Modal Deduction in Second-Order Logic and Set Theory - I

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Abstract

We investigate modal deduction through translation into standard logic and set theory. In a previous paper, using a set-theoretic translation method, we proved that derivability in the minimal modal logic K_s corresponds precisely to derivability in a weak, computationally attractive set theory Ω. In this paper, this approach is shown equivalent to working with standard first-order translations of modal formulae in a theory of general frames. The employed techniques are mainly model-theoretic and set-theoretic, and they admit extensions to richer languages and modal deductive systems than that of basic modal logic. Some of these extensions are discussed in the last part of the paper.

1 Introduction

We are interested in analyzing general deduction for modal formulae [4]. The standard systems used for this purpose are the so-called "minimal modal logic" K or, if one wants to work with general frames (as we do), the system K_s obtained from K adding a substitution rule. We do not consider special purpose calculi for stronger modal logics over K_s, such as $S4$ or $S5$, although we expect that our results and method can be specialized to frame classes obeying their additional specific constraints. Such logics are treated as particular sets of assumptions in our general deduction mechanisms.

Our aim is to analyze general modal deduction via translations into standard logics. What already exists for this purpose in the logical tradition are "standard trans-

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lations" into first-order and monadic second-order logic (for their extensive theory, cf. [4]). In recent years, the computer-science community has also become interested in this approach (see, for example, [18]), with results that are mainly restricted to cases where specific modal logics have "nice" translations (first-order, or well-behaved second-order). A new theme in the latter approach has been variations in translation formats, e.g., special operator-based 'path languages' have been used to profit optimally from resolution methods [16, 17].

This paper takes its point of departure in [11], where a general *set-theoretic* translation, whose main idea is to treat the universal modality as a *powerset* operator, was proposed for (poly)modal logics. The engine driving soundness and completeness of this translation was a weak set theory Ω, having pleasant computational properties [10, 9]. The method was shown adequate only for frame-complete modal logics, using an argument heavily relying on non-well-founded set theory [1].

Against this background, the present paper makes the following new contributions. First, we completely clear up the relationship between the set-theoretic translation and the standard modal translation. Standard modal deduction via the standard translation is supported by a simple two-sorted first-order theory μ of general frames. Comparing μ with Ω, we establish an effective equivalence between the latter and a suitable extension of μ, that we call μ^+. Such an extension is obtained adding to μ an axiom reflecting the "uniformity" of our set-theoretic approach which uses just $\in$ for both set-membership and the accessibility relation. We then prove that μ^+ is non-conservative over μ. On the ground of the equivalence between Ω and μ^+, this allows us to conclude that the translation method given in [11] is not adequate for logics which are not frame complete. By varying our translation, however, we manage to obtain a full equivalence between general modal deduction in K_s and Ω-deduction. This is the desired general result, which is based on standard set-theoretic methods, including a suitable adaptation of Fraenkel-Mostowski permutations of the universe [15]. In the last part of the paper we briefly consider what happens when we vary the source language for these translations and deductive engines, from the basic modal language to stronger (e.g. temporal or second-order) formalisms.

The paper is organized as follows. Section 2 contains basic results about the standard translation method and the set-theoretic one. Section 3 compares the two methods providing the basic equivalence result. In Section 4 the full equivalence between general modal deduction in K_s and Ω is proved. Finally, Section 5 presents some variations of the proposed framework suitable to deal with languages other than the basic modal one.

2 Translation methods for modal logic

Let us briefly recall some notation and basic results about modal logic (see [4]) for details). *Propositional modal formulae* are constructed using propositional letters $P_1, P_2, \ldots$, boolean operators $\vee, \neg$ and the $\Box$ operator. The minimal modal logic K_s consists of a set of propositional axioms complete for classical logic, the modal axiom $\Box(P \to Q) \to (\Box P \to \Box Q)$, and the rules of substitution, modus ponens and necessitation (infer $\Box\phi$ from ϕ). Deducibility of ψ from ϕ in K_s, denoted by $\phi \vdash_{K_s} \psi$, is defined as deducibility of ψ from $K_s \cup \{\phi\}$ using substitution, modus ponens, and necessitation.

A *frame* is a pair $F = (W, R)$, where W is a set (of *worlds*) and R is a binary relation on W (the *accessibility relation*). A *valuation* $\models$ in the frame (W, R) is a subset of $W \times \{P_1, \ldots\}$. For all $w \in W$, $w \models \phi$ is defined by induction on the structural complexity of the modal formula ϕ, as usual. A formula ϕ is *valid* in a frame (W, R) if and only if, for all $w \in W$ and for all valuation $\models$, $w \models \phi$ holds. A formula ψ is a *frame logical consequence* of a formula ϕ ($\phi \models_f \psi$) if and only if, for all frames F, if ϕ is valid in F, then ψ is valid in F. A formula ϕ is said to be *complete* if and only if, for all ψ, $\phi \vdash_{K_s} \psi \Leftrightarrow \phi \models_f \psi$. Examples of incomplete formulae can be found in the literature (see for example [4]).

A semantic characterization of K_s is obtained via general frames. A *general frame* is a pair $(F, \mathcal{W})$, where F is a frame and $\mathcal{W}$ is a set of subsets of W, which is closed under the boolean operations of union, complementation with respect to W, and $\Box(X) = \{w \in W : \forall v\ (wRv \rightarrow v \in X)\}$. Validity in general frames is defined as for frames, except for the fact that we only consider valuations such that the set of worlds at which a given propositional variable holds belongs to $\mathcal{W}$. Logical consequence in general frames ($\phi \models_{gf} \psi$) is defined accordingly. For all modal formulae ϕ, ψ we have

$$\phi \vdash_{K_s} \psi \Leftrightarrow \phi \models_{gf} \psi.$$

The *standard translation* of modal formulae into formulae of a second-order language containing equality, a binary constant R and unary predicate variables (L_2-formulae [4, 7]) is defined as follows:

- $ST(P_i) = \mathtt{P_i}(x)$,
- $ST(\phi \vee \psi) = ST(\phi) \vee ST(\psi)$,
- $ST(\neg\phi) = \neg ST(\phi)$,
- $ST(\Box\phi) = \forall y\ (xRy \rightarrow ST(\phi)(y|x))$,

where $y|x$ denotes uniform substitution of the variable y for the variable x.

