Satisfiability and Model Checking for the Logic of Sub-Intervals under the Homogeneity Assumption

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The temporal logic D of the sub-interval relation

One modality only, $\langle D \rangle$, corresponding to the Allen relation during

- Consider the property: "there is always a banquet during a conference"

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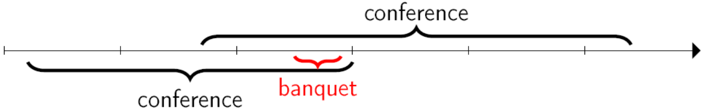


- In D:

$$conference \longrightarrow \langle D \rangle banquet$$

- In LTL:

$$conference_{start} \longrightarrow X(\neg conference_{end} U(banquet_{start} \wedge (\neg conference_{end} U(banquet_{end} \wedge \neg conference_{end} \wedge F\, conference_{ends}))))$$

Motivations (partial list)

- The temporal logic of sub-intervals comes into play in the study of temporal prepositions in natural language [Pratt-Hartmann 2005]
- The connections between the temporal logic of (strict) sub-intervals and the logic of Minkowski space-time have been explored by Shapirovsky and Shehtman [Shapirovsky and Shehtman 2003].
- The temporal logic of reflexive sub-intervals has been studied first by van Benthem, who proved that, when interpreted over dense linear orderings, it is equivalent to the standard modal logic S4 [van Benthem 1991].

What is known about D satisfiability?

The logic D is a real character:

- The satisfiability problem for D is **PSPACE**-complete over the class of dense linear orders [Shapirovsky 2004, Bresolin et al. 2010]
- It is undecidable when interpreted over the classes of finite and discrete linear orders [Marcinkowski and Michaliszyn 2011]
- Unknown over the class of all linear orders

Remark: three variables are needed to encode D in first-order logic (the two-variable property is a sufficient condition for decidability, but it is not a necessary one. . .)

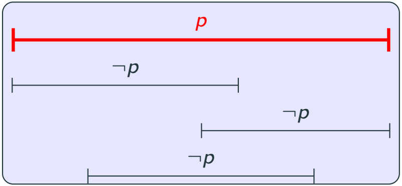
What we prove in the present paper

We show that:

- the satisfiability problem for D over finite linear orders (under the homogeneity assumption) belongs to **PSPACE**
- the model checking problem for D formulas over finite Kripke structures (under the homogeneity assumption) is in **PSPACE** as well
- Both problems are **PSPACE**-complete

General vs. homogeneous semantics

The general case: truth of a proposition letter is defined over intervals (not points), with no restriction.



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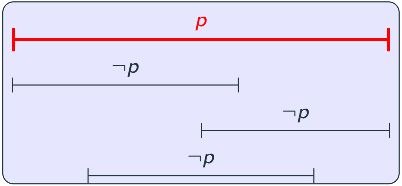
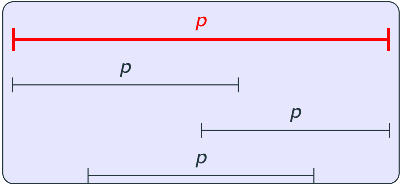


The homogeneous case: a proposition letter holds over an interval iff it holds over all its points/subintervals (a reasonable assumption in various application domains).



Syntax and semantics of D under homogeneity ($\mathsf{D}|_{\mathcal{H}om}$ logic)

Syntax

- $\mathsf{D}|_{\mathcal{H}om}$-formulas are defined by the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle D\rangle\varphi \qquad ([D]\varphi = \neg\langle D\rangle\neg\varphi)$$

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Semantics

Let $\mathcal{M} = \langle \mathbb{I}(\mathbb{S}), \sqsubset, \mathcal{V}\rangle$, where

- $\mathbb{I}(\mathbb{S})$ is the set of intervals over the linear order $\mathbb{S} = \langle S, <\rangle$;
- $\sqsubset$ is the proper sub-interval relation (it is not reflexive);
- $\mathcal{V} : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{S})}$ assigns to every proposition letter $p \in \mathcal{AP}$ the set of intervals $\mathcal{V}(p)$ over which p holds in such a way that $[x, y] \in \mathcal{V}(p)$ iff $[x', x'] \in \mathcal{V}(p)$ for every $x \leq x' \leq y$ (homogeneity).

