{G}$, with $r > 0$, it holds that $\mathtt{s_0}(\mathtt{s_1}(n_r)) = \mathtt{s_1}(\mathtt{s_0}(n_r))$. In MFO$_\mathcal{P}[<, \downarrow_0, \downarrow_1, \oplus 1]$ we can define a monadic predicate $\texttt{grid}$ such that $\texttt{grid}(x)$ if and only if x belongs to the octant grid domain $\mathcal{G}$. Let $P_{lp}, Q_{grid} \in \mathcal{P}$. For all x, we have that $\texttt{grid}(x)$ if and only if

$$
\begin{aligned}
& x \in Q_{grid} \wedge 0_0 \in P_{lp} \wedge \forall y((y \in P_{lp} \to \exists z(\downarrow_0 (z) = y \wedge \\
& z \in P_{lp} \wedge \downarrow_1 (z) \notin P_{lp})) \wedge (y \notin P_{lp} \wedge \neg T^0(y) \to \\
& \downarrow_0 (y) \notin P_{lp} \wedge \downarrow_1 (y) \notin P_{lp})) \wedge \forall y((y \in P_{lp} \to y \in Q_{grid}) \wedge \\
& (y \in Q_{grid} \wedge y \in P_{lp} \wedge \neg T^0(y) \to \downarrow_0 (y) \in Q_{grid} \wedge \downarrow_1 (y) \in Q_{grid}) \wedge \\
& (y \in Q_{grid} \wedge y \notin P_{lp} \wedge \neg T^0(y) \to \downarrow_0 (y) \notin Q_{grid} \wedge \downarrow_1 (y) \in Q_{grid}) \wedge \\
& \\
& (y \notin Q_{grid} \wedge \neg T^0(y) \to \downarrow_0 (y) \notin Q_{grid} \wedge \downarrow_1 (y) \notin Q_{grid})).
\end{aligned}
$$

Moreover, the horizontal successor $\mathtt{s_0}$ is $\oplus 1$, and the vertical successor $\mathtt{s_1}$ is $\downarrow_1$. We further need to impose the tiling conditions on the grid. To this end, we take advantage of monadic predicates in $\{P_1, \dots P_k\} \subset \mathcal{P}$ corresponding to the tile types in $\{T_1, \dots T_k\}$:

1. exactly one tile is placed at each node ($\phi_1(x)$)

$$\bigvee_{i=1}^{i=k} x \in P_i \bigwedge_{1 \leq i < j \leq k} \neg(x \in P_i \wedge x \in P_j);$$

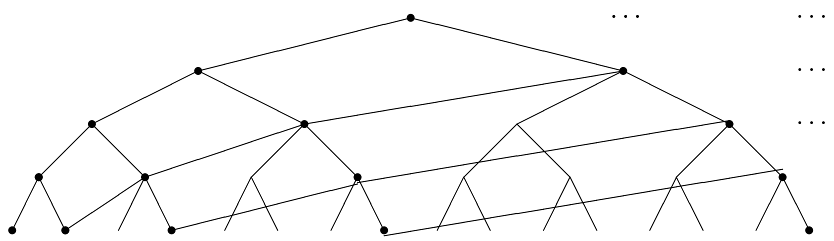

Figure 6: Shaping the octant over the UULS.

2. colors match going right ($\phi_2(x)$)

$$\bigvee_{right(T_i)=left(T_j)} x \in P_i \wedge \oplus 1(x) \in P_j;$$

3. colors match going up ($\phi_3(x)$)

$$\neg T^0(x) \rightarrow \bigvee_{up(T_i)=down(T_j)} x \in P_i \wedge \downarrow_1 (x) \in P_j$$

We define

$$\varphi_{\mathcal{T}} = \forall x(\mathtt{grid}(x) \rightarrow \phi_1(x) \wedge \phi_2(x) \wedge \phi_3(x)).$$

It is not difficult to show that $\mathcal{T}$ tiles $\mathcal{O}$ if and only if $\varphi_{\mathcal{T}}$ is satisfiable over the binary UULS. $\square$

We do not know whether the addition of $+1$ or T to $\mathrm{MFO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ produces the same effect. The decidability problems for both $\mathrm{MFO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}, +1]$ and $\mathrm{MFO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}, T]$ over the UULS are indeed open.

The following theorem shows that both $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}, +1]$ and $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}, T]$ are undecidable. The proof reduces the decidability problem for $\mathrm{MSO}[<, \mathtt{adj}]$ over positive natural numbers ($\langle \mathbb{N}^+, < \rangle$), which has been shown to be undecidable [21], to the decidability problem for $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}, +1]$. The predicate $\mathtt{adj}$ over $\langle \mathbb{N}^+, < \rangle$ is defined as follows: $\mathtt{adj}(x, y)$ if and only if $x = 2^{k_n} + 2^{k_{n-1}} + \ldots + 2^{k_0}$, with $k_n > k_{n-1} > \ldots > k_0 > 0$, and $y = x + 2^{k_0} + 2^{k_0 - 1}$. For instance, if $x = 12 = 2^3 + 2^2$, then $k_0 = 2$ and $y = 12 + 2^2 + 2^1 = 18$, while if $x = 13 = 2^3 + 2^2 + 2^0$, then $k_0 = 0$ and thus there is no y such that $\mathtt{adj}(13, y)$.

Theorem 4.2 *Both* $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}, +1]$ *and* $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}, T]$ *over the* k*-ary UULS are undecidable.*

Proof. We prove the thesis for the binary UULS; the generalization to the k-ary UULS is straightforward. To show that $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}, +1]$ is undecidable, we embed $\mathrm{MSO}[<, \mathtt{adj}]$ over $\langle \mathbb{N}^+, < \rangle$ into $\mathrm{MSO}[<, \downarrow_0, \downarrow_1, +1]$ over the binary UULS. Since T and $+1$ are inter-definable, the undecidability result holds for $\mathrm{MSO}[<, \downarrow_0, \downarrow_1, T]$ as well. We define the binary predicate $\mathtt{adj}$ over the UULS as follows (we are overloading the symbol $\mathtt{adj}$): for every $n_r \in \mathcal{U}$, we have $\mathtt{adj}(n_r, m_s)$ if and only if $r > 0$, $s = r - 1$, and $m = 2(n + 1)$.

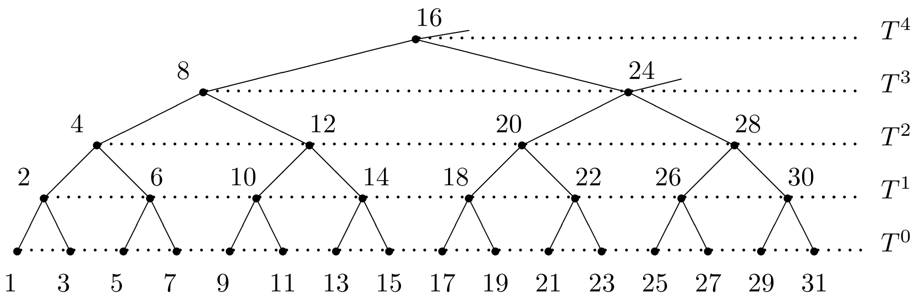

Figure 7: The 2-refinable UULS over the natural numbers.