The *closed standard translation* $\overline{ST}(\phi)$ of a modal formula ϕ is defined as the L_2-sentence $\forall \mathtt{P_1} \ldots \mathtt{P_n} \forall x ST(\phi)$, and it is is easy to see that

$$\phi \models_f \psi \Longleftrightarrow \overline{ST}(\phi) \models \overline{ST}(\psi),$$

where the $\models$ on the right-hand side denotes second-order logical consequence.

The *set-theoretic translation* ($\Box$*-as-Pow translation*) represents any Kripke frame as a set, with the accessibility relation modeled using the membership relation $\in$ (cf. [11]). The theory in which the translation is carried out is a very weak[1] *finitely* (first-order) axiomatizable set theory called Ω. The axioms, in the language with relational symbols $\in$ and $\subseteq$, and functional symbols $\cup, \setminus$, and Pow, are the following:

$x \in y \cup z \leftrightarrow x \in y \vee x \in z$;

[1] Compare this theory with more classical finite axiomatizations of set theory, such as NBG [14].

$x \in y \setminus z \leftrightarrow x \in y \wedge x \notin z$;

$x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y)$;

$x \in Pow(y) \leftrightarrow x \subseteq y$.

Notice that neither the extensionality axiom nor the axiom of foundation are in Ω.

Given a modal formula $\phi(P_1, ..., P_n)$, with propositional variables $P_1, ..., P_n$, we define its translation as the set-theoretic *term* $\phi^*(x, x_1, ..., x_n)$, with set variables $x, x_1, ..., x_n$, built using $\cup$, $\setminus$, and Pow. Intuitively the term $\phi^*(x, x_1, ..., x_n)$ represents the set of those worlds (in the frame x) in which the formula ϕ holds. The inductive definition of $\phi^*(x, x_1, ..., x_n)$ is the following:

- $P_i^* = x_i$;
- $(\phi \vee \psi)^* = \phi^* \cup \psi^*$;
- $(\neg\phi)^* = x \setminus \phi^*$;
- $(\Box\phi)^* = Pow(\phi^*)$.

For all modal formulae ϕ, ψ, the following results showing the adequacy of the translation hold [11]:

completeness:

$$\phi \vdash_{K_s} \psi \;\Rightarrow \Omega \vdash \forall x(Trans(x) \wedge$$
$$\wedge \forall x_1 \ldots x_m(x \subseteq \phi^*(x, x_1, \ldots, x_m)) \rightarrow \forall x_1 \ldots x_n(x \subseteq \psi^*(x, x_1, \ldots, x_n))),$$

soundness:

$$\Omega \vdash \forall x(Trans(x) \wedge \forall x_1 \ldots x_m(x \subseteq \phi^*(x, x_1, \ldots, x_m)) \rightarrow$$
$$\rightarrow \forall x_1 \ldots x_n(x \subseteq \psi^*(x, x_1, ... x_n))) \;\Rightarrow \phi \models_f \psi,$$

where $Trans(x)$ stands for $\forall z\ (z \in x \rightarrow z \subseteq x)$ and expresses the fact that x is a *transitive* set (cf. [14]).

Completeness is proved (proof-theoretically) by induction on the length of the derivation in K_s, while soundness is proved (semantically) by constructing a model of Ω from a Kripke frame using an argument relying on non-well-founded set theory. It is worth noting that, for frame-complete theories, the above translation captures exactly the notion of K_s-derivability.

We conclude this section recalling that representing the accessibility relation R by the (unique) membership relation $\in$ does not restrict the field of applicability of the translation to monomodal logics. This is essentially because the set-theoretic structure of worlds can be used to represent *multiple* accessibility relations (in a completely *symmetric* way, as a matter of fact) as was proved in [11].

3 A comparison of the two methods

In this section, we compare the two translation methods discussed above. To this end, consider a two-sorted first-order language, with *worlds* (denoted by small letters) and *sets* (denoted by capital letters), with binary predicates R (on worlds) and $\in$ (between worlds and sets), as well as operations $-, \cup$ and $\Box$ on the set sort. This language is adequate for the description of the general frame semantics. The minimal logic describing it is obtained by considering the following theory μ:

- First-Order Principles for both sorts;

- $\forall \mathtt{P} \forall w (w \in -\mathtt{P} \leftrightarrow \neg w \in \mathtt{P})$;

- $\forall \mathtt{PQ} \forall w (w \in \mathtt{P} \cup \mathtt{Q} \leftrightarrow w \in \mathtt{P} \vee w \in \mathtt{Q})$;

- $\forall \mathtt{P} \forall w (w \in \Box \mathtt{P} \leftrightarrow \forall v (wRv \rightarrow v \in \mathtt{P}))$

- $\forall \mathtt{PQ} (\forall w (w \in \mathtt{P} \leftrightarrow w \in \mathtt{Q}) \rightarrow \mathtt{P} = \mathtt{Q})$ (extensionality).

Since Ω models both set membership and the accessibility relation in terms of $\in$-relationship, let μ^+ be the theory obtained by adding to μ the axiom

- $\forall w \exists \mathtt{P} \forall v \ (v \in \mathtt{P} \leftrightarrow wRv)$

that links R and $\in$, guaranteeing that, for each world w, there exists the set of its R-successors..

Notice that L_2-formulae can be viewed as formulae of the two-sorted first-order language μ (μ^+), with unary predicate variables corresponding to variables of the set sort and atomic formulae of the form $\mathtt{P_i}(x)$ corresponding to the formulae of the form $x \in \mathtt{P_i}$. For the sake of brevity, in both the language of L_2 and μ (μ^+), we denote the standard translation of a modal formula ϕ by $\overline{ST}(\phi)$.