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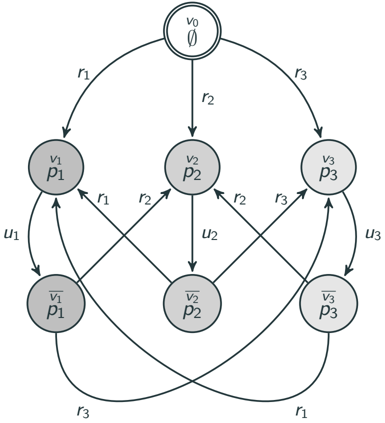
(i) $\mathcal{M}, [x, y] \models p$ if and only if $[x, y] \in \mathcal{V}(p)$;
(ii) Boolean connectives are standard;
(iii) $\mathcal{M}, [x, y] \models \langle D \rangle \psi$ if and only if
there is an interval $[x', y'] \in \mathbb{I}(\mathbb{S})$ s.t. $[x', y'] \sqsubset [x, y]$ and $\mathcal{M}, [x', y'] \models \psi$

The logic $D|_{\mathcal{H}om}$ at work: model checking



Model Checking

$\mathcal{K} \models \psi \iff$ for all *initial* traces ρ of $\mathcal{K}$, $\mathcal{K}, \rho \models \psi$

Possibly infinitely many traces!

At least 2 processes witnessed in any sub-interval of length ≥ 5 of an initial trace:

$$\mathcal{K}_{Sched} \models [D]\big(len_{\geq 5} \rightarrow \bigvee_{1 \leq i < j \leq 3} (\langle D \rangle p_i \ \wedge \ \langle D \rangle p_j)\big)$$

In any sub-interval of length at ≥ 11 of an initial trace, process 3 is executed at least once in some states:

$$\mathcal{K}_{Sched} \not\models [D]\big(len_{\geq 11} \rightarrow \langle D \rangle p_3\big)$$

In any sub-interval of length ≥ 7 of an initial trace, p_1, p_2, and p_3 are all witnessed:

$$\mathcal{K}_{Sched} \not\models [D]\big(len_{\geq 7} \rightarrow (\langle D \rangle p_1 \wedge \langle D \rangle p_2 \wedge \langle D \rangle p_3)\big)$$

The PSPACE satisfiability proof — Atoms

Definition

Given a $D|_{\mathcal{H}om}$-formula φ, a φ-atom A is a subset of the closure of φ, denoted by $CL(\varphi)$, such that:

- for every $\psi \in CL(\varphi)$, $\psi \in A$ iff $\neg\psi \notin A$, and
- for every $\psi_1 \vee \psi_2 \in CL(\varphi)$, $\psi_1 \vee \psi_2 \in A$ iff $\psi_1 \in A$ or $\psi_2 \in A$.

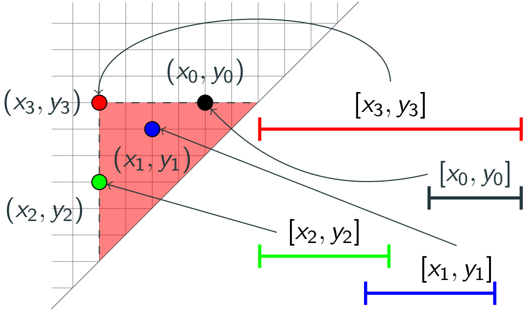
An atom enforces a "local" form of consistency among the formulas it contains.

For "global consistency" (among atoms), we introduce the binary relation D_φ.

Definition

For each pair of atoms $A, A' \in \mathcal{A}_\varphi$, $A \; D_\varphi \; A'$ holds iff both $\psi \in A'$ and $[D]\psi \in A'$ for each formula $[D]\psi \in A$.