It is easy to check that $\mathtt{adj}(x, y)$ if and only if y is the *adoptive* son of x, that is, y is the horizontal successor of the right son of x. As an example, we have that $\mathtt{adj}(1_2, 4_1)$ (cf. Figure 2). The predicate adj is thus definable in MSO$[<, \downarrow_0, \downarrow_1, +1]$ over the 2-refinable UULS as follows:

$$\mathtt{adj}(x, y) \quad = \quad \exists z(\downarrow_1 (x) = z \wedge +1(z, y)).$$

Consider now the bijection $\tau : \mathcal{U} \to \mathbb{N}^+$ defined as follows: for every $n_r \in \mathcal{U}$, $\tau(n_r) = 2^r + n2^{r+1}$ (τ is graphically depicted in Figure 7). It is easy to see that τ is an isomorphism between $\langle \mathcal{U}, <, \mathtt{adj} \rangle$ and $\langle \mathbb{N}^+, <, \mathtt{adj} \rangle$. From the undecidability of MSO$[<, \mathtt{adj}]$ over $\langle \mathbb{N}^+, < \rangle$, it follows that MSO$[<, \mathtt{adj}]$ over the UULS is undecidable. Since MSO$[<, \mathtt{adj}] \to$ MSO$[<, \downarrow_0, \downarrow_1, +1]$, we have the thesis. $\square$

From Theorem 4.1, it immediately follows that MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ extended with either $\oplus 1$ or D is undecidable. Hence, putting together Theorems 4.1 and 4.2, we obtain the following corollary.

Corollary 4.3 *Global predicates cannot be defined in* MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ *over the k-ary UULS and the extension of* MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ *with any global predicate is undecidable.*

The following theorem and its corollary provide information about the relationships between horizontal and vertical predicates over the UULS (we shall take advantage of such relationships in subsequent theorems). More precisely, Theorem 4.4 shows that $+1$ is first-order definable in terms of D over the binary UULS.

Theorem 4.4 MFO$[<, \downarrow_0, \downarrow_1, +1]$ *is embedable in* MFO$[<, \downarrow_0, \downarrow_1, D]$ *over the binary UULS.*

Proof. First, the horizontal successor $+1$ can be defined in terms of the predicate adj over the UULS as follows:

$$+1(x, y) \quad = \quad \exists z((\downarrow_0 (z) = x \wedge \downarrow_1 (z) = y) \vee (\downarrow_1 (z) = x \wedge \mathtt{adj}(z, y))).$$

Next, we encode the predicate adj in MFO$[<, \downarrow_0, \downarrow_1, D]$. We claim that $\mathtt{adj}(x, y)$ if and only if $\phi(x, y)$, where

$$\begin{aligned}\phi(x, y) \quad = \quad & \neg T^0(x) \wedge \exists z_1 \exists z_2 \exists z_3 \exists z_4 \exists z_5 (T^0(z_1) \wedge D(z_1, x) \wedge +_0 1(z_1, z_2) \wedge \\ & \oplus 1(z_2, z_3) \wedge \downarrow_0 (z_3) = z_4 \wedge D(z_4, z_5) \wedge \forall w(D(z_4, w) \wedge x \leq w \to \\ & z_5 \leq w) \wedge (D^0(x) \wedge x \neq 0_1 \to \oplus 1(z_5, y)) \wedge \\ & ((\neg D^0(x) \vee x = 0_1) \to z_5 = y)).\end{aligned}$$

We prove that the above definition captures the predicate $\mathtt{adj}$. Let $x = n_r$, $y = m_s$, and x does not belong to T^0 (that is, $r \geq 1$). Suppose that $\phi(x, y)$ holds. Then, there exist $z_1, \ldots, z_5$, such that $z_1 = n_0$, $z_2 = (n+1)_0$, $z_3 = (n+1)_1$, $z_4 = (2(n+1))_0$, and $z_5 = \min\{w \mid w = (2(n+1))_i \wedge i \geq 0 \wedge n_r \leq w\}$. We show that $z_5 = y = (2(n+1))_{r-1}$, and thus $\mathtt{adj}(x, y)$ holds, whenever $\neg D^0(x)$ or $x = 0_1$, and $z_5 = (2(n+1))_{r-2}$, $y = (2(n+1))_{r-1}$, and thus $\mathtt{adj}(x, y)$ holds, whenever $D^0(x)$ and $x \neq 0_1$ (remind that $\mathtt{adj}(x, y)$ if and only if $r \geq 1$, $s = r - 1$ and $m = 2(n+1)$). Suppose $\neg D^0(x)$, that is, $n \geq 1$. Since, for every $i, j \geq 0$, $i_j < i_{j+1}$, we only have to prove that $(2(n+1))_{r-2} < n_r < (2(n+1))_{r-1}$. To conclude that $(2(n+1))_{r-2} < n_r$, it suffices to prove that $(2(n+1))_{r-2} \leq (4n)_{r-2}$, and this follows from the fact that $2(n+1) \leq 4n$ whenever $n \geq 1$. Moreover, $(4n)_{r-2} < n_r$, since $(4n)_{r-2} = \downarrow_0 (\downarrow_0 (n_r))$, and, for every point v, $\downarrow_0 (v) < v$. To complete the argument for the $\neg D^0(x)$ case, we must prove that $n_r < (2(n+1))_{r-1}$. It immediately follows from the fact that $(2(n+1))_{r-1} = +1(\downarrow_1 (n_r))$, and, for every v, $v <\downarrow_1 (v)$ and $v < +1(v)$. The other two cases ($x = 0_1$ and $D^0(x) \wedge x \neq 0_1$) are easier. Similarly, we can prove that if $\mathtt{adj}(x, y)$ holds, then $\phi(x, y)$ holds. Hence the thesis. $\square$

Since $+1$ and T are inter-definable in the MSO language over the (binary) UULS, and the same holds for $\oplus$ and D, we have the following corollary.

Corollary 4.5 *Let $o \in \{+1, T\}$ and $v \in \{\oplus 1, D\}$.* MSO$[<, \downarrow_0, \downarrow_1, o]$ *is embedable into* MSO$[<, \downarrow_0, \downarrow_1, v]$ *over the binary UULS.*

To complete the picture, we consider the decidability problem for the extensions of MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ fragments with global predicates. Surprisingly, it turns out that the addition of global predicates to the chain fragment MCL$[<, (\downarrow_i)_{i=0}^{k-1}]$ preserves decidability (global predicates are obviously *not* definable in such a fragment). More precisely, we prove that the decidability problem for both MCL$[<, (\downarrow_i)_{i=0}^{k-1}, T, +1, \oplus 1]$ and MCL$[<, (\downarrow_i)_{i=0}^{k-1}, D, +1, \oplus 1]$ can be reduced to the decidability problem for MSO$[<]$ over natural numbers by exploiting two different encodings. As a matter of fact, we do not know whether the same holds for MCL$[<, (\downarrow_i)_{i=0}^{k-1}, T, D, +1, \oplus 1]$ or not.