In the following, we will show that the standard translation method with respect to μ^+ and the set-theoretic one with respect to Ω are equivalent. More precisely, we will prove that, for all modal formulae ϕ, ψ,

$$\mu^+ \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi) \Longleftrightarrow \Omega \vdash \forall x \ (Trans(x) \wedge$$

$$\wedge \forall x_1 \dots x_m (x \subseteq \phi^*(x, x_1, \dots, x_m)) \rightarrow \forall x_1 \dots x_n (x \subseteq \psi^*(x, x_1, \dots, x_n))).$$

We begin proving that derivability in μ^+ implies derivability in Ω.

Given an Ω-model $\mathbf{U}$ and a transitive set x in the support U of that model, we show how to construct a model for μ^+. For the sake of brevity, from now on we will denote any Ω-model $\mathbf{U}$ by a pair $(U, \in^{\mathbf{U}})$, omitting the function symbols.

Definition 1 Let $\mathbf{U} = (U, \in^{\mathbf{U}})$ be a model for Ω, and let x be a transitive element of U. The general frame $(W, R, \mathcal{W})$ corresponding to $\mathbf{U}$ is defined as follows:

- the *world* sort W is given by all elements (with respect to the model $\mathbf{U}$) of x ;

- the *set* sort $\mathcal{W}$ is given by equivalence classes in U with respect to the equivalence relation of "having the same elements of the world sort"[2]. More precisely, we state that $[y] = [z]$ if and only if, $w \in^{\mathbf{U}} y \Leftrightarrow w \in^{\mathbf{U}} z$, for every $w \in^{\mathbf{U}} x$. An element of the set sort is an equivalence class $[y]$, for $y \in U$;

- the operations over $\mathcal{W}$ are defined as follows: $[y] \cup [z] = [y \cup^{\mathbf{U}} z]$, $-[y] = [x \setminus^{\mathbf{U}} y]$, and $\Box[y] = [Pow^{\mathbf{U}}(y)]$.

- for worlds y, z we define yRz iff $z \in^{\mathbf{U}} y$, while if y is a world and $[z]$ is a set, we define $y \in [z]$ iff $y \in^{\mathbf{U}} z$.

It is easy to show that the general frame corresponding to a model of Ω verifies all principles of μ^+.

Theorem 2 For each pair of formulae ϕ, ψ,

$$\mu^+ \vdash \overline{ST}(\phi) \to \overline{ST}(\psi) \Rightarrow \Omega \vdash \forall x \ (Trans(x) \wedge$$

$$\wedge \forall x_1 \dots x_m (x \subseteq \phi^*(x, x_1, \dots, x_m)) \to \forall x_1 \dots x_n (x \subseteq \psi^*(x, x_1, \dots, x_n))).$$

Proof Let $\mathbf{U}$ be a model of Ω and $(W, R, \mathcal{W})$ be the corresponding general frame as in Definition 1. We first prove that, for any modal formula $\theta(P_1, .., P_n)$, each $y \in x$, and $y_1, ..., y_n \in U$,

$$\mathbf{U} \models y \in \theta^*(x, y_1, ..., y_n) \text{ if and only if } ST(\theta)(y, [y_1], ..., [y_n]) \text{ holds in}(W, R, \mathcal{W}).$$

The proof is by induction on the structure of θ.

- if $\theta = P_i$, then $\mathbf{U} \models y \in \theta^*(x, y_i) \Leftrightarrow \mathbf{U} \models y \in y_i \Leftrightarrow y \in [y_i]$ holds in $(W, R, \mathcal{W})$ $\Leftrightarrow ST(\theta)(y, [y_i])$ holds;

- if $\theta = \neg\rho$, then $\mathbf{U} \models y \in \theta^*(x, y_1, ..., y_n) \Leftrightarrow \mathbf{U} \models y \in x \setminus \rho^*(x, y_1, ..., y_n) \Leftrightarrow$ $\mathbf{U} \models y \in x \wedge y \notin \rho^*(x, y_1, ..., y_n) \Leftrightarrow ST(\rho)(y, [y_1], ..., [y_n])$ does not hold in $(W, R, \mathcal{W}) \Leftrightarrow ST(\theta)(y, [y_1], ..., [y_n])$ holds;

- the case of $\cup$ is left to the reader;

- if $\theta = \Box\rho$ then $\mathbf{U} \models y \in \theta^*(x, y_1, ..., y_n) \Leftrightarrow \mathbf{U} \models y \in Pow(\rho^*(x, y_1, ..., y_n)) \Leftrightarrow$ $\mathbf{U} \models \forall w \ (w \in y \to w \in \rho^*(x, y_1, ..., y_n)) \Leftrightarrow \forall w \ (yRw \to ST(\rho)(w, [y_1], ..., [y_n]))$ holds in $(W, R, \mathcal{W}) \Leftrightarrow ST(\theta)(y, [y_1], ..., [y_n])$ holds.

Therefore, for any modal formula $\theta(P_1, ..., P_n)$,

$$\mathbf{U} \models \forall y_1 \dots y_n \ (x \subseteq \theta^*(x, y_1, \dots, y_n)) \text{ if and only if } \overline{ST}(\theta) \text{ holds in}(W, R, \mathcal{W}).$$

If x is transitive and $\mathbf{U} \models \forall x_1 \dots x_m (x \subseteq \phi^*(x, x_1, \dots, x_m))$, then, from the above statement, it follows that $\overline{ST}(\phi)$ holds in $(W, R, \mathcal{W})$. Thus, by hypothesis, we obtain $\overline{ST}(\psi)$ and hence $\mathbf{U} \models \forall x_1 \dots x_n (x \subseteq \psi^*(x, x_1, \dots, x_n))$.

[2]We cannot restrict ourselves to x, because x may be not closed under the set operations. For instance, the transitive set $\{\emptyset, \{\emptyset\}\}$ is not closed under the Pow operation.