A spatial representation of interval models: compasses



Definition

Given a finite $\mathbb{S} = \langle S, < \rangle$ and φ, a compass φ-structure is $\mathcal{G} = (\mathbb{P}_{\mathbb{S}}, \mathcal{L})$, where

- $\mathbb{P}_{\mathbb{S}}$ is the (finite) set of points (x, y), with $x, y \in S$ and $x \leq y$, and
- $\mathcal{L}$ is a function that maps $(x, y) \in \mathbb{P}_{\mathbb{S}}$ to a φ-atom $\mathcal{L}(x, y)$

such that for all pairs $(x, y), (x', y') \in \mathbb{P}_{\mathbb{S}}$,
$x \leq x' \leq y' \leq y \wedge (x, y) \neq (x', y') \Longrightarrow \mathcal{L}(x, y)\, D_{\varphi}\, \mathcal{L}(x', y')$ (temporal consistency)

The PSPACE satisfiability proof — Fulfilling compasses

Definition

$\mathcal{G} = (\mathbb{P}_{\mathbb{S}}, \mathcal{L})$ is fulfilling if for every $(x, y) \in \mathbb{P}_{\mathbb{S}}$ and each formula $\langle D \rangle \psi \in \mathcal{L}(x, y)$, there exists $(x', y') \sqsubset (x, y)$ in $\mathbb{P}_{\mathbb{S}}$ such that $\psi \in \mathcal{L}(x', y')$.

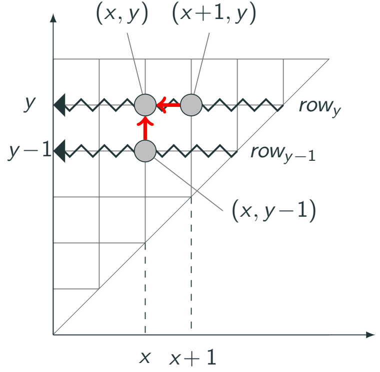
Proposition

A $\mathrm{D}|_{\mathcal{H}om}$-formula φ is satisfiable if and only if there is a fulfilling compass φ-structure such that $\varphi \in \mathcal{L}(x, y)$, for some $(x, y) \in \mathbb{P}_{\mathbb{S}}$.

The PSPACE satisfiability proof — the labeling rule

The ingredient #1: the labeling rule

We define a rule (a ternary relation over φ-atoms) that determines the φ-atoms labeling all the points of $\mathcal{G}$, starting from the ones on the diagonal (homogeneity plays a key role here)



If the above rule holds among all atoms in consecutive positions of a compass φ-structure, then the structure is fulfilling, and vice versa.

The PSPACE satisfiability proof — the contraction rule

The ingredient #2: the contraction rule

We define an equivalence relation $\sim$ between rows of a compass φ-structure that

- relates pairs of rows with the same "shape" (the same atoms in the same order and with the same multiplicity up to a certain threshold);
- has a finite index.

Since $\sim$ preserves the fulfillment of compasses, it is possible to "contract" the structure between two equivalent rows

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Outcome:

- a (non-deterministic) satisfiability algorithm for $\mathsf{D}|_{\mathcal{H}om}$-formulas which makes use of polynomial working space only, because
  1. all rows satisfy some nice properties that make it possible to succinctly encode them
  2. it only needs to keep track of two rows of a compass at a time
  3. it guesses the most compact elements of the equivalence classes of $\sim$

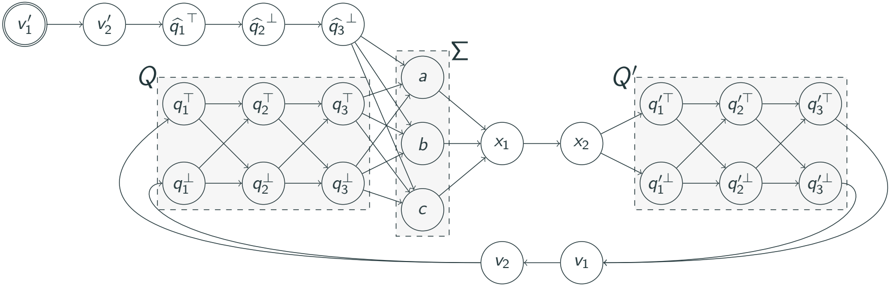
Model Checking $\mathsf{D}|_{\mathcal{Hom}}$-formulas over finite Kripke structures

- We consider some finite linear orders — precisely those corresponding to the initial traces of the finite Kripke structure $\mathcal{K}$ — checking whether $\neg\varphi$ holds over them (if this is the case, counterexample found: $\mathcal{K} \not\models \varphi$).
- "satisfiability driven by the traces of $\mathcal{K}$": for any initial trace ρ, we build a compass φ-structure induced by ρ
  - ρ can be viewed as the diagonal of the compass structure; the labeling rule allows us to generate the whole structure from the diagonal

The model checking procedure: a simple variant of the satisfiability algorithm, still working in polynomial space.