Theorem 4.6 MCL$[<, (\downarrow_i)_{i=0}^{k-1}, T, +1, \oplus 1]$ *over the k-ary UULS is decidable.*

Proof. The proof is given for $k = 2$, and it can be easily extended to the general case of an arbitrary k. As a preliminary result, we prove that MCL$[<, \downarrow_0, \downarrow_1, T, +1, \oplus 1] \rightarrow$ MCL$[\downarrow_0, \downarrow_1, T, \oplus 1]$. First of all, the horizontal successor $+1$ is first-order definable in terms of T and $<$. Next, we prove that $<$ can be removed. We have that $x < y$ can be defined as follows:

$$x < y \quad = \quad x \neq y \wedge (x \in t_{\downarrow 0(y)} \vee y \in t_{\downarrow 1(x)} \vee \exists z(x \in t_{\downarrow 0(z)} \wedge y \in t_{\downarrow 1(z)})),$$

where $y \in t_{\downarrow i(x)}$ stands for 'y belongs to the tree rooted at the i-th son of x'($y \leq_P \downarrow_i (x)$ for short). In its turn, the formula $x <_P y$ can be defined as follows:

$$\begin{aligned} x <_P y \quad = \quad & x \neq y \wedge \exists X(\mathtt{dcPath}(X) \wedge x \in X \wedge y \in X \wedge \\ & \exists z(\textstyle\bigvee_{i \in \{0,1\}} \downarrow_i (z) = x \wedge z \notin X)), \end{aligned}$$

where $\mathtt{dcPath}(X)$ is the MCL$[\downarrow_0, \downarrow_1, T, \oplus 1]$ formula that constrains the chain X to be a downward closed path:

$$\mathtt{dcPath}(X) \quad = \quad \forall x((x \in X \wedge x \notin T^0) \rightarrow (\downarrow_0 (x) \in X \vee \downarrow_1 (x) \in X)).$$

Finally, it is not difficult to show that MCL[$\downarrow_0, \downarrow_1, T, \oplus 1$] is equivalent to a version of chain logic in which only second-order variables occur and atomic formulas are of the forms $X_1 \subseteq X_2$ (chain X_1 is included in chain X_2), $\mathtt{Sing}(X)$ (chain X is a singleton), $\mathtt{proj}_i(X_1, X_2)$ (chain $X_1 = \{x_1\}$, chain $X_2 = \{x_2\}$ and $\downarrow_i (x) = y$), $\mathtt{equiL}(X_1, X_2)$ (chain $X_1 = \{x_1\}$, chain $X_2 = \{x_2\}$, and $T(x, y)$), and $\mathtt{vsucc}(X_1, X_2)$ (chain $X_1 = \{x_1\}$, chain $X_2 = \{x_2\}$, and $\oplus 1(x, y)$).

It is possible to prove that every formula of the resulting chain logic can be encoded into an equi-satisfiable formula of MSO[$<$], over natural numbers. The proof is by induction on the structure of the formulas of the chain logic (the idea is partly borrowed from [23]).

Any second-order variable X interpreted as a chain is encoded by a pair of set variables Ch_X and Lv_X over the natural numbers. Ch_X is interpreted as a set of natural numbers encoding the leftmost upward unbounded path (starting from the first layer) containing the chain X, i.e., $i \in Ch_X$ if and only if the element of the i-th layer of the mentioned path is a right-hand side son. Lv_X is interpreted as a set of natural numbers describing the elements of the path actually belonging to the chain, i.e., $i \in Lv_X$ if and only if the element of the i-th layer of the path belongs to the chain X. To guarantee that a chain X corresponds to a unique pair (Ch_X, Lv_X), we impose the condition $\mathtt{unique}(Ch_X, Lv_X) = \forall y(y \in Ch_X \rightarrow \mathtt{min}(Lv_X) \leq y)$, where $\mathtt{min}(X)$ is the minimum of the set X of natural numbers with respect to the usual ordering relation $<$. As for atomic formulas, $X_1 \subseteq X_2$ is encoded as $Ch_{X_1} \subseteq Ch_{X_2} \wedge Lv_{X_1} \subseteq Lv_{X_2}$; $\mathtt{Sing}(X)$ in encoded as 'Lv_X is a singleton', that is, $\exists Y(Y \subseteq Lv_X \wedge Lv_X \neq Y \wedge \neg\exists Z(Z \subseteq Lv_X \wedge Z \neq Lv_X \wedge Z \neq Y))$; $\mathtt{proj}_0(X_1, X_2)$ is encoded as $Ch_{X_1} = Ch_{X_2}$, 'Lv_{X_1} is a singleton $\{x_1\}$', 'Lv_{X_2} is a singleton $\{x_2\}$', and $x_1 = x_2 + 1$; $\mathtt{proj}_1(X_1, X_2)$ is encoded as 'Lv_{X_1} is a singleton $\{x_1\}$', 'Lv_{X_2} is a singleton $\{x_2\}$', $x_1 = x_2 + 1$, and $Ch_{X_1} = Ch_{X_2} \cup \{x_2\}$; $\mathtt{equiL}(X_1, X_2)$ is encoded as 'Lv_{X_1} is a singleton $\{x_1\}$', 'Lv_{X_2} is a singleton $\{x_2\}$', and $x_1 = x_2$; $\mathtt{vsucc}(X_1, X_2)$ is encoded as 'Lv_{X_1} is a singleton $\{x_1\}$', 'Lv_{X_2} is a singleton $\{x_2\}$', $x_1 = x_2 + 1$, and $Ch_{X_2} = \{n + 1 \mid n \in Ch_{X_1}\}$. The inductive cases $\wedge$ and $\neg$ are trivial. The second-order existentially quantified chain formula $\exists X \phi(X)$ is translated into the formula $\exists Ch_X \exists Lv_X(\mathtt{unique}(Ch_X, Lv_X) \wedge \phi^\tau(Ch_X, Lv_X))$, where ϕ^τ is the translation of ϕ. $\square$

	T	$+1$	D	$\oplus 1$
MFO	Decidable	Decidable	Decidable	Decidable
MPL	Decidable	Decidable	Decidable	Decidable
MCL	Decidable	Decidable	Decidable	Decidable
MFO$_\mathcal{P}$	?	?	Undecidable	Undecidable
MPL$_\mathcal{P}$	?	?	Undecidable	Undecidable
MCL$_\mathcal{P}$	?	?	Undecidable	Undecidable
MSO = MSO$_\mathcal{P}$	Undecidable	Undecidable	Undecidable	Undecidable

Table 1: Decidability results for the UULS.

Theorem 4.7 MCL[$<, (\downarrow_i)_{i=0}^{k-1}, D, +1, \oplus 1$] *over the k-ary UULS is decidable.*

Proof. The proof is given for $k = 2$, and it can be easily extended to any k. Moreover, since $\oplus 1$ is first-order definable in terms of D, we have that MCL$[<, \downarrow_0, \downarrow_1, D, +1, \oplus 1] \to$ MCL$[<, \downarrow_0, \downarrow_1, D]$ thus allowing us to focus on the decidability of the latter.