We now prove that derivability in Ω implies derivability in μ^+. Let $(W, R, \mathcal{W})$ be a general frame satisfying μ^+. This means that $\mathcal{W}$ includes all subsets of W defined by the additional axiom characterizing μ^+, plus the subsets that can be obtained from them by union, (relative) complementation, and the $\Box$ operation. Moreover, without loss of generality, we suppose that the elements of W are sets of the same rank $\alpha \geq \omega$. An Ω-model can be defined starting from $(W, R, \mathcal{W})$ in the following way:

Definition 3 Let $(W, R, \mathcal{W})$ be a general frame for μ^+. The model $\mathbf{U} = (U, \in^{\mathbf{U}})$ generated by $(W, R, \mathcal{W})$ is defined as follows.

Let $U_0 = W$, $U_n = Pow(U_{n-1}) \cup U_{n-1}$, for $n > 0$, and $U = \bigcup_{n=0}^{\omega} U_n$.

Define a function $F : U \to U$ as:

$$F(x) \quad = \quad \begin{cases} \{v \in W : xRv\} & \text{if } x \in W, \\ x \setminus W & \text{if } x \notin W \wedge x \cap W \notin \mathcal{W}, \\ x & \text{if } x \notin W \wedge x \cap W \in \mathcal{W}, \end{cases}$$

and let

$$x \in^{\mathbf{U}} y \text{ if and only if } x \in F(y).$$

The definition of $\in^{\mathbf{U}}$ is very similar to that given in the so-called Fraenkel-Mostowski's permutations of the universe [15].

Notice that, since $(W, R, \mathcal{W})$ is a model of μ^+, the set $\{v \in W : xRv\}$ is in $\mathcal{W}$, for any $x \in W$.

The following properties are easily verified:

- if $y \in W$, then $x \in^{\mathbf{U}} y$ if and only if yRx;
- if $y \in \mathcal{W}$, then $x \in^{\mathbf{U}} y$ if and only if $x \in y$.

Lemma 4 The model $\mathbf{U} = (U, \in^{\mathbf{U}})$, generated by a general frame $(W, R, \mathcal{W})$ satisfying μ^+, is an Ω-model.

Proof To show that $\mathbf{U}$ is a model of Ω, it is enough to prove that

$$\begin{array}{lcl} \mathbf{U} & \models & \forall xy \exists z \forall u\ (u \in z \leftrightarrow u \in x \vee u \in y); \\ \mathbf{U} & \models & \forall xy \exists z \forall u\ (u \in z \leftrightarrow u \in x \wedge u \notin y); \\ \mathbf{U} & \models & \forall x \exists z \forall y\ (y \in z \leftrightarrow \forall s\ (s \in y \to s \in x)). \end{array}$$

Since all the elements of W have the same rank $\alpha \geq \omega$, it is easily seen that each $x \in U$ contains only finitely many elements of rank (strictly) less than α. From this, it follows that the intersection of the range of F and W is empty. More precisely,

$$range(F) = \{y \in U \setminus W : y \cap W \in \mathcal{W}\}.$$

For any $x, y \in U$, the set $F(x) \cup F(y)$ is in the range of F. Indeed, $(F(x) \cup F(y)) \cap W = (F(x) \cap W) \cup (F(y) \cap W)$, and both $(F(x) \cap W)$ and $(F(y) \cap W)$ belong to $\mathcal{W}$, which is closed under union. Therefore, take a z such that $F(z) = F(x) \cup F(y)$, and observe that

$$t \in^{\mathbf{U}} z \leftrightarrow t \in F(z) \leftrightarrow t \in F(x) \vee t \in F(y) \leftrightarrow t \in^{\mathbf{U}} x \vee t \in^{\mathbf{U}} y.$$

Similarly, we show that the set $F(x) \setminus F(y)$ is in the range of F. In fact, $(F(x) \setminus F(y)) \cap W = (F(x) \cap W) \setminus (F(y) \cap W)$, and both $(F(x) \cap W)$ and $(F(y) \cap W)$ belong to $\mathcal{W}$, which is closed under difference. Any z such that $F(z) = F(x) \setminus F(y)$ will verify the axiom for set difference.

Consider now the set $\{y \in U : F(y) \subseteq F(x)\}$. This set is in the range of F because

$$(\{y \in U : F(y) \subseteq F(x)\}) \cap W = \{y \in W : F(y) \subseteq F(x)\} =$$

$$\{y \in W : \forall z(yRz \to z \in F(x))\} = \Box(F(x) \cap W).$$

Since $F(x) \cap W$ is in $\mathcal{W}$ and $\mathcal{W}$ is closed under $\Box$, the set $\Box(F(x) \cap W)$ is in $\mathcal{W}$. Any z such that $F(z) = \{y \in U : F(y) \subseteq F(x)\}$ will verify the axiom for the powerset, because

$$F(y) \subseteq F(x) \Leftrightarrow \forall s(s \in^{\mathbf{U}} y \to s \in^{\mathbf{U}} x).$$

Theorem 5 For each pair of formulae ϕ, ψ,

$$\Omega \vdash \forall x\ (Trans(x) \wedge \forall x_1 \ldots x_m(x \subseteq \phi^*(x, x_1, \ldots, x_m)) \to$$

$$\to \forall x_1 \ldots x_n(x \subseteq \psi^*(x, x_1, \ldots, x_n))) \Rightarrow \mu^+ \vdash \overline{ST}(\phi) \to \overline{ST}(\psi).$$

Proof Let $(W, R, \mathcal{W})$ be a general frame for μ^+ and $\mathbf{U} = (U, \in^{\mathbf{U}})$ be the model generated by $(W, R, \mathcal{W})$ (cf. Definition 3).

For any modal formula $\theta(P_1, ..., P_n)$, we prove that, for each $y \in W$ and $y_1, ..., y_n$ in U,

$$\mathbf{U} \models y \in \theta^*(W, y_1, .., y_n)$$

if and only if

$$ST(\theta)(y, F(y_1) \cap W, .., F(y_n) \cap W) \text{ holds in } (W, R, \mathcal{W}),$$

where in the left-hand side we view W as an element of U and in the right-hand side $F(y_i) \cap W \in \mathcal{W}$.