PSPACE-hardness of satisfiability and MC for $\mathsf{D}|_{\mathcal{H}om}$

We reduce the **PSPACE**-complete problem of *(non-)universality of the language of an NFA* to the MC problem for $\mathsf{D}|_{\mathcal{H}om}$ over finite Kripke structures.



- The Kripke structure + the $\mathsf{D}|_{\mathcal{H}om}$ formula encode legal computations of the NFA

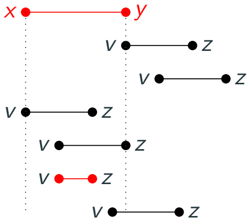
This proves the **PSPACE**-hardness of model checking.

As for the **PSPACE**-hardness of satisfiability, for any Kripke structure there is a polynomial-size $\mathsf{D}|_{\mathcal{H}om}$-formula encoding its initial traces.

Where are we? $D|_{\mathcal{H}om}$ is a small fragment of the logic HS

HS features a modality for any Allen ordering relation between pairs of intervals (except for equality)

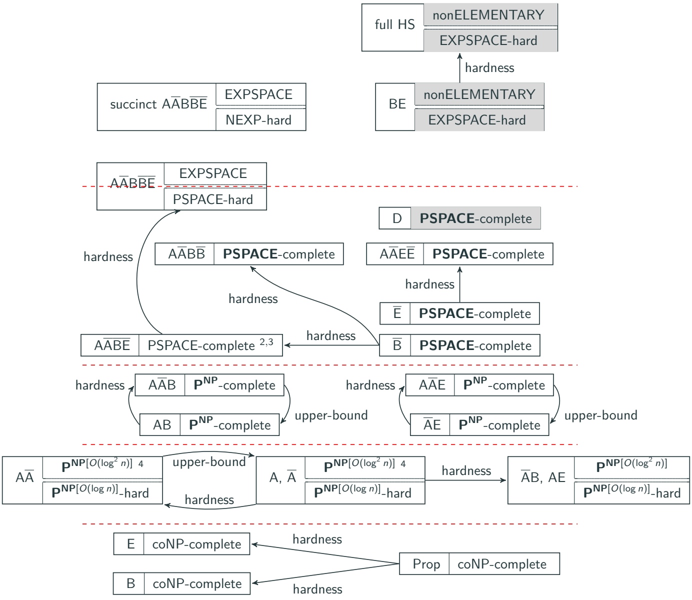
Allen rel.	HS	Definition	Example
meets	$\langle A \rangle$	$[x, y]\mathcal{R}_A[v, z] \iff y = v$	
before	$\langle L \rangle$	$[x, y]\mathcal{R}_L[v, z] \iff y < v$	
started-by	$\langle B \rangle$	$[x, y]\mathcal{R}_B[v, z] \iff x = v \wedge z < y$	
finished-by	$\langle E \rangle$	$[x, y]\mathcal{R}_E[v, z] \iff y = z \wedge x < v$	
contains	$\langle D \rangle$	$[x, y]\mathcal{R}_D[v, z] \iff x < v \wedge z < y$	
overlaps	$\langle O \rangle$	$[x, y]\mathcal{R}_O[v, z] \iff x < v < y < z$	



$\langle D \rangle$ can be easily defined by means of modality $\langle B \rangle$ and $\langle E \rangle$:

$$\langle D \rangle \varphi = \langle B \rangle \langle E \rangle \varphi = \langle E \rangle \langle B \rangle \varphi$$

Future work: BE model checking (under homogeneity)



Future work: BE satisfiability (under homogeneity)

Known results:

- The satisfiability problem for BE is undecidable over the class of dense linear orders [Lodaya 2000] (for D is **PSPACE**-complete)
- It is undecidable also over the classes of finite and discrete linear orders [Marcinkowski and Michaliszyn 2011] (it immediately follows from undecidability of D)

Open issue: exact complexity of the satisfiability problem for BE over finite linear orders (under homogeneity)