We prove the decidability of MCL$[<, \downarrow_0, \downarrow_1, D]$ by encoding any formula ϕ of MCL$[<, \downarrow_0, \downarrow_1, D]$ into an equi-satisfiable formula of $MSO[<]$ over natural numbers.

We start by showing the encoding of individual variables and set variables (interpreted as chains). Any individual variable x is mapped into a pair of individual variables s_x (shift) and l_x (length), and a set variable Ch_x, which satisfy the constraint $Point_x = l_x \leq s_x \wedge \forall z(z \in Ch_x \to 0 < z \leq l_x) \wedge (l_x > 0 \to 1 \in Ch_x)$. Such a triplet can be viewed as the specification of a point in the UULS as follows: it identifies the point of layer $s_x - l_x$ which can be reached from the point of layer s_x belonging to the leftmost branch of the tree by following the path of length l_x codified in Ch_x ($i \in Ch_x$ if and only if the i-th step of the path leads from a point z to the point $\downarrow_1 (z)$, and thus $i \notin Ch_x$ if and only if the i-th step of the path leads from a point z to the point $\downarrow_0 (z)$). Notice that the points belonging to the leftmost branch of the tree are characterized by $l_x = 0$ and thus $Ch_x = \emptyset$. Any set variable X is mapped into an individual variable s_X and three set variables Ch_X, Up_X, and $Down_X$ which satisfy the constraint $Chain_X = \forall z(z \in Ch_X \to 0 < z \leq s_X) \wedge \forall z(z \in Down_X \to z \leq s_X) \wedge \forall z(z \in Up_X \to z > s_X) \wedge (s_X > 0 \to 1 \in Ch_X)$. Such a tuple determines a chain in the UULS as follows: s_X and Ch_X identify the full path lying on the leftmost branch of the structure until layer s_X and then following the path specified by Ch_X from the layer s_X to layer 0; Up_X (resp. $Down_X$) identifies the subset of layers greater than (resp. less than or equal to) s_X to which the elements of the chain belong.

On the basis of the given correspondence of variables, we inductively define the translation τ of $MSO[<, \downarrow_0, \downarrow_1, D]$ formulas into $MSO[<]$ formulas as follows:

$$\tau(x = y) \quad \text{is} \quad Ch_x = Ch_y \wedge l_x = l_y \wedge s_x = s_y;$$

$$\begin{array}{lll} \tau(x < y) & \text{is} & s_x < s_y \vee (s_x = s_y \wedge (\exists k(k \in Ch_y \wedge k \notin Ch_x \wedge \\ & & \forall k'(k' < k \to (k' \in Ch_y \leftrightarrow k' \in Ch_x))) \vee \\ & & (l_y < l_x \wedge l_y + 1 \notin Ch_x \wedge \forall k(k \in Ch_y \to k \in Ch_x \wedge \\ & & (k \leq l_y \wedge k \in Ch_x) \to k \in Ch_y)))); \end{array}$$

$$\tau(\downarrow_0 (x) = y) \quad \text{is} \quad s_x \geq l_x + 1 \wedge Ch_x = Ch_y \wedge l_x + 1 = l_y \wedge s_x = s_y;$$

$$\begin{array}{lll} \tau(\downarrow_1 (x) = y) & \text{is} & s_x \geq l_x + 1 \wedge Ch_x \cup \{l_x + 1\} = Ch_y \wedge l_x + 1 = l_y \wedge \\ & & s_x = s_y; \end{array}$$

$$\tau(D(x, y)) \quad \text{is} \quad Ch_x = Ch_y \wedge l_x = l_y;$$

$$\begin{array}{lll} \tau(x \in Y) & \text{is} & (s_y < s_x \wedge l_x = 0 \wedge s_x \in Up_Y) \vee (s_y = s_x \wedge l_x \in Down_Y \wedge \\ & & \forall z(0 < z \leq l_x \to (z \in Ch_x \leftrightarrow z \in Ch_Y))); \end{array}$$

$$\begin{array}{lll} \tau(\phi \wedge \psi) & \text{is} & \tau(\phi) \wedge \tau(\psi); \\ \tau(\neg \phi) & \text{is} & \neg \tau(\phi); \\ \tau(\exists x \phi) & \text{is} & \exists Ch_x, l_x, s_x(Point_x \wedge \tau(\phi)); \end{array}$$

$$\tau(\exists X \phi) \quad \text{is} \quad \exists Ch_X, s_X, Down_X, Up_X(Chain_X \wedge \tau(\phi)).$$

The equi-satisfiability of ϕ and $\tau(\phi)$ can be easily proved by induction on the structure of ϕ. $\square$

$$
\begin{array}{ccccc}
\mathcal{S}L_\omega(\text{Trellis}) & \subseteq & \text{MSO}[<, 2\times] & = & \text{MSO}[<, \downarrow_0, \downarrow_1, D] \\
\cup\nmid & & \cup\mid & & \cup\mid \\
\mathcal{S}L_\omega(\text{Y-Trees}) & \subsetneq & \text{MSO}[<, \texttt{adj}] & = & \text{MSO}[<, \downarrow_0, \downarrow_1, T] \\
\cup\nmid & & \cup\nmid & & \cup\nmid \\
\mathcal{S}L_\omega(\text{B-Trees}) & = & \text{MSO}[<, \texttt{flip}] & = & \text{MSO}[<, \downarrow_0, \downarrow_1]
\end{array}
$$

Figure 8: Systolic ω-languages and monadic second-order theories.

The decidability results for the UULS are summarized in Table 1 (a question mark stands for an open problem). Such results allow us to connect MSO languages over the binary UULS (Figure 8, right column) to both MSO languages over $\mathbb{N}^+$ (Figure 8, middle column) and systolic ω-languages over binary trees, Y-trees, and trellis [20, 21] (Figure 8, left column)[1]. They establish a connection between MSO$[<, 2\times]$ (resp. MSO$[<, \texttt{adj}]$) over $\mathbb{N}^+$ and MSO$[<, \downarrow_0, \downarrow_1, D]$ (resp. MSO$[<, \downarrow_0, \downarrow_1, T]$) over the binary UULS. Moreover, we have that MSO$[<, \downarrow_0, \downarrow_1, T]$ is a proper extension of MSO$[<, \downarrow_0, \downarrow_1]$, and that it can be embedded into MSO$[<, \downarrow_0, \downarrow_1, D]$. One advantage of such a connection is a different and more intuitive characterization of Y-tree ω-automata: every Y-tree automaton A can be associated with an MSO$[<, \downarrow_0, \downarrow_1, T]$-formula φ_A interpreted over the binary UULS such that the models of φ_A are, modulo an isomorphism, all and only the ω-words accepted by A. The opposite embedding does not hold, since Y-tree automata are not closed under complementation. Similarly, trellis ω-automata, whose expressive power is greater than that of Y-tree automata, can be embedded into MSO$[<, \downarrow_0, \downarrow_1, D]$-formulas. In this case, the opposite embedding is an open problem interestingly related to the closure under complementation of the well-known computational complexity class NP [21].