The proof is by induction on the structure of θ:

- if $\theta = P_i$, then $\mathbf{U} \models y \in \theta^*(W, y_i) \Leftrightarrow \mathbf{U} \models y \in y_i \Leftrightarrow y \in F(y_i) \Leftrightarrow y \in F(y_i) \cap W \Leftrightarrow ST(\theta)(y, F(y_i) \cap W)$ holds in $(W, R, \mathcal{W})$;

- if $\theta = \neg\rho$, then $\mathbf{U} \models y \in \theta^*(W, y_1, \ldots, y_n) \Leftrightarrow \mathbf{U} \models y \in W \setminus \rho^*(W, y_1, \ldots, y_n) \Leftrightarrow \mathbf{U} \models y \in W \wedge y \notin \rho^*(W, y_1, \ldots, y_n) \Leftrightarrow ST(\rho)(y, F(y_1) \cap W, \ldots, F(y_n) \cap W)$ does not hold in $(W, R, \mathcal{W}) \Leftrightarrow \neg ST(\rho)(y, F(y_1) \cap W, \ldots, F(y_n) \cap W)$ holds $\Leftrightarrow ST(\theta)(y, F(y_1) \cap W, \ldots, F(y_n) \cap W)$ holds;

- the case of $\cup$ is left to the reader;

- if $\theta = \Box\rho$, then $\mathbf{U} \models y \in \theta^*(W, y_1, \ldots, y_n) \Leftrightarrow \mathbf{U} \models y \in Pow(\rho^*(W, y_1, \ldots, y_n)) \Leftrightarrow \mathbf{U} \models \forall v(v \in y \to v \in \rho^*(W, y_1, \ldots, y_n)) \Leftrightarrow$ for all $v \in W$, if yRv then $\mathbf{U} \models v \in \rho^*(W, y_1, \ldots, y_n) \Leftrightarrow$ for all $v \in W$, if yRv then $ST(\rho)(v, F(y_1) \cap W, \ldots, F(y_n) \cap W)$ holds in $(W, R, \mathcal{W}) \Leftrightarrow ST(\theta)(y, F(y_1) \cap W, \ldots, F(y_n) \cap W)$ holds.

This implies that, for any modal formula $\theta(P_1, \ldots, P_n)$,

$$\mathbf{U} \models \forall y_1 \ldots y_n \ (W \subseteq \theta^*(W, y_1, \ldots, y_n)) \text{ if and only if } \overline{ST}(\theta) \text{ holds in} (W, R, \mathcal{W}).$$

Let us prove that W is transitive in $\mathbf{U}$, that is:

$$\mathbf{U} \models \forall x (x \in W \rightarrow \forall y (y \in x \rightarrow y \in W)).$$

From $x \in^{\mathbf{U}} W$, it follows that x is an element of the world sort in $(W, R, \mathcal{W})$, and from $y \in^{\mathbf{U}} x$, it follows that xRy in $(W, R, \mathcal{W})$; therefore, $y \in^{\mathbf{U}} W$.

Thus, if $\overline{ST}(\phi)$ holds in $(W, R, \mathcal{W})$, then $\mathbf{U} \models \forall x_1 \ldots x_m (W \subseteq \phi^*(W, x_1, \ldots, x_m))$ and W is transitive. By hypothesis, we have $\mathbf{U} \models \forall x_1 \ldots x_n (W \subseteq \psi^*(W, x_1, \ldots, x_n))$ and hence $\overline{ST}(\psi)$.

It is worth noting that it is possible to remove the extensionality axiom from μ^+ without affecting the validity of Theorems 2 and 5.

4 Capturing derivability in K_s

In Section 3, we have proved that Ω and μ^+ are *modally equivalent*, in the sense that they derive the same 'modal formulae' modulo the translations defined in Section 2. Moreover μ is complete for capturing general modal derivability in the sense of K_s. It is indeed easy to see that:

$$\mu \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi) \Leftrightarrow \phi \models_{gf} \psi \Leftrightarrow \phi \vdash_{K_s} \psi.$$

Hence, our obvious next question is: what about Ω's power in this respect? If μ^+ were conservative over μ, then Ω would exactly correspond to K_s.

In Section 4.1, we will show that μ^+ is a non-conservative extension of μ. This gives rise to a natural question: can we change the translation to make Ω and μ *modally equivalent*? In such a case, we would be able to capture derivability in K_s not only for complete formulae, as in [11], but in general. The answer is positive, and the generalization of the proposed set-theoretic translation method to any (possibly incomplete) modal formula is given in Section 4.2.

4.1 Non-conservativity of μ^+ over μ.

Proposition 6 The theory $\mu^+ = \mu + \forall x \exists \mathtt{P} \forall y \ (y \in \mathtt{P} \leftrightarrow xRy)$ is non-conservative (with respect to modal formulae) over μ.

Proof The proof is a variant of the proof given in [3], where it is proved that the theory μ^s equal to μ plus the axiom $\forall x \exists \mathtt{P} \forall y \ (y \in \mathtt{P} \leftrightarrow x = y)$ is non-conservative over μ. The modal incomplete logic and the formula showing its incompleteness are indeed the same as in [3].

We prove that, if $\phi = \Box \diamond \top \rightarrow \Box(\Box(\Box P \rightarrow P) \rightarrow P)$ and $\psi = \Box \diamond \top \rightarrow \Box \perp$, then

$$\mu \not\vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi) \ \ \text{(or, equivalently, } \phi \not\models_{gf} \psi),$$

but

$$\mu^+ \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi).$$

The proof that $\phi \not\models_{gf} \psi$ is given in [3]. Let us prove now that $\mu^+ \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi)$. The proof is model-theoretic: we suppose that a general frame $(W, R, \mathcal{W})$ satisfying μ^+ is given in which ψ is not valid, and then prove that such a general frame invalidates ϕ too.

If ψ is not valid, then there exists a world x in the general frame such that $\Box\Diamond\top$ is true at x and $\exists y\ xRy$. There are two possible cases:

- all y such that xRy are reflexive;
- there exists y such that xRy and y is not reflexive.

Let us consider the first case. We know that there exists $X = \{y : xRy\}$, together with its relative complement $-X$. Let us take a valuation $\models$ such that $\{w \in W : w \models \neg P)\} = X$. Consider a world y such that xRy. For all z such that yRz, either $z \in -X$, or xRz and z is reflexive; in both cases $z \models \Box P \rightarrow P$. Thus, $y \models \Box(\Box P \rightarrow P)$, but $y \not\models P$.

Consider now the second case. Let $\models$ be such that $\{w \in W : w \models P\} = \{z : yRz\}$. Since y is not reflexive, again $y \not\models P$; moreover, for all z such that yRz, $z \models P$, and then $y \models \Box(\Box P \rightarrow P)$.

In both cases, there exists at least one y such that xRy and $y \not\models \Box(\Box P \rightarrow P) \rightarrow P$. Thus,

$$x \not\models \Box\Diamond\top \rightarrow \Box(\Box(\Box P \rightarrow P) \rightarrow P).$$

We can actually prove a slightly stronger result, namely, that μ^+ plus $\overline{ST}(\phi)$ is complete with respect to the class of frames for ϕ.

Proposition 7 For any formula θ,

$$\phi \models_f \theta \Longrightarrow \mu^+ \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\theta)$$

Proof Consider a formula ψ as in the proof of Proposition 6. We already proved that $\mu^+ \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi)$. Therefore, in order to prove that $\phi \models_f \theta$ implies $\mu^+ \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\theta)$, it suffices to show that $\mu \vdash \overline{ST}(\psi) \rightarrow \overline{ST}(\theta)$. Since μ^+ is an extension of μ, the thesis indeed follows from $\mu^+ \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi)$.

Let $(W, R, \mathcal{W})$ be a general frame satisfying ψ. Since ψ does not involve any propositional variable, the underline frame (W, R) validates ψ too. Since $\psi \models_f \phi$, (W, R) validates ϕ and, from $\phi \models_f \theta$, it follows that it validates θ. This implies that the general frame $(W, R, \mathcal{W})$ validates θ.

4.2 A modified translation for K_s

In this section, we show how to change the translation to capture derivability in K_s. To mimic what is done in passing from frames to general frames, we modify the translation by introducing a new variable y denoting a *set of possible valuations* (closed under the set-theoretic counterparts of standard modal operations). Formally, let $Cl(y, x)$ be the conjunction of the following formulae:

- $\forall z\ (z \in y \rightarrow Pow(z) \cap x \in y)$;
- $\forall z\ (z \in y \rightarrow x \setminus z \in y)$;

- $\forall zu\ (z \in y \wedge u \in y \rightarrow z \cup u \in y)$.

We also modify the translation of $\Box$:

$$(\Box\phi)^* = Pow(\phi^*) \cap x.$$

We prove that, for all modal formulae ϕ, ψ, the following holds:

$$\phi \vdash_{K_s} \psi$$

$$\Longleftrightarrow$$

$$\Omega \vdash \forall xy\ (Trans(x) \wedge Cl(y,x) \wedge \forall x_1 \ldots x_m \in y\ (x \subseteq \phi^*(x, x_1, \ldots, x_m)) \rightarrow$$
$$\forall x_1 \ldots x_n \in y\ (x \subseteq \psi^*(x, x_1, \ldots, x_n))),$$

where for any formula α, $\forall x_1 \ldots x_m \in y\ (\alpha)$ stands for $\forall x_1 \ldots x_m\ (x_1 \in y \wedge \ldots \wedge x_m \in y \rightarrow \alpha)$.

The proof of completeness is similar to the proof of completeness for frame-complete logics given in [11].

Theorem 8 (Completeness of the translation method) For each pair of formulae ϕ, ψ,

$$\phi \vdash_{K_s} \psi \Rightarrow \Omega \vdash \forall xy\ (Trans(x) \wedge Cl(y,x) \wedge \forall x_1 \ldots x_m \in y\ (x \subseteq \phi^*(x, x_1, \ldots, x_m)) \rightarrow$$
$$\forall x_1 \ldots x_n \in y\ (x \subseteq \psi^*(x, x_1, \ldots, x_n))).$$

Proof The proof is by induction on the derivation $\phi \vdash_{K_s} \psi$. We skip the verification of closure under modus ponens, tautologies, necessitation rule, and K_s-axiom. We only prove the case of closure under substitution rule.

Let $\psi(P_1, ..., P_n)$ such that

$$\Omega \vdash \forall xy\ (Trans(x) \wedge Cl(y,x) \wedge \forall x_1 \ldots x_m \in y\ (x \subseteq \phi^*(x, x_1, \ldots, x_m)) \rightarrow$$
$$\forall x_1 \ldots x_n \in y\ (x \subseteq \psi^*(x, x_1 \ldots x_n))).$$

and let $\tau_1(P_1, .., P_k), ..., \tau_n(P_1, .., P_k)$ be modal formulae. We want to prove that

$$\Omega \vdash \forall xy\ (Trans(x) \wedge Cl(y,x) \wedge \forall x_1 \ldots x_m \in y\ (x \subseteq \phi^*(x, x_1, \ldots, x_m)) \rightarrow$$
$$\forall x_1 \ldots x_k \in y\ (x \subseteq \psi(\tau_1, \ldots, \tau_n)^*(x, x_1, \ldots, x_k))).$$

This is easily shown because:

(i) For any modal formula $\sigma(P_1, ..., P_k)$ it holds that

$$\Omega \vdash Cl(y,x) \rightarrow \forall x_1 \ldots x_k \in y\ (\sigma^*(x_1, \ldots, x_k) \in y);$$

(ii) $\psi(\tau_1, \ldots, \tau_n)^*$ is the same (syntactically) as $\psi^*(\tau_1^*|x_1, \ldots, \tau_n^*|x_n)$ (both are easily proved by induction).