5 Definability and decidability over the n-LS

In this section, we investigate the definability and decidability of local and global binary predicates in monadic languages interpreted over the n-LS. We start with the i-th equi-level predicate T^i which can be inductively defined as follows:

$$
\begin{array}{lcl}
T^0(x,y) & = & \neg\exists z_1(\downarrow(z_1) = x) \wedge \neg\exists z_2(\downarrow(z_2) = y); \\
T^{i+1}(x,y) & = & \exists z_1 \exists z_2 (T^i(z_1, z_2) \wedge \downarrow(z_1) = x \wedge \downarrow(z_2) = y),
\end{array}
$$

where $\downarrow(x) = y$ is a shorthand for $\bigvee_{j=0}^{k-1} \downarrow_j(x) = y$.
The equi-level predicate $T(x,y)$ can be defined as $\bigvee_{i=0}^{n-1}(T^i(x,y))$. Horizontal successors $+_i 1$ and $+1$ are definable in terms of T^i and T, respectively. Consider now the equi-column predicate D^i. Let $+_i j(x,y)$ be a shorthand for $(+_i 1)^j(x) = y$ and 0_0 be the first-order definable origin of T^0. $D^0(x,y)$ can be defined as follows:

$$
D^0(x,y) = \bigvee_{i=0}^{n-1}(\downarrow_{0^i}(0_0) = x) \wedge \bigvee_{i=0}^{n-1}(\downarrow_{0^i}(0_0) = y).
$$

Then, for $i > 0$, we define $D^i(x,y)$ as follows:

[1] A survey on systolic computations can be found in [11].

$$D^i(x,y) \quad = \quad \bigvee_{j=0}^{n-1} \exists z(T^j(z) \wedge D^0(z) \wedge +_j i(z,x)) \wedge \bigvee_{j=0}^{n-1} \exists z(T^j(z) \wedge D^0(z) \wedge +_j i(z,y)).$$

Once more, the vertical successor predicate $\oplus_i 1$ can be defined in terms of D^i. Since all the above definitions do not exploit second-order quantification, we can conclude that all local predicates and the global predicates T and $+1$ are definable in MFO$[<, (\downarrow_i)_{i=0}^{k-1}]$ over the n-LS.

We now turn to the global predicates equi-column D and vertical successor $\oplus 1$. We show that D is not definable in MSO$[<, \downarrow_0, \downarrow_1]$ over the binary 2-layered structure, and, even worse, the addition of D to MSO$[<, \downarrow_0, \downarrow_1]$ yields undecidability. Since D and $\oplus 1$ are inter-definable, the same holds for $\oplus 1$. Moreover, it is easy to show that MSO$[<, \downarrow_0, \downarrow_1]$ over the binary 2-layered structure can be embedded in MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ over the k-ary n-layered structures (for any k and n), and thus all the above results generalize to MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ over the k-ary n-layered structure.
We begin with an auxiliary lemma. Let us define the predicate D over natural numbers as the reflexive and symmetric closure of the following set:

$$\{(3k, \frac{3k+2}{2}) \mid k \text{ even}\} \cup \{(3k, \frac{3k+1}{2}) \mid k \text{ odd}\}.$$

The following lemma proves that MSO$[<]$ over the natural numbers cannot be extended with such a predicate preserving decidability.

Lemma 5.1 MSO$[<, D]$ *over* $\langle \mathbb{N}, < \rangle$ *is undecidable.*

Proof. Let $P_0 = \{3n \mid n \geq 0\}$, $P_1 = \{3n+1 \mid n \geq 0\}$ and $P_2 = \{3n+2 \mid n \geq 0\}$ be three unary predicates over natural numbers representing the congruence classes modulo 3. They can be easily defined in MSO$[<]$. For instance, $P_0(x)$ is defined as follows:

$$P_0(x) \quad = \quad \exists X(x \in X \wedge 0 \in X \wedge \forall y, v, z, w(y \in X \rightarrow ((+1(y,v) \rightarrow v \notin X) \wedge (+2(y,z) \rightarrow z \notin X) \wedge (+3(y,w) \rightarrow w \in X))),$$

where 0 is the first-order definable constant representing the natural number 0 and $+1$, $+2$ and $+3$ are the first-order definable predicates defining the first, the second, and the third successor of a point, respectively. By exploiting $P_0(x)$, $P_1(x)$, and $P_2(x)$ and the relation D, we are able to define the relation $2\times$ such that $2\times(x,y)$ if and only if $y = 2x$ as follows:

$$\begin{aligned} 2\times(x,y) \quad = \quad & (x = 0 \rightarrow y = 0) \wedge \exists z, w(\\ & (P_0(x) \wedge +1(x,w) \wedge D(w,z) \wedge w \neq z) \rightarrow y = z \wedge \\ & (P_1(x) \wedge +2(z,w) \wedge D(x,z) \wedge x \neq z) \rightarrow y = w \wedge \\ & (P_2(x) \wedge +1(z,w) \wedge D(x,z) \wedge x \neq z) \rightarrow y = w). \end{aligned}$$

It is well known that MSO$[<, 2\times]$ over natural numbers is undecidable, since it allows one to interpret full first-order arithmetic. This allows us to conclude that MSO$[<, D]$ over the natural numbers is undecidable. □

To prove our thesis, it suffices to show now that MSO$[<, D]$ over natural numbers can be embedded into MSO$[<, \downarrow_0, \downarrow_1, D]$ over the binary 2-LS.

Theorem 5.2 *The predicate D (resp. $\oplus 1$) is not definable in* MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ *over the k-ary n-LS and the extension of* MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ *with D (resp. $\oplus 1$) is undecidable.*

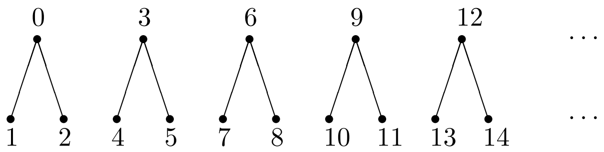

Figure 9: The binary 2-layered structure over the natural numbers.