By simultaneous substitution, it holds that

$$\Omega \vdash \forall x_1 \ldots x_n\ (x_1 \in y \wedge \ldots \wedge x_n \in y \rightarrow x \subseteq \psi^*(x, x_1, \ldots, x_n)) \rightarrow$$

$$\forall x_1 \ldots x_k \; (\tau_1^* \in y \wedge \ldots \wedge \tau_n^* \in y \rightarrow x \subseteq \psi^*(x, \tau_1^*, \ldots, \tau_n^*)),$$

By applying *(i)* to terms $\tau_1^*, .., \tau_n^*$, it follows that

$$\Omega \vdash Cl(y, x) \wedge \forall x_1 \ldots x_n \in y \; (x \subseteq \psi^*(x, x_1, \ldots, x_n)) \rightarrow$$

$$\forall x_1 \ldots x_k \in y \; (x \subseteq \psi^*(x, \tau_1^*, \ldots, \tau_n^*)),$$

and from *(ii)* the desired result follows.

Let us prove now the soundness of the translation method. First, we show how an Ω-model can be obtained from $(W, R, \mathcal{W})$. Without loss of generality, we suppose that the elements of W are non-empty sets of the same rank α.

Definition 9 The (Ω-)model $\mathbf{U} = (U, \in^{\mathbf{U}})$ generated by $(W, R, \mathcal{W})$ is defined as follows.

Let $U_0 = W$, $U_n = Pow(U_{n-1}) \cup U_{n-1}$, and $U = \bigcup_{n=0}^{\omega} U_n$.

Define a function $F : U \rightarrow U$ as

$$F(x) \quad = \quad \begin{cases} \{v \in W : xRv\} & \text{if } \; x \in W, \\ x & \text{otherwise,} \end{cases}$$

and let

$$x \in^{\mathbf{U}} y \text{ if and only if } \; x \in F(y).$$

First notice that, if $y \in W$, then we have:

$$x \in^{\mathbf{U}} y \text{ if and only if } \; x \in W \wedge yRx.$$

Moreover, by the hypothesis on the rank of the elements of W, it is immediate to see that the range of F is the set $U \setminus W$. This set is closed under union and difference, and F is the identity on $U \setminus W$. Therefore, we can define

$$x \cup^{\mathbf{U}} y = F(x) \cup F(y) \qquad x \setminus^{\mathbf{U}} y = F(x) \setminus F(y),$$

and the axioms for $\cup$ and $\setminus$ will be satisfied. Finally, the set $\{y \in U : F(y) \subseteq F(x)\}$ is in $U \setminus W$ too, and we can safely define

$$Pow^{\mathbf{U}}(x) = \{y \in U : F(y) \subseteq F(x)\}.$$

Theorem 10 (Soundness of the translation method) For each pair of formulae ϕ, ψ,

$$\Omega \vdash \forall x \forall y \; (Trans(x) \wedge Cl(y, x) \wedge \forall x_1 \ldots x_m \in y \; (x \subseteq \phi^*(x, x_1, \ldots, x_m)) \rightarrow$$

$$\forall x_1 \ldots x_n \in y \; (x \subseteq \psi^*(x, x_1, \ldots, x_n))) \Rightarrow \phi \vdash_{K_s} \psi$$

Proof We prove that $\phi \vdash_{K_s} \psi$ by showing that, for any general frame $(W, R, \mathcal{W})$, if ϕ is valid in $(W, R, \mathcal{W})$, then ψ is valid as well.

Let $\mathbf{U} = (U, \in^{\mathbf{U}})$ be the Ω-model generated by $(W, R, \mathcal{W})$ (cf. Definition 9). As in Theorem 5, it is easy to prove that $\mathbf{U} \models Trans(W)$. Moreover, it is straightforward to show that $Closure(\mathcal{W}, W)$ holds in $\mathbf{U}$: closure under union and complementation with respect to W follows from the corresponding closure properties of $\mathcal{W}$, while, for all $x \in \mathcal{W}$,

$$Pow^{\mathbf{U}}(x) \cap W = \{y \in W : F(y) \subseteq F(x)\} =$$
$$= \{y \in W : \{v \in W : yRv\} \subseteq x\} = \Box(x) \in \mathcal{W}.$$

It is now easy to show that, for all $y_1, ..., y_n$ in $\mathcal{W}$ and for any modal formula $\theta(P_1, \ldots, P_n)$,

$$\mathbf{U} \models W \subseteq \theta^*(W, y_1, ..., y_n)) \text{ if and only if } \theta(P_1, .., P_n) \text{ holds in } (W, R),$$

where $\{w \in W : w \models P_i\} = y_i$, for all $i = 1, ..., n$.

Using this correspondence, one easily shows that if ϕ is valid in $(W, R, \mathcal{W})$, then ψ is valid as well.

5 Generalizations of the method

There are two possible ways for generalizing the previous results: we can either extend the modal language, without changing the deductive engine, or consider more powerful modal deductive systems. In this section, we give examples of both such generalizations, discussing the case of temporal languages (minimal tense logic) and the case of weak second-order logic.

5.1 Translating temporal languages

Let us consider the case of minimal tense logic TL [8]. TL extends propositional logic with two one-place operators: F (future-tense) and P (past-tense). A sound and complete axiomatic system for TL consists of the usual axioms and rules of propositional logic plus the axioms:

$$G(p \to q) \to (Gp \to Gq) \qquad H(p \to q) \to (Hp \to Hq)$$
$$p \to GPp \qquad p \to HFp$$

where G and H stand for $\neg F \neg$ and $\neg P \neg$, respectively, and the rules of (temporal) necessitation: from ϕ to infer $G\phi$ and $H\phi$.

As shown in [20], it is possible to reduce frame validity in tense logic to that in modal logic. Even though somewhat complicated, the composition of such a reduction and our translation method for modal logic would allow us to encode minimal tense logic into Ω as it stands.