Proof. We first show that MSO$[<, D]$ over natural numbers can be embedded into MSO$[<, \downarrow_0, \downarrow_1, D]$ over 2-layered binary structures (for notational simplicity, we are overloading the symbols $<$ and D). Let us consider the bijection $\tau : \mathcal{U} \rightarrow \mathbb{N}$ (depicted in Figure 9) defined as follows: $\tau(n_0) = 3n$, $\tau(n_1) = (3n+2)/2$ if n is even, and $\tau(n_1) = (3n+1)/2$ if n is odd. It is easy to see that τ is an isomorphism between $\langle \mathcal{U}, <, D\rangle$ and $\langle \mathbb{N}, <, D\rangle$. It follows that, for every $\varphi \in$ MSO$[<, D]$, φ is satisfiable over $\langle \mathbb{N}, <, D\rangle$ if and only if φ is satisfiable over $\langle \mathcal{U}, <, D\rangle$. From Lemma 5.1, it immediately follows that MSO$[<, \downarrow_0, \downarrow_1, D]$ is undecidable. Moreover, it is easy to show that MSO$[<, \downarrow_0, \downarrow_1]$ over the binary 2-LS is embedable in MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ over the k-ary n-LS. Hence, MSO$[<, (\downarrow_i)_{i=0}^{k-1}, D]$ and, thus, MSO$[<, (\downarrow_i)_{i=0}^{k-1}, \oplus]$ are undecidable. Since MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ over the k-ary n-LS is decidable (Theorem 2.2), we have that D and $\oplus$ are not definable in MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$. $\square$

Finally, it is possible to show that the extensions of the chain, path, and first-order fragments of MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ with the undefinable predicate D are decidable. The decidability of MCL$[<, (\downarrow)_{i=0}^{k-1}, D]$ is a consequence of Theorem 4.7, since the n-LS can be easily embedded into the UULS. The decidability of both MPL$[<, (\downarrow)_{i=0}^{k-1}, D]$ and MFO$[<, (\downarrow)_{i=0}^{k-1}, D]$ immediately follows. Since $\oplus 1$ is first-order definable in terms of D, it follows that the extensions of the chain, path, and first-order fragments of MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ with $\oplus 1$ are decidable as well. The decidability results for the n-layered structure are summarized in Table 2.

	T	$+1$	D	$\oplus 1$
MFO	Decidable	Decidable	Decidable	Decidable
MPL	Decidable	Decidable	Decidable	Decidable
MCL	Decidable	Decidable	Decidable	Decidable
MFO$_\mathcal{P}$	Decidable	Decidable	?	?
MPL$_\mathcal{P}$	Decidable	Decidable	?	?
MCL$_\mathcal{P}$	Decidable	Decidable	?	?
MSO = MSO$_\mathcal{P}$	Decidable	Decidable	Undecidable	Undecidable

Table 2: Decidability results for the n-LS.

6 Definability and decidability over the DULS

We conclude the paper by investigating the definability and decidability of the given binary predicates in monadic languages interpreted over the DULS. The local predicates T^i and $+_i 1$ can be expressed as in the case of the n-LS. On the contrary, the definition of the local predicate D^i given in the case of the n-LS does not work anymore since we have to cope with an infinite number of layers. We first define $\mathtt{D}^0(x, y)$ as follows:

$$\begin{aligned} D^0(x,y) \quad = \quad & \exists X(x \in X \wedge y \in X \wedge 0_0 \in X \wedge \forall z(T^0(z) \wedge z \neq 0_0 \rightarrow \\ & z \notin X) \wedge \forall z((z \in X \rightarrow (\downarrow_0 (z) \in X \wedge \textstyle\bigwedge_{i=1}^{k-1} \downarrow_i (z) \notin X)) \wedge \\ & (z \notin X \rightarrow \textstyle\bigwedge_{i=0}^{k-1} \downarrow_i (z) \notin X))), \end{aligned}$$

where 0_0 is the first-order definable origin of layer T^0. For $i > 0$, let $a_n k^n + \ldots + a_0 k^0$ be the k-ary representation of i. We define $D^i(x, y)$ as follows:

$$\begin{aligned} D^i(x,y) \quad = \quad & (\textstyle\bigvee_{j=0}^{\lfloor \log_k(i) \rfloor} \exists z(D^0(z) \wedge T^j(z) \wedge +_j i(z) = x) \vee \exists z(D^0(z) \wedge \\ & \downarrow_{a_0,\ldots,a_n} (z) = x)) \wedge (\textstyle\bigvee_{j=0}^{\lfloor \log_k(i) \rfloor} \exists z(D^0(z) \wedge T^j(z) \wedge \\ & +_j i(z) = y) \vee \exists z(D^0(z) \wedge \downarrow_{a_0,\ldots,a_n} (z) = y)). \end{aligned}$$

As we have already shown, $\oplus_i 1$ can be defined in terms of D^i. Moreover, as in the case of the UULS, second-order quantification is needed to define D^0 only, and the semantics of D^0 does not change if we interpret the second-order variable X as a path. Hence, both the i-th equi-column D^i and the i-th vertical successor $\oplus_i 1$ can be encoded in MPL$[<, (\downarrow_i)_{i=0}^{k-1}]$, while the i-th equi-level T^i and the i-th horizontal successor $+_i 1$ can be encoded in MFO$[<, (\downarrow_i)_{i=0}^{k-1}]$.

Let us consider now the global predicates. As in the case of the UULS, the addition of the vertical predicates $\oplus 1$ or D to MFO$_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ leads to undecidability. The proof takes advantage of a reduction of the $\mathbb{N} \times \mathbb{N}$ tiling problem to the satisfiability problem for MFO$_{\mathcal{P}}[<, \downarrow_0, \downarrow_1, \oplus 1]$ (as a matter of fact, the proof of Theorem 6.1 does not exploit the whole DULS, but only its first tree).

Theorem 6.1 *Both* MFO$_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}, D]$ *and* MFO$_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}, \oplus 1]$ *over the k-ary DULS are undecidable.*

Proof. We show that both theories are undecidable over the binary infinite tree. Since the binary infinite tree is embedable into the k-ary DULS, we have the thesis. We show that MFO$_{\mathcal{P}}[<, \downarrow_0, \downarrow_1, \oplus 1]$ over the binary infinite tree is undecidable by embedding the $\mathbb{N} \times \mathbb{N}$ tiling problem into it [2]. Since MFO$_{\mathcal{P}}[<, \downarrow_0, \downarrow_1, \oplus 1] \rightarrow$ MFO$_{\mathcal{P}}[<, \downarrow_0, \downarrow_1, D]$, it follows that MFO$_{\mathcal{P}}[<, \downarrow_0, \downarrow_1, D]$ is undecidable as well. The $\mathbb{N} \times \mathbb{N}$ tiling problem is the problem of establishing whether, given a finite set of tile types $\mathcal{T}$, $\mathcal{T}$ can tile $\mathbb{N} \times \mathbb{N}$. For every tile type $t \in \mathcal{T}$, let $\mathtt{right}(t)$, $\mathtt{left}(t)$, $\mathtt{up}(t)$, and $\mathtt{down}(t)$ be the colors of the corresponding sides of t. We must find a function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathcal{T}$ such that $\mathtt{right}(f(n, m)) = \mathtt{left}(f(n + 1, m))$ and $\mathtt{up}(f(n, m)) = \mathtt{down}(f(n, m + 1))$. The embedding is accomplished as follows. Let $\mathcal{T} = \{T_1, \ldots, T_k\}$ be the set of tile types. We construct a formula $\varphi_{\mathcal{T}} \in$ MFO$_{\mathcal{P}}[<, \downarrow_0, \downarrow_1, \oplus 1]$ such that $\mathcal{T}$ tiles $\mathbb{N} \times \mathbb{N}$ if and only if $\varphi_{\mathcal{T}}$ is satisfiable over the binary infinite tree.