As an alternative, a specific translation method for TL can be devised that slightly extends Ω, but keeps the translation simpler. In view of the one-way character of $\in$, the main issue to consider is the treatment of the *backward looking* operator P. To this end, we extend the language of Ω with a binary operator $Pow^{-y}(x)$, and add the following axiom (defining it) to Ω:

$$x \in Pow^{-y}(z) \quad \text{if and only if} \quad x \in y \wedge]x, y[\subseteq z$$

where $]x, y[\subseteq z$ is a shorthand for the formula $\forall t \ (x \in t \wedge t \in y \to t \in z)$. Let us call the resulting theory Ω_{TL}. Notice that the additional axiom can be expressed via a particular instance of the *comprehension* schema.

The translation of propositional letters and boolean connectives is the usual one. The translation of G is the same as the translation of $\Box$ in basic modal logic. The translation of H can be defined in terms of the $Pow^{-y}(x)$ operator as follows:

$$(H\phi)^* \quad = \quad Pow^{-W}(\phi^*).$$

As an example, consider the translation of the formula HP_i, which is the term $Pow^{-W}(x_i)$. By definition, $x \in Pow^{-W}(x_i)$ if and only if $x \in W \wedge \forall t(x \in t \wedge t \in W \to t \in x_i)$, and this is the intended meaning of HP_i when tRx corresponds to $x \in t$.

More generally, we have that

$$w \models H\phi \Leftrightarrow \forall t \in W\ (tRw \to t \models \phi)$$

corresponds to

$$w \in Pow^{-W}(\phi^*) \Leftrightarrow w \in W \wedge\ ]w, W[\ \subseteq \phi^* \Leftrightarrow w \in W \wedge \forall t\ (w \in t \wedge t \in W \to t \in \phi^*).$$

It is worth noting that, unlike the case of the operator G, we do not need to require any "(inverse) transitivity" for the operator H, because if $W \subseteq \phi^*$, then we always have $W \subseteq Pow^{-W}(\phi^*)$.

Proposition 11 For any TL-formula $\phi(P_1, \ldots, P_n)$, the following holds:

$$\vdash_{TL} \phi \ \Leftrightarrow\ \Omega_{TL} \vdash \forall x(Trans(x) \to \forall x_1 \ldots x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))).$$

5.2 Strengthening of minimal modal deduction

One can consider various principled strengthenings of minimal modal deduction. In [3], axiomatic theories for deduction in the L_2-language are introduced. In particular, a system of weak second-order logic is defined by means of a suitable form of substitution. Let an L_0-formula be an L_2-formula with a free variable x but without occurrences of second-order quantifiers. Denote by $\alpha(\gamma|\mathtt{P})$ the L_2-formula obtained from the L_2-formula α by replacing subformulae of the form $\mathtt{P}(u)$ by an L_0-formula $\gamma(u|x)$ (modulo some technicalities about free variables [3]). L_0-substitution is expressed by the axiom

$$\forall \mathtt{P}\alpha \to \alpha(\gamma|P).$$

Weak second-order logic contains a set of axioms complete for first-order predicate logic, plus L_0-substitution. Let us denote derivability in weak second-order logic with $\vdash_{L_2}$. It is a non-conservative extension of K_s, namely, $\phi \vdash_{K_s} \psi \Rightarrow \overline{ST}(\phi) \vdash_{L_2} \overline{ST}(\psi)$, but there exist ϕ and ψ such that $\overline{ST}(\phi) \vdash_{L_2} \overline{ST}(\psi)$ but $\phi \not\vdash_{K_s} \psi$ [3]. Weak second-order logic turns out to be rather expressive; in particular, it allows one to derive the first-order correspondents of Sahlqvist formulae.

The semantics of weak second-order logic employs a straightforward modification of general frames. A general frame is said to be *closed under L_0-definitions* if, given an L_0-formula φ, possibly containing individual constants for some specific worlds, the set of worlds satisfying φ is a possible value for a proposition letter. L_2-derivable formulae are exactly those valid in all general frames closed under L_0-definitions ([3]).

In [5], we showed how to match derivability of modal formulae in weak second-order logic with derivability in a suitably modified version Ω_c of Ω. Ω_c is obtained from Ω, essentially, by requiring closure under the well-known eight operations originally introduced by Gödel to characterize the constructible universe (singleton, cartesian product, domain, range, etc.) [2, 13]. The relevant axioms impose closure conditions on the models of Ω_c guaranteeing sufficient richness to represent the semantics of weak second-order logic. In particular, in any model of Ω_c, any general frame closed under L_0-definitions has a set-theoretic counterpart.

This result, besides demonstrating the modal equivalence between Ω_c and weak second-order logic, also provides an alternative solution to the problem of generalizing the method to more expressive languages and more powerful modal deductive systems. As an example, in the case of minimal tense logic one can look for the proper subset of the operations of constructible set theory corresponding to the new operator $Pow^{-y}(x)$. This would allow us to single out the set theory for minimal tense logic somewhere between Ω and Ω_c. In [6], we showed that similar generalizations of the proposed translation method can be devised to obtain a set-theoretic characterization of other extensions of basic modal logic such as the Gabbay's extension of K with the irreflexivity rule [12] and the difference logic [19].

Conclusions and further directions

The main purpose of this paper has been to build bridges between modal deduction and set-theoretic reasoning. Let us collect some points concerning this enterprise already mentioned in our text. Our program may be extended in several ways, by systematic variation in several 'parameters' of this linkage. For instance, given a fixed translation, one can run through known modal logics extending the minimal one, and see what set theories these involve - and the same may be done in the opposite direction. In the same spirit, one could analyze the set-theoretic strength of currently popular 'additional modal rules of inference'. But also, one can extend the language of the basic modal formalism, all the way up to the full classical logic of frames, as we have seen, obtaining richer reductions. Finally, there is also a constructive aspect. Our semantical arguments can probably be reworked so as to yield effective purely combinatorial transformations between formal derivations in modal logic and our set theory. This will allow a more finely structured comparison of reasoning in both perspectives.

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