The first step consists in the construction of the grid over the binary infinite tree $\langle \{0, 1\}^*, < \rangle$. The grid is given by the domain $\{0^* 1^*\}$, and the horizontal (resp. vertical) successor $\mathtt{s}_0$ (resp. $\mathtt{s}_1$) is such that $\mathtt{s}_0(x) = x1$ (resp. $\mathtt{s}_1(x) = 0x$). For every

$x \in \{0^*1^*\}$, it holds that $\mathtt{s}_0(\mathtt{s}_1(x)) = \mathtt{s}_1(\mathtt{s}_0(x))$. In $\mathrm{MFO}_{\mathcal{P}}[<, \downarrow_0, \downarrow_1, \oplus 1]$, we can define a monadic predicate $\mathtt{grid}$ such that $\mathtt{grid}(x)$ if and only if x belongs to the grid domain $\{0^*1^*\}$. Notice that $\{0^*1^*\} = \bigcup_{i\geq 0}\{0^i1^*\}$, and thus $\mathtt{grid}(x)$ if and only if x is reachable through a rightmost path rooted at some point in $\{0^*\}$. Let $P_{lp}, Q_{grid} \in \mathcal{P}$. For every x, we have that $\mathtt{grid}(x)$ if and only if

$$
\begin{aligned}
&x \in Q_{grid} \wedge \epsilon \in P_{lp} \wedge \forall y((y \in P_{lp} \rightarrow \downarrow_0 (y) \in P_{lp} \wedge \downarrow_1 (y) \notin P_{lp}) \wedge \\
&(y \notin P_{lp} \rightarrow \downarrow_0 (y) \notin P_{lp} \wedge \downarrow_1 (y) \notin P_{lp})) \wedge \forall y((y \in P_{lp} \rightarrow y \in Q_{grid}) \wedge \\
&(y \in Q_{grid} \wedge y \in P_{lp} \rightarrow \downarrow_0 (y) \in Q_{grid} \wedge \downarrow_1 (y) \in Q_{grid}) \wedge (y \in Q_{grid} \wedge \\
&y \notin P_{lp} \rightarrow \downarrow_0 (y) \notin Q_{grid} \wedge \downarrow_1 (y) \in Q_{grid}) \wedge (y \notin Q_{grid} \rightarrow \downarrow_0 (y) \notin Q_{grid} \wedge \\
&\downarrow_1 (y) \notin Q_{grid}))
\end{aligned}
$$

Once we have shaped the grid, we can encode the horizontal and vertical successors as $\downarrow_1$ and $\oplus 1$, respectively, and we can write the tiling constraints on the grid. To this end, we make use of monadic predicates in $\{P_1, \dots P_k\} \subset \mathcal{P}$ corresponding to the tile types in $\{T_1, \dots T_k\}$:

1. exactly one tile is placed at each node ($\phi_1(x)$)

$$\bigvee_{i=1}^{i=k} x \in P_i \bigwedge_{1\leq i<j\leq k} \neg(x \in P_i \wedge x \in P_j);$$

2. colors match going right ($\phi_2(x)$)

$$\bigvee_{right(T_i)=left(T_j)} x \in P_i \wedge \downarrow_1 (x) \in P_j;$$

3. colors match going up ($\phi_3(x)$)

$$\bigvee_{up(T_i)=down(T_j)} x \in P_i \wedge \oplus 1(x) \in P_j.$$

We define

$$\varphi_{\mathcal{T}} = \forall x(\mathtt{grid}(x) \rightarrow \phi_1(x) \wedge \phi_2(x) \wedge \phi_3(x)).$$

It is not difficult to see that $\mathcal{T}$ tiles $\mathbb{N} \times \mathbb{N}$ if and only if $\varphi_{\mathcal{T}}$ is satisfiable over the binary infinite tree. $\square$

As for the equi-level predicate T, it is possible to show that its addition to $\mathrm{MPL}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ does not preserve decidability. Once more, the proof exploits an embedding of the $\mathbb{N} \times \mathbb{N}$ tiling problem.

Theorem 6.2 $\mathrm{MPL}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}, T]$ *over the k-ary DULS is undecidable.*

Proof. We prove the theorem for the binary DULS; the generalization to the k-ary DULS is straightforward. We show that $\mathrm{MPL}_{\mathcal{P}}[<, \downarrow_0, \downarrow_1, T]$ over the binary DULS is undecidable by reducing to it the $\mathbb{N} \times \mathbb{N}$ tiling problem. Suppose that $\mathcal{T} = \{T_1, \dots, T_k\}$ is the given set of tile types. We shall construct an $\mathrm{MPL}_{\mathcal{P}}[<, \downarrow_0, \downarrow_1, T]$-formula $\varphi_{\mathcal{T}}$ such that $\mathcal{T}$ tiles $\mathbb{N} \times \mathbb{N}$ if and only if $\varphi_{\mathcal{T}}$ is satisfiable over the binary DULS.

The first step consists in forcing the grid over the binary DULS $\langle \mathcal{U}, \downarrow_0, \downarrow_1, < \rangle$. We define the grid domain as the set $\mathcal{G} = \bigcup_{i \geq 0} \{(i2^j)_j \mid j \geq 0\} \subset \mathcal{U}$, the horizontal successor $\mathtt{s}_0(n_r)$ as $(n + 2^r)_r$, and the vertical successor $\mathtt{s}_1(n_r)$ as $(2n)_{r+1}$. Note that, for every $n_r \in \mathcal{G}$, $\mathtt{s}_0(\mathtt{s}_1(n_r)) = \mathtt{s}_1(\mathtt{s}_0(n_r))$. Moreover, it is easy to define in $\mathrm{MPL}_{\mathcal{P}}[<, \downarrow_0, \downarrow_1, T]$ a monadic predicate $\mathtt{grid}$ such that $\mathtt{grid}(x)$ if and only if x belongs to the grid domain $\mathcal{G}$. This predicate is true over x if and only if x is reachable along a leftmost path rooted at some point belonging to layer T^0. We have

$$\mathtt{grid}(x) = \exists y (T^0(y) \wedge \mathtt{LP}(y, x)),$$

where $\mathtt{LP}(y, x)$ if and only if $\exists X (x \in X \wedge y \in X \wedge \forall z (z \in X \rightarrow \downarrow_0 (z) \in X))$ (x and y belongs to the same leftmost path). Moreover, the vertical successor can be defined as $\downarrow_0$ and the horizontal successor as $\rightarrow$, where

$$\rightarrow (x, y) = \exists v, w (T(x, y) \wedge \mathtt{LP}(v, x) \wedge T^0(v) \wedge +_0(v, w) \wedge \mathtt{LP}(w, y)).$$

The proof proceeds as the proof of Theorem 6.1. $\square$

Unlike the proof of Theorem 6.1, the proof of Theorem 6.2 involves the whole DULS. We do not know whether the addition of the horizontal successor $+1$ to $\mathrm{MPL}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ has the same effect. However, since T and $+1$ are inter-definable in the MSO language, it follows that $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}, +1]$ is undecidable. Summing up, we have the following corollary.

Corollary 6.3 *Global predicates cannot be defined in* $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}]$ *over the k-ary DULS and the extension of* $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}]$ *with any global predicate is undecidable.*

The following theorem and its corollary provide information about the relationships between horizontal and vertical predicates over the DULS. More precisely, Theorem 6.4 shows that $+1$ is first-order definable in terms of D over the binary DULS.

Theorem 6.4 $\mathrm{MFO}[<, \downarrow_0, \downarrow_1, +1]$ *is embedable in* $\mathrm{MFO}[<, \downarrow_0, \downarrow_1, D]$ *over the binary DULS.*

Proof.

First, the horizontal successor $+1$ can be defined in terms of the predicate $\mathtt{adj}$ over the DULS as follows:

$$\begin{aligned} +1(x, y) \quad \text{iff} \quad & \exists z ((\downarrow_0 (z) = x \wedge \downarrow_1 (z) = y) \vee (\downarrow_1 (z) = x \wedge \mathtt{adj}(z, y))) \vee \\ & (T^0(x) \wedge +_0 1(x, y)). \end{aligned}$$

Next, we encode the predicate $\mathtt{adj}$ into $\mathrm{MFO}[<, \downarrow_0, \downarrow_1, D]$. We claim that $\mathtt{adj}(x, y)$ if and only if $\phi(x, y)$, where

$$\begin{aligned} \phi(x, y) \quad \text{iff} \quad & \exists z_1 \exists z_2 \exists z_3 \exists z_4 (T^0(z_1) \wedge D(z_1, x) \wedge +_0 1(z_1, z_2) \wedge \downarrow_0 (z_2) = z_3 \wedge \\ & D(z_3, z_4) \wedge \forall w (D(z_3, w) \wedge x \leq w \rightarrow z_4 \leq w) \wedge ((D^0(x) \vee \\ & D^1(x)) \rightarrow \oplus 1(y, z_4)) \wedge (\neg(D^0(x) \vee D^1(x)) \rightarrow z_4 = y)). \end{aligned}$$

We prove that the above definition captures the predicate $\mathtt{adj}$. Let $x = n_r$ and $y = m_s$. Note that $\mathtt{adj}(x, y)$ if and only if $s = r + 1$ and $m = 2(n + 1)$. Suppose that $\phi(x, y)$ holds. Then, there exist $z_1, \ldots, z_4$ such that $z_1 = n_0$, $z_2 = (n + 1)_0$, $z_3 = (2(n + 1))_1$, and $z_4 = \min\{w \mid w = (2(n + 1))_i \wedge i \geq 0 \text{ and } n_r \leq w\}$. We claim that $z_4 = y = (2(n + 1))_{r+1}$, and thus $\mathtt{adj}(x, y)$ holds, whenever neither $D^0(x)$ nor $D^1(x)$, and $z_4 = (2(n+1))_{r+2}$, $y = (2(n+1))_{r+1}$, and thus $\mathtt{adj}(x, y)$ holds, whenever ($D^0(x)$ or $D^1(x)$). Suppose that neither $D^0(x)$ nor $D^1(x)$, that is, $n > 1$. Since, for every $i, j \geq 0$, $i_j < i_{j+1}$, we only have to prove that $(2(n+1))_{r+2} < n_r < (2(n+1))_{r+1}$. To show that $(2(n+1))_{r+2} < n_r$, consider the set $\{i_{r+2} \mid i \geq 0\}$. It is easy to verify that, for every $i \geq 4n$, $n_r < i_{r+2}$, and, for every $i < 4n$, $i_{r+2} < n_r$. Hence, it suffice to prove that $2(n+1) < 4n$, which is true for $n > 1$. The inequality $n_r < (2(n+1))_{r+1}$ follows from the fact that $(2(n+1))_{r+1} = +1(\downarrow_1 (n_r))$, since, for every v, $v <\downarrow_1 (v)$ and $v < +1(v)$. The case ($D^0(x)$ or $D^1(x)$) is easier. Similarly, we can prove that if $\mathtt{adj}(x, y)$ holds, then $\phi(x, y)$ holds. Hence the thesis. $\square$

Since $+1$ and T are inter-definable in the MSO language over the (binary) DULS, and the same holds for $\oplus$ and D, we have the following corollary.

Corollary 6.5 *Let $o \in \{+1, T\}$ and $v \in \{\oplus 1, D\}$.* MSO$[<, \downarrow_0, \downarrow_1, o]$ *is embedable in* MSO$[<, \downarrow_0, \downarrow_1, v]$ *over the binary DULS.*

The only positive result is the decidability of MCL$[<, (\downarrow_i)_{i=0}^{k-1}, T, +1]$ over the k-ary DULS. Its proof is similar to that of Theorem 4.6, and thus omitted.

Theorem 6.6 MCL$[<, (\downarrow_i)_{i=0}^{k-1}, T, +1]$ *over the k-ary DULS is decidable.*

The decidability results for the DULS are summarized in Table 3.

	T	$+1$	D	$\oplus 1$
MFO	Decidable	Decidable	?	?
MPL	Decidable	Decidable	?	?
MCL	Decidable	Decidable	?	?
MFO$_{\mathcal{P}}$	?	?	Undecidable	Undecidable
MPL$_{\mathcal{P}}$	Undecidable	?	Undecidable	Undecidable
MCL$_{\mathcal{P}}$	Undecidable	?	Undecidable	Undecidable
MSO = MSO$_{\mathcal{P}}$	Undecidable	Undecidable	Undecidable	Undecidable

Table 3: Decidability results for the DULS.

7 Conclusions

The outcomes of the research work presented in this paper can be summarized as follows. We first showed that all the considered binary predicates are not definable in the MSO language over the DULS and the UULS, and that their addition immediately leads the MSO theories of such structures to undecidability. As for the n-LS, we pointed out the different status of the horizontal (equi-level and horizontal successor) and vertical (equi-column and vertical successor) predicates: while horizontal predicates are easily definable, vertical ones are undefinable and their addition yields undecidability. Then, we studied the effects of adding the above predicates to suitable *fragments* of the MSO language, such as its first-order, path, and chain fragments, possibly admitting free set variables.

We systematically explored all the possibilities, and gave a number of positive and negative results. From a technical point of view, (un)definability and (un)decidability results are obtained by reduction from/to a wide spectrum of undecidable/decidable problems. We are still missing the complete picture, because some decidability problems are open. However, the achieved results suffice to formulate some general statements. We proved that all predicates can be added to monadic first-order, path, and chain fragments, devoid of free set variables, over the n-LS and the UULS preserving decidability. In the case of the DULS, we proved the same result for the equi-level and horizontal successor predicates, while we do not know yet whether the same holds for the equi-column and vertical successor predicates. Moreover, we proved that the addition of the equi-column or vertical successor predicates to monadic first-order fragments over the ω-layered structures, with free set variables, makes the resulting theories undecidable. The effect of such additions to the n-layered structure is not known yet. As for the equi-level predicate, we only proved that adding it to the monadic path fragment over the DULS, with free set variables, leads to undecidability. Finally, as far as the MSO language over the UULS is concerned, we established an interesting connection between its extension with the equi-level (resp. equi-column) predicate and systolic ω-languages over Y-trees (resp. trellis) [11].